Physical and Human Capital over the Business Cycle

Accolley, Delali

Universite Laval

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Delali Accolley
accolleyd@aim.ca
Université Laval

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Abstract

The available disaggregated capital data are across industries. What one needs *inter alia* when calibrating multi-sector neoclassical growth models, are not industries' capital endowments but the ones used in producing commodities, particularly consumption and investment goods. To fill this gap, following the existing literature on capital measurement and input-output analysis, we have sequentially produced these estimates for the US economy over the period 1998-2007.

We have then used our estimates to calibrate and solve numerically a three-sector optimal growth model of physical and human capital accumulation. Using the right capital shares and stocks has improved the ability of the three-sector model to explain business cycle fluctuations.

JEL classification: E01, E10, E32, E37
Keywords: Capital measurement, Macroeconomics, Business Cycles

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1 Introduction

Since Uzawa (1961, 1963) and subsequent research including Drandakis (1963), Hahn (1965) and Kuo (1977), to explain growth and business cycles, the profession, more and more, turns to two-sector neoclassical models of capital accumulation that instead attribute the production of the final good to two different sectors: the consumption and investment sectors. Each of these sectors combines its own capital and labor inputs to produce its output. In addition to handling difference in input intensities, these models take into account the decline observed, over the long-run, in the price of investment goods relative to the price of consumption goods and the rise in the investment per capita. Over the business cycle, unlike standard one-sector models, they generate fluctuations that are as persistent as those observed (Cogley and Nason, 1995; Benhabib, Perli, and Sakellaris, 2005). A reason for that is they are endowed with more driving forces and propagation mechanisms.

Calibrating and solving numerically two-sector growth models in order to assess their ability to replicate key business cycle facts necessitate *inter alia* data on the physical capital stocks used in the consumption and investment sectors. But, the available disaggregated capital stock data, particularly those on the United States (US) economy, released by the Commerce Department's Bureau of Economic Analysis (BEA), are across industries. What one really needs are not industries' capital endowments but their shares of capital used in producing both consumption and investment goods.

We have developed and implemented techniques enabling the sequential conversion of the existing industry annual data into capital stocks disaggregated across the consumption and investment sectors. These techniques are based on an existing result in the input-output analysis literature: the transformation of industries' capital shares of income (henceforth: capital shares), computed from "use tables", into commodity capital shares (Basu, Fernald, Fisher, and Kimball, 2010; Ten Raa, 2007). We have, first, proven that the just mentioned relationship holds under the assumption that firms within all industries operate Cobb-Douglas technologies exhibiting constant returns to scale. Then, using some aggregation rules provided by Diewert (1980) and Hulten (1991), we have established that the capital stock used in producing a given consumption or investment good is a share-weighted sum of industries' capital endowments.

We have used the estimates we produced to calibrate a three-sector neoclassical growth model of physical and *human capital* accumulation. The sectors the model comprises are: the consumption, the investment, and the human capital sectors. In addition to a neutral technological change that indistinctly boosts productivity across the economy, the investment sector experiences specifically a technological change that lowers the price of its output. We build on the model used by DeJong and Ingram (2001) allowing‘ the capital share to vary across the economy and long-run growth to occur. Long-run growth is driven exogenously by the technological changes and endogenously by human capital accumulation. Human capital, which is the ability to perform labor, is accumulated by allocating time to education. We have modeled it following Lucas Jr (1988). Introducing human capital significantly improves growth models' ability to explain the short-run dynamics of the labor market (Einarsson and

Marquis, 1997; Accolley, 2016). Einarsson and Marquis (1997) also showed that this helps explain the correlation observed, over the business cycle, between investments in physical capital across different sectors. The model that they augmented with a human capital accumulation sector was put forward by Greenwood and Hercowitz (1991) and Benhabib, Rogerson, and Wright (1991) and comprised a market and a household production sectors.

The rest of this paper is organized as follows. In Section 2, we have set up the analytical framework for estimating the capital stocks used in producing commodities and established some results. In Section 3, we have produced, out of industries data, commodities' capital shares and nominal stocks. The dataset used are: BEA's commodity "make and use tables", and net private fixed assets by industry over 1998-2007. Two stylized facts emerged. First, the distribution of capital shares across commodities is stable over time. Second, the consumption sector is more capital intensive than the investment sector. The first stylized fact shed light on one of Kaldor (1957)'s findings while the second one has implications for business cycle models' ability to properly replicate the short-run dynamics of consumption and investment. Section 4 sketches the three-sector growth model that we have calibrated using the estimates we produced. It results from simulating this model that using the right capital shares and stocks improves the ability of the three-sector model to explain business cycle fluctuations. Particularly, the cyclical fluctuations in the aggregate output, consumption, and total hours worked the model generated are close to observations. It also appeared that because the consumption sector is capital intensive, labor in that sector is negatively correlated with the business cycle. Education and human capital are are also negatively related with the business cycle. Finally, Section 5 concludes suggesting the introduction of adjustment cost in the investment sector in order to reduce volatility in its output.

2 The Framework for Estimating Commodities' Physical Capital

The economy produces q commodities labeled $Q_1, \ldots, Q_q$ using, in addition to the capital and labor inputs, commodities produced in various industries. Firms within an industry $i = 1, \ldots q$, besides producing the good or service i the industry is named after, possibly produce other commodities $j \neq i$ as Table 2.1, the *make table*, illustrates. A make table indicates the nominal amount $p_j Y_{ij}$ of a commodity j produced by all firms operating within industry i. The sum of its row elements, pY_i, is the total amount of goods and services produced within industry i whereas the sum of its column elements $p_j Q_j$ is the total production of commodity j. The variable p_j in Table 2.1 denotes the price of commodity j and p is the general price index. From the make table, let's define the $(q \times q)$ matrix $\mathbf{M}$ whose elements $m_{ij} = \frac{p_j Y_{ij}}{pY_i}$ are the shares of commodity j in industry i's total production.

The final uses of the commodities, *viz.* consumption, investment, government purchases, exports, and imports, as well as their uses as intermediate inputs are described in the *use table*. Table 2.2 depicts a use table. The variables r_i and w_i in Table 2.2 des-

Table 2.1: A Make Table

Com / Ind	1	...	q	Scrap	Inventory	Total Ind
1	p_1Y_{11}	...	p_qY_{1q}	$p_{q+1}Y_{1q+1}$	pY_{1q+2}	pY_1
...	...	$\ddots$	...	...	...	...
q	p_1Y_{q1}	...	p_qY_{qq}	$p_{q+1}Y_{qq+1}$	pY_{qq+2}	pY_q
Total Com	p_1Q_1	...	p_qQ_q	$p_{q+1}Q_{q+1}$	pQ_{q+2}	pQ

Table 2.2: A Use Table

Ind / Com	1	...	q	C	I	...	Total Com
1	p_1Q_{11}	...	p_1Q_{1q}	p_1C_1	p_1I_1	...	p_1Q_1
...	...	$\ddots$	...	...	...	$\ddots$	...
q	p_qQ_{q1}	...	p_qQ_{qq}	p_qC_q	p_qI_q	...	p_qQ_q
...	...	$\ddots$	...	...	...	$\ddots$	...
K	$r_1\tilde{K}_1$	...	$r_q\tilde{K}_q$				
L	$w_1\tilde{L}_1$	...	$w_q\tilde{L}_q$				
Total Ind	pY_1	...	pY_q	pC	pI	...	pQ

ignate respectively the nominal rental price of physical capital and wage while $\tilde{K}_i$ and $\tilde{L}_i$ are the physical capital and labor inputs used in industry i. From the use table, the matrix of *technical coefficients* is defined. These coefficients are the proportions of the intermediate commodities used, the wage bill, and capital remuneration in the overall production of industry i. Let's consider the $(1 \times q)$ vector $\tilde{\boldsymbol{\alpha}} = \left\{\frac{r_i\tilde{K}_i}{pY_i}\right\}$ extracted from this latter matrix. The vector $\tilde{\boldsymbol{\alpha}}$ contains the *capital shares* across industries. We have defined $r_i\tilde{K}_i$, the capital income, as the sum of the gross operating surplus $\tilde{S}_i$ and the amount of the ambiguous incomes apportioned to the capital input. These ambiguous incomes are the taxes on production and imports less subsidies $\tilde{T}_i$. Following Cooley and Prescott (1995) and Gomme and Rupert (2007), the row vector $\tilde{\boldsymbol{\alpha}}$ is equivalent to $\left\{\frac{\tilde{S}_i}{pY_i-\tilde{T}_i}\right\}$.

The main purpose of our project is to estimate the stocks of physical capital used to produce private personal consumption and fixed investment goods. This requires data on capital stock disaggregated according to commodities. But the available disaggregated data on capital stock are industry ones. Both the matrix $\mathbf{M}$ obtained from the make table and the row vector $\tilde{\boldsymbol{\alpha}}$ computed from the use table help convert such data into commodity data . It is assumed, for this purpose, that firms within industries are all endowed with a Cobb-Douglas technology.

Assumption 2.1 (Cobb–Douglas technology). *The underlying production technology in each industry is Cobb-Douglas and exhibits constant returns to scale.*

Ten Raa (2006, pp. 105-6) demonstrated how a Cobb-Douglas technology can stem from the input-output analysis.

Let's now define K_{ji} the nominal amount of capital used in producing commodity j by firms in industry i, and α_j, the capital share in commodity j's sector.

Proposition 2.2 (Capital shares). *Under the Cobb-Douglas technology assumption, the capital share in commodity j's producing sector is related to industry ones as follows:*

$$\boldsymbol{\alpha} = \tilde{\boldsymbol{\alpha}}\mathbf{M}'^{-1}.$$

Proof. The technology underlying the production of commodity j by firms in industry i is:

$$p_j Y_{ij} = A_j K_{ji}^{\alpha_j} L_{ji}^{\beta_j} E_{ji}^{\gamma_j} M_{ji}^{1-\alpha_j-\beta_j-\gamma_j}, \qquad 0 < \alpha_j, \beta_j, \gamma_j < 1$$

where L_{ji}, E_{ji} and M_{ji} are respectively the labor, energy, and material inputs, and β_j, γ_j and $1-\alpha_j-\beta_j-\gamma_j$ their respective shares in the production of commodity j. The parameter A_j is the Hicks neutral technology parameter (also known as total factor productivity) specific to commodity j's sector. A necessary condition for the firms' profit maximization problem is:

$$K_{ji}: \qquad \alpha_j p_j Y_{ij} = r_i K_{ji}$$

Summing the above relation over j and dividing by the industry's output, pY_i, yields the desired result

$$\sum_{j=1}^{q} \alpha_j \frac{p_j Y_{ij}}{pY_i} = r_i \frac{\sum_{j=1}^{q} K_{ji}}{pY_i}$$

$$\sum_{j=1}^{q} \alpha_j m_{ij} = \tilde{\alpha}_i \Longrightarrow \boldsymbol{\alpha}\mathbf{M}' = \tilde{\boldsymbol{\alpha}} \quad \text{for } i = 1, \ldots, q.$$

□

Basu, Fernald, Fisher, and Kimball (2010, pp. 24-6) and Ten Raa (2007, p. 456) made a claim similar to Proposition 2.2.

Corollary 2.3 (Disaggregated capital stocks). *Under the Cobb-Douglas technology assumption, the aggregate stock of capital used in producing a given commodity is the sum of industries' capital endowments weighted by their respective capital shares attributed to that commodity.*

$$K_j = \alpha_j \sum_{i=1}^{q} \frac{m_{ij}}{\tilde{\alpha}_i} \tilde{K}_i \tag{2.1}$$

Proof. From the aforementioned first-order condition, one has:

$$\frac{K_{1i}}{\alpha_1 p_1 Y_{i1}} = \cdots = \frac{K_{qi}}{\alpha_q p_q Y_{iq}} = \frac{1}{r_i}.$$

Moreover, $\tilde{K}_i = \sum_{j=1}^{q} K_{ji}$. Solving this system of linear equations yields:

$$\begin{aligned} K_{ji} &= \frac{\alpha_j p_j Y_{ij}}{\sum_{j=1}^{q} \alpha_j p_j Y_{ij}} \tilde{K}_i \\ &= \frac{\alpha_j m_{ij}}{\tilde{\alpha}_i} \tilde{K}_i. \end{aligned}$$

It then turns out that:

$$K_j = \sum_{i=1}^{q} K_{ji} = \alpha_j \sum_{i=1}^{q} \frac{m_{ij}}{\tilde{\alpha}_i} \tilde{K}_i$$

□

The aggregate production of commodity j is:

$$Q_j = \sum_{i}^{q} F\left(K_{ji}, L_{ji}, E_{ji}, M_{ji}\right) = \sum_{i}^{q} A_j K_{ji}^{\alpha_j} L_{ji}^{\beta_j} E_{ji}^{\gamma_j} M_{ji}^{1-\alpha_j-\beta_j-\gamma_j}.$$

All firms producing commodity j are competitive profit maximizers and end up facing the same price. The aggregate production specified above can then be rewritten as if the commodity were produced by a single firm (see Diewert, 1980, pp. 464-5):

$$Q_j = F\left(K_j, L_j, E_j, M_j\right) = A_j K_j^{\alpha_j} L_j^{\beta_j} E_j^{\gamma_j} M_j^{1-\alpha_j-\beta_j-\gamma_j}$$

Since the above production function exhibits constant returns to scale or, in other words, is homogeneous of degree one, for any $d_j > 0$, one has

$$d_j Q_j = F\left(d_j K_j, d_j L_j, d_j E_j, d_j M_j\right). \tag{2.2}$$

Let's define $d_{jC} = \frac{C_j}{Q_j}$ and $d_{jI} = \frac{I_j}{Q_j}$, where C_j and I_j respectively denote the amounts of commodity j used as consumption and investment goods. For $d_j = \{d_{jC}, d_{jI}\}$, it transpires from (2.2) that

$$\begin{aligned} C_j &= A_j K_{jC}^{\alpha_j} L_{jC}^{\beta_j} E_{jC}^{\gamma_j} M_{jC}^{1-\alpha_j-\beta_j-\gamma_j} \\ I_j &= A_j K_{jI}^{\alpha_j} L_{jI}^{\beta_j} E_{jI}^{\gamma_j} M_{jI}^{1-\alpha_j-\beta_j-\gamma_j}, \end{aligned}$$

where *inter alia*

$$\begin{aligned} K_{jC} &= d_{jC} K_j \\ K_{jI} &= d_{jI} K_j. \end{aligned}$$

It follows from the above relations that K_C and K_I, the total capital stock the economy allocates to the production of consumption and investment goods, are:

$$\begin{aligned} K_C &= \sum_{j=1}^{q} d_{jC} K_j \\ K_I &= \sum_{j=1}^{q} d_{jI} K_j. \end{aligned} \tag{2.3}$$

Finally, the aggregate consumption and investment are composites of q commodities that we represent with the generalized Cobb-Douglas technologies $C = \prod C_j^{b_{jC}}$, and $I = \prod I_j^{b_{jI}}$ where $b_{jC} = \frac{p_j C_j}{pC}$ and $b_{jI} = \frac{p_j I_j}{pI}$. It follows that the respective total capital shares in the production of these two goods are:

$$\begin{aligned} \alpha_C &= \sum_{j=1}^{q} \alpha_j b_{jC} \\ \alpha_I &= \sum_{j=1}^{q} \alpha_j b_{jI}. \end{aligned} \tag{2.4}$$

3 The Estimates of Commodities' Physical Capital

We have used BEA's annual tables on the use and make of commodities as well as the current cost net stock of private fixed assets by industry in the US over the period 1998-2007 to estimate the nominal stocks of capital allocated to the production of private consumption and investment goods. The steps established in the previous section are followed to produce the estimates.

Figure 3.1 plots the distributions of the capital shares across industries and commodities over the sample period. These distributions appear to be stable from one year to the other. The capital shares of two industries/commodities: (1) the federal government enterprises and (2) the state and local government enterprises are negative. These are enterprises providing public goods or services financed out of taxes raised on the private sector.

To test for the stability over time of the capital share distributions, we have used the following econometric models

$$\begin{aligned} &\tilde{\alpha}_{it} = \tilde{a}_i + \tilde{\zeta}_t + \tilde{\varepsilon}_{it}, \\ &\alpha_{jt} = a_j + \zeta_t + \varepsilon_{jt}, \quad \tilde{\varepsilon}_{it}, \varepsilon_{it} \sim \mathcal{N}(0, \sigma^2). \end{aligned} \tag{3.1}$$

Models (3.1) postulate the observed fluctuations in a given industry's (commodity's) capital share are either aggregate $\tilde{\zeta}_t$ (ζ_t) or *idiosyncratic* $\tilde{\varepsilon}_{it}$ (ε_{it}). An idiosyncratic change is one that is specific to a given industry or commodity. It is thus uncorrelated with changes occurring elsewhere. It is normally and identically distributed with mean 0

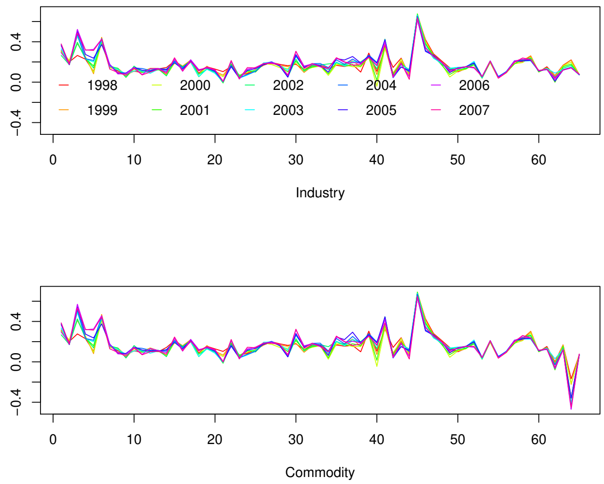

Figure 3.1: The Distributions of Capital Share across Industries and Commodities, US, 1998-2007

Table 3.1: Capital Share Equations: OLS Estimates of the Aggregate Changes, US, 1998-2007

t	Estimate of $\tilde{\zeta}_t$	t-ratio	Estimate of ζ_t	t-ratio
1999	-.0003	-.068	-.0007	-.122
2000	-.0097	-2.019	-.0097	-1.72
2001	-.0092	-1.896	-.011	-1.914
2002	-.0021	-.443	-.0035	-.624
2003	.001	.214	-.001	-.171
2004	.004	.821	.0012	.209
2005	.0035	.725	.0025	.45
2006	.0057	1.178	.0027	.482
2007	.0043	.901	.0012	.218
	$\bar{R}^2 = .981$		$\bar{R}^2 = .976$	
	$F = 2.507$		$F = 1.458$	

$t_{.025}(576)= 1.964$ $F_{.05}(9,576)=1.896$

and variance σ^2. The aggregate shifts $\tilde{\zeta}_t$ and ζ_t, $t = 1999, \ldots, 2007$ are expressed as deviations from the state of the economy in 1998, the starting period.

Table 3.1 displays the ordinary least squares (OLS) estimates of these aggregate changes. We have run each regression pooling 650 observations on capital shares and using 74 dummy variables as regressors; 65 of these dummy variables are specific to the industries (commodities). Comparing the t- and F-ratios to their critical values indicates there is no significant aggregate shift in the distribution of capital share across commodities. As far as its distribution across industries is concerned, the test results indicate a significant downward shift took place, at the aggregate level, in 2000.

The determination coefficients reported in Table 3.1 indicate models (3.1) explains about 98% of the fluctuations in capital shares. It also follows that:

$$\mathrm{E}\left(\alpha_{jt}\right) = a_j,$$

which means the best estimator of commodities' capital shares are their averages. This result sheds light on Nicholas Kaldor's stylized fact relating to the stability of the share of capital in total income (see Kaldor 1957 or Cooley and Prescott 1995, p. 3). As a matter of fact, the aggregate capital share is stable because commodities' capital shares are.

Stylized Fact 3.1 (Capital shares)**.** *The distribution of capital share across commodities is stable over time.*

Estimating relations (2.4) shows that the capital share in the consumption sector is higher than in the investment sector.

Stylized Fact 3.2 (Capital intensity)**.** *The consumption sector is more capital intensive than the investment sector.*

Figure 3.2 shows the capital shares in these two sectors. The average share of capital income in the value added by the consumption and investment sectors are respectively .439 and .308. The capital share in the former sector is higher than the one at the aggregate level, which is .406. If the capital share were the same in the two sectors, it would not be necessary distinguishing between them. The production technology of consumption and investment goods could then be aggregated into one.

Finally, the estimated nominal stocks of physical capital allocated to the consumption and investment sectors are plotted in Figure 3.3. The capital stock used in the consumption sector is about 10 times as high as the one used in the investment sector.

4 A Three-Sector Model of Physical and Human Capital Accumulation

The economy consists of three sectors: a human capital, a consumption good, and an investment good sectors. The first sector is operated by households and the two others by firms. Households produce human capital by allocating time to education.

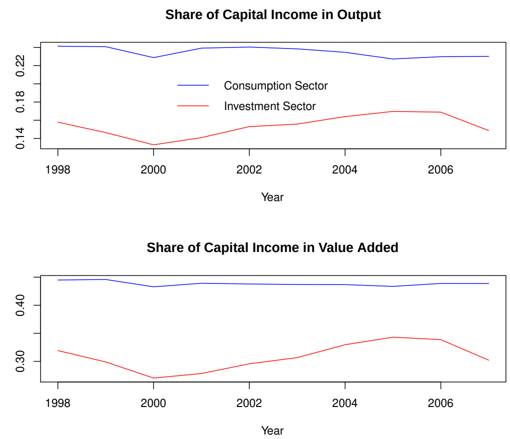

Figure 3.2: Consumption and Investment Sectors' Capital Share, US, 1998-2007

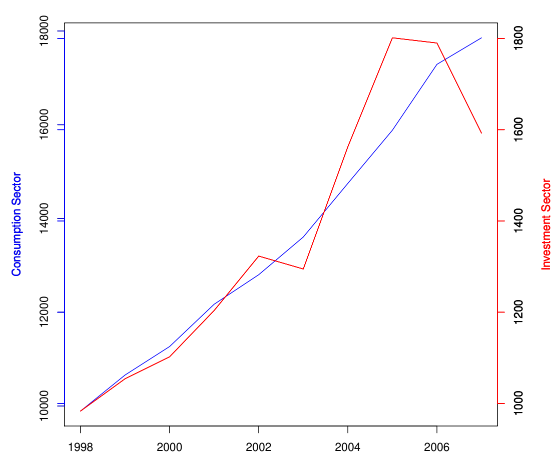

Figure 3.3: Consumption and Investment Sectors' Nominal Net Physical Capital Stocks, US, Billion US$, 1998-2007

To produce the consumption and investment goods, each sector uses its own physical capital and "effective labor". Both households and firms are rational agents seeking to maximize respectively their welfare and profit. Thus, in addition to accumulating human capital, households have to choose how to optimally allocate capital and labor across the consumption and investment sectors.

The model is similar to the one used by DeJong and Ingram (2001), a part from the fact: the capital share differs across sectors and the economy experiences long-run growth.

4.1 The Decision Making and General Equilibrium

Households are identical and infinitely-lived. The representative household is endowed with logarithmic preferences defined over consumption and leisure. His program is to maximize his lifetime utility

$$\mathrm{E}_0 \sum_{t=0}^{\infty} \beta^t \left[\ln c_t + \sigma \ln(1 - e_t - l_{ct} - l_{it}) \right], \quad 0 < \beta, < 1, \ \sigma > 0. \tag{4.1}$$

The parameters β and σ in the utility function are respectively the discount factor and the relative weight of leisure. The variables c_t, e_t, l_{ct}, and l_{it} designate respectively consumption, the shares of time allocated to education, labor in the consumption and investment sectors. Thus, $1 - e_t - l_{ct} - l_{it}$ represents the share of time allocated to leisure.

The representative household faces the following resource constraints

$$c_t = z_t k_{ct}^{\alpha_c} (h_t l_{ct})^{1-\alpha_c} \tag{4.2a}$$
$$i_t = z_t q_t k_{it}^{\alpha_i} (h_t l_{it})^{1-\alpha_i} \tag{4.2b}$$
$$k_t = k_{ct} + k_{it} \tag{4.2c}$$
$$k_{t+1} = (1-\delta) k_t + i_t \tag{4.2d}$$
$$h_{t+1} = (1 + \psi_t e_t) h_t \tag{4.2e}$$
$$\psi_t = \bar{\psi} \exp(\tilde{\psi}_t) \tag{4.2f}$$
$$\tilde{\psi}_t = \rho_\psi \tilde{\psi}_{t-1} + \epsilon_{\psi t}, \quad \epsilon_{\psi t} \sim \mathcal{N}(0, \sigma_\psi^2) \tag{4.2g}$$
$$z_t = \gamma_{zt}^t \exp(\tilde{z}_t) \tag{4.2h}$$
$$\tilde{z}_t = \rho_z \tilde{z}_{t-1} + \epsilon_{zt}, \quad \epsilon_{zt} \sim \mathcal{N}(0, \sigma_z^2) \tag{4.2i}$$
$$q_t = \gamma_q^t \exp(\tilde{q}_t) \tag{4.2j}$$
$$\tilde{q}_t = \rho_q \tilde{q}_{t-1} + \epsilon_{qt}, \quad \epsilon_{qt} \sim \mathcal{N}(0, \sigma_q^2), \ \mathrm{E}(\epsilon_{qt}\epsilon_{zt}) = \sigma_{qz}. \tag{4.2k}$$

The constraints (4.2a) and (4.2b) describe respectively how consumption c_t and investment i_t goods are produced. The technologies for producing these goods are Cobb-Douglas. Each sector uses as inputs its own physical capital, k_{ct} and k_{it} respectively, and its own labor l_{ct} and l_{it}. The representative household supplies his labor jointly with his human capital h_t. The interaction of these two variables is referred to as effective labor. Both consumption and investment sectors experience a technological change that equally raises their total factor productivity (TFP) z_t. The investment good is measured in terms of the consumption good. The variable q_t thus designates the price of the consumption good relative to the investment good. The investment sector experiences specifically a technological change that lowers the price of its output, which increases by so doing q_t. According to (4.2c), the aggregate physical capital stock k_t is the sum of the capital stocks used in the two sectors.

Relations (4.2d) through (4.2k) describe the laws of motion of the state variables k_t, h_t, z_t and q_t. It appears in (4.2d) that all physical capital stocks depreciate at the constant rate δ. This is motivated by the fact that the two sectors own the same type of physical capital. [1] Relation (4.2e) shows human capital is produced by allocating time to education. Its parameter $\psi_t > 0$ is the household's ability to learn. It is also known as human capital productivity parameter. All the productivity parameters and the relative price are subject to disturbances that are normally distributed. The parameters ρ_ψ, ρ_z, and ρ_q indicate how persistent these disturbances are. In the absence of stochastic disturbances, the TFP and the relative price q_t are respectively expected to grow at the rates γ_z and γ_q.

The following relations are derived from maximizing (4.1) subject to (4.2):

$$\frac{\sigma}{1 - e_t - l_{ct} - l_{it}} = \frac{1-\alpha_c}{l_{ct}} \tag{4.3a}$$

$$\frac{1-\alpha_c}{1-\alpha_i}\frac{k_{ct}}{k_{it}} = \frac{\alpha_c}{\alpha_i}\frac{l_{ct}}{l_{it}} \tag{4.3b}$$

$$\beta \mathrm{E}_t \left[\left(1 + \alpha_i \frac{i_{t+1}}{k_{i,t+1}} - \delta \right) \frac{k_{ct}}{k_{c,t+1}} \frac{k_{i,t+1}}{k_{it}} \frac{i_t}{i_{t+1}} \right] = 1 \tag{4.3c}$$

$$\beta \mathrm{E}_t \left\{ \left[(e_{t+1} + l_{c,t+1} + l_{i,t+1})\psi_t + \frac{\psi_t}{\psi_{t+1}} \right] \frac{l_{ct}}{l_{c,t+1}} \frac{h_t}{h_{t+1}} \right\} = 1. \tag{4.3d}$$

The relation (4.3a) is the labor supply equation. Condition (4.3b) ensures each input is remunerated at the same rate across the economy, which prevents specialization in the production of a single good. Condition (4.3c) governs the decision to invest in the consumption or investment sector while (4.3d) governs the allocation of time to education.

Definition 4.1 (Dynamic System). *The dynamic system describing the general equilibrium model consists of relations* (4.3) *resulting from the representative household's optimization problem and the constraints in* (4.2).

Along the balanced growth path (BGP), the shares of time allocated to education and labor are stationary. So is the household ability to learn. Human capital is expected to grow at the rate $\nu = 1 + \psi e$. Capital stocks and investment grow at the gross rate g. It follows from (4.2b) that

$$\mathrm{g} = (\gamma_q \gamma_z)^{\frac{1}{1-\alpha_i}} \nu. \tag{4.4}$$

[1] There is no sector-specific capital.

It also follows from (4.2a) that consumption grows at the gross rate

$$\mathrm{g}_c = \left(\gamma_q^{\alpha_c}\gamma_z^{1+\alpha_c-\alpha_i}\right)^{\frac{1}{1-\alpha_i}}\nu. \tag{4.5}$$

It turns out that the share of the neutral technological change in the consumption sector's long-run growth, $1+\alpha_c/(1-\alpha_i)$, is greater than its share in the investment sector, $1/(1-\alpha_i)$. The difference in the two growth rates depends on two factors: inflation and capital intensity. While g is a nominal rate, g_c is a real rate: for $\alpha_c = \alpha_i$, $\mathrm{g} = \gamma_q \mathrm{g}_c$. The sector with the highest capital share experiences a higher growth in real terms.

4.2 Calibration

As it turns out in the previous section, α_c and α_i respectively equal .439 and .308. The ratio k_c/k_i equals 9.982. [2] The value of the other parameters will be set using data observed over the sample period 1980-2014.

The physical capital depreciation rate— Since the model assumes this rate is constant across the economy, we have computed it using aggregate data. It has been estimated as the ratio of current period depreciation to the previous period aggregate gross stock of private fixed assets. This gives an annual average rate of 5.44% over the sample period, or a quarterly rate of 1.36%.

The neutral technological change— Since this change affects all the sectors in the same way, we have computed it doing an aggregate production function growth accounting

$$\Delta \ln z_t = \Delta \ln y_t - \alpha \Delta \ln k_t - (1-\alpha)\Delta \ln l_t.$$

The variables y_t and l_t are respectively measured by the private sector's quarterly real gross domestic product (GDP) and total hours weekly worked by all employees. The measure of k_t used is the annual chain-type index for net stock of private fixed assets. As Cooley and Prescott (1995), we have transformed the annual capital stock series into quarterly data assuming no change from one quarter to the other within a year. The average share of capital income in GDP, α, is .406. Since $z_0 = 1$, it follows from the above relation that $\ln z_t = \sum_{s=1}^{t} \Delta \ln z_s$. Fitting the relations in (4.2h) and (4.2i) with the TFP series we have thus produced gives

$$\begin{aligned} \ln z_t &= \underset{(157)}{.0033}t + \widehat{\tilde{z}}_t \\ \bar{R}^2 &= .994 \;\; t_{2.5\%}(138) = 1.95 \\ \widehat{\tilde{z}}_t &= \underset{(26.67)}{.916}\tilde{z}_{t-1} \end{aligned} \tag{4.6a}$$

[2]Instead of using the ratio k_c/k_i to calibrate the model, one can alternatively use the ratio of consumption to investment.

$$\bar{R}^2 = .837 \ \sigma_z = .008, \tag{4.6b}$$

where $\widehat{\tilde{z}}_t$ designates the estimated regression residuals. It turns out from (4.6a) that the neutral technological growth rate is significantly positive, *i.e.*, the test statistic (t-ratio) reported in parentheses is greater than the 2.5% critical value. We have therefore set γ_z to 1.003

The investment-specific technological change— We have run a log-linear regression on the inverse of the relative price of investment goods to estimate the gross growth rate γ_q, the persistence parameter ρ_q, and the standard deviation of the innovation. The OLS estimates of relations (4.2j) and (4.2k) are reported below

$$\begin{aligned}
\ln q_t &= -.294 + .0027t + \widehat{\tilde{q}}_t \\
&\quad\ (-59.5) \quad (44.5) \\
\bar{R}^2 &= .934 \ t_{2.5\%}(138) = 1.95 && (4.7a) \\
\widehat{\tilde{q}}_t &= .99\tilde{q}_{t-1} \\
&\quad (60.82) \\
\bar{R}^2 &= .964 \ \sigma_q = .006. && (4.7b)
\end{aligned}$$

The value of γ_q is therefore 1.003. The correlation between ϵ_q and ϵ_z, $\mathrm{E}(\epsilon_q\epsilon_z)$, equals .235.

The nominal long-run growth rate— We have computed the nominal investment series multiplying the real gross private fixed investment by the personal consumption implicit price. The private sector's nominal GDP has been computed in the same way. We have then estimated two log-linear models using these nominal series.

$$\begin{aligned}
\widehat{\ln i_t} &= 10.74 + .015t \\
&\quad (416.4) \quad (45.8) \\
\bar{R}^2 &= .938 \ t_{2.5\%}(138) = 1.95 && (4.8a) \\
\widehat{\ln y_t} &= 12.32 + .014t \\
&\quad (937.6) \quad (85.1) \\
\bar{R}^2 &= .981 \ t_{2.5\%}(138) = 1.95 && (4.8b)
\end{aligned}$$

The estimates of the two long-run nominal growth rates reported in (4.8) are equal. We have then set g to 1.015.

The other parameters— According to the American Time Use survey, a household allocates, on average, 9.44 hours a day to personal care. This average and the series on the average annual hours worked by workers suggest that labor represents about 32% of the time households do not allocate to personal care. Given this information, evaluating the rest of the model along the BGP gives the values reported in Table 4.1. The share of time allocated to education suggested byt the model is .029. This exactly matches the share of the discretionary time allocated to education, *viz* attending class and doing homework, by the civilian population. Finally, we set ρ_ψ and σ_ψ respectively to .95 and .001 and normalized human capital along the BGP to unity.

Table 4.1: The Parameters of the Three-Sector Model

Households	β	Discount factor	.937
	σ	Leisure weight	1.318
	ψ	Household ability to learn	.207
Consumption Sector	α_c	Capital share	.439
	δ	Physical capital depreciation rate	.014
	γ_z	TFP growth rate	1.003
	ρ_z	Persistence parameter	.916
	σ_z	Standard deviation of innovation ϵ_{zt}	.008
	g_c	Real growth rate	1.013
Investment Sector	α_i	Capital share	.308
	γ_q	Technological growth rate	1.003
	ρ_q	Persistence parameter	.99
	σ_q	Standard deviation of innovation ϵ_{qt}	.006
	$\sigma_{qz}/\sigma_q\sigma_z$	Correlation between innovation ϵ_{qt} and ϵ_{zt}	.235
	g	Nominal long-run growth	1.015

4.3 Numerical Solution and Findings

To solve numerically the system described in Definition 4.1, we have normalized all the trended variables dividing them by their expected long-run growth factors. We have then simulated the model over 140 periods, which corresponds to the length of our sample, assuming the economy simultaneously, repeatedly, and randomly experiences TFP, investment-specific, and human capital productivity changes. We have carried out hundred times these simulations using the package Dynare in Matlab. After the simulations, we have reconstructed the original time series and removed inflation from nominal variables dividing them by their implicit prices. Output is therefore defined as $y_t = c_t + i_t/q_t$. We have then extracted cyclical components from these series using the Hodrick and Prescott filter. Table 4.2 displays some summary statistics from the model and the US economies.

The fluctuations in the aggregate output generated by the model almost match those observed over the business cycle in the business sector's real GDP. The simulated standard deviation of cyclical consumption and total hours worked are also very close to observations. However the model exhibits more volatility in investment than what has been observed. Introducing adjustment cost in the investment sector may help fix this issue.

We now compare our findings to those of related works that used different calibration. In a two-sector model, Benhabib, Perli, and Sakellaris (2005) set the capital share in the consumption and investment sectors respectively to .2 and .4. The consumption series they simulated were almost twice as volatile as in the US data. In a three-sector model that comprises a consumption sector and two investment sectors, they set the capital share in the consumption sector higher, .32 precisely. The capital shares in the

Table 4.2: Cyclical Behavior of the US and the Three-Sector Economies, Percentage Deviation from Trend of Key Variables, 140 Observations

	Canadian Economy			Three-Sector Economy		
Variable	(1)	(2)	(3)	(1)	(2)	(3)
GDP (Business)	1.76	1	.87	1.72	1	.73
Consumption	1.08	.88	.87	.99	.67	.9
Investment	6.18	.93	.84	14.2	.85	.64
Relative Price	92	-.08	.84	.77	-.3	.71
Total Hours	1.75	.88	.94	1.76	.86	.64
Consumption				.37	-.36	.71
Investment				13.57	.81	.64
Education				19.12	-.88	.68
Human Capital				.31	-.25	.95

Columns (1) Percentage standard deviations, columns (2) Correlation coefficient with GD, and columns (3) First-order autocorrelation coefficient.

two investment sector were .24 and .12. The standard deviation of the simulated cyclical consumption became 1.42 as high as its true value. The model used by DeJong and Ingram (2001) is similar to ours but assumes capital share to be equal in the consumption and investment sectors and set it to .29. Fluctuations in the cyclical consumption they generated were 65% higher than observations.

Education and human capital turn out to be countercyclical, *i.e.* negatively correlated with cyclical aggregate output. Whereas labor in the investment sector is procyclical, *i.e.* positively correlated with cyclical aggregate output, labor in the consumption sector is countercyclical. As a matter of fact, the investment sector is more labor intensive than the consumption sector.

The correlation between cyclical consumption and investment in the US is .73. The model generates a positive but lower correlation (.2). The persistence of the fluctuations in consumption and aggregate output the model has generated is close to observations.

Figures 4.1, 4.2, and 4.3 compare the response of the three sectors to the three shocks. In summary, the impacts of a one-off shock to the human capital productivity is the opposite of those of the neutral and investment-specific technological change.

In the consumption sector, the immediate impacts of a one-off positive neutral technological change on consumption, physical capital, and hours are stronger than those occasioned by the investment-specific shock but they fade out faster. Whereas the former shock immediately raises both consumption and investment, the latter lowers consumption, which has become more expensive, and raises investment. Both shocks shift labor from the consumption to the investment sector because the latter sector produces the additional physical capital needed by the former to meet the increased demand. In the human capital sector, both shocks cause households to reduce the time the allocate to education. Whereas the neutral technological change reduces for good

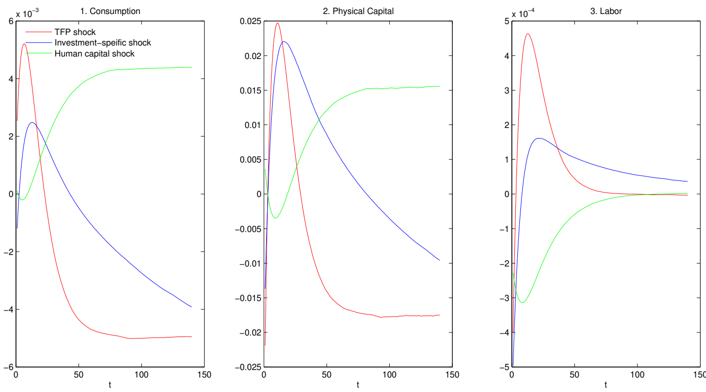

Figure 4.1: Impulse Responses, Deviation from BGP, the Three-Sector Model, Consumption Sector

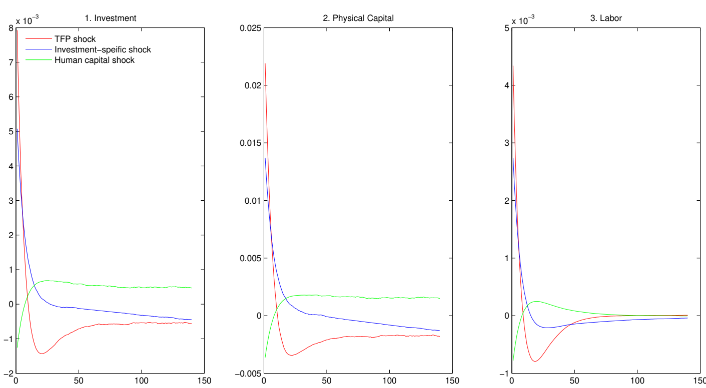

Figure 4.2: Impulse Responses, Deviation from BGP, the Three-Sector Model, Investment Sector

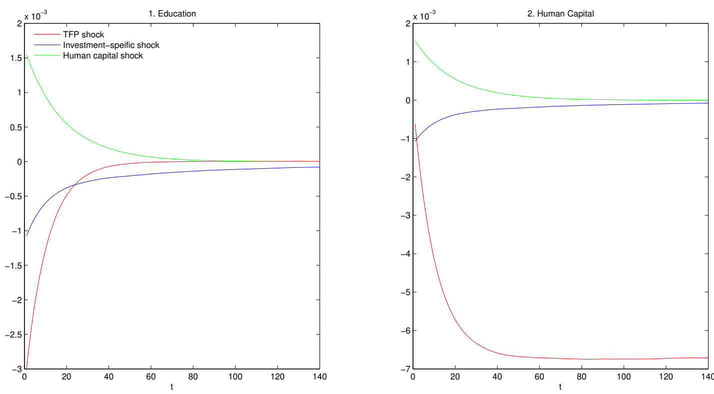

Figure 4.3: Impulse Responses, Deviation from BGP, the Three-Sector Model, Human Capital Sector

the level of human capital, when a one-off positive investment-specific shock occurs human capital slowly returns to its initial level after falling. An innovation in the human capital sector ends up having a positive effect, especially in the consumption sector.

5 Conclusion

The purpose of this paper is to produce annual estimates of the capital shares and stocks used in producing private consumption and investment goods in the US. The capital stock allocated to the production of the former turns out to be about tenfold the one used in producing the latter. These estimates are then used to calibrate a three-sector optimal growth model. The third sector of this model produces human capital. The business cycle summary statistics we have produced after simulating this model are closer to observations than those produced in related works that used different calibration. Having used the right capital shares and stocks has played an important role in that.

It has turned out that because the consumption sector is capital intensive, the labor input it uses is countercyclical. The fact is that the capital needed to produce additional consumption goods during periods of expansion are manufactured in the investment sector. Households reduce therefore the time they allocate to both education and labor in the consumption sector to increase their commitment in the investment sector. Thus, whereas investment and the labor input in that sector are pro-cyclical, the time allocated to education is countercyclical.

The volatility in investment generated by our model is larger than observation. A way to improve the model's ability to explain the fluctuations in the output of this sector would be to introduce adjustment costs, *viz* imposing a cost on changing the level of investment.

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Appendices

A The Optimization Problem

Maximizing (4.1) subject to (4.2) is equivalent to the following program

$$\begin{aligned}\max\ \mathcal{V}(\mathcal{S}_t) &= \ln c_t + \sigma \ln(1 - e_t - l_{ct} - l_{it}) + \beta \mathrm{E}_t \mathcal{V}(\mathcal{S}_{t+1}) \\ &+ \mu_{1t}\left[z_t k_{ct}^{\alpha_c}(h_t l_{ct})^{1-\alpha_c} - c_t\right] + \mu_{2t}\left[z_t q_t k_{it}^{\alpha_i}(h_t l_{it})^{1-\alpha_i} - i_t\right] \\ &+ \mu_{3t}\left[k_{ct} + k_{it} - k_t\right] + \mu_{4t}\left[(1-\delta)k_t + i_t - k_{t+1}\right] \\ &+ \mu_{5t}\left[(1 + \psi_t e_t)h_t - h_{t+1}\right], \end{aligned} \tag{A.1}$$

where $\mathcal{S}_t = (k_t, h_t, z_t, q_t)$ designates the states variables, $\mathcal{V}$ designates the value function, and μ the Lagrange multiplier (see for further details, Stokey, Lucas Jr, and Prescott, 1989; Ljungqvist and Sargent, 2004; Sargent, 2009).

The FOCs

$$c_t : \quad \frac{1}{c_t} = \mu_{1t} \tag{A.2a}$$

$$e_t : \quad \frac{\sigma}{1 - e_t - l_{ct} - l_{it}} = \mu_{5t}\psi_t h_t \tag{A.2b}$$

$${}_{ct} : \quad \frac{\sigma}{1 - e_t - l_{ct} - l_{it}} = \mu_{1t}(1 - \alpha_c)\frac{c_t}{l_{ct}} \tag{A.2c}$$

$$l_{it} : \quad \frac{\sigma}{1 - e_t - l_{ct} - l_{it}} = \mu_{2t}(1 - \alpha_i)\frac{i_t}{l_{it}} \tag{A.2d}$$

$$i_t : \quad \mu_{2t} = \mu_{4t} \tag{A.2e}$$

$$k_{ct} : \quad \alpha_c \frac{c_t}{k_{ct}}\mu_{1t} = -\mu_{3t} \tag{A.2f}$$

$$k_{it} : \quad \alpha_i \frac{i_t}{k_{it}}\mu_{2t} = -\mu_{3t} \tag{A.2g}$$

$$k_{t+1} : \quad \beta \mathrm{E}_t \frac{\partial \mathcal{V}(\mathcal{S}_{t+1})}{\partial k_{t+1}} = \mu_{4t} \tag{A.2h}$$

$$h_{t+1} : \quad \beta \mathrm{E}_t \frac{\partial \mathcal{V}(\mathcal{S}_{t+1})}{\partial h_{t+1}} = \mu_{5t} \tag{A.2i}$$

The Envelope Conditions

$$k_t : \frac{\partial \mathcal{V}(\mathcal{S}_t)}{\partial k_t} = -\mu_{3t} + \mu_{4t}(1 - \delta) \tag{A.3a}$$

$$h_t : \quad \frac{\partial \mathcal{V}(\mathcal{S}_t)}{\partial h_t} = \mu_{1t}(1 - \alpha_c)\frac{c_t}{h_t} + \mu_{2t}(1 - \alpha_i)\frac{i_t}{h_t} + \mu_{5t}(1 + \psi_t e_t) \tag{A.3b}$$

It follows that:

$$\frac{\sigma}{1-e_t-l_{ct}-l_{it}} = \frac{1-\alpha_c}{l_{ct}} \tag{A.4a}$$

$$\frac{1-\alpha_c}{1-\alpha_i}\frac{k_{ct}}{k_{it}} = \frac{\alpha_c}{\alpha_i}\frac{l_{ct}}{l_{it}} \tag{A.4b}$$

$$\beta \mathrm{E}_t\left[\left(1+\alpha_i\frac{i_{t+1}}{k_{i,t+1}}-\delta\right)\frac{k_{ct}}{k_{c,t+1}}\frac{k_{i,t+1}}{k_{it}}\frac{i_t}{i_{t+1}}\right] = 1 \tag{A.4c}$$

$$\beta \mathrm{E}_t\left\{\left[(e_{t+1}+l_{c,t+1}+l_{i,t+1})\psi_t+\frac{\psi_t}{\psi_{t+1}}\right]\frac{l_{ct}}{l_{c,t+1}}\frac{h_t}{h_{t+1}}\right\} = 1. \tag{A.4d}$$

To get (A.4a), plug (A.2a) into (A.2c). To have (A.4b), express‘μ_{2t} as a function of μ_{1t} using, on the one hand, (A.2c) and (A.2d) and, on the other hand, (A.2f) into (A.2g). For (A.4c), first update at time $t+1$ the envelope condition (A.3a) using (A.2e) and (A.2g). Then, plug the result into (A.2h), which will give

$$\beta \mathrm{E}_t\left[\left(1+\alpha_i\frac{i_t}{k_{it}}-\delta\right)\mu_{2,t+1}\right] = \mu_{2t}$$

To get rid of μ_{2t} in the above expression, you can either use (A.2d) or a combination of (A.2a), (A.2f), and (A.2g).
The Euler equation (A.4d) is obtained updating at time $t+1$ the envelope condition (A.3b) using (A.2b) though (A.2d). One then plug the result into (A.2i). One can also eliminate leisure using (A.4a).

B The Normalized DSGE Model

Let $\mathbf{x}_t$ designates a vector of variables, $\hat{\mathbf{x}}_t = \frac{\mathbf{x}_t}{\mathrm{g}^t}$, with $\mathbf{x}_t = (i_t, k_t, k_{ct}, k_{it})$. The variables $\tilde{c}_t$ and $\bar{h}_t$ are respectively defined as c_t/g_x^t and h_t/ν^t.

$$\frac{\sigma}{1-e_t-l_{ct}-l_{it}} = \frac{1-\alpha_c}{l_{ct}} \tag{B.1a}$$

$$\frac{1-\alpha_c}{1-\alpha_i}\frac{\hat{k}_{ct}}{\hat{k}_{it}} = \frac{\alpha_c}{\alpha_i}\frac{l_{ct}}{l_{it}} \tag{B.1b}$$

$$\frac{\beta}{\mathrm{g}}\mathrm{E}_t\left[\left(1+\alpha_i\frac{\hat{i}_{t+1}}{\hat{k}_{i,t+1}}-\delta\right)\frac{\hat{k}_{ct}}{\hat{k}_{c,t+1}}\frac{\hat{k}_{i,t+1}}{\hat{k}_{it}}\frac{\hat{i}_t}{\hat{i}_{t+1}}\right] = 1 \tag{B.1c}$$

$$\frac{\beta}{\nu}\mathrm{E}_t\left\{\left[(e_{t+1}+l_{c,t+1}+l_{i,t+1})\psi_t+\frac{\psi_t}{\psi_{t+1}}\right]\frac{l_{ct}}{l_{c,t+1}}\frac{\bar{h}_t}{\bar{h}_{t+1}}\right\} = 1 \tag{B.1d}$$

$$\exp(\tilde{z}_t)\hat{k}_{ct}^{\alpha_c}(\bar{h}_t l_{ct})^{1-\alpha_c} = \tilde{c}_t \tag{B.1e}$$

$$\exp(\tilde{z}_t+\tilde{q}_t)\hat{k}_{it}^{\alpha_i}(\bar{h}_t l_{it})^{1-\alpha_i} = \hat{i}_t \tag{B.1f}$$

$$\hat{k}_{ct}+\hat{k}_{it} = \hat{k}_t \tag{B.1g}$$

$$(1-\delta)\hat{k}_t+\hat{i}_t = \mathrm{g}\hat{k}_{t+1} \tag{B.1h}$$

$$(1+\psi_t e_t)\bar{h}_t = \nu \bar{h}_{t+1} \tag{B.1i}$$

$$\bar{\psi}\exp(\tilde{\psi}_t) = \psi_t \tag{B.1j}$$

$$\rho_\psi \tilde{\psi}_{t-1} + \epsilon_{\psi t} = \tilde{\psi}_t \tag{B.1k}$$

$$\rho_z \tilde{z}_{t-1} + \epsilon_{zt}, = \tilde{z}_t \tag{B.1l}$$

$$\rho_q \tilde{q}_{t-1} + \epsilon_{qt} = \tilde{q}_t \tag{B.1m}$$