Risk aversion and the dynamics of optimal liquidation strategies in illiquid markets

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Abstract

We consider the infinite-horizon optimal portfolio liquidation problem for a von Neumann-Morgenstern investor in the liquidity model of Almgren (2003). Using a stochastic control approach, we characterize the value function and the optimal strategy as classical solutions of nonlinear parabolic partial differential equations. We furthermore analyze the sensitivities of the value function and the optimal strategy with respect to the various model parameters. In particular, we find that the optimal strategy is aggressive or passive in-the-money, respectively, if and only if the utility function displays increasing or decreasing risk aversion. Surprisingly, only few further monotonicity relations exist with respect to the other parameters. We point out in particular that the speed by which the remaining asset position is sold can be decreasing in the size of the position but increasing in the liquidity price impact.

1 Introduction

A standard service of investment banks is the execution of large trades. Unlike for small trades, the liquidation of a large portfolio is a very complex task: an immediate execution is often not possible or only at a very high cost due to insufficient liquidity. Significant added value therefore lies in the experience in exercising an order in a way that minimizes execution costs for the client. Triggered by the introduction of electronic trading systems by many exchanges, automatic order execution has become an alternative to manually worked orders.

Our goal in this paper is to determine the adaptive trading strategy that maximizes the expected utility of the proceeds of an asset sale[1]. We address this question in the continuous-time liquidity model introduced by Almgren (2003) with an infinite time horizon and linear price impact (see also Bertsimas and Lo (1998), Almgren and Chriss (1999), and Almgren and Chriss (2001) for discrete-time precursors of this model). Since we consider a wide range of utility functions, we cannot hope to find closed-form solutions for the optimal trading strategies. Instead, we pursue a stochastic control approach and show that the value function and optimal control satisfy certain nonlinear parabolic partial differential equations. These PDEs can be solved numerically, thus providing a computational solution of the problem. But perhaps even more importantly, the PDE characterization facilitates a qualitative sensitivity analysis of the optimal strategy and the value function.

It turns out that the absolute risk aversion of the utility function is the key parameter that determines the optimal strategy by defining the initial condition for the PDE of the optimal strategy. The optimal strategy thus inherits monotonicity properties of the absolute risk aversion. In particular, we show that investors with increasing absolute risk aversion (IARA) should sell faster when the asset price rises than when it falls. The optimal strategy is hence "aggressive in-the-money" (AIM). On the other hand, investors with decreasing absolute risk aversion (DARA) should sell slower when asset prices rise, i.e., should pursue a strategy that is "passive in-the-money" (PIM). In general, adaptive liquidation strategies can realize higher expected utility than static liquidation strategies which do not react to asset price changes: static strategies are optimal only for investors with constant absolute risk aversion.

The preceding characterization of AIM and PIM strategies is a consequence of the more general fact that the optimal trading strategy is increasing in the absolute risk aversion of the investor. Surprisingly, however, very few monotonicity relation exists with respect to the other model parameters. For example, a larger asset

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[1]The focus on sell orders is for convenience only; our approach and symmetric statements hold for the case of buy orders.

position can lead to a reduced liquidation speed. Moreover, reducing liquidity by increasing the temporary price impact can result in an increased liquidation speed. The occurrence of the preceding anomaly, however, depends on the risk profile of the utility function, and we show that it cannot happen in the IARA case.

Our approach to the PDE characterizations of the value function and the optimal strategy deviates from the standard paradigm in control theory. Although our strategies are parameterized by the time rate of liquidation, it is the remaing asset position that plays the role of a "time" variable in the parabolic PDEs. As a consequence, the HJB equation for the value function is nonlinear in the "time" dervative. We therefore do not follow the standard approach of first solving the HJB equation and then identifying the optimal control as the corresponding maximizer or minimizer. Instead we reverse these steps. We first find that a certain transformation $\tilde{c}$ of the optimal strategy can be obtained as the unique bounded classical solution of a fully nonlinear but classical parabolic PDE. Then we show that the solution of a first-order transport equation with coefficient $\tilde{c}$ yields a smooth solution of the HJB equation. A verification theorem finally identifies this function as the value function. Our qualitative results are proved by combining probabilistic and analytic arguments.

Building on empirical investigations of the market impact of large transactions, a number of theoretical models of illiquid markets have emerged. One part of these models focuses on the underlying mechanisms for illiquidity effects, e.g., Kyle (1985) and Easley and O'Hara (1987). We follow a second line that takes the liquidity effects as given and derives optimal trading strategies within such an stylized model market. Several market models have been proposed for this purpose, e.g., Bertsimas and Lo (1998), Almgren and Chriss (2001), Almgren (2003), Obizhaeva and Wang (2006) and Alfonsi, Schied, and Schulz (2007). While the advantages and disadvantages of these models are still a topic of ongoing research, we apply the market model introduced by Almgren (2003) in this paper for the following reasons. First, it captures both the permanent and temporary price impacts of large trades, while being sufficiently simple to allow for a mathematical analysis. It has thus become the basis of several theoretical studies, e.g,. Rogers and Singh (2007), Almgren and Lorenz (2007), Carlin, Lobo, and Viswanathan (2007) and Schöneborn and Schied (2007). Second, it demonstrated reasonable properties in real world applications and serves as the basis of many optimal execution algorithms run by practitioners (see e.g., Kissell and Glantz (2003), Schack (2004), Abramowitz (2006), Simmonds (2007) and Leinweber (2007)).

Within the optimal liquidation literature, most research was directed to finding the optimal *deterministic* or *static* liquidation strategy[2]. Some real-world investors however prefer aggressive in-the-money or passive in-the-money strategies, which are provided by many sell side firms (see e.g., Kissell and Malamut (2005) and Kissell and Malamut (2006)). Only recently, academic research has started to investigate the optimization potential of aggressive in-the-money strategies in a mean-variance setting (Almgren and Lorenz (2007)). By using the expected utility maximization framework, we can explain both aggressive in-the-money and passive in-the-money strategies as being rational for investors with different absolute risk aversion profiles.

The remainder of this paper is structured as follows. In Section 2, we introduce the market model. We consider two questions in this market model: optimal liquidation (Section 3.1) and maximization of asymptotic portfolio value (Section 3.2). The solution to these two problems is presented in Section 4. All proofs are given in Section 5.

2 Market model

We consider a large investor who trades in one risky asset and the risk free asset. The investor chooses a trading strategy that we describe by the number X_t of shares held at time t. We assume that $t \mapsto X_t$ is absolutely continuous with derivative $\dot{X}_t$, i.e.,

$$X_t = x + \int_0^t \dot{X}_s \, ds. \tag{1}$$

Due to insufficient liquidity, the investor's trading rate $\dot{X}_t$ is moving the market price. We follow the linear market impact model of Almgren (2003) and assume that an incremental order of $\dot{X}_t \, dt$ shares induces a *permanent price impact* $\gamma \dot{X}_t \, dt$, which accumulates over time, and a *temporary impact* $\lambda \dot{X}_t$, which vanishes instantaneously and only effects the incremental order $\dot{X}_t$ itself. In addition to the large investor's impact, the price process P is driven by a Brownian motion with volatility σ, similar to a Bachelier model. The resulting

[2]Notable exceptions describing optimal adaptive strategies include Submaranian and Jarrow (2001), He and Mamaysky (2005), Almgren and Lorenz (2007) and Çetin and Rogers (2007).

stock price dynamics are

$$P_t = P_0 + \sigma B_t + \gamma(X_t - X_0) + \lambda \dot{X}_t \tag{2}$$

for a standard Brownian motion B starting at $B_0 = 0$ and positive constants σ (volatility), γ (permanent impact parameter), λ (temporary impact parameter), and P_0 (price at time 0).

This model is one of the standard models for dealing with the price impact of large liquidations and is the basis for optimal execution algorithms that are widely used in practice. The idealization of instantaneous recovery from the temporary impact is derived from the well-known resilience of stock prices after order placement. It approximates reality reasonably well as long as the time intervals between physical order placements are longer than a few minutes; see, e.g., Bouchaud, Gefen, Potters, and Wyart (2004), Potters and Bouchaud (2003) and Weber and Rosenow (2005) for empirical studies on resilience in order books and Obizhaeva and Wang (2006) and Alfonsi, Schied, and Schulz (2007) for corresponding market impact models. At first sight, it might seem to be a shortcoming of this model that it allows for negative asset prices. In reality, however, even very large asset positions are almost completely liquidated within days or even hours. In Section 4, we find that this is also true in our model (we find an exponentially decreasing upper bound for the optimal asset position X_t at time t). Hence for the liquidation of the largest part of the asset position, negative prices only occur with negligible probability. Moreover, on the scale we are considering, the price process is a random walk on an equidistant lattice and thus perhaps better approximated by an arithmetic rather than, e.g., a geometric Brownian motion.

We parameterize strategies with $\xi(t) := -\dot{X}(t)$ such that $X_t = X_0 - \int_0^t \xi_s \, ds$ with a progressively measurable process ξ such that $\int_0^t \xi_s^2 \, ds < \infty$ for all $t > 0$. We assume in addition that our strategies are *admissible* in the sense that the resulting position in shares, $X_t(\omega)$, is bounded uniformly in t and ω with upper and lower bounds that may depend on the choice of ξ. Economically, there is clearly no loss of generality in doing so as the total amount of shares available for any stock is always bounded, i.e., X is always a bounded process in practice. By $\mathcal{X}$ we denote the class of all admissible strategies ξ.

In the following we assume that the investor is a von-Neumann-Morgenstern investor with a utility function u with absolute risk aversion $A(R)$ that is bounded away from 0 and ∞:

$$A(R) := -\frac{u_{RR}(R)}{u_R(R)} \tag{3}$$

$$0 < \inf_{R \in \mathbb{R}} A(R) =: A_{min} \leq \sup_{R \in \mathbb{R}} A(R) =: A_{max} < \infty \tag{4}$$

Furthermore, we assume that the utility function u is sufficiently smooth (C^6). Most of the theorems that we provide are also valid under weaker smoothness conditions, but to keep things simple we only discuss the C^6-case explicitly.

3 Liquidation and optimal investment

We now define the problems of optimal liquidation and optimal investment in the illiquid market model.

3.1 Optimal liquidation

We consider a large investor who needs to sell a position of $X_0 > 0$ shares of a risky asset and already holds r units of cash. When following an admissible trading strategy ξ, the investor's total cash position is given by

$$\mathcal{R}_t(\xi) = r + \int_0^t \xi_s P_s \, ds \tag{5}$$

$$= r + P_0 X_0 - \frac{\gamma}{2} X_0^2 + \underbrace{\sigma \int_0^t X_s^\xi \, dB_s}_{\Phi_t} - \lambda \int_0^t \xi_s^2 \, ds \underbrace{- P_0 X_t^\xi - \frac{\gamma}{2}\left((X_t^\xi)^2 - 2X_0 X_t^\xi\right) - \sigma X_t^\xi B_t}_{\Psi_t}. \tag{6}$$

On the time scale we are interested in, the accumulation of interest can clearly be neglected. Since the large investor intends to sell the asset position, we expect the liquidation proceeds to converge $\mathbb{P}$-a.s. to a (possibly infinite) limit as $t \to \infty$. Convergence of Φ_t follows if

$$\mathbb{E}\Big[\int_0^\infty (X_s^\xi)^2 \, ds\Big] < \infty \tag{7}$$

and a.s. convergence of Ψ_t is guaranteed if a.s.

$$\lim_{t\to\infty} (X_t^\xi)^2 t \ln\ln t = 0. \tag{8}$$

Note that these conditions do not exclude buy orders (negative ξ_t) or short sales (negative X_t^ξ). We will regard strategies admissible for optimal liquidation if they satisfy the preceding two conditions in addition to the assumptions in Section 2. We then have

$$\begin{aligned} R_\infty^\xi &:= \lim_{t\to\infty} \mathcal{R}_t(\xi) && (9)\\ &= \underbrace{r + P_0 X_0 - \frac{\gamma}{2} X_0^2}_{=:R_0} + \sigma \int_0^\infty X_s^\xi \, dB_s - \lambda \int_0^\infty \xi_s^2 \, ds. && (10) \end{aligned}$$

All of the five terms adding up to R_∞^ξ can be interpreted economically. The number r is simply the initial cash endowment of the investor. $P_0 X_0$ is the face value of the initial position. The term $\frac{\gamma}{2} X_0^2$ corresponds to the liquidation costs resulting from the permanent price impact of ξ. Due to the linearity of the permanent impact function, it is independent of the choice of the liquidation strategy. The stochastic integral corresponds to the volatility risk that is accumulated by selling throughout the interval $[0, \infty[$ rather than liquidating the portfolio instantaneously. The integral $\lambda \int_0^\infty \xi_t^2 \, dt$ corresponds to the transaction costs arising from temporary market impact.

We assume that the investor wants to maximize the expected utility of her cash position after liquidation:

$$v_1(X_0, R_0) := \sup_{\xi \in \mathcal{X}} \mathbb{E}[u(R_\infty^\xi)] \tag{11}$$

3.2 Maximization of asymptotic portfolio value

Now consider an investor holding x units of the risky asset and r units of cash at time t. In a liquid market, the value of this portfolio is simply $xP_t + r$. If the market is illiquid, there is no canonical portfolio value. The effect of the temporary price impact depends on the liquidation strategy and can be very small for traders with small risk aversion who liquidate the position at a very slow rate. The permanent impact however cannot be avoided, and its impact on a liquidation return is independent of the trading strategy. We therefore suggest to value the portfolio as

$$r + x\left(P_t - \frac{\gamma}{2} x\right) \tag{12}$$

where P_t is the market price at time t including permanent but not temporary impact. In practice, P_t can be observed whenever the large investor does not trade. We can think of the portfolio value as the expected liquidation value when the asset position x is sold infinitely slowly. One advantage of this approach is that the portfolio value cannot be permanently manipulated by moving the market; any such market movement is directly accounted for.

When the trading strategy ξ is pursued, the portfolio value[3] in the above sense evolves over time as

$$R_t^\xi = r + P_0 X_0 - \frac{\gamma}{2} X_0^2 + \sigma \int_0^t X_s^\xi \, dB_s - \lambda \int_0^t \xi_s^2 \, ds. \tag{13}$$

We assume that the investor trades the risky asset in order to maximize the asymptotic expected utility of portfolio value:

$$v_2(X_0, R_0) := \sup_{\xi \in \mathcal{X}} \lim_{t\to\infty} \mathbb{E}[u(R_t^\xi)]. \tag{14}$$

The existence of the limit will be established in Lemma 15. Note that our assumptions on strategies admissible for the maximization of asymptotic portfolio value are weaker than those for optimal liquidation. In particular, we do not require that R_t^ξ or X_t^ξ converge.

[3] Note that R_t denotes the portfolio value (including risky assets) at time t, while $\mathcal{R}_t$ denotes only the cash position at time t.

4 Statement of results

Theorem 1. *The value functions $v = v_1$ for optimal liquidation and v_2 for maximization of asymptotic portfolio value are equal and are classical solutions of the Hamilton-Jacobi-Bellman equation*

$$\inf_c \left[-\frac{1}{2}\sigma^2 X^2 v_{RR} + \lambda v_R c^2 + v_X c \right] = 0 \tag{15}$$

with boundary condition

$$v(0, R) = u(R) \text{ for all } R \in \mathbb{R}. \tag{16}$$

The a.s. unique optimal control $\hat{\xi}_t$ is Markovian and given in feedback form by

$$\hat{\xi}_t = c(X_t^{\hat{\xi}}, R_t^{\hat{\xi}}) = -\frac{v_X}{2\lambda v_R}(X_t^{\hat{\xi}}, R_t^{\hat{\xi}}). \tag{17}$$

For the value functions, we have convergence:

$$v(X_0, R_0) = \lim_{t\to\infty} \mathbb{E}[u(R_t^{\hat{\xi}})] = \mathbb{E}[u(R_\infty^{\hat{\xi}})] \tag{18}$$

Note that the HJB equation in the preceding theorem is fully nonlinear in all partial derivatives of v, even in the "time" derivative, v_X. This can best be observed in the corresponding reduced-form equation:

$$v_X^2 = -2\lambda\sigma^2 X^2 v_R v_{RR}. \tag{19}$$

In the following we will use the term "optimal control" to refer to the optimal admissible strategy $\hat{\xi}$ or the optimal feedback function c, depending on the circumstances. At the heart of the above theorem lies the transformed optimal control

$$\tilde{c}(Y, R) := c(\sqrt{Y}, R)/\sqrt{Y}. \tag{20}$$

The existence of a solution to the HJB equation in Theorem 1 will be derived from the existence of a smooth solution to the fully nonlinear parabolic PDE given in the following theorem.

Theorem 2. *The transformed optimal control $\tilde{c}$ is a classical solution of the fully nonlinear parabolic PDE*

$$\tilde{c}_Y = -\frac{3}{2}\lambda\tilde{c}\tilde{c}_R + \frac{\sigma^2}{4\tilde{c}}\tilde{c}_{RR} \tag{21}$$

with initial condition

$$\tilde{c}(0, R) = \sqrt{\frac{\sigma^2 A(R)}{2\lambda}}. \tag{22}$$

The bounds of the absolute risk aversion give bounds for the transformed optimal control:

$$\inf_{(Y,R)\in\mathbb{R}_0^+\times\mathbb{R}} \tilde{c}(Y, R) = \inf_{R\in\mathbb{R}} \tilde{c}(0, R) =: \tilde{c}_{min} = \sqrt{\frac{\sigma^2 A_{min}}{2\lambda}} \tag{23}$$

$$\sup_{(Y,R)\in\mathbb{R}_0^+\times\mathbb{R}} \tilde{c}(Y, R) = \sup_{R\in\mathbb{R}} \tilde{c}(0, R) =: \tilde{c}_{max} = \sqrt{\frac{\sigma^2 A_{max}}{2\lambda}} \tag{24}$$

Figure 1 shows a numerical example of c and $\tilde{c}$.

Corollary 3. *The asset position $X_t^{\hat{\xi}}$ at time t under the optimal control $\hat{\xi}$ is given by*

$$X_t^{\hat{\xi}} = X_0 \exp\left(-\int_0^t \tilde{c}((X_s^{\hat{\xi}})^2, R_s^{\hat{\xi}})\, ds \right) \tag{25}$$

and is bounded by

$$X_0 \exp(-t\tilde{c}_{max}) \le X_t^{\hat{\xi}} \le X_0 \exp(-t\tilde{c}_{min}). \tag{26}$$

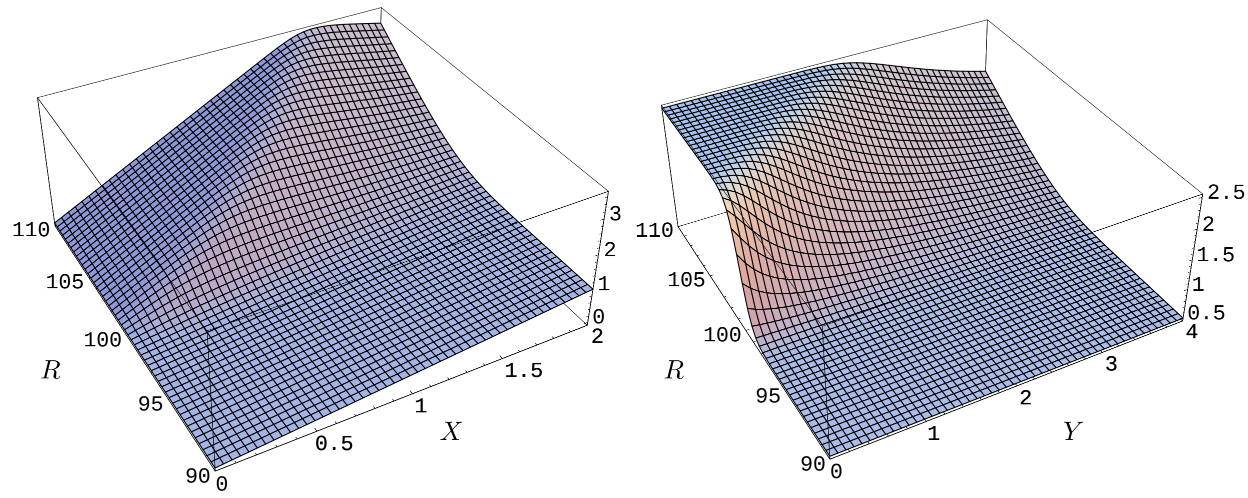

Figure 1: Optimal control $c(X,R)$ (left hand figure) and transformed optimal control $\tilde{c}(Y,R)$ (right hand figure) for the utility function with absolute risk aversion $A(R) = 2(1.5 + \tanh(R-100))^2$ and parameter $\lambda = \sigma = 1$.

Although we did not a priori exclude intermediate buy orders or short sales, the preceding theorem and corollary reveal that these are never optimal. For investors with constant absolute risk aversion $A = A_{min} = A_{max}$, Corollary 3 yields the following explicit formula for the optimal strategy. It is identical to the optimal strategy for mean-variance investors (see Almgren (2003)) and is the limit of optimal execution strategies for finite time horizons (see Schied and Schöneborn (2007)).

Corollary 4. *Assume that the investor has a utility function $u(R) = -e^{-AR}$ with constant risk aversion $A(R) \equiv A$. Then her optimal adaptive liquidation strategy is static and is given by*

$$X_t^{\hat{\xi}} = X_0 \exp\left(-t\sqrt{\frac{\sigma^2 A}{2\lambda}}\right) \tag{27}$$

Given the optimal control $c(X,R)$ (or the transformed optimal control $\tilde{c}(X,R)$), we can identify the optimal strategy as aggressive in-the-money (AIM), neutral in-the-money (NIM) and passive in-the-money (PIM). If prices rise, then R rises. A strategy with an optimal control c that is increasing in R (everything else held constant) sells fast in such a scenario, i.e., is aggressive in-the-money; if c is decreasing in R, it is passive in-the-money, and if c is independent of R, then the strategy is neutral in-the-money. The initial value specification for $\tilde{c}$ given in Theorem 2 shows that there is a tight relation between the absolute risk aversion and the optimal adaptive trading strategy: If A is an increasing function, i.e., the utility function u exhibits increasing absolute risk aversion (IARA), then the optimal strategy is aggressive in-the-money at least for small values of X. The next theorem states that such a monotonicity of $\tilde{c}$ propagates to all values of X, not only to small values of X.

Theorem 5. *$c(X,R)$ is increasing (decreasing) in R for all values of X if and only if the absolute risk aversion $A(R)$ is increasing (decreasing) in R. In particular, $A(R)$ determines the characteristics of the optimal strategy:*

Utility function		*Optimal trading strategy*
Decreasing absolute risk aversion (DARA)	$\Leftrightarrow$	*Passive in-the-money (PIM)*
Constant absolute risk aversion (CARA)	$\Leftrightarrow$	*Neutral in-the-money (NIM)*
Increasing absolute risk aversion (IARA)	$\Leftrightarrow$	*Aggressive in-the-money (AIM)*

Note that in the numerical example in Figure 1, A is increasing. The figure confirms that c and $\tilde{c}$ are also increasing in R. Figure 2 shows two sample paths of $X_t^{\hat{\xi}}$. As expected, the asset position is decreased quicker when the asset price is rising than when it is falling.

We now turn to the dependence of the optimal control c on the problem parameters u, X, λ and σ. The following theorem describes the dependence on u. Theorem 5 is in fact a corollary to the following general result.

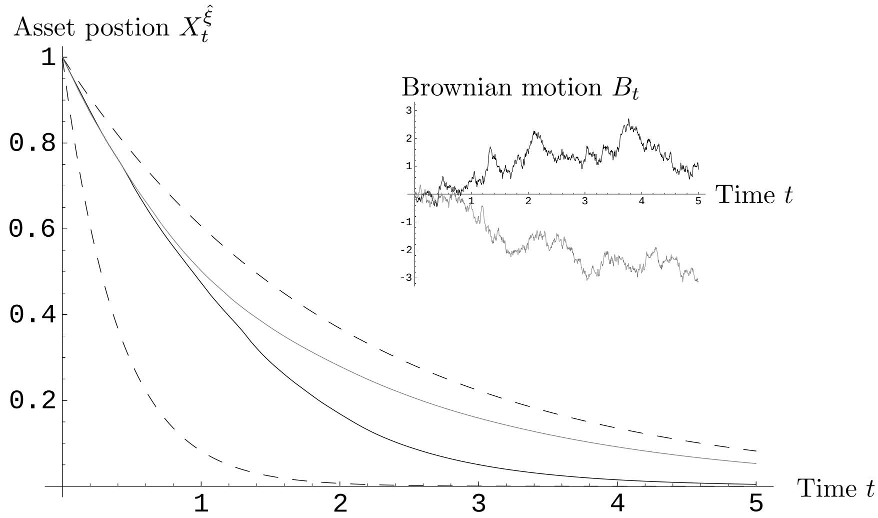

Figure 2: Two sample optimal execution paths $X_t^{\hat{\xi}}$ corresponding to the sample paths of the Brownian motion B_t in the inset. The dashed lines represent the upper and lower bounds on $X_t^{\hat{\xi}}$. Parameters are $\lambda = \gamma = \sigma = 1$, $X_0 = 1$, $R_0 = 0$, $P_0 = 100$ and the utility function with absolute risk aversion $A(R) = 2(1.5 + \tanh(R - 100))^2$. 1000 simulation steps were used covering the time span $[0, 5]$.

Theorem 6. *Suppose u^0 and u^1 are two utility functions such that u^1 has a higher absolute risk aversion than u^0, i.e., $A^1(R) \geq A^0(R)$ for all R. Then an investor with utility function u^1 liquidates the same portfolio X_0 faster than an investor with utility function u^0. More precisely, the corresponding optimal strategies satisfy*

$$c^1 \geq c^0 \qquad \text{and} \qquad \hat{\xi}_t^1 \geq \hat{\xi}_t^0 \qquad \mathbb{P}\text{-a.s.} \tag{28}$$

An increase of the asset position X has two effects on the optimal liquidation strategy. First, it increases overall risk, leading to a desire to increase the selling speed. Second, it changes the distribution of total proceeds R_∞: it increases its dispersion due to increased risk, and it moves it downwards due to increased temporary impact liquidation cost. This change in return distribution can lead to a reduction in relevant risk aversion and thus a desire to reduce the selling speed. In Figure 1 one can make the surprising observation that the second effect can outweigh the first, i.e., that the optimal strategy $c(X, R)$ need not be increasing in X. That is, an *increase* of the asset position may lead to a *decrease* of the liquidation rate.

We now turn to the dependence of c on the impact parameters. Perhaps surprisingly, neither the value function v nor the optimal control $\hat{\xi}$ respectively c depend directly on the permanent impact parameter γ. However, γ influences the portfolio value state variable $R = r + X\left(P - \frac{\gamma}{2}X\right)$ and therefore indirectly also the optimal control. For the temporary impact parameter λ, we intuitively expect that the optimal control c decreases when λ increases, since fast trading becomes more expensive. Figure 3 shows that this is not necessarily the case: in this example, an increased temporary impact cost leads to faster selling. This counterintuitive behavior cannot occur for IARA utility functions:

Theorem 7. *If the utility function u exhibits increasing absolute risk aversion (IARA), then the optimal control c is decreasing in the temporary impact parameter λ.*

We conclude our sensitivity analysis with the following Theorem that links the dependence on σ to the dependence on λ and X.

Theorem 8 (Relation between σ, λ and X)**.** *Let $c(X, R, \lambda, \sigma)$ be the optimal control in a market with temporary impact parameter λ and volatility σ. Then*

$$c(X, R, \lambda, \sigma_1) = \frac{\sigma_2}{\sigma_1} c\left(\frac{\sigma_1}{\sigma_2}X, R, \frac{\sigma_2^2}{\sigma_1^2}\lambda, \sigma_2\right) \tag{29}$$

By the boundary condition, we know that $v(0, R) = u(R)$ is a utility function. The next theorem states that for each value of X, $v(X, R)$ can be regarded as a utility function in R.

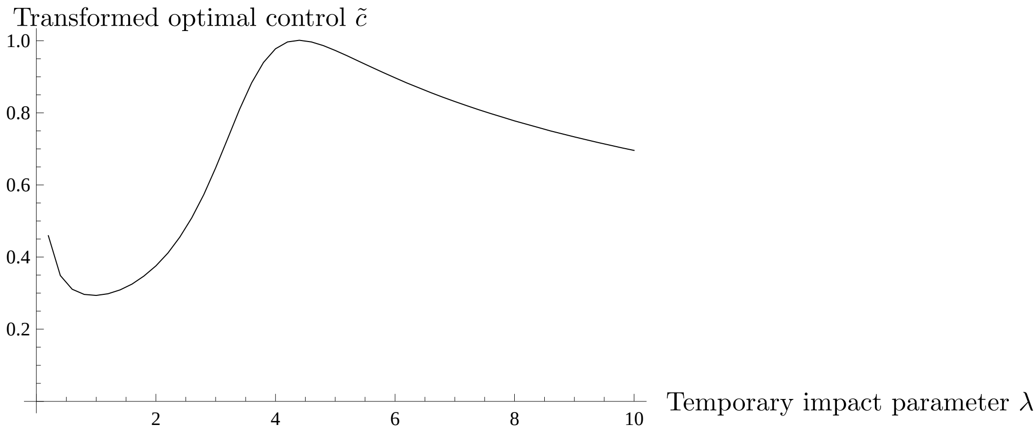

Figure 3: Transformed optimal control $\tilde{c}(Y, R, \lambda, \sigma)$ depending on the temporary impact parameter λ. Parameters are $Y = 0.5$, $R = 2$, $\sigma = 1$ and the utility function u with absolute risk aversion $A(R) = 2(1.2 - \tanh(15R))^2$.

Theorem 9. *The value function $v(X, R)$ is strictly concave, jointly in X and R, increasing in R and decreasing in X. In particular, for every $X > 0$, the value function $v(X, R)$ is again a utility function in R. Moreover, for all X and R, $\tilde{c}(X^2, R)$ is proportional to the square root of the absolute risk aversion $A(X, R) := -v_{RR}(X, R)/v_R(X, R)$ of $v(X, R)$:*

$$\tilde{c}(X^2, R) = \sqrt{\frac{\sigma^2 A(X, R)}{2\lambda}}. \tag{30}$$

The value function $v(X, R)$ is only *decreasing* in X when the portfolio value R is kept constant. In this case, increasing X shifts value from the cash account toward the risky asset, which always decreases utility for a risk-averse investor.

In view of non-concave utility functions suggested, e.g., by the prospect theory of Kahneman and Tversky (1979), one might ask to what extend the concavity of u is an essential ingredient of our analysis. Which of our results may carry over to 'utility functions' u that are strictly increasing but not concave? Let us suppose that v is defined as in Equations 11 or 14. Then it follows immediately that $R \mapsto v(X, R)$ is strictly increasing. If v also satisfies the HJB equation, Equation 15, then Equation 19 yields

$$v_{RR} = -\frac{v_X^2}{2\sigma^2 \lambda v_R} \leq 0. \tag{31}$$

Hence, $R \mapsto v(X, R)$ is concave for every $X > 0$. Therefore v cannot be a solution of the initial value problem in Equations 15 and 16 unless $v(0, R) = u(R)$ is also concave. This shows that the concavity of u is essential to our approach. Note that the preceding argument can also be used to give an alternative proof of the assertion of concavity in Theorem 9.

5 Proof of results

This section consists of three parts. First we show that a smooth solution of the HJB equation exists and provide some of its properties. This is achieved by first obtaining a solution of the PDE for the transformed optimal strategy, $\tilde{c}$, and then solving a transport equation with coefficient $\tilde{c}$. In the second part, we apply a verification argument and show that this solution of the HJB equation must be equal to the value function. Theorems 1 and 2 are direct consequences of the propositions in these two subsections. In the last subsection we prove the qualitative properties of the optimal adaptive strategy and the value function given in Theorems 5, 6, 7, 8 and 9.

5.1 Existence and characterization of a smooth solution of the HJB equation

As a first step, we observe that $\lim_{R\to\infty} u(R) < \infty$ due to the boundedness of the risk aversion, and we can thus assume without loss of generality that

$$\lim_{R\to\infty} u(R) = 0. \tag{32}$$

Proposition 10. *There exists a smooth ($C^{2,4}$) solution* $\tilde{c} : (Y, R) \in \mathbb{R}_0^+ \times \mathbb{R} \to \tilde{c}(Y, R) \in \mathbb{R}$ *of*

$$\tilde{c}_Y = -\frac{3}{2}\lambda\tilde{c}\tilde{c}_R + \frac{\sigma^2}{4\tilde{c}}\tilde{c}_{RR} \tag{33}$$

with initial value

$$\tilde{c}(0, R) = \sqrt{\frac{\sigma^2 A(R)}{2\lambda}}. \tag{34}$$

The solution satisfies

$$\tilde{c}_{min} := \inf_{R\in\mathbb{R}} \sqrt{\frac{\sigma^2 A(R)}{2\lambda}} \leq \tilde{c}(Y, R) \leq \sup_{R\in\mathbb{R}} \sqrt{\frac{\sigma^2 A(R)}{2\lambda}} =: \tilde{c}_{max}. \tag{35}$$

The function $\tilde{c}$ is $C^{2,4}$ in the sense that it has a continuous derivative $\frac{\partial^{i+j}}{\partial Y^i \partial R^j}\tilde{c}(Y, R)$ if $2i + j \leq 4$. In particular, $\tilde{c}_{YRR}$ and $\tilde{c}_{RRR}$ exist and are continuous.

The statement follows from the following auxiliary theorem from the theory of parabolic partial differential equations. We do not establish the uniqueness of $\tilde{c}$ directly in the preceding proposition. However, it follows from Proposition 18.

Theorem 11 (Auxiliary theorem: Solution of Cauchy problem)**.** *There is a smooth solution ($C^{2,4}$)*

$$f : (t, x) \in \mathbb{R}_0^+ \times \mathbb{R} \to f(t, x) \in \mathbb{R} \tag{36}$$

for the parabolic partial differential equation[4]

$$f_t - \frac{d}{dx}a(x, t, f, f_x) + b(x, t, f, f_x) = 0 \tag{37}$$

with initial value condition

$$f(0, x) = \psi_0(x) \tag{38}$$

if all of the following conditions are satisfied:

- $\psi_0(x)$ *is smooth (C^4) and bounded*
- a *and* b *are smooth (C^3 respectively C^2)*
- *There are constants* b_1 *and* $b_2 \geq 0$ *such that for all* x *and* u*:*

$$\left(b(x, t, u, 0) - \frac{\partial a}{\partial x}(x, t, u, 0)\right) u \geq -b_1 u^2 - b_2. \tag{39}$$

- *For all* $M > 0$*, there are constants* $\mu_M \geq \nu_M > 0$ *such that for all* x*,* t*,* u *and* p *that are bounded in modulus by* M*:*

$$\nu_M \leq \frac{\partial a}{\partial p}(x, t, u, p) \leq \mu_M \tag{40}$$

  and

$$\left(|a| + \left|\frac{\partial a}{\partial u}\right|\right)(1 + |p|) + \left|\frac{\partial a}{\partial x}\right| + |b| \leq \mu_M(1 + |p|)^2. \tag{41}$$

Proof. The theorem is a direct consequence of Theorem 8.1 in Chapter V of Ladyzhenskaya, Solonnikov, and Ural'ceva (1968). In the following, we outline the last step of its proof because we will use it for the proof of subsequent propositions.

The conditions of the theorem guarantee the existence of solutions f_N of Equation 37 on the strip $\mathbb{R}_0^+ \times [-N, N]$ with boundary conditions

$$f_N(0, x) = \psi_0(x) \text{ for all } x \in [-N, N] \tag{42}$$

and

$$f_N(t, \pm N) = \psi_0(\pm N) \text{ for all } t \in \mathbb{R}_0^+. \tag{43}$$

These solutions converge smoothly as N tends to infinity: $\lim_{N\to\infty} f_N = f$. □

[4] Here, f_t refers to $\frac{d}{dt}f$ and not $f(t)$.

Proof of Proposition 10. We want to apply Theorem 11 and set

$$a(x,t,u,p) := h_1(u)p \tag{44}$$

$$b(x,t,u,p) := \frac{3}{2}\lambda h_2(u)p + h_1'(u)p^2 \tag{45}$$

$$\psi_0(x) := \sqrt{\frac{\sigma^2 A(R)}{2\lambda}} \tag{46}$$

with smooth functions $h_1, h_2 : \mathbb{R} \to \mathbb{R}$. With $h_1(u) = \frac{\sigma^2}{4u}$ and $h_2(u) = u$, Equation 37 becomes Equation 33 by relabeling the coordinates from t to Y and from x to R. All conditions of Auxiliary Theorem 11 are fulfilled, except for the last boundedness condition. In order to fulfill these, we take h_1 and h_2 to be smooth nonnegative bounded functions fulfilling $h_1(u) = \frac{\sigma^2}{4u}$ and $h_2(u) = u$ for $\tilde{c}_{min} \le u \le \tilde{c}_{max}$. Now all conditions of Theorem 11 are fulfilled and there exists a smooth solution to

$$f_t = -\frac{3}{2}\lambda h_2(f) f_x + h_1(f) f_{xx}. \tag{47}$$

We now show that this solution f also fulfills

$$f_t = -\frac{3}{2}\lambda f f_x + \frac{\sigma^2}{4f} f_{xx} \tag{48}$$

by using the maximum principle to show that $\tilde{c}_{min} \le f \le \tilde{c}_{max}$. First assume that there is a (t_0, x_0) such that $f(t_0, x_0) > \tilde{c}_{max}$. Then there is an $N > 0$ and $\gamma > 0$ such that also $\tilde{f}_N(t_0, x_0) := f_N(t_0, x_0)e^{-\gamma t_0} > \tilde{c}_{max}$ with f_N as constructed in the proof of Theorem 11. Then $\max_{t\in[0,t_0], x\in[-N,N]} \tilde{f}_N(t,x)$ is attained at an interior point (t_1, x_1), i.e., $0 < t_1 \le t_0$ and $-N < x_1 < N$. We thus have

$$\tilde{f}_{N,t}(t_1, x_1) \ge 0 \tag{49}$$

$$\tilde{f}_{N,x}(t_1, x_1) = 0 \tag{50}$$

$$\tilde{f}_{N,xx}(t_1, x_1) \le 0. \tag{51}$$

We furthermore have that

$$\tilde{f}_{N,t} = e^{-\gamma t} f_{N,t} - \gamma e^{-\gamma t} f_N \tag{52}$$

$$= -\frac{3}{2} e^{-\gamma t}\lambda h_2(f_N) f_{N,x} + e^{-\gamma t} h_1(f_N) f_{N,xx} - \gamma e^{-\gamma t} f_N \tag{53}$$

$$= -\frac{3}{2}\lambda h_2(f_N)\tilde{f}_{N,x} + h_1(f_N)\tilde{f}_{N,xx} - \gamma \tilde{f}_N \tag{54}$$

and therefore that

$$\tilde{f}_N(t_1, x_1) \le 0. \tag{55}$$

This however contradicts $\tilde{f}_N(t_1, x_1) \ge \tilde{f}_N(t_0, x_0) \ge \tilde{c}_{max} > 0$.

By a similar argument, we can show that if there is a point (t_0, x_0) with $f(t_0, x_0) < \tilde{c}_{min}$, then the interior minimum (t_1, x_1) of a suitably chosen $\tilde{f}_N := f_N - \tilde{c}_{max} < 0$ satisfies $\tilde{f}_N(t_1, x_1) > 0$ and thus causes a contradiction. □

Proposition 12. *There exists a $C^{2,4}$-solution $\tilde{w} : \mathbb{R}_0^+ \times \mathbb{R} \to \mathbb{R}$ of the transport equation*

$$\tilde{w}_Y = -\lambda \tilde{c} \tilde{w}_R \tag{56}$$

with initial value

$$\tilde{w}(0, R) = u(R). \tag{57}$$

The solution satisfies

$$0 \ge \tilde{w}(Y, R) \ge u(R - \lambda \tilde{c}_{max} Y) \tag{58}$$

and is increasing in R and decreasing in Y.

Proof. The proof uses the method of characteristics. Consider the function $P : (Y, S) \in \mathbb{R}_0^+ \times \mathbb{R} \to P(Y, S) \in \mathbb{R}$ satisfying the ODE

$$P_Y(Y, S) = \lambda \tilde{c}(Y, P(Y, S)) \tag{59}$$

with initial value condition $P(0, S) = S$. Since $\tilde{c}$ is smooth and bounded, a solution of the above ODE exists for each fixed S. For every Y, $P(Y, \cdot)$ is a diffeomorphism mapping $\mathbb{R}$ onto $\mathbb{R}$ and has the same regularity as $\tilde{c}$, i.e., belongs to $C^{2,4}$. We define

$$\tilde{w}(Y, R) = u(S) \quad \text{iff} \quad P(Y, S) = R. \tag{60}$$

Then $\tilde{w}$ is a $C^{2,4}$-function satisfying the initial value condition. By definition, we have

$$\begin{aligned} 0 &= \frac{d}{dY} \tilde{w}(Y, P(Y, S)) && (61)\\ &= \tilde{w}_R(Y, P(Y, S)) P_Y(Y, S) + \tilde{w}_Y(Y, P(Y, S)) && (62)\\ &= \tilde{w}_R(Y, P(Y, S)) \lambda \tilde{c}(Y, P(Y, S)) + \tilde{w}_Y(Y, P(Y, S)). && (63) \end{aligned}$$

Therefore $\tilde{w}$ fulfills the desired partial differential equation. Since $\tilde{c} \leq \tilde{c}_{max}$, we know that $P_Y \leq \lambda \tilde{c}_{max}$ and hence $P(Y, S) \leq S + Y \lambda \tilde{c}_{max}$ and thus $\tilde{w}(Y, R) \geq u(R - \lambda \tilde{c}_{max} Y)$.

The monotonicity statements in the proposition follow because the family of solutions of the ODE above do not cross and since $\tilde{c}$ is positive. □

Proposition 13. *The function $w(X, R) := \tilde{w}(X^2, R)$ solves the HJB equation*

$$\min_c \left[-\frac{1}{2} \sigma^2 X^2 w_{RR} + \lambda w_R c^2 + w_X c \right] = 0. \tag{64}$$

The unique minimum is attained at

$$c(X, R) := \tilde{c}(X^2, R) X. \tag{65}$$

Proof. Assume for the moment that

$$\tilde{c}^2 = -\frac{\sigma^2 \tilde{w}_{RR}}{2 \lambda \tilde{w}_R}. \tag{66}$$

Then with $Y = X^2$:

$$\begin{aligned} 0 &= -\lambda X^2 \tilde{w}_R \left(\frac{\sigma^2 \tilde{w}_{RR}}{2 \lambda \tilde{w}_R} + \tilde{c}^2 \right) && (67)\\ &= -\lambda X^2 \tilde{w}_R \left(\frac{\sigma^2 \tilde{w}_{RR}}{2 \lambda \tilde{w}_R} + \frac{\tilde{w}_Y^2}{\lambda^2 \tilde{w}_R^2} \right) && (68)\\ &= -\frac{1}{2} \sigma^2 X^2 w_{RR} - \frac{w_X^2}{4 \lambda w_R} && (69)\\ &= \inf_c \left[-\frac{1}{2} \sigma^2 X^2 w_{RR} + \lambda w_R c^2 + w_X c \right] && (70) \end{aligned}$$

and Equation 65 follows from Equations 56 and 64.

We now show that Equation 66 is fulfilled for all R and $Y = X^2$. First, observe that it holds for $Y = 0$. For general Y, consider the following two equations:

$$\begin{aligned} \frac{d}{dY} \tilde{c}^2 &= -3 \lambda \tilde{c}^2 \tilde{c}_R + \frac{\sigma^2}{2} \tilde{c}_{RR} && (71)\\ -\frac{d}{dY} \frac{\sigma^2 \tilde{w}_{RR}}{2 \lambda \tilde{w}_R} &= \sigma^2 \tilde{c} \frac{d}{dR} \frac{\tilde{w}_{RR}}{2 \tilde{w}_R} + \sigma^2 \tilde{c}_R \frac{\tilde{w}_{RR}}{2 \tilde{w}_R} + \frac{\sigma^2}{2} \tilde{c}_{RR} && (72) \end{aligned}$$

The first of these two equations holds because of Equation 33 and the second one because of Equation 56. Now we have

$$\begin{aligned} \frac{d}{dY} \left(\tilde{c}^2 + \frac{\sigma^2 \tilde{w}_{RR}}{2 \lambda \tilde{w}_R} \right) &= -3 \lambda \tilde{c}^2 \tilde{c}_R + \frac{\sigma^2}{2} \tilde{c}_{RR} - \sigma^2 \tilde{c} \frac{d}{dR} \frac{\tilde{w}_{RR}}{2 \tilde{w}_R} - \sigma^2 \tilde{c}_R \frac{\tilde{w}_{RR}}{2 \tilde{w}_R} - \frac{\sigma^2}{2} \tilde{c}_{RR} && (73)\\ &= -\lambda \tilde{c} \frac{d}{dR} \left(\tilde{c}^2 + \frac{\sigma^2 \tilde{w}_{RR}}{2 \lambda \tilde{w}_R} \right) - \lambda \tilde{c}_R \left(\tilde{c}^2 + \frac{\sigma^2 \tilde{w}_{RR}}{2 \lambda \tilde{w}_R} \right). && (74) \end{aligned}$$

Hence, the function $f(Y,R) := \tilde{c}^2 + \frac{\sigma^2 \tilde{w}_{RR}}{2\lambda \tilde{w}_R}$ satisfies the linear PDE

$$f_Y = -\lambda \tilde{c} f_R - \lambda \tilde{c}_R f \tag{75}$$

with initial value condition $f(0,R) = 0$. One obvious solution to this PDE is $f(Y,R) \equiv 0$. By the method of characteristics this is the unique solution to the PDE, since $\tilde{c}$ and $\tilde{c}_R$ are smooth and hence locally Lipschitz. □

The next auxiliary lemma will prove useful in the following.

Lemma 14 (Auxiliary Lemma)**.** *There are positive constants α, a_1, a_2, a_3 and a_4 such that*

$$u(R) \quad \geq \quad w(X,R) \quad \geq \quad u(R)\exp(\alpha X^2) \tag{76}$$

$$0 \quad \leq \quad w_R(X,R) \quad \leq \quad a_1 + a_2 \exp(-a_3 R + a_4 X^2) \tag{77}$$

for all $(X,R) \in \mathbb{R}_0^+ \times \mathbb{R}$.

Proof of Lemma 14. The left hand side of the first inequality follows by the boundary condition for w and the monotonicity of w with respect to X as established in Proposition 12. Since the risk aversion of u is bounded from above by $2\lambda\tilde{c}^2_{max}$, we have

$$u(R-\Delta) \geq u(R) e^{2\lambda \tilde{c}^2_{max} \Delta} \tag{78}$$

and thus by Proposition 12

$$w(X,R) \geq u(R - \lambda \tilde{c}_{max} X^2) \geq u(R) e^{2\lambda^2 \tilde{c}^3_{max} X^2} \tag{79}$$

which establishes the right hand side of the first inequality with $\alpha = 2\lambda^2 \tilde{c}^3_{max}$.

For the second inequality, we will show the equivalent inequality

$$0 \leq \tilde{w}_R(Y,R) \leq a_1 + a_2 \exp(-a_3 R + a_4 Y). \tag{80}$$

The left hand side follows since $\tilde{w}$ is increasing in R by Proposition 12. For the right hand side, note that $\tilde{w}$ has "bounded absolute risk aversion" due to Equation 66 and the bound on $\tilde{c}$ established by Proposition 10:

$$-\frac{\tilde{w}_{RR}}{\tilde{w}_R} < \frac{2\lambda \tilde{c}^2_{max}}{\sigma^2} =: \tilde{A} \tag{81}$$

Then

$$\tilde{w}(Y,R_0) \geq \tilde{w}(Y,R) + \frac{\tilde{w}_R(Y,R)}{\tilde{A}} \left(1 - e^{-\tilde{A}(R_0 - R)}\right). \tag{82}$$

Since

$$\lim_{R_0 \to \infty} \tilde{w}(Y,R_0) = \lim_{R_0 \to \infty} u(R_0) = 0 \tag{83}$$

we have

$$0 \geq \tilde{w}(Y,R) + \frac{\tilde{w}_R(Y,R)}{\tilde{A}} \tag{84}$$

and thus

$$\tilde{w}_R(Y,R) \leq -\tilde{w}(Y,R)\tilde{A} \leq -u(R - \lambda \tilde{c}_{max} Y)\tilde{A}. \tag{85}$$

Since u is bounded by an exponential function, we obtain the desired bound on $\tilde{w}_R$. □

5.2 Verification argument

We now connect the PDE results from Subsection 5.1 with the optimal stochastic control problem introduced in Section 3. For any admissible strategy $\xi \in \mathcal{X}$ and $k \in \mathbb{N}$ we define

$$\tau_k^\xi := \inf \Big\{ t \geq 0 \,\Big|\, \int_0^t \xi_s^2 \, ds \geq k \Big\}. \tag{86}$$

We proceed by first showing that $u(R_t^\xi)$ and $w(X_t^\xi, R_t^\xi)$ fulfill local supermartingale inequalities. Thereafter we show that $w(X_0,R_0) \geq \lim_{t\to\infty} \mathbb{E}[u(R_t^\xi)]$ with equality for $\xi = \hat{\xi}$. The next lemma in particular justifies our definition of $v_2(X_0,R_0)$ in Equation 14.

Lemma 15. *For any admissible strategy ξ the expected utility $\mathbb{E}[\,u(R_t^\xi)\,]$ is decreasing in t. Moreover, we have $\mathbb{E}[\,u(R^\xi_{t\wedge\tau_k^\xi})\,] \geq \mathbb{E}[\,u(R_t^\xi)\,]$.*

Proof. Since $R_t^\xi - R_0$ is the difference of the true martingale $\int_0^t \sigma X_s^\xi\,dB_s$ and the increasing process $\lambda\int_0^t \xi_s^2\,ds$, it satisfies the supermartingale inequality $\mathbb{E}[\,R_t^\xi\,|\,\mathcal{F}_s\,] \leq R_s^\xi$ for $s \leq t$ (even though it may fail to be a supermartingale due to the possible lack of integrability). Hence $\mathbb{E}[\,u(R_t^\xi)\,]$ is decreasing according to Jensen's inequality.

For the second assertion, we first take $n = k$ and write for $\tau_m := \tau_m^\xi$

$$\mathbb{E}[\,u(R^\xi_{t\wedge\tau_k})\,] = \mathbb{E}\Big[\,u\Big(R_0 + \sigma\int_0^{t\wedge\tau_n} X_s^\xi\,dB_s - \lambda\int_0^{t\wedge\tau_k}\xi_s^2\,ds\Big)\,\Big]. \tag{87}$$

When sending n to infinity, the right-hand side decreases to

$$\mathbb{E}\Big[\,u\Big(R_0 + \sigma\int_0^{t} X_s^\xi\,dB_s - \lambda\int_0^{t\wedge\tau_k}\xi_s^2\,ds\Big)\,\Big], \tag{88}$$

by dominated convergence because u is bounded from below by an exponential function, the integral of ξ^2 is bounded by k, and the stochastic integrals are uniformly bounded from below by $\inf_{s\leq K^2t} W_s$, where W is the DDS-Brownian motion of $\int X_s^\xi\,dB_s$ and K is an upper bound for $|X^\xi|$. Finally, the term in Equation 88 is clearly larger than or equal to $\mathbb{E}[\,u(R_t^\xi)\,]$. □

Lemma 16. *For any admissible strategy ξ, $w(X_t^\xi, R_t^\xi)$ is a local supermartingale with localizing sequence (τ_k^ξ).*

Proof. We use a verification argument similar to the one in Schied and Schöneborn (2007). For $T > t \geq 0$, Itô's formula yields that

$$w(X_T^\xi, R_T^\xi) - w(X_t^\xi, R_t^\xi) = \int_t^T w_R(X_s^\xi, R_s^\xi)\sigma X_s^\xi\,dB_s - \int_t^T \Big[\lambda w_R\xi_s^2 + w_X\xi_s - \frac{1}{2}(\sigma X_s^\xi)^2 w_{RR}\Big](X_s^\xi, R_s^\xi)\,ds. \tag{89}$$

By Proposition 13 the latter integral is nonnegative and we obtain

$$w(X_t^\xi, R_t^\xi) \geq w(X_T^\xi, R_T^\xi) - \int_t^T w_R(X_s^\xi, R_s^\xi)\sigma X_s^\xi\,dB_s. \tag{90}$$

We will show next that the stochastic integral in Equation 90 is a local martingale with localizing sequence $(\tau_k) := (\tau_k^\xi)$. For some constant C_1 depending on t, k, λ, σ, and on the upper bound K of $|X^\xi|$ we have for $s \leq t\wedge\tau_k$

$$R_s^\xi = R_0 + \sigma B_s X_s^\xi + \int_0^s (\sigma\xi_q B_q - \lambda\xi_q^2)\,dq \geq -C_1\big(1 + \sup_{q\leq t}|B_q|\big). \tag{91}$$

Using Lemma 14, we see that for $s \leq t\wedge\tau_k$

$$0 \leq w_R(X_s^\xi, R_s^\xi) \leq a_1 + a_2\exp\Big(a_3C_1\big(1 + \sup_{q\leq t}|B_q|\big) + a_4K^2\Big). \tag{92}$$

Since $\sup_{q\leq t}|B_q|$ has exponential moments of all orders, the martingale property of the stochastic integral in Equation 90 follows. Taking conditional expectations in Equation 90 thus yields the desired supermartingale property

$$w(X^\xi_{t\wedge\tau_k}, R^\xi_{t\wedge\tau_k}) \geq \mathbb{E}[\,w(X^\xi_{T\wedge\tau_k}, R^\xi_{T\wedge\tau_k})|\mathcal{F}_t\,]. \tag{93}$$

The integrability of $w(X^\xi_{t\wedge\tau_k}, R^\xi_{t\wedge\tau_k})$ follows from Lemma 14 and Equation 78 in a similar way as in Equation 92. □

Lemma 17. *Let $\hat{\xi}$ be defined by*

$$\hat{\xi}_t := c(X_t^{\hat{\xi}}, R_t^{\hat{\xi}}). \tag{94}$$

Then $\hat{\xi}$ is admissible for optimal liquidation and maximization of asymptotic portfolio value and satisfies $\int_0^\infty \hat{\xi}_t^2\,dt < K$ for some constant K. Furthermore, $w(X_t^{\hat{\xi}}, R_t^{\hat{\xi}})$ is a martingale and

$$w(X_0, R_0) = \lim_{t\to\infty}\mathbb{E}[\,u(R_t^{\hat{\xi}})\,] \leq v_2(X_0, R_0). \tag{95}$$

Proof. By Equations 35 and 65, $X_t^{\hat{\xi}} > 0$ is bounded from above by an exponentially decreasing function of t. Therefore $\hat{\xi}$ is also bounded by such a function and $\int_0^\infty \hat{\xi}_t^2\,dt < K$ for some constant K, showing that $\hat{\xi}$ is admissible both for optimal liquidation and maximization of asymptotic portfolio value. Next, with the choice $\xi = \hat{\xi}$ the rightmost integral in Equation 89 vanishes, and we get equality in Equation 93. Since $\tau_K^{\hat{\xi}} = \infty$, this proves the martingale property of $w(X_t^{\hat{\xi}}, R_t^{\hat{\xi}})$. Furthermore, we obtain from Equation 76 that

$$u(R_t^{\hat{\xi}}) \geq w(X_t^{\hat{\xi}}, R_t^{\hat{\xi}}) \geq u(R_t^{\hat{\xi}}) \exp(\alpha (X_t^{\hat{\xi}})^2). \tag{96}$$

Since $X_t^{\hat{\xi}}$ is bounded by an exponentially decreasing function, we obtain Equation 95. □

Proposition 18. *Consider the case of the asymptotic maximization of the portfolio value. We have $v_2 = w$ and the a.s. unique optimal strategy is given by $\hat{\xi}$ respectively c.*

Proof. By Lemma 17, we already have $w \leq v_2$. Hence we only need to show that $v_2 \leq w$. Let ξ be any admissible strategy such that

$$\lim_{t\to\infty} \mathbb{E}[u(R_t^{\xi})] > -\infty. \tag{97}$$

By Lemmas 16 and 14 we have for all k, t and $(\tau_k) := (\tau_k^{\xi})$

$$w(X_0, R_0) \geq \mathbb{E}[w(X_{t\wedge\tau_k}^{\xi}, R_{t\wedge\tau_k}^{\xi})] \geq \mathbb{E}\left[u(R_{t\wedge\tau_k}^{\xi}) \exp(\alpha (X_{t\wedge\tau_k}^{\xi})^2)\right]. \tag{98}$$

As in the proof of Lemma 15 one shows that

$$\liminf_{k\to\infty} \mathbb{E}\left[u(R_{t\wedge\tau_k}^{\xi}) \exp(\alpha (X_{t\wedge\tau_k}^{\xi})^2)\right] \geq \liminf_{k\to\infty} \mathbb{E}\left[u(R_t^{\xi}) \exp(\alpha (X_{t\wedge\tau_k}^{\xi})^2)\right] = \mathbb{E}\left[u(R_t^{\xi}) \exp(\alpha (X_t^{\xi})^2)\right]. \tag{99}$$

Hence,

$$w(X_0, R_0) \geq \mathbb{E}[u(R_t^{\xi})] + \mathbb{E}\left[u(R_t^{\xi})(\exp(\alpha (X_t^{\xi})^2) - 1)\right]. \tag{100}$$

Let us assume for a moment that the second expectation on the right attains values arbitrarily close to zero. Then

$$w(X_0, R_0) \geq \lim_{t\to\infty} \mathbb{E}[u(R_t^{\xi})]. \tag{101}$$

Taking the supremum over all admissible strategies ξ gives $v_2 \leq w$. The optimality of $\hat{\xi}$ follows from Lemma 17, its uniqueness from the fact that c is the unique solution to the HJB Equation 64.

We now show that $\mathbb{E}\left[u(R_t^{\xi})(\exp(\alpha (X_t^{\xi})^2) - 1)\right]$ attains values arbitrarily close to zero. By Lemma 15 and the same line of reasoning as in the proof of Lemma 16, we have for all k, t and $(\tau_k) := (\tau_k^{\xi})$

$$-\infty < \lim_{s\to\infty} \mathbb{E}[u(R_s^{\xi})] \leq \mathbb{E}[u(R_t^{\xi})] \leq \mathbb{E}[u(R_{t\wedge\tau_k}^{\xi})] \tag{102}$$

$$= u(R_0) + \mathbb{E}\left[\int_0^{t\wedge\tau_k} u_R(R_s^{\xi})\sigma X_s^{\xi}\,dB_s\right] - \mathbb{E}\left[\int_0^{t\wedge\tau_k} \left[\lambda u_R \xi_s^2 - \frac{1}{2}(\sigma X_s^{\xi})^2 u_{RR}\right](R_s^{\xi})\,ds\right] \tag{103}$$

$$= u(R_0) - \mathbb{E}\left[\int_0^{t\wedge\tau_k} \left[\lambda u_R \xi_s^2 - \frac{1}{2}(\sigma X_s^{\xi})^2 u_{RR}\right](R_s^{\xi})\,ds\right]. \tag{104}$$

Sending k and t to infinity yields

$$\int_0^\infty \mathbb{E}\left[(X_s^{\xi})^2 u_{RR}(R_s^{\xi})\right]\,ds > -\infty. \tag{105}$$

Next we observe that

$$0 \geq u(R) \geq a_5 u_{RR}(R) \tag{106}$$

for a constant $a_5 > 0$, due to the boundedness of the risk aversion of u, and that

$$\exp(\alpha (X_t^{\xi})^2) - 1 \leq a_6 \alpha (X_t^{\xi})^2, \tag{107}$$

due to the bound on X_t^{ξ}. We now have

$$0 \geq \mathbb{E}\left[u(R_t^{\xi})(\exp(\alpha (X_t^{\xi})^2) - 1)\right] \geq \mathbb{E}[\alpha a_5 a_6 u_{RR}(R_t^{\xi})(X_t^{\xi})^2]. \tag{108}$$

Therefore the right hand side of the above equation attains values arbitrarily close to zero. □

Proposition 19. *Consider the case of optimal liquidation. Then $v_1 = w$ and the a.s. unique optimal strategy is given by $\hat{\xi}$ respectively c.*

Proof. For any strategy ξ that is admissible for optimal liquidation, the martingale $\sigma \int_0^t X_s dB_s$ is uniformly integrable due to the requirement in Equation 7. Therefore $\mathbb{E}[u(R_t^\xi)] \geq \mathbb{E}\left[u(R_\infty^\xi)\right]$ follows as in the proof of Lemma 15. Hence, Proposition 18 yields

$$\mathbb{E}[u(R_\infty^\xi)] = \lim_{t\to\infty} \mathbb{E}[u(R_t^\xi)] \leq v_2(X_0, R_0) \leq w(X_0, R_0). \tag{109}$$

Taking the supremum over all admissible strategies ξ gives $v_1 \leq w$. The converse inequality follows from Lemma 16, since $\hat{\xi}$ is admissible for optimal liquidation. □

5.3 Characterization of the optimal adaptive strategy

Proof of Theorem 6. We prove the equivalent inequality $\tilde{c}^1 \geq \tilde{c}^0$. Fix $N > 0$ and let f^i denote the function $\tilde{f}_N$ constructed in the proof of Proposition 10 when the parabolic boundary condition is given by $\tilde{f}_N(Y, R) = \sqrt{\sigma^2 A^i(R)/(2\lambda)}$ for $Y = 0$ or $|R| = N$. The result follows if we can show that $g := f^1 - f^0 \geq 0$. A straightforward computation shows that g solves the linear PDE

$$g_Y = -\frac{3}{2}\lambda\big(f^1 g_R + f_R^0 g\big) + \frac{\sigma^2}{4} f_{RR}^1\Big(\frac{1}{f^1} - \frac{1}{f^0}\Big) + \frac{\sigma^2}{4f^0} g_{RR} \tag{110}$$

$$= \frac{1}{2} a g_{RR} + b g_R + V g, \tag{111}$$

where the coefficients a and b and the potential V are given by

$$a = \frac{\sigma^2}{2f^0}, \qquad b = -\frac{3}{2}\lambda f^1, \qquad \text{and} \qquad V = -\frac{\sigma^2 f_{RR}^1}{4 f^0 f^1} - \frac{3}{2}\lambda f_R^0. \tag{112}$$

The parabolic boundary condition of g is

$$g(Y, R) = \sqrt{\frac{\sigma^2 A^1(R)}{2\lambda}} - \sqrt{\frac{\sigma^2 A^0(R)}{2\lambda}} =: h(R) \qquad \text{for } Y = 0 \text{ or } |R| = N. \tag{113}$$

The functions a, b, V, and h are smooth and (at least locally) bounded on $\mathbb{R}_+ \times [-N, N]$, and a is bounded away from zero. Next, take $T > 0$, $R \in]-N, N[$, and let Z be the solution of the stochastic differential equation

$$dZ_t = \sqrt{a(T - t, Z_t)}\, dB_t + b(T - t, Z_t)dt, \qquad Z_0 = R, \tag{114}$$

which is defined up to time

$$\tau := \inf\big\{t \geq 0 \,\big|\, |Z_t| = N \text{ or } t = T\big\}. \tag{115}$$

By a standard Feynman-Kac argument, g can then be represented as

$$g(T, R) = \mathbb{E}\Big[h(Z_\tau)\exp\Big(\int_0^\tau V(T - t, Z_t)\, dt\Big)\Big]. \tag{116}$$

Hence $g \geq 0$ as $h \geq 0$ by assumption. □

Proof of Theorem 5. In Theorem 6 take $u^0(x) := u(x)$ and $u^1(x) := u(x+r)$. If u exhibits IARA, then $A^1 \geq A^0$ if $r > 0$ and hence $c^1 \geq c^0 = c$. But we clearly have $c^1(X, R) = c(X, R + r)$. The result for decreasing A follows by taking $r < 0$. □

The following proof follows the same setup as the proof of Theorem 6. The line of argument however is analytic and not probabilistic.

Proof of Theorem 7. Let $\lambda^1 > \lambda^0$ be two positive constants. Fix $N > 0$ and let f^i denote the function $\tilde{f}_N$ constructed in the proof of Proposition 10 with $\lambda = \lambda^i$. The result follows if we can show that $g := f^0 - f^1 \geq 0$. Let us assume by way of contradiction that (Y_0, R_0) is a root of g with minimal Y_0. The point (Y_0, R_0) does not lie on the boundary of the strip $\mathbb{R}_0^+ \times [-N, N]$ since $g > 0$ on the boundary due to Equation 34. We therefore have that (Y_0, R_0) is a local minimum in $]0, Y_0] \times]-N, N[$ and a root. Hence

$$g(Y_0, R_0) = 0 \qquad \Rightarrow f^0 = f^1 \tag{117}$$
$$g_Y(Y_0, R_0) \leq 0 \tag{118}$$
$$g_R(Y_0, R_0) = 0 \qquad \Rightarrow f_R^0 = f_R^1 \tag{119}$$
$$g_{RR}(Y_0, R_0) \geq 0 \tag{120}$$

By Equation 33, we now have

$$0 \geq g_Y(Y_0, R_0) \tag{121}$$
$$= f_Y^0 - f_Y^1 \tag{122}$$
$$= \left(-\frac{3}{2}\lambda^0 f^0 f_R^0 + \frac{\sigma^2}{4f^0} f_{RR}^0\right) - \left(-\frac{3}{2}\lambda^1 f^1 f_R^1 + \frac{\sigma^2}{4f^1} f_{RR}^1\right) \tag{123}$$
$$= -\frac{3}{2}(\lambda^0 - \lambda^1) f^0 f_R^0 + \frac{\sigma^2}{4f^0} g_{RR} \tag{124}$$
$$> 0. \tag{125}$$

The last inequality uses that $f_R^0 > 0$, which holds for IARA utility function u by Theorem 5. The established contradiction leads us to conclude that g does not have any roots and thus that $f^0 > f^1$. □

Proof of Theorem 8. Equation 29 holds since $\tilde{d}(Y, R) = \tilde{c}\left(\frac{\sigma_1^2}{\sigma_2^2} Y, R, \frac{\sigma_2^2}{\sigma_1^2}\lambda, \sigma_2\right)$ is a solution of Equation 21 with $\sigma = \sigma_1$. □

Proof of Theorem 9. First, it follows immediately from the definition of v in Equation 11 that $R \mapsto v(X, R)$ is strictly increasing. Next, take distinct pairs (R_1, X_1), (R_2, X_2) and let $0 < \alpha < 1$ be given. Select the optimal strategies $\hat{\xi}^1, \hat{\xi}^2 \in \mathcal{X}$ such that $v(X_i, R_i) = \mathbb{E}[u(R_\infty^{\hat{\xi}^i})]$ for $i = 1, 2$. Define $\xi := \alpha\hat{\xi}^1 + (1-\alpha)\hat{\xi}^2$. Then

$$v(\alpha X_1 + (1-\alpha)X_2, \alpha R_1 + (1-\alpha)R_2) \geq \mathbb{E}[u(R_\infty^{\xi})] \tag{126}$$
$$> \mathbb{E}[u(\alpha R_\infty^{\hat{\xi}^1} + (1-\alpha)R_\infty^{\hat{\xi}^2})] \tag{127}$$
$$> \alpha\mathbb{E}[u(R_\infty^{\hat{\xi}^1})] + (1-\alpha)\mathbb{E}[u(R_\infty^{\hat{\xi}^2})] \tag{128}$$
$$= \alpha v(X_1, R_1) + (1-\alpha) v(X_2, R_2). \tag{129}$$

Hence v is strictly concave. By Proposition 12, we know that v is decreasing in X. Equation 30 follows immediately from Equation 66. □

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