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Abstract

I prove existence of stationary recursive competitive equilibrium in Bewley economies with production under specifications in which (i) utility function is allowed to be unbounded, and (ii) the underlying discrete idiosyncratic productivity process can take any form, aside from mild restrictions. Some of the intermediate results provide theoretical basis for assumptions often made in the quantitative macroeconomics literature. By providing an example, I illustrate that equilibrium is not necessarily unique, even under typical specifications of the model, and discuss the underlying reasons for multiplicity.

1 Introduction

A large body of macroeconomics literature is devoted to the study of income and wealth inequality and their impact on macroeconomic variables. While economic models without frictions proved useful in answering many macroeconomic questions of interest, due to their stark prediction of "no cross-sectional heterogeneity", they fell short of analyzing any sort of inequality.[1] Stepping outside the boundaries of representative-agent/complete-markets paradigm, many economists working on inequality used Bewley (1984, 1986) models to capture the interplay between financial frictions, cross-sectional heterogeneity, and macroeconomic variables. This class of models had been used extensively to analyze excess sensitivity of

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[1]Unless, of course, ex ante heterogeneity is exogenously imposed, for instance, by using heterogeneous preferences.

consumption to temporary changes in income, equity-premium and "low risk-free rate" puzzles, the relationship between micro and macro estimates of labor supply elasticity, welfare costs of inflation and business cycles, as well as normative and positive implications of income taxation.

Surprisingly, despite this vast literature that dates back to the 1970s, the issue of existence of equilibrium in these models remains unresolved except under restrictive assumptions that do not correspond to those commonly made in actual applications. To the best of my knowledge, a complete proof of existence of stationary recursive competitive equilibrium in the canonical Bewley model with production, i.e. Aiyagari (1994) model, is still missing from the literature. Some proofs are available in slightly different environments and/or under restrictive assumptions such as bounded utility, and i.i.d. shocks. However, even for a "textbook version" of the model that features constant-relative-risk-aversion (CRRA) utility function, or productivity shocks that exhibit some persistence, the existence of an equilibrium is simply *assumed*, but not rigorously established.

Motivated by this shortcoming, in this paper I provide a proof of existence of stationary recursive competitive equilibrium for Aiyagari (1994) model. I relax two of the most restrictive assumptions for practical purposes: (i) the utility function is allowed to be unbounded, both from above, and from below, and (ii) the discrete idiosyncratic productivity process can take any form, aside from the mild restriction that the lowest idiosyncratic productivity state exhibits some persistence.

Several technical challenges are addressed. Since utility function is not necessarily bounded, I do not use traditional Bellman equation-based methods to establish the existence and uniqueness of the solution to the households' problem. Building on several theoretical results in a recent paper by Li and Stachurski (2014), I use Euler equation-based methods, motivated by Coleman (1990) "policy iteration" approach. This approach has the advantage that one can focus on the properties of the marginal utility function, effectively eliminating the need to impose strong restrictions on the *level* of the utility function.

The first step I take to establish existence and uniqueness of a stationary distribution is to generalize a well-known early result of boundedness of state space under i.i.d. shocks, by Schechtman and Escudero (1977). I show that the same property holds under *weaker* assumptions for arbitrary Markov processes. Second, since the Markov process is allowed to be non-monotone, the proof of existence and uniqueness of stationary distribution does not (and cannot) rely on monotonicity of policy functions with respect to idiosyncratic productivity. The key result used in the several steps of the proof is the fact that every household with finite wealth level is eventually borrowing constrained with positive probability. I show that every agent runs down assets in the lowest productivity state independent of whether the household is impatient with respect to the interest rate or not. This follows as a simple consequence of the fact that in the least productive state, the uncertainty faced by the agents only has an upside potential. As long as this state exhibits some persistence, a positive mass of agents must be borrowing constrained in the long run. This property alone imposes a lot of structure on the joint Markov process over assets and labor productivity, in fact, it is sufficient for the existence and uniqueness of the stationary distribution.

An intermediate result that is crucial for the main theorem in this paper is that the aggregate supply of capital diverges to infinity when interest rate approaches the inverse of the discount rate. In a seminal paper, Chamberlain and Wilson (2000) proved this result when interest rate (or its expectation) *equals* the inverse of the discount rate. In the literature, it is often stated that the former result follows from the latter, since the stationary distribution is continuous in prices. In the following sections, I argue that this argument, although intuitively correct, is not technically accurate, and I provide a constructive proof that does not rely on the main theorem in Chamberlain and Wilson (2000). The proof isolates the importance of occasionally-binding borrowing constraint and its role in precautionary savings while highlighting the irrelevance of prudence (convex marginal utility) for this outcome. Avoiding Martingale Convergence Theorem makes the divergence result of Chamberlain and Wilson (2000) less of a "black box" by rendering the underlying economic forces more transparent. To the best of my knowledge, this is the first attempt to establish this limit result rigorously without imposing any curvature properties on the marginal utility function.

Next, using very standard neoclassical assumptions on the representative firm's production function, I prove that there exist prices that clear all markets, in particular, that there exists an interest rate that equates firm's demand for capital and supply of capital by the households. The main challenge is that the continuity of the stationary distributions with respect to prices is not sufficient to guarantee that capital supply function is continuous in prices, because the state space is not compact *uniformly* over all prices. To deal with this problem, I find an interval for prices over which the desired uniformity requirement is met, and which must contain an equilibrium interest rate if it exists. Existence of equilibrium then follows by standard continuity arguments.

After presenting the main theorem of existence and cover some of the immediate corollaries, I discuss the relevance of these theoretical results for applications. In particular, I provide closed-form expressions for upper bounds on policy and value functions under CRRA-type utility functions. Using a common specification of the model, I illustrate that there can be multiple equilibria and provide an extensive discussion of the underlying reasons for multiplicity. Even though this possibility was well-known, to the best of my knowledge, this is the first numerical example in the literature.

1.1 Literature Review

I view my work as complementary to some of the earlier results. For some cases, I provide strict generalizations. An important building block of Bewley models is households' income fluctuation problem. Classical references include Schechtman and Escudero (1977), Sotomayor (1984), Laitner (1979, 1992), Clarida (1987, 1990), Zeldes (1989), Kimball (1990), Deaton (1991), Carroll (1992) among many others.

In the general equilibrium vein, Bewley (1984, 1986) proved existence of monetary equilibrium in an economy with continuum of agents who face idiosyncratic income shocks. The advance of numerical methods, and the availability of computational power led to a few very influential papers in the general equilibrium tradition. Huggett (1993) takes a general equilibrium approach in a Bewley model without production. He assumes a two-state Markov process that is restricted to be monotone, and provides rigorous proofs related to existence of stationary distribution. His analysis of general equilibrium is numerical. Aiyagari (1994), in a seminal paper, provided an informal proof of existence of recursive competitive equilibrium under the assumptions of bounded utility and i.i.d. shocks. His numerical implementation features an unbounded utility function and a Markov process for which his results do not apply.

Miao (2002), in a more recent paper, considers a continuous monotone Markov process with a strong smoothness condition. He relaxes "boundedness from below" assumption for the utility function while imposing other curvature restrictions. Among most notable contributions is his careful investigation of whether law of large numbers can be readily applied to a continuum of agents, a point that was largely ignored in the earlier literature on Bewley models. He provides an extensive comparative statics analysis that extend to cases of ex ante heterogeneity among the households.

Marcet, Obiols-Homs, and Weil (2007) incorporate endogenous labor supply choice and show that precautionary motive for savings is dampened for the wealth-rich agents due to a dominant wealth effect on labor supply. In their analysis, they assume that the choice set for assets is exogenously bounded from above and labor productivity is restricted to follow a two-state monotone Markov process.[2] Taking this as a starting point, Zhu (2013) proves existence of equilibrium in this environment, imposing bounded utility, relaxing the two-state and monotonicity assumptions on the Markov process, but instead, assuming the transition matrix is positive everywhere. To complement these results, in section 4, I show that an equilibrium exists even when there is no wealth effect on labor supply. In fact, existence of a stationary equilibrium in an economy with Greenwood-Hercowitz-Huffman (GHH) preferences follows as a corollary to my main theorem.[3]

This is not the first attempt on a proof of existence using unbounded utility. Kuhn (2013) uses the specific, but widely used CRRA utility to establish the existence of recursive competitive equilibrium in a different environment with *permanent*, but i.i.d. income shocks, where stationarity is recovered by an exogenous probability of death.[4]

Acemoglu and Jensen (2015) take a different approach and provide a very inclusive proof of existence that not only applies to Aiyagari (1994) model, but also to models of industry dynamics. However, their theorem of existence only applies to the case of bounded utility and exogenously bounded choice set for assets. On the other hand, their main emphasis is on presenting many novel comparative statics in this

[2]The fact that agents optimally choose not to supply labor for large wealth levels eliminates stochasticity of earnings for wealth-rich agents. Interestingly, endogenous labor supply simplifies the existence proof significantly, because the state space is compact even when the interest rate equals the inverse of the discount rate. One can then resort to standard continuity arguments to establish the existence of equilibrium without going through most of the arguments in the next section.

[3]GHH utility index is a monotonic transformation of a quasi-linear function of consumption and leisure, and hence features zero wealth effect on labor supply.

[4]Similar to Marcet, Obiols-Homs, and Weil (2007) and Zhu (2013), in Kuhn's (2013) model, due to exogenous probability of exit, a stationary distribution exists even when the interest rate equals the inverse of the discount rate.

model which most likely apply under more general assumptions.

In the next section, I discuss the baseline model in detail and prove some intermediate results related to households' and representative firm's problem. In section 3, I formally define an equilibrium and present the main theorem of its existence. Section 4 discusses some useful extensions that arise frequently in practice. Section 5 provides a discussion of the relevance of the theoretical results for applications. This is followed by an example of non-uniqueness of equilibrium in section 6. Section 7 concludes.

2 Model

Time $t \in \{0, 1, 2, \dots\}$ is discrete. There is a continuum of ex-ante identical households of measure one, and a representative competitive firm. There are no aggregate shocks.

2.1 Household's Problem

I consider a standard optimal savings/income fluctuation problem. In every period, each household is subject to an idiosyncratic labor productivity shock $e_t \in E = \{e^1, e^2, ..., e^s\}$ with $0 < \underline{e} = e^1 < e^2 < ... < e^s = \bar{e}$ that follows a discrete, first-order Markov process with the transition matrix P. Let $(E, \mathcal{E})$ denote the measurable space for labor productivity where $\mathcal{E}$ denotes all subsets of E. Let $(E^t, \mathcal{E}^t)$ denote the product space of labor productivity shocks up to and including period t.

Financial markets are incomplete and agents only have access to a single risk-free asset a_t. Agents are not allowed to borrow, so that constraint $a_{t+1} \geq 0$ holds for all $t \geq 0$.[5] Let $A = [0, \infty)$ denote the space for assets. Households discount future at a geometric rate $\beta \in (0, 1)$.

Given an initial level of assets a, labor productivity e, exogenous and constant interest rate r, and a wage rate w, the household's problem can be represented as

$$V(a, e) = \sup_{\{c_t, a_{t+1}\}_{t=0}^{\infty}} \mathbb{E} \sum_{t=0}^{\infty} \beta^t u(c_t) \tag{1}$$

subject to the constraints

$$c_t + a_{t+1} \leq (1 + r)a_t + we_t \text{ for all } t \geq 0$$

$$a_{t+1} \geq 0 \text{ for all } t \geq 0$$

$$c_t \text{ and } a_{t+1} \text{ are } \mathcal{E}^t\text{-measurable for all } t \geq 0$$

$$\text{Given } a_0 = a \text{ and } e_0 = e$$

I make the following assumption on the utility function:

[5] In section 4.1, I relax this assumption as an extension to the baseline model.

Assumption 1 *Utility function* $u : \mathbb{R}_{++} \to \mathbb{R}$ *is continuously differentiable, strictly increasing, strictly concave with* $\lim_{c \downarrow 0} u'(c) = \infty$ *and* $\lim_{c \to \infty} u'(c) = 0$.

Arguably, assumption 1 is strong, and is critical for many of the results below. On the other hand, most of the utility functions considered in the applied macroeconomics literature satisfy it. Most importantly, it allows for the utility function to be unbounded both from above and from below. Most of the theoretical literature imposes a strong boundedness assumption, which is violated by some of the most widely used utility functions including CRRA.

Assumption 2 *Interest rate and wage level satisfy* $w > 0$, $r > -1$ *and* $\beta(1 + r) < 1$.

It is straightforward to show, as in Deaton (1991), that the first-order necessary conditions for the households' problem (1) can be written compactly as

$$u'(c_t) = \max\{\beta(1 + r)\mathbb{E}_t u'(c_{t+1}), u'((1 + r)a_t + we_t)\} \tag{2}$$

I also impose the following transversality condition

$$\lim_{t \to \infty} \beta^t \mathbb{E}[u'(c_t)a_{t+1}] = 0 \tag{3}$$

Also consider the functional Euler equation for consumption policy $c(a, e)$

$$u'(c(a, e)) = \max\left\{\beta(1 + r)\mathbb{E}\big\{u'\big[c((1 + r)a + we - c(a, e), e')\big]\big|e\big\}, u'((1 + r)a + we)\right\} \tag{4}$$

In a recent paper, Li and Stachurski (2014) established some of the results below under the assumption that $r > 0$ and with continuous Markov processes with an increasing kernel. I provide a proof in the appendix that follows a very similar methodology, however (i) assuming that $r > -1$, and (ii) without any restrictions on the (discrete) Markov process.[6] Let $\mathcal{C}$ denote the set of continuous functions $c : A \times E \to \mathbb{R}_+$ that are weakly increasing in $a \in A$, satisfy $0 < c(a, e) \leq (1 + r)a + we$ for all $(a, e) \in A \times E$, and $\sup |u'(c(.)) - u'((1 + r)a + we)| < \infty$.

Proposition 1 *Under assumptions 1 and 2,*

1. *For any initial state* (a, e), $V(a, e)$ *is finite, i.e.* $|V(a, e)| < \infty$.
2. *There exists a unique solution* $c \in \mathcal{C}$ *to the functional equation* (4).
3. *(Li and Stachurski (2014)) If a feasible plan satisfies the Euler equation* (2) *and the transversality condition* (3)*, then it is the unique optimal plan.*

[6]Extension to $r > -1$ case is important for the existence proof for completeness. Due to precautionary savings motives, under extreme parameter values on labor productivity process and capital demand function, one can in principle support an equilibrium with $r < 0$. This case typically does not arise under reasonable parameterizations of the model.

4. *(Li and Stachurski (2014)) Consumption time series generated by $c(a,e)$ solves the household's sequential problem* (1).

As part of their proof (and also key to the proof in this paper), Li and Stachurski (2014) use the implicit Coleman (1990) operator

$$u'(Kc(a,e)) = \max\left\{\beta(1+r)\mathbb{E}\{u'\left[c((1+r)a+we-Kc(a,e),e')\right]\big|e\},u'((1+r)a+we)\right\}.$$

Clearly the fixed point $Kc = c$ solves the functional equation (4). Define the metric $\rho(c,d) \equiv \sup|u'(c)-u'(d)|$. The proof uses the fact that this operator maps $\mathcal{C}$ into itself. Furthermore, the pair $(\mathcal{C},\rho)$ is a complete metric space, and operator K is a contraction mapping of modulus $\beta(1+r) < 1$. The uniqueness of the solution in $\mathcal{C}$ then follows by Banach's Contraction Mapping Theorem.

Next, I characterize properties of the policy functions. For what is to follow, let $\omega(e) \equiv \frac{r}{1+r}we + \frac{1}{1+r}w\underline{e}$ denote the annuitized present value of lifetime earnings under the "worst-case scenario", i.e. the profile $\{we, w\underline{e}, w\underline{e}...\}$. Clearly, this function satisfies $\omega(\underline{e}) = w\underline{e}$ and $\omega(e) > \omega(e')$ if and only if $e > e'$. Also define asset demand (saving) policy $a' = g(a,e) \equiv (1+r)a + we - c(a,e)$.

Proposition 2 *Under assumptions 1 and 2*

1. *The consumption function $c(a,e)$ is continuous and weakly increasing in a. Moreover, it satisfies $c(a,e) \geq w\underline{e}$ for all $r > -1$, and $c(a,e) \geq ra + \omega(e)$ when $r > 0$.*
2. *The saving function $g(a,e)$ is continuous and weakly increasing in a.*
3. *For each $e \in E$, $\lim_{a\to\infty} g(a,e) = \lim_{a\to\infty} c(a,e) = \infty$.*

Some of the results in proposition 2 were established earlier in the literature. The fact that consumption and saving policy are continuous and increasing in wealth were covered in many papers, e.g. Schechtman and Escudero (1977), Laitner (1992), Aiyagari (1994), Miao (2002). The lower bound for consumption function, i.e. $ra + \omega(e)$, is in fact the analytical solution for a permanent income consumer in an alternative model with $\beta(1+r) = 1$ and the certain sequence of earnings $\{we, w\underline{e}, w\underline{e}...\}$. Since the agent is impatient in the current model, and $\omega(e)$ is the worst-case scenario, current consumption level in this environment must be higher in comparison. To the best of my knowledge this lower bound is novel. Limit results in item 3 in proposition 2 were proven by Chamberlain and Wilson (2000) under bounded utility assumption.

The following result, which I present as a separate proposition, is used repeatedly to prove many of the results. In this environment, households always run down assets in the lowest productivity state and this property holds regardless of whether the agent is impatient or not. In fact, the proof I provide in the appendix only imposes the *weak* inequality $\beta(1+r) \leq 1$ for the case in which $r > 0$. This result is a consequence of the fact that there is a positive probability of reaching a better state next period and no possibility of reaching a lower productivity state. If the agent does not run down assets, consumption would be weakly higher in all states in the next period, and if it is possible to move up in productivity, *strictly* higher in at least one accessible state. The proof exploits the fact that this cannot be optimal for a consumption-smoothing agent even if the agent is patient: Agent would be better off by marginally running down assets, consuming more this period and less in the following period.

Proposition 3 *Suppose $P_{1j} > 0$ for some $j > 1$, and assumptions 1 and 2 hold. Then assets always decline in the lowest productivity state, i.e. $g(a, \underline{e}) < a$ for all $a > 0$.*

Next, I impose additional structure on the utility function to derive some key properties. The assumption below states that the degree of absolute risk aversion converges to zero as consumption tends to infinity.

Assumption 3 *Utility function is twice continuously differentiable and satisfies $\liminf_{c\to\infty} -\frac{u''(c)}{u'(c)} = 0$.*

I use assumption 3 to establish compactness of the state space. It is weaker than the "asymptotic exponent" assumption made originally by Brock and Gale (1969) and subsequently used by Schechtman and Escudero (1977) to establish the compactness result for the case of i.i.d. process.[7] Rabault (2002) used an analogue of assumption 3 to prove boundedness of the state space in the case of i.i.d. earnings process, and I generalize it to arbitrary Markov processes. This assumption ensures that as wealth level gets large, influence of stochastic earnings on consumption/savings gets arbitrarily small. CARA utility violates this assumption due to absence of wealth effect, and not surprisingly, when there is sufficient stochasticity in labor earnings, assets blow up to infinity even if the agent is impatient relative to the interest rate. (See Schechtman and Escudero (1977) for an example.) The proof in the appendix highlights the fact that, provided that E is bounded, compactness of the state space is a consequence of the preferences, and it is independent of the underlying earnings process. Calibrated versions of these models typically use non-i.i.d. Markov processes some of which do not even satisfy monotonicity, and in this sense, this proposition provides theoretical foundations for the compactness assumption implicitly made in this literature.

Proposition 4 *Under assumptions 1, 2 and 3, state space for the household's problem can be chosen to be compact, i.e. there exists a finite $\bar{a} \geq 0$ such that $g(a, e) < a$ for all $a > \bar{a}$ and all $e \in E$.*

Let $\mathcal{A}$ represent the Borel σ-algebra over $[0, \bar{a}]$ and Σ represents the product σ-algebra over $S = [0, \bar{a}] \times E$. Define the following transition function $Q : S \times \Sigma \to \mathbb{R}_+$ for the Markov process over S.

$$Q((a, e), C) = \begin{cases} Pr(e' \in C_E | e) & g(a, e) \in C_A \\ 0 & g(a, e) \notin C_A \end{cases} \tag{5}$$

for all $a \in [0, \bar{a}]$, $e \in E$, $C \in \Sigma$, where $C_A \in \mathcal{A}$ and $C_E \in \mathcal{E}$ represent the projection of C on $[0, \bar{a}]$ and E respectively.

[7] Asymptotic exponent assumption states: Utility function satisfies $\lim_{c\to\infty} -\frac{\log u'(c)}{\log c} = \sigma$ for some $\sigma > 0$. It is easy to show that this assumption includes all marginal utility functions that satisfy $u'(c) = c^{-\sigma}\phi(c)$ where $\phi(c)$ is any continuous function that satisfies $\lim_{c\to\infty} \frac{\log \phi(c)}{\log(c)} = 0$. Clearly, any CRRA utility function satisfies it. Another example that is not economically motivated is $u'(c) = c^{-\sigma} \log(c+1)$, which also satisfies assumption 1 when $\sigma > 1$.

The assumptions made so far are sufficient to ensure that a stationary distribution exists since $Q(.,.)$ has Feller property and the state space is compact (See Stokey, Lucas, and Prescott (1989) Theorem 12.10). We need to impose more discipline on the labor productivity process to make sure that the stationary distribution is unique. It turns out the only nontrivial assumption we require is that the lowest productivity state exhibits some persistence.

Assumption 4 *Markov chain for labor productivity $e \in E$ is irreducible, aperiodic, and the transition matrix satisfies $P_{11} > 0$.*

It is well known that irreducibility and aperiodicity assumptions together are equivalent to the following statement: There exists $m_0 > 0$ such that $[P^m]_{ij} > 0$ for all i, j and all $m \geq m_0$, there is a strictly positive probability of reaching any state from any other state in m_0 (or more) periods. This implies, in particular, that the unique limiting distribution has full support.

The following proposition establishes the uniqueness of the stationary distribution. The key to the proof is the fact that state $(0, \underline{e})$ is an accessible state with positive mass.[8] This property follows from proposition 3 and the persistence of the lowest earnings state. Due to ergodicity of the earnings Markov process, every agent reaches the state with lowest earnings with positive probability. Moreover, provided that this state is persistent, and that agents run down assets in this state (by proposition 3), there is a strictly positive probability of hitting the borrowing constraint starting from any state. When the transition function exhibits this property, state $(0, \underline{e})$ must be in the support of all stationary distributions. If, in addition, it has a positive mass (as in this case), there can only be one such distribution. For this final step, I use a uniform ergodicity theorem by Meyn and Tweedie (2009), which establishes this idea formally. The steps involved are similar in spirit to those in Benhabib, Bisin, and Zhu (2015) and Zhu (2013).[9]

Proposition 5 *Under assumptions 1,2, 3, and 4, there exists a unique stationary distribution for the Markov process with the transition function Q.*

Remark: Condition $P_{11} > 0$ in assumption 4 is not necessary. All we require is the existence of a "worst" sequence of productivities originating from the lowest productivity state. For instance it can be replaced by the following assumption: The sequence of lowest accessible states from $\underline{e}$, $\underline{\mathbf{e}} \equiv \{\underline{e}_0, \underline{e}_1 ... \underline{e}_t, ...\}$ is dominated pointwise by all sample paths originating from $\underline{e}$.[10] This assumption is weaker, however it comes at the cost of expanding the proof by several steps without providing any new insight.

[8] In an exchange economy, Krebs (2004) proved that *if* the state space is compact, borrowing constraint must bind for some agents at a stationary equilibrium. This property is not sufficient for uniqueness even when the Markov process on E is ergodic. Using the additional structure imposed by $P_{11} > 0$, the proof in the appendix finds a uniform lower bound on the probability $\underline{q} > 0$, and an upper bound on number of periods T such that any agent in the economy reaches state $(0, \underline{e})$ in T periods with probability of at least $\underline{q}$.

[9] Recently, Foss et al. (2015) provided a proof of uniqueness using an alternative approach that applies under CRRA utility and potentially unbounded, but monotone Markov processes.

[10] To clarify, we let $\underline{e}_0 = \underline{e}$ and inductively define $\underline{e}_t = \min\{e \in E | Pr(e|\underline{e}_{t-1}) > 0\}$. Let $\{e_t\}$ be any sample path with $e_0 = \underline{e}$. We require that $e_t \geq \underline{e}_t$ for all t almost surely. It is trivial to show that if the assumption holds, the deterministic sequence $\underline{\mathbf{e}}$ is either constant, which is the case under my stronger assumption $P_{11} > 0$ above, or exhibits a cycle of finite length (under the assumption that the process is ergodic), i.e. $\underline{e}_T = \underline{e}$ for some $T > 1$. In the latter case, we can show that assets decline over each cycle of length T, i.e. $a_{t+T} < a_t$ whenever $e_t = e_{t+T} = \underline{e}$.

Given interest rate r let $g(.;r)$, $Q(.;r)$ and $\mu(.;r)$ represent the policy function for saving, the associated transition function and the (unique) stationary distribution respectively. Let $A(r) \equiv \int a d\mu(.;r)$ be the aggregate supply of capital (demand for assets) at the stationary distribution and define $\bar{r} \equiv \frac{1}{\beta} - 1$.

Main existence result in this paper relies on the following lemma and proposition, which state that asset demand diverges to infinity as the interest rate approaches the inverse of the discount rate. At first glance, looking at the literature, this should not come as a surprise. In a seminal paper, Chamberlain and Wilson (2000) proved that assets blow up to infinity under very mild assumptions when equality holds, i.e. when $r = \bar{r}$. However, to a large extent, this result remained a "black box", since they invoked the powerful Martingale Convergence Theorem to establish it. Their paper did not feature a motivation of this result, as the authors themselves were puzzled by the fact that (i) it does not depend on prudence ($u''' > 0$), an assumption typically made to generate precautionary savings, and that (ii) assets get arbitrarily large only under infinite horizon. The proof I provide in the appendix is constructive and deliberately avoids having to use Martingale Convergence Theorem. In particular, it highlights the fact that contingency of being borrowing constrained in the future is responsible for this result, even if marginal utility is not convex. The point that in the presence of liquidity constraints, agents engage in precautionary savings independent of the curvature of the marginal utility was made earlier by Deaton (1991), Huggett and Ospina (2001), Carroll and Kimball (2005) among others.[11] The proof in the appendix reveals that little else matters for the divergence of assets. Obviously, under finite horizon, this motive is absent for large initial wealth levels because borrowing constraint never binds even in the worst-case scenario. It is not surprising that under the extreme case of quadratic utility for which marginal utility is linear, when $r = \bar{r}$ and for large initial wealth levels, a finite horizon model predicts expected value of consumption to be constant and equal to time-0 income, whereas infinite time horizon version of the same model predicts a tendency of consumption to rise over time.

A second caveat is that main theorem by Chamberlain and Wilson (2000) is silent about the behavior of stationary aggregate asset demand as $r \uparrow \bar{r}$. Some earlier literature argued that this limit result follows as a corollary due to Theorem 12.13 by Stokey, Lucas, and Prescott (1989) on the parametric continuity of the stationary distributions. Unfortunately, this theorem does *not* apply as claimed, since it requires the state space to be *uniformly* compact over all prices (r, w), which is clearly violated in this model. In fact, I claim that *no* continuity theorem would help establish this result, because the stationary distribution does not even exist when $r = \bar{r}$. Technically, stationary aggregate asset demand function $A(r)$ is only defined on the open set $r \in (-1, \frac{1}{\beta} - 1)$, therefore even if one establishes continuity of $A(r)$, it does not readily

[11] Huggett and Ospina (2001) proved the following statement: If there exists an equilibrium, then the aggregate precautionary savings is positive if and only if there is a positive mass of agents who are borrowing constrained at the stationary distribution. My chain of arguments, in some sense, go in the opposite direction: I exploit the property that any agent with finite wealth, independent of impatience, (i.e. as long as *weak inequality* $\beta(1+r) \leq 1$ holds) has a precautionary motive due to presence of borrowing constraints to prove the existence of equilibrium. Carroll and Kimball (2005) take a different approach: They show that under quadratic utility, otherwise linear consumption function becomes concave due to the occasionally binding liquidity constraints. They show that this non-linearity of the policy function is intimately related to the precautionary savings motive.

imply $\lim_{r\uparrow\bar{r}} A(r) = \infty$. In this sense, the limit result I establish below is not just an alternative proof to that of Chamberlain and Wilson (2000), it is *essential* for the main existence theorem. Laitner (1992) established this limit result without the continuity argument, however, his argument relied on the positive third derivative of the utility function. To the best of my knowledge, this result has not been established rigorously without making a reference to curvature of marginal utility function.

A critical step for proving the divergence result is the following lemma, which states that as $r \uparrow \bar{r}$, the measure of agents with low assets at the stationary distribution gets arbitrarily small.

Lemma 1 *Under assumptions 1,2, 3, and 4, for any $L > 0$, $\lim_{r\uparrow\bar{r}} \mu\big([0, L] \times E; r\big) = 0$*

The proof is technical and can be found in the appendix. An interpretation of the proof is as follows: Let θ_t be the value of the Lagrange multiplier for the borrowing constraint in period t. Then for any period t and $T > 0$, we can derive the following T-period-ahead Euler equation inductively

$$u'(c_t) = \beta(1+r)\mathbb{E}_t\big(u'(c_{t+1})\big) + \theta_t = [\beta(1+r)]^T \mathbb{E}_t\big(u'(c_{t+T})\big) + \sum_{j=0}^{T-1} [\beta(1+r)]^j \mathbb{E}_t(\theta_{t+j})$$

Observe that if the agent expects to hit the borrowing constraint sometime in the next T periods, she would consume less and save *more* today, than she would otherwise, since the last term is positive. From this perspective, I interpret the last term as a measure of the strength of the precautionary savings motive that arises from the contingency of hitting the borrowing constraint sometime in the next T periods. Consider an agent with asset level $a_t \in [0, L]$ where L is some positive number. In a nutshell, I show that there is a period $t + T$ for which the strength of the precautionary motive for all such agents is bounded from below by some $\underline{\eta}$ uniformly over all large interest rates $r < \bar{r}$, i.e. we can write

$$u'(c_t) \geq [\beta(1+r)]^T \mathbb{E}_t\big(u'(c_{t+T})\big) + [\beta(1+r)]^{T-1} \underline{\eta} \text{ for all large } r < \bar{r}$$

for some constant $\underline{\eta} > 0$. When t is large enough so that I can treat all variables as ergodic, the strength of precautionary motive is positive at an *aggregate* level. Not surprisingly this measure is at least $p(r)\underline{\eta}$, where $p(r)$ is the stationary measure of agents with $a \in [0, L]$ when interest rate is r. The very reason this model admits a non-trivial stationary distribution is the fact that impatience and precautionary motive for savings are two opposing forces. At any non-trivial stationary distribution, these two forces cancel each other out *exactly*, otherwise either assets wander off to infinity, or all agents run assets down to the liquidity constraint. Since assets remain bounded for all large $r < \bar{r}$, it must be the case that the strength of the aggregate precautionary motive of at least $p(r)\underline{\eta}$ is weakly dominated by the level of impatience, $1 - \beta(1+r)$. Then, as impatience level converges to zero when $r \uparrow \bar{r}$, so must $p(r)$. This is shown rigorously in the appendix.

The next proposition establishes that the asset demand diverges to infinity as $r \uparrow \bar{r}$ and it follows immediately from Lemma 1.

Proposition 6 *Under assumptions 1,2, 3, and 4,* $\lim_{r\uparrow\bar{r}} A(r) = \infty$

Proof. Take any $L > 0$. By Markov inequality $1 - \mu\big([0, L] \times E; r\big) = Pr(a > L; r) \leq \frac{A(r)}{L}$, or equivalently, $L\big(1 - \mu\big([0, L] \times E; r\big)\big) \leq A(r)$ holds. The measure $\mu(.; r)$ converges to zero as $r \uparrow \bar{r}$ by Lemma 1, therefore we have $\liminf_{r\uparrow\bar{r}} A(r) \geq L$. Since L is arbitrary, the result follows.■

Remark: Proofs of lemma 1 and proposition 6 require neither the uniqueness of the stationary distribution, nor boundedness of the state space. In this sense, assumption 3 is too strong as a sufficiency condition. Essentially, along with assumptions 1, 2, and 4, any condition that ensures existence of a stationary distribution for all relevant prices can replace assumption 3.

2.2 Firm's Problem

There is a representative firm renting capital at rate r_t and employing labor at rate w_t. The representative firm produces output with technology $f : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$. Define output net of depreciation

$$F(K, N) = f(K, N) - \delta K$$

where $\delta \in (0, 1)$.

Assumption 5 *Production function* f *is constant-returns-to-scale(CRS), strictly increasing, strictly concave, continuously differentiable, and satisfies* $\lim_{K\to 0} f(K, 1) = 0$*,* $\lim_{K\to 0} f_1(K, 1) = \infty$*,* $\lim_{K\to\infty} f_1(K, 1) < \delta$*.*

Profit maximization implies

$$r = F_1(K, 1) \tag{6}$$
$$w = F_2(K, 1)$$

Let $K(r) \equiv F_1^{-1}(r, 1)$ represent the demand for capital when the interest rate equals r and let $w(r) \equiv F_2(K(r), 1)$ represent the corresponding wage level. The following properties follow trivially from assumption 5 and I state them without proof.

Lemma 2 *Under assumption 5,*

1. $K(r)$ *and* $w(r)$ *are strictly decreasing and continuous,*
2. $\lim_{r\uparrow\bar{r}} K(r) = \bar{K} < \infty$*,*
3. *There exists* $\underline{r} \in [-\delta, 0)$ *such that* $\lim_{r\downarrow\underline{r}} K(r) = \infty$*,*
4. $0 < w(r) < \infty$ *for all* $r \in (\underline{r}, \bar{r}]$*.*

3 Equilibrium

I define a stationary recursive competitive equilibrium in the standard way.

Definition 1 *A stationary recursive competitive equilibrium (RCE) consists of prices* (r, w)*, value function* $V : A \times E \to \mathbb{R}$*, policy function* $g : A \times E \to A$*, and a probability measure* $\mu : \Sigma \to [0, 1]$ *such that,*

1. *Given prices* (r, w)*, the value function* $V(a, e)$ *and policy function* $g(a, e)$ *solve the household's problem.*
2. *Given prices* (r, w)*, the representative firm maximizes profits, i.e., capital demand* K *satisfies conditions* (6).
3. *Prices* (r, w) *clear markets:* $N = \mathbb{E}(e) = 1$ *and*

$$A(r, w) \equiv \int a d\mu = K(r)$$

4. *The probability measure* μ *is invariant with respect to the transition function* (5)*, i.e*

$$\mu(C) = \int Q((a, e), C) d\mu \text{ for all } C \in \Sigma$$

Next I present the main theorem of existence. Many steps of the proof are technical and the proof can be found in the appendix. Broadly speaking, the proof involves showing that there exist prices (r, w) that clear markets. The main challenge is that continuity of the stationary distribution with respect to prices does not imply continuity of the means of these distributions, i.e. aggregate capital supply function $A(r)$, because the state space is not uniformly compact over all prices.[12] To deal with this problem, I find an interval for r over which the state space is uniformly compact, and which must contain an equilibrium interest rate if it exists. The existence of an equilibrium then follows by standard continuity arguments.

Theorem 1 *Under assumptions 1, 3, 4, and 5, there exists a stationary recursive competitive equilibrium.*

4 Extensions

In this section, I consider 3 different extensions with various features that arise frequently in practice, and show that the existence results in these alternative environments hold as simple corollaries to theorem 1. The first one relaxes the "no-borrowing" assumption and allows for limited borrowing opportunities. The second extension introduces ex ante heterogeneity in preferences. The last extension introduces endogenous labor supply in a very specific environment with Greenwood-Hercowitz-Hoffman preferences.

[12] To be more precise, the integrand is continuous, but not bounded, and weak* continuity of the distributions does not put any discipline on convergence of integrals with unbounded integrands.

4.1 Relaxed Borrowing Limits

In the baseline model, borrowing is not allowed, but limited borrowing can be readily accommodated under some technical conditions. Suppose variable b_t represents assets, and households are allowed to borrow up to some $\underline{b} \geq 0$. The constraints of the household are $c_t + b_t \leq b_{t+1}(1+r) + we_t$, and $b_{t+1} \geq -\underline{b}$. Using a monotonic transformation of variables, let households choose $a_t \equiv b_t + \underline{b}$ instead. Then the constraint set becomes $c_t + a_{t+1} \leq a_t(1+r) + (we_t - r\underline{b})$ and $a_{t+1} \geq 0$. Let $\underline{w} = w(\bar{r})$ represent the lowest possible equilibrium wage level as the interest rate takes values in $(\underline{r}, \bar{r}]$. I impose the following additional assumption to make sure that over the relevant space for prices, borrowing limit is always tighter than the natural borrowing limit[13]:

Assumption 6 *Borrowing limit $\underline{b} \geq 0$ satisfies $\underline{we} > \bar{r}\underline{b}$.*

It is straightforward to show that a recursive competitive equilibrium exists after making suitable redefinitions of variables. For any pair of prices (r, w), define $y^i \equiv we^i - r\underline{b}$. Obviously y^i exhibits the same qualitative properties as the discrete Markov process on E, having the same transition matrix. Moreover it is continuous in (r, w). Replacing all occurances of we^i with y^i, all results in section 2.1 hold. The proof of existence also follows the same steps, if in addition, we use the transformed excess demand for capital, $(K(r) + \underline{b}) - A(r)$.

Corollary 1 *Under assumptions 1, 3, 4, 5, and 6, there exists a stationary recursive competitive equilibrium in the economy with limited borrowing opportunities.*

4.2 Heterogeneous Preferences

We can introduce ex ante heterogeneity in preferences in a straightforward way. Suppose there are n types of agents with measure q_j (with $\sum_{j=1}^{n} q_j = 1$) having utility indices $u_j(.)$ and discount rates β_j. Assume, in addition, that assumptions 1, 2 and 3 hold for each type j individually. It is easy to verify that all propositions hold in this environment with minor changes in notation. In particular, given prices (r, w), stationary distributions are unique for each type j separately, which can be aggregated into an economy-wide distribution. Most importantly a recursive competitive equilibrium exists with an interest rate that is lower than $\bar{r} \equiv \frac{1}{\max_j \beta_j} - 1$ since capital supply (asset demand) for the type that has the highest discount rate diverges to infinity faster than other types as interest rate goes up.

Corollary 2 *Suppose assumptions 1, 3 (for each type j), 4 and 5 hold. Then, there exists a stationary recursive competitive equilibrium in the economy with heterogeneous preferences.*

[13] Natural borrowing limit is defined as the tightest borrowing constraint that never binds over an optimal solution. (See, for instance Aiyagari (1994).) Under my assumptions, given wage $w > 0$ and interest rate $r > 0$, this limit is $\underline{b} = \frac{w\underline{e}}{r}$.

4.3 Endogenous Labor Supply without Wealth Effect

All the proofs so far can be modified in a simple way to prove the existence of an equilibrium in an economy populated with households that have Greenwood-Hoffman-Hercowitz (GHH) preferences. Consider the household's problem with the additional choice variable $n_t \in [0,1]$ for labor supply.

$$V(a,e) = \sup_{\tilde{c}_t, n_t, a_{t+1}} \sum_{t=0}^{\infty} \beta^t u\big(\tilde{c}_t + H(1-n_t)\big)$$

subject to

$$\tilde{c}_t + a_{t+1} \leq a_t(1+r) + we_t n_t \text{ for all } t$$

$$a_{t+1} \geq 0, n_t \in [0,1] \text{ for all } t$$

$$\tilde{c}_t, n_t, a_{t+1} \text{ are } \mathcal{E}^t\text{-measurable for all } t$$

Given $a_0 = a$, $e_0 = e$.

Suppose utility function $u(.)$ satisfies assumptions 1 and 3. In addition, impose the following for $H(.)$.

Assumption 7 *Function $H : \mathbb{R}_+ \to \mathbb{R}$ is continuously differentiable, strictly increasing, strictly concave, $H(0) = 0$ and $\lim_{n \uparrow 1} H'(1-n) = \infty$.*

Now re-define consumption as follows $c_t \equiv \tilde{c}_t + H(1-n_t)$, also let $y(e,w) \equiv \max_{n \in [0,1]}\{wen + H(1-n)\}$ and $n(e,w) \in [0,1]$ is the associated labor supply function. By assumption 7, function $y(e,w)$ is well-defined, continuous and increasing in e and w (strictly when $n \in (0,1]$), and $y(0,w) = H(1) > 0$. Similarly $n(e,w)$ possesses the same continuity and monotonicity properties. Then it is clear that the following problem is isomorphic to the original problem above:

$$V(a,e) = \sup_{c_t, a_{t+1}} \mathbb{E} \sum_{t=0}^{\infty} \beta^t u(c_t)$$

subject to

$$c_t + a_{t+1} \leq a_t(1+r) + y(e_t, w) \text{ for all } t$$

$$a_{t+1} \geq 0, \text{ for all } t$$

$$c_t, a_{t+1} \text{ are } \mathcal{E}^t\text{-measurable for all } t$$

Given $a_0 = a$, $e_0 = e$.

Given the properties of $y(e_t, w)$, for a given $w > 0$, the inequalities $0 < y(e^1, w) \leq y(e^2, w) \leq \ldots \leq y(e^s, w)$ hold, where the inequalities are strict whenever $n > 0$ is optimal. Therefore, the discrete Markov process on E induces a discrete Markov process on $y^i \equiv y(e^i, w)$ with the same transition matrix, and $y^i \geq y^j$ if and only if $e^i \geq e^j$. Moreover, due to continuity of y, this process is continuous in w. Replacing all occurances of we^i with y^i, it is easy to check that *all* propositions in section 2.1 also hold.[14]

[14]One caveat is that in my baseline model, $e^i > e^j$ holds if and only if $i > j$. With endogenous labor supply, these inequalities are

Define the aggregate labor supply function $N(w) \equiv \sum_{i=1}^{s} \pi_i e^i n(e^i, w)$ where π is the unique limiting distribution of transition matrix P. This function is weakly increasing, $N(w) < 1$, and $\lim_{w\to\infty} N(w) = 1$ under assumption 7. As in the previous section, I define $\underline{w} \equiv w(\bar{r})$ to be the lowest possible equilibrium wage level as r takes values in $(-\underline{r}, \bar{r}]$. The proof of existence requires no other assumptions except for $H'(1) > \underline{w}\bar{e}$. This assumption ensures that labor supply is strictly positive for the agents with highest labor productivity even when wage level obtains its minimum value over the range of relevant prices. If this condition holds, the earnings process y^i is sufficiently stochastic for all interest rates. Then $A(r) \to \infty$ and therefore $A(r)/N(w(r)) \to \infty$ as $r \uparrow \bar{r}$. The following result whose proof is omitted, follows as a corollary to theorem 1 and the existence can be established by seeking prices that equate capital per labor demanded by the representative firm, K/N, and aggregate capital per labor supplied $A(r, w)/N(w)$, i.e. defining the excess demand as $(K/N)(r) - \frac{A(r)}{N(w(r))}$.

Corollary 3 *Suppose $H'(1) > \underline{w}\bar{e}$ and assumptions 1, 3, 4, 5, and 7 hold. Then there exists a stationary recursive competitive equilibrium in the economy with GHH preferences.*

5 Practical Considerations

In this section, I discuss relevance of my results for applications. Typical steps taken to find a RCE in calibrated versions of this model can be summarized as follows: (i) Given prices $(r, w(r))$, find policy functions using a value function iteration algorithm; (ii) Either simulate a large number of agents over many periods, or iterate on the distribution directly until convergence to find the stationary supply of capital $A(r, w(r))$; (iii) Check if $K(r) = A(r, w(r))$ is approximately satisfied. If it is, then prices $(r, w(r))$ support a RCE and stop the algorithm, if not, repeat the procedure going back to the beginning with an updated r.

Aside from existence of equilibrium (step (iii) above), my theoretical results address the validity of some of the procedures involved in step (i). Numerical value function iteration approach necessitates exogenously bounding the state space and the choice set. The upper bound on the space for assets is chosen with the premise that it will not bind over an optimal solution when it is sufficiently large. To the best of my knowledge, under arbitrary discrete Markov processes, existence of such an upper bound (Proposition 3) has not been established earlier. Note that unless the objective function is bounded, compactness of the state-space also rationalizes the value function iteration approach in the first place, since otherwise there is no guarantee that the fixed point found corresponds to the solution of the sequential problem. Boundedness of the state space implies value function $V : [0, \bar{a}] \times E \to \mathbb{R}$ satisfies

$$\underline{V} \equiv \frac{u(w\underline{e})}{1-\beta} \leq V(a, e) \leq \frac{u((1+r)\bar{a} + w\bar{e})}{1-\beta} \equiv \bar{V}$$

replaced by weak ones due to the possibility of $n = 0$ for at least two states i, j. However, as long as there is at least one state of the world in which $n > 0$ is optimal, all proofs apply with minor modifications.

where $\bar{a}$ is the upper bound on assets. Hence, standard fixed point arguments can be applied to the Bellman operator defined over continuous and bounded functions that satisfy the above inequality to find the value/policy functions, i.e. this confirms the validity of the typical approach taken in practice.[15] Due to the abstract nature of my results, we cannot proceed any further without the exact specifications of preferences and production technology. For instance, there is yet no guidance on how to choose the upper bound on assets in practice. Next, I give some results that are specific to CRRA class of utility functions that are used frequently in applications.

Proposition 7 *Suppose that utility function satisfies constant-relative-risk-aversion with coefficient $\sigma > 0$, and assumption 2 holds. Define $\kappa(r) \equiv [\beta(1+r)]^{1/\sigma} < 1$. Then,*

1. *Stationary aggregate supply of capital is homogeneous of degree 1 in w, i.e. $A(r, tw) = tA(r, w)$ holds for all $t, w > 0$.*
2. *Consumption and saving policies satisfy $c(a, e; r, w) \geq (1+r-\kappa(r))a + \omega(e)$ and $g(a, e; r, w) \leq \kappa(r)a + we - \omega(e) = \kappa(r)a + w\frac{e-\underline{e}}{1+r}$ when $r > 0$.*

Remark: Item 1 in proposition 7 generalizes to the case where borrowing limit $a' \geq -\underline{a}$ satisfies $\underline{a} = \phi w$ for some $\phi \geq 0$, i.e. when it scales with the wage level.[16] This generalization essentially follows from the change of variables approach taken in the extensions section.

These results have some theoretical and practical implications. First of all, it provides a closed-form upper bound for assets: Given (r, w), item 2 implies that state space is effectively bounded from above by $\bar{a} = w\frac{\bar{e}-\underline{e}}{(1+r)(1-\kappa(r))}$.

Second, Item 1 in proposition 7 is equivalent to the property that $wA(r, 1) = A(r, w)$ for all $w > 0$. Note that this implies equilibrium condition $A(r, w(r)) = K(r)$ can be re-written as

$$A(r, 1) = \frac{K(r)}{w(r)} \tag{7}$$

This approach allows one to ignore the general equilibrium effect of the interest rate on $w(.)$ and focus on $A(r, 1)$ in isolation. In particular, it completely detaches representative firm's influence on prices in a convenient way since right-hand side of this condition can be written in closed form.

Third, in my own numerical experiments, I find that the policy upper bound in item 2 is "tight" for reasonable parameterizations of the model, i.e. the actual policy function is very close to the given upper bound. Hence, for computational tasks, $\hat{g}(a, e; r, w) \equiv \kappa(r)a + w\frac{e-\underline{e}}{1+r}$ makes for an excellent initial guess for

[15] Define the Bellman operator as

$$TV(a, e) = \max_{a' \in [0, \bar{a}]} u((1+r)a + we - a') + \beta \mathbb{E}[V(a', e')|e]$$

It is easy to show that $T : \mathcal{F} \to \mathcal{F}$ satisfies monotonicity and discounting (with modulus β) where $\mathcal{F}$ is the space of continuous and bounded functions $f : [0, \bar{a}] \times E \to \mathbb{R}$ that satisfy $\underline{V} \leq f \leq \bar{V}$, endowed with the sup-norm metric. Hence, it is a contraction mapping.

[16] I would like to thank Dirk Krueger for pointing this out.

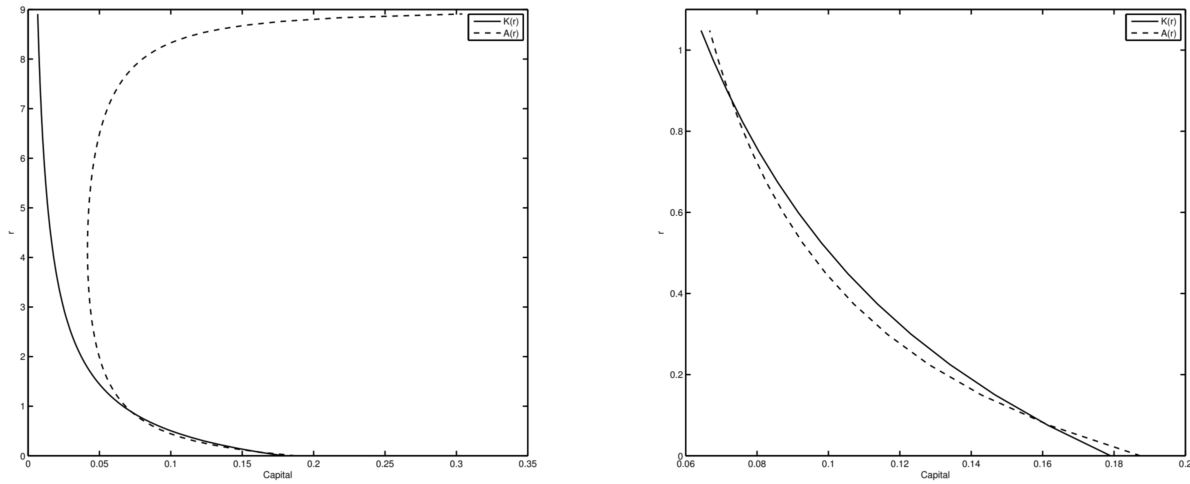

Figure 1: Example of Multiple Equilibria

the policy function.[17]

By integrating the policy inequality in proposition 7 item 2 with the stationary distribution, we immediately obtain an upper bound on the *aggregate* supply of capital:

$$A(r,w) \leq w\frac{\mathbb{E}(e) - \underline{e}}{(1-\kappa(r))(1+r)}. \tag{8}$$

where $\mathbb{E}(e)$ represents the unconditional mean of labor productivity. Moreover, this inequality also provides us with a rate at which the supply of capital diverges to infinity: Note that $A(r,1) \to \infty$ at rate $O\big(((1-\kappa(r))(1+r))^{-1}\big)$ as $r \uparrow \bar{r}$.

6 On the Non-uniqueness of Equilibrium

In models of heterogeneous agents and incomplete markets, there is no good reason why we should expect the equilibrium to be unique. In fact, figure below illustrates an example where there is multiplicity of equilibria under a standard specification of the model with CRRA preferences and Cobb-Douglas production function, however using rather extreme parameter values. The right panel zooms into the low-interest-rate region for clarity.[18]

Even though demand for capital is well-behaved and monotone decreasing under the given assumptions,

[17] This function works well due to two reasons: First, $\kappa(r)$ is the asymptotic derivative of the policy function under CRRA, i.e. $\lim_{a\to\infty}\frac{g(a,e)}{a} = \kappa(r)$ for all $e \in E$. Second, even though policy function is convex even in the presence of borrowing constraints (see, for instance Jensen (2015)), it has curvature only for very low wealth levels under reasonable parameter values. Hence the policy function is mostly linear with slope $\kappa(r)$. This initial guess is derived from the optimal solution of a pessimistic consumer who expects $\underline{e}$ with probability 1 for all future periods, hence it might do a poor job in terms of its intercept term, especially when $\underline{e}$ is small. I find that an even better initial choice that is not theoretically motivated is $\hat{g}(a,e;r,w) = \max\{0, \kappa(r)a + w\frac{e-\mathbb{E}(e'|e)}{1+r}\}$.

[18] The parameter values used in this example are: CRRA coefficient $\sigma = 6.5$, $\beta = 0.1$, Cobb-Douglas share of capital $\alpha = 0.3$, depreciation rate $\delta = 1.0$. For the idiosyncratic productivity, I discretized the process $\log e' = \rho \log e + \varepsilon$ with $\rho = 0.82$, $\sigma_\varepsilon = 0.29$ into 7 states using Tauchen method, then replaced the lowest realization of the shock by $\underline{e} = 10^{-6}$.

capital supply function might not be monotone as shown in the figure. We can think of at least two reasons why non-monotonicity might arise. (i) Due to complementarities in production, an increase in the interest rate reduces how much firms are willing to pay for labor, i.e. $w(r)$ is decreasing in r. Therefore any tendency of capital supply function to increase due to the interest rate might be more than offset by the reduction in wages. (ii) With fixed w, an increase in the interest rate might induce conflicting income and substitution effects leading to a decline in supply of capital. Moreover, it is not clear how interest rate interacts with the precautionary motive for saving in the presence of borrowing constraints.

It turns out, under CRRA utility and Cobb-Douglas production (as in this example), the first effect *cannot* be responsible for multiplicity of equilibria even though it can lead to a backward bending capital supply function. (See Kuhn (2013) for an example of a backward bending supply curve due to first effect.) This is clear from expression (7) in the previous section. Under Cobb-Douglas production, right-hand side of this alternative statement of equilibrium condition, $K(r)/w(r)$, is a monotone decreasing function in r.[19] Hence, if there is multiplicity, it is due to potential non-monotonicity of $A(r,1)$, i.e. the second effect must be present.

To have a better understanding of the case of non-uniqueness in figure 1, it is useful to look at some of the bounds I derived in the previous section. Since the general equilibrium effect is not responsible for multiplicity of equilibrium, I will assume that w is fixed for this analysis. The upper bound on the policy function derived in proposition 7 gives us some idea about why there can be non-monotonicity in capital supply function even when the general equilibrium effect on wage level w is neutralized. Let us assume for a moment that the upper bound is the actual saving policy. With a constant asset level a, and a constant wage level w, an increase in r induces two effects: First, a substitution effect shows up through the slope $\kappa(r) = [\beta(1+r)]^{1/\sigma}$, which represents marginal propensity to save out of current resources. With a fixed lifetime earnings profile, agent wants to save more and consume less today as r goes up and the gap between agent's discount rate β and market discount rate $1/(1+r)$ diminishes, i.e. as impatience $(1-\beta(1+r))$ declines. The second effect is an income effect: Observe that worst-case annuitized lifetime earnings, $\omega(e) = w\frac{re+\underline{e}}{1+r}$, is increasing in the interest rate since current earnings we is always higher than lowest level of earnings $w\underline{e}$. But then the the expression for the upper bound of the policy function reveals that conditional on having assets a, an increase in the interest rate leads to a *decline* in savings. The upper bound on the aggregate capital supply function in expression (8) is derived from the upper bound on the policy function and the two conflicting effects show up in the denominator term $(1-\kappa(r))(1+r)$. It turns out this term is not monotone in r with some combinations of β and σ, i.e. there are values of r for which income effect dominates the substitution effect.

Why is it the case that my analysis above on the upper bound is relevant for the actual policy function?

[19] Assumption 5 is not sufficient for $K(r)/w(r)$ to be downward sloping. An extra condition on the second derivative of the production function must be imposed. I do not take this route, because this condition is hard to motivate. On the other hand, Cobb-Douglas function under any parameterization, and CES production function under some parameterizations satisfy this property. I would like to thank Adrien Auclert for clarifying the CES case.

The upper bound in proposition 7 is derived from the analytical solution to the problem of an impatient consumer who faces the worst-case earnings draw $\omega(e)$ with certainty. When there is uncertainty, agents attach implicit weights to different realizations of lifetime earnings and make saving choices accordingly. However, if the worst-case scenario is particularly undesirable (e.g. low $\underline{e}$), and agents are sufficiently prudent, they attach a high weight on these worse realizations. Hence, the income effect described above does not cancel out at an aggregation even though it is likely to have different magnitudes and different signs for agents over the cross section.[20] My example above supports this intuition. In my parameterization, I chose β and σ to be such that function $(1-\kappa(r))(1+r)$ is non-monotone in r and I picked the labor efficiency process to feature an extremely low value for $\underline{e}$.

Non-uniqueness of equilibrium is a cause for concern for applications if it arises under reasonable parameterizations, since the model loses its predictive power. This is not entirely bad news. Typical calibration strategy employed for this class of models uses capital/output ratio K/Y as a calibration target. The unobservable discount rate β is adjusted until the stationary equilibrium features the given capital/output ratio. It is easy to show that this procedure is bound to find a *unique* such equilibrium. Under CRS technology assumption, K/Y pins down candidate equilibrium prices $(r^*, w(r^*))$ and capital stock $K^* \equiv K(r^*)$. Given these prices, the equilibrium condition is a simple expression that involves only the discount rate: $A(\beta; r^*, w^*) = K^*$. Miao (2002) and more recently Acemoglu and Jensen (2015) prove that capital supply function is monotone in β, which follows by an application of Topkis (1978) monotonicity results. Moreover, it is easy to show that $A(\beta) \to 0$ as $\beta \downarrow 0$ and $A(\beta) \to \infty$ as $\beta \uparrow \frac{1}{1+r^*}$.[21] Since policy functions, and hence $A(\beta)$, are continuous in β (see proof of Theorem 1), by intermediate value theorem, there exists a unique $\beta^* \in [0, \frac{1}{1+r^*})$ that makes the given capital output ratio consistent with a stationary recursive competitive equilibrium.

7 Conclusion

The proof of existence of RCE in this paper covers many cases of interest, including the canonical benchmark with CRRA utility and arbitrary discrete Markov processes. Most of the results in this paper can be extended to the case in which Markov process for earnings is continuous, provided that analogous restrictions and standard continuity assumptions are imposed on the transition function.[22]

There are many open questions for further research. The key requirement for existence of RCE is the existence of a stationary distribution for all relevant price levels. Boundedness of state space and its prereq-

[20] To clarify, this means, if there were no impatience ($\kappa = 1$), the "representative agent" (an aggregation of all saving decisions in the economy) would have been a net saver due to prudence ($u''' > 0$) and inequality in marginal utility realizations across different states. Borrowing constraints amplify the precautionary motives further, but it is not clear how interest rate interacts with the latter effect.

[21] The former result follows trivially from the fact that agents do not have an incentive to save when $\beta = 0$. The latter result follows from my existence results above, essentially by replacing all limit results for $r \uparrow \bar{r}$ with limit results for $\beta \uparrow \frac{1}{1+r^*}$, I omit these details.

[22] Although I did not explore this avenue, an additional assumption on the lowest realization of the shock is most likely to be necessary: It must have a positive mass in order to apply some of the results that depend on persistence of the lowest state.

uisites constitute strong sufficiency conditions for existence of a stationary distribution. In fact, stationary distribution and equilibrium might exist even for the cases of unbounded state space. Although theoretical results in this direction are limited, an extension that dispenses with these assumptions would be a promising next step.[23]

It is still an open question under what conditions the stationary RCE is unique. As shown in the previous section, multiplicity of equilibrium can arise under some parameterizations of the model, but it is not clear whether this problem arises under reasonable calibrations. It would be illuminating to characterize the set of parameter values for which aggregate supply of capital is monotone increasing in interest rate, which would lead to the uniqueness of equilibrium.

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[23] See, for example, Szeidl (2013) and Kamihigashi and Stachurski (2012).

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Appendices

A Proof of Proposition 1

Define $\psi(a,e) \equiv u'(a(1+r)+we)$. Let $\mathcal{P}$ be the set of continuous functions $p : A \times E \to \mathbb{R}$ that are decreasing in A, $p \geq \psi(a,e)$, and $\sup|p-\psi| < \infty$. Let $d(p,q) \equiv \sup|p-q|$ represent sup-norm (uniform) metric. Define the functional equation

$$p(a,e) = \max\left\{\beta(1+r)\mathbb{E}\left[p\left((1+r)a + we - (u')^{-1}(p(a,e)), e'\right)|e\right], \psi(a,e)\right\}.$$

Consider mapping T defined on $\mathcal{P}$ as follows:

$$Tp(a,e) = \max\left\{\beta(1+r)\mathbb{E}\left[p\left((1+r)a + we - (u')^{-1}(Tp(a,e)), e'\right)|e\right], \psi(a,e)\right\}. \tag{9}$$

Lemma A.1 *Tp is a well-defined function, i.e. for any $(a,e) \in A \times E$ and any $p \in \mathcal{P}$, there exists a unique $Tp(a,e) \geq \psi(a,e)$ that solves* (9).

Proof. Fix $(a,e) \in A \times E$, $p \in \mathcal{P}$, and let $\tilde{p} \equiv Tp(a,e)$. Define

$$\phi(\tilde{p}) \equiv \max\left\{\beta(1+r)\mathbb{E}\left[p\left((1+r)a + we - (u')^{-1}(\tilde{p}), e'\right)|e\right], \psi(a,e)\right\}.$$

We need to show that $\phi(.)$ has a unique fixed point. Function $\phi(.)$ is weakly decreasing and maps $[\psi(a,e),\infty)$ into itself. Since $\tilde{p} \geq \psi(a,e)$,

$$\phi(\tilde{p}) \leq \phi(\psi(a,e)) = \max\left\{\beta(1+r)\mathbb{E}[p(0,e')|e], \psi(a,e)\right\}$$

It is easy to see that $\phi(.)$ is always bounded if $\mathbb{E}[p(0,e')|e]$ is bounded. I show the boundedness of the latter. Since $\sup|p-\psi| < \infty$, there exists a $K < \infty$ such that $p \leq \psi + K$. Boundedness then follows from $\mathbb{E}[p(0,e')|e] \leq \mathbb{E}[\psi(0,e')|e] + K = \mathbb{E}[u'(we')|e] + K \leq u'(w\underline{e}) + K$, and the fact that $u'(w\underline{e})$ is finite by assumptions 1 and 2. Morevoer, ϕ is a continuous function in $\tilde{p}$, this follows trivially from the fact that p and $u'()$ are continuous functions. We have $\phi(\psi(a,e)) - \psi(a,e) \geq 0$ and since $\phi(\tilde{p})$ is bounded, $\lim_{\tilde{p}\to\infty}\phi(\tilde{p}) - \tilde{p} = -\infty$. By intermediate value theorem there exists a fixed point $\tilde{p} = \phi(\tilde{p})$ and since ϕ is weakly decreasing, the fixed point is unique.■.

Lemma A.2 *Operator T maps P into P.*

Proof. Take any $p \in \mathcal{P}$. It is obvious from the previous lemma, that $Tp \geq \psi$.

(i) Continuity of Tp in (a,e): Proof of continuity essentially follows the same steps in Li and Stachurski (2014). As shown in the proof of lemma A.1, $\tilde{p} \to \phi(\tilde{p};a,e)$ takes values in a closed interval $I(a,e) \equiv$

$[\psi(a,e), u'(w\underline{e}) + K]$ where $K > 0$ satisfies $p \leq \psi + K$ for all $(a,e) \in A \times E$. Correspondence $I(a,e)$ is non-empty, compact-valued, and continuous. Define function $f : gr(I) \to \mathbb{R}_+$ as $f(a,e,\tilde{p}) \equiv \phi(\tilde{p}; a, e)$. Clearly $f(a,e,\tilde{p}) \in I(a,e)$ for all (a,e). Function f is continuous on $gr(I)$, then, the fixed point of $\phi(.;a,e)$ is continuous on $A \times E$ by a variation of Maximum Theorem. (See Theorem B.1.4 in Stachurski (2009) on the parametric continuity of fixed points.)

(ii) *Tp is weakly decreasing in $a \in A$*: Suppose, to get a contradiction, Tp is strictly increasing in A over some interval, so that there exist $e \in E$, $a, \tilde{a} \in A$ for which $\tilde{a} > a$ and $Tp(\tilde{a}, e) > Tp(a,e)$. Then, using equation (9), we have

$$\begin{aligned} Tp(\tilde{a},e) &> \max\left\{\beta(1+r)\mathbb{E}\left[p\big((1+r)a + we - (u')^{-1}(Tp(a,e)), e'\big)|e\right], \psi(a,e)\right\} = Tp(a,e) \\ &\geq \max\left\{\beta(1+r)\mathbb{E}\left[p\big((1+r)a + we - (u')^{-1}(Tp(a,e)), e'\big)|e\right], \psi(\tilde{a},e)\right\} \\ &\geq \max\left\{\beta(1+r)\mathbb{E}\left[p\big((1+r)\tilde{a} + we - (u')^{-1}(Tp(\tilde{a},e)), e'\big)|e\right], \psi(\tilde{a},e)\right\} \\ &= Tp(\tilde{a},e) \end{aligned}$$

where the second inequality follows from $\psi(a,e) \geq \psi(\tilde{a},e)$ and the third follows from the fact that $u'^{-1}(.)$ is strictly decreasing and p is decreasing in A. This is a contradiction.

(iii) $\sup|Tp - \psi| < \infty$ *for all* $p \in \mathcal{P}$: We have

$$\begin{aligned} |Tp(a,e) - \psi(a,e)| &= Tp(a,e) - \psi(a,e) \\ &\leq \beta(1+r)\mathbb{E}\left[p\big((1+r)a + we - (u')^{-1}(p(a,e)), e'\big)|e\right] \\ &\leq \mathbb{E}[p(0,e')|e] \\ &\leq \mathbb{E}[\psi(0,e')|e] + K = \mathbb{E}[u'(we')|e] + K \\ &\leq u'(w\underline{e}) + K \equiv \bar{K} < \infty \end{aligned}$$

where the first 3 lines follow from $Tp \geq \psi$ (from the lemma above) and $\beta(1+r) < 1$. Line 4 follows from the fact that $\sup|p - \psi| < \infty$, so that there exists a $K < \infty$ that satisfies $p - \psi < K$. The last line follows from the fact that E is a finite set and $u'(.)$ is a decreasing function that is finite at $w\underline{e}$ by assumptions 1 and 2. Since $|Tp(a,e) - \psi(a,e)| \leq \bar{K} < \infty$, $\sup|Tp(a,e) - \psi(a,e)| < \infty$.■

Lemma A.3 *Metric space $(\mathcal{P}, d)$ is complete.*

Proof. Let $\mathcal{P}_0$ be the set of all functions $p : A \times E \to \mathbb{R}$ such that $d(p,\psi) = \sup|p - \psi| < \infty$. Let $\mathcal{F} \subset \mathcal{P}_0$ represent the set of bounded functions in $\mathcal{P}_0$. Clearly $(\mathcal{F}, d)$ is a complete metric space. I first show that (P_0, d) is complete. Take any Cauchy sequence $\{p_n\}_{n=0}^{\infty}$ in $(\mathcal{P}_0, d)$ and define $q_n \equiv p_n - \psi$. Sequence $\{q_n\}_{n=0}^{\infty}$ is Cauchy, and moreover q_n is bounded for all n, therefore $q_n \in \mathcal{F}$. Since $(\mathcal{F}, d)$ is complete, $q_n \to q \in \mathcal{F} \subset \mathcal{P}_0$. It is easy to check that $p \equiv q + \psi \in \mathcal{P}_0$. Moreover, Cauchy sequence $\{p_n\}_{n=0}^{\infty}$ converges to p since $d(p_n, p_m) = d(q_n, q_m)$ for all n, m. This proves that $(\mathcal{P}_0, d)$ is complete. $\mathcal{P}$ is a closed subset of

$\mathcal{P}_0$ and is therefore complete. ■

Lemma A.4 *The mapping $T : \mathcal{P} \to \mathcal{P}$ is a contraction with modulus $\beta(1+r) < 1$ on $(\mathcal{P}, d)$.*

Proof. Blackwell Sufficiency conditions do not apply directly since $\mathcal{P}$ is not a subset of the space of bounded functions. Nevertheless, we show that Blackwell's Monotonicity and Discounting conditions hold and then show that they are indeed sufficient conditions for T to be a contraction mapping on $(\mathcal{P}, d)$.

(i) Monotonicity: Take any $p, \tilde{p} \in \mathcal{P}$ such that $\tilde{p} \geq p$. Assume, to get a contradiction that $T\tilde{p}(a,e) < Tp(a,e)$ for some (a,e). Then we have

$$\begin{aligned}
T\tilde{p}(a,e) &= \max\left\{\beta(1+r)\mathbb{E}\big[\tilde{p}\big((1+r)a + we - (u')^{-1}(T\tilde{p}(a,e)), e'\big)|e\big], \psi(a,e)\right\} \\
&\geq \max\left\{\beta(1+r)\mathbb{E}\big[p\big((1+r)a + we - (u')^{-1}(T\tilde{p}(a,e)), e'\big)|e\big], \psi(a,e)\right\} \\
&\geq \max\left\{\beta(1+r)\mathbb{E}\big[p\big((1+r)a + we - (u')^{-1}(Tp(a,e)), e'\big)|e\big], \psi(a,e)\right\} \\
&= Tp(a,e)
\end{aligned}$$

where the first inequality follows from $\tilde{p} \geq p$, the second inequality follows from our assumption that $T\tilde{p}(a,e) < Tp(a,e)$ and that $u'^{-1}(.)$ and p are weakly decreasing functions. The last line establishes $T\tilde{p}(a,e) \geq Tp(a,e)$, this is a contradiction.

(ii) Discounting: Take any $p \in \mathcal{P}$ and $\lambda \geq 0$. Clearly $p + \lambda \in \mathcal{P}$. Since T is monotone, $T(p+\lambda) \geq Tp$. Then we have

$$\begin{aligned}
T(p(a,e)+\lambda) &= \max\left\{\beta(1+r)\mathbb{E}\big[p\big((1+r)a + we - (u')^{-1}(T(p(a,e)+\lambda)), e'\big)|e\big] + \beta(1+r)\lambda, \psi(a,e)\right\} \\
&\leq \max\left\{\beta(1+r)\mathbb{E}\big[p\big((1+r)a + we - (u')^{-1}(Tp(a,e), e'\big)|e\big] + \beta(1+r)\lambda, \psi(a,e)\right\} \\
&\leq \max\left\{\beta(1+r)\mathbb{E}\big[p\big((1+r)a + we - (u')^{-1}(Tp(a,e), e'\big)|e\big], \psi(a,e)\right\} + \beta(1+r)\lambda \\
&= Tp(a,e) + \beta(1+r)\lambda.
\end{aligned}$$

(iii) Sufficiency of Monotonicity and Discounting: Let T satisfy monotonicity and discounting (with a constant $\beta(1+r) \in [0,1)$) on $\mathcal{P}$. Take any $p, q \in \mathcal{P}$. Observe that $|p - q| = |(p-\psi) - (q-\psi)| \leq |p-\psi| + |q-\psi| \leq d(p,\psi) + d(q,\psi) < \infty$, hence $d(p,q) < \infty$ for all $p, q \in \mathcal{P}$.

Inequalities $p \leq q + d(p,q)$ and $q \leq p + d(p,q)$ hold. First applying monotonicity, and then discounting, we obtain $Tp \leq T(q + d(p,q)) \leq Tq + \beta(1+r)d(p,q)$ and $Tq \leq T(p + d(p,q)) \leq Tp + \beta(1+r)d(p,q)$. Therefore $|Tp - Tq| \leq \beta(1+r)d(p,q)$ and hence $d(Tp, Tq) \leq \beta(1+r)d(p,q)$.

I have shown that monotonicity and discounting on $(\mathcal{P}, d)$ are sufficient conditions for T to be a contraction mapping with modulus $0 \leq \beta(1+r) < 1$.■

Proof of Proposition 1: The proofs of items 3 and 4 are technical and I refer the readers to Li and Stachurski (2014) as the steps involved are identical. The proof of part 2 of the proposition follows a very similar methodology to Li and Stachurski (2014), however many steps (some of which are already covered in lemmas above) need to be re-written to allow for negative interest rate and the notation needs to be changed due to the fact that the productivity process is discrete in the current model. I provide an alternative proof for the finiteness of the value function in item 1.

1. Consider an alternative problem where the constraint that "c_t and a_{t+1} are $\mathcal{E}^t$-measurable for all t", is replaced with the much weaker constraint "c_t and a_{t+1} are $\mathcal{E}^\infty$-measurable for all t". This alternative problem represents the environment in which the *entire* sequence of income shocks is revealed in time zero. Let $\tilde{V}(a, e; r)$ represent the value of the alternative problem under the interest rate r. Let $\bar{r} = \frac{1}{\beta} - 1$. Since $a_{t+1} \geq 0$, $\tilde{V}(a, e; \bar{r}) \geq \tilde{V}(a, e; r)$ for all $r \leq \bar{r}$. This follows from the fact that any feasible plan for problem $\tilde{V}(a, e; r)$ is feasible in $\tilde{V}(a, e; \bar{r})$.[24]

   Since uncertainty is resolved in period 0, if borrowing constraints never bind, the analytical solution to problem $\tilde{V}(a, e; \bar{r})$ would be $c_t = \bar{r}a_0 + \omega(\mathbf{e})$ for all $t \geq 0$ where $\omega(\mathbf{e})$ denotes the *annuitized* present value of the realized lifetime earnings $\mathbf{e} = \{we_0, we_1, we_2, ...\}$. (It is easy to check that this solution satisfies Euler equation, budget constraint, and the transversality condition). The "luckiest" agent receives a productivity sequence of $e_t = \bar{e}$, therefore $c_t = \bar{r}a_0 + w\bar{e}$ and $a_{t+1} = a_0$ for all $t \geq 0$. Following this plan, this agent never hits the borrowing constraint and enjoys a constant consumption. Hence $\tilde{V}(a, e; r) \leq \tilde{V}(a, e; \bar{r}) \leq \frac{u(\bar{r}a + w\bar{e})}{1-\beta} < \infty$ where finiteness follows from assumption 1.[25] Since additional measurability constraints are imposed on problem $V(a, e)$, its value cannot exceed $\tilde{V}(a, e; r)$. Therefore $V(a, e) \leq \frac{u(\bar{r}a + w\bar{e})}{1-\beta}$. This establishes an upper bound on $V(a, e)$. Establishing the lower bound on $V(a, e)$ is trivial and follows from the fact that $c_t = w\underline{e}$ for all t is a feasible plan. Therefore $V(a, e) \geq \frac{u(w\underline{e})}{1-\beta}$.[26] ■

2. Lemmas above jointly imply that mapping T has a unique fixed point p^* by Banach Fixed-Point Theorem. Moreover $p^*(a, e)$ is continuous and weakly decreasing in a. Define the Coleman operator $K : \mathcal{C} \to \mathcal{C}$ where $\mathcal{C}$ is the set of continuous functions $c : A \times E \to \mathbb{R}$ that are weakly increasing in a, satisfy $0 < c(a, e) \leq (1 + r)a + we$, and $\sup |u'(c(.)) - \psi(.)| < \infty$.

$$u'(Kc(a, e)) = \max \left\{ \beta(1 + r)\mathbb{E}\big\{u'\big[c((1 + r)a + we - Kc(a, e), e')\big]\big|e\big\}, \psi(a, e) \right\} \tag{10}$$

   Marginal utility function $u'(.)$ is a continuous bijection with a continuous inverse, therefore there is a homeomorphism H between $(\mathcal{P}, d)$ and the functional space $\mathcal{C}$ endowed with the metric $\tilde{d}(c, d) =$

[24] Observe that $(1 + r)a + we \leq (1 + \bar{r})a + we$ for all $r < \bar{r}$ and all $a \geq 0, e \in E$ holds. Therefore increasing the interest rate effectively expands the choice set.

[25] Assumption 1 ensures that there exists a finite L such that $u(c) \leq c + L$ for all $c \geq 0$.

[26] Here is a less intuitive alternative proof: Assumption 1 implies, there exists $0 < L < \infty$ such that $u(c) \leq c + L$ for all $c > 0$. By consolidating the budget constraints, one can show that $c_t \leq (1 + r)^{t+1}a + w\sum_{j=0}^{t}(1 + r)^{t-j}e_j \leq (1 + r)^{t+1}a + w\bar{e}\frac{1-(1+r)^{t+1}}{r}$. Then we have

$$V(a, e) = \mathbb{E}\sum_{t=0}^{\infty}\beta^t u(c_t) \leq \mathbb{E}\sum_{t=0}^{\infty}\beta^t c_t + \frac{L}{1-\beta} \leq a\frac{(1 + r)}{(1 - \beta(1 + r))} + w\bar{e}\frac{1}{(1 - \beta)(1 - \beta(1 + r))} + \frac{L}{1-\beta} < \infty.$$

$\sup |u'(c) - u'(d)|$, where $Hp(a, e) \equiv (u')^{-1}(p(a, e))$ for all (a, e) and all $p \in \mathcal{P}$. Then $K : \mathcal{C} \to \mathcal{C}$ is a contraction mapping with modulus $\beta(1 + r)$ in $(\mathcal{C}, \tilde{d})$. This implies K has the unique fixed point $c^* = Hp^*$.■

B Proof of Proposition 2

1. As part of proposition 1, it has been shown that Coleman operator maps continuous, weakly increasing functions in a, to continuous, weakly increasing functions in a. Therefore, the fixed point of the operator, i.e. the consumption policy function $c(a, e)$, inherits the same properties since this property is preserved under pointwise convergence. Next we show that $c(a, e) \geq ra + \omega(e)$ when $r \geq 0$. Define $\tilde{c}(a, e) \equiv ra + \omega(e)$. Clearly $\tilde{c}(a, e) \in \mathcal{C}$ since it is weakly increasing and $0 < \tilde{c}(a, e) \leq (1+r)a + we$ for all (a, e). Below, we prove that $K\tilde{c}(a, e) \geq \tilde{c}(a, e)$ for all (a, e). Since K is monotone by Proposition 1, this will prove that the fixed point also satisfies $c(a, e) \geq ra + \omega(e)$.

   Assume, to get a contradiction, that $K\tilde{c}(a, e) < ra + \omega(e)$ for some (a, e). Then the following must hold

$$\begin{aligned}
u'(K\tilde{c}(a, e)) &= \max\left\{\beta(1+r)\mathbb{E}\big[u'\big(r(a(1+r) + we - K\tilde{c}(a, e)) + \omega(e')\big)|e\big], \psi(a, e)\right\} \\
&\leq \max\left\{\mathbb{E}\big[u'\big(ra + \omega(e) + \omega(e') - w\underline{e}|e\big], \psi(a, e)\right\} \\
&\leq \max\left\{\mathbb{E}\big[u'\big(ra + \omega(e)\big)|e\big], \psi(a, e)\right\} \\
&= u'(ra + \omega(e))
\end{aligned}$$

   The first inequality follows from $\beta(1 + r) < 1$, $K\tilde{c}(a, e) < ra + \omega(e)$ and that utility function is concave. The second inequality follows from concavity of the utility function and $e' \geq \underline{e}$ for all $e' \in E$. The equality in the last line follows from $ra + \omega(e) \leq (1 + r)a + we$ for all (a, e), and the fact that expectation operator is redundant due to non-stochasticity. Then we have $K\tilde{c}(a, e) \geq ra + \omega(e)$, a contradiction.

   The proof for $c(a, e) \geq w\underline{e}$ for all $r > -1$ trivial, and follows essentially the same steps, and is therefore omitted.■

2. Continuity of $g(a, e) = (1+r)a + we - c(a, e)$ in a follows from continuity of $c(a, e)$. Next, I show that $g(a, e)$ is weakly increasing.

   If $g(a, e) = 0$ for all a, the property is trivially satisfied. For other cases, suppose, to get a contradiction, $g(a, e)$ is strictly decreasing in a in an open neighborhood of some a^* where $a'^* \equiv g(a^*, e) > 0$. Therefore Euler equation is satisfied with equality at (a^*, e), i.e. $u'(c(a^*, e)) = \beta(1 + r)\mathbb{E}[u'(c(a'^*, e'))|e]$ holds. By continuity of the policy function, there exists $\tilde{a} > a^*$ such that $g(a^*, e) > g(\tilde{a}, e)$ and $\tilde{a}' \equiv g(\tilde{a}, e) > 0$. This implies Euler equation is satisfied with equality at $(\tilde{a}, e)$ as well. Since $u(.)$ is strictly concave and consumption satisfies $c(\tilde{a}, e) = \tilde{a}(1+r) + we - g(\tilde{a}, e) > a^*(1+r) + we - g(a^*, e) = c(a^*, e)$, $u'(c(a^*, e)) > u'(c(\tilde{a}, e))$ must hold. Similarly, since $c(a', e')$ is a weakly increasing function and $u(.)$ is strictly concave, $\beta(1+r)\mathbb{E}[u'(c(\tilde{a}', e'))|e] \geq \beta(1+r)\mathbb{E}[u'(c(a'^*, e'))|e]$. Combining these results, we have the following inequalities

$$\beta(1+r)\mathbb{E}[u'(c(\tilde{a}', e'))|e] \geq \beta(1+r)\mathbb{E}[u'(c(a'^*, e'))|e] = u'(c(a^*, e)) > u'(c(\tilde{a}, e))$$

This is a contradiction to the fact that the Euler equation holds with equality at $\tilde{a}$.■

3. First, we show that $\lim_{a\to\infty} g(a, e) = \infty$ for all $e \in E$. Suppose, to get a contradiction, that for some $e \in E$, there exists $\bar{a} < \infty$ such that $g(a, e) \leq \bar{a}$ for all $a \in A$. Since budget constraint of the household is satisfied with equality, we have $\lim_{a\to\infty} c(a, e) = \infty$. Euler equation implies $u'(c(a, e)) \geq \beta(1+r)\mathbb{E}[u'(c(a', e'))|e] \geq \beta(1+r)\mathbb{E}[u'(c(\bar{a}, e'))|e] \geq \beta(1+r)u'((1+r)\bar{a} + w\bar{e}) \equiv M > 0$ where the second inequality follows from the fact that $\bar{a} \geq a'$ and the fact that $c(., e)$ is weakly increasing in a. The third follows from feasibility constraint $c(a, e) \leq (1+r)a + we$ for all (a, e), $\bar{e} \geq e$, and assumptions 1 and 2. Taking the limit as $a \to \infty$, using assumption 1 and $\lim_{a\to\infty} c(a, e) = \infty$, we obtain $\lim_{a\to\infty} u'(c(a, e)) = 0 \geq M > 0$, a contradiction.

   Now we show that $\lim_{a\to\infty} c(a, e) = \infty$. Suppose this were not the case for some states $e^i \in \bar{E} \subset E$ and for these states, define $\bar{c}_i \equiv \lim_{a\to\infty} c(a, e^i) < \infty$. By assumption 1, $0 < u'(\bar{c}^i) < \infty$ for all $e^i \in \bar{E}$. For all large a,

$$u'(c(a, e^i)) = \beta(1+r)\sum_j P_{ij} u'(c(g(a, e^i), e^j))$$

   As $a \to \infty$, $u'(c(a, e^i)) \to u'(\bar{c}^i) > 0$, therefore right-hand side must also converge to a finite limit. In fact, since $g(a, e_i) \to \infty$ as $a \to \infty$ as established above, $u'(c(g(a, e^i), e^j)) \to u'(\bar{c}_j) > 0$ for all $j \in \bar{E}$, for all other states, limits equal zero. Then the following limit equality holds:

$$u'(\bar{c}_i) = \beta(1+r)\sum_{j\in\bar{E}} P_{ij} u'(\bar{c}_j)$$

   But inductively, going forward, one can obtain

$$u'(\bar{c}_i) = [\beta(1+r)]^t \sum_{j\in\bar{E}} [P^t]_{ij} u'(\bar{c}_j) \leq [\beta(1+r)]^t u'(\bar{c}) \text{ for all } e_i \in \bar{E} \text{ and } t \geq 1$$

   where $\bar{c} \equiv \min_{j\in\bar{E}} \bar{c}_j$. Since $\beta(1+r) < 1$, taking the limit as $t \to \infty$, we obtain the contradiction $u'(\bar{c}_i) \leq 0$ for all $e_i \in \bar{E}$.■

C Proof of Proposition 3

Lemma C.1 *Suppose $r > 0$. Then*

1. *There exists a unique* $\underline{a} > -\frac{w\underline{e}}{1+r}$ *that solves* $u'(\underline{a}(1+r)+w\underline{e}) = \mathbb{E}[u'(\omega(e'))|\underline{e}]$*, and it satisfies* $\underline{a} > 0$*.*

2. *For all* $0 < r \leq \frac{1}{\beta} - 1$*, saving policy satisfies* $g(a, \underline{e}) = 0$ *for all* $a < \underline{a}$*.*

Proof.

1. Existence and uniqueness follows trivially from continuity and strict monotonicity of $u'(.)$ since we can then express $\underline{a}$ explicitly as $\underline{a} = \frac{(u')^{-1}\left(\mathbb{E}[u'(\omega(e'))|\underline{e}]\right) - w\underline{e}}{1+r}$.

   Now we show that $\underline{a} > 0$ for all $r > 0$. Since $P_{1j} > 0$ for some $j > 1$ and $\omega(e^j) > w\underline{e}$, we have $\mathbb{E}[u'(\omega(e'))|\underline{e}] < u'(w\underline{e})$ by strict concavity of the utility function. Then we have $u'(\underline{a}(1+r)+w\underline{e}) < u'(w\underline{e})$. The claim that $\underline{a} > 0$ follows from strict concavity of the utility function. ■

2. Take any $a < \underline{a}$, we have

$$u'(a(1+r)+w\underline{e}) > u'(\underline{a}(1+r)+w\underline{e}) = \mathbb{E}[u'(\omega(e'))|\underline{e}] \geq \beta(1+r)\mathbb{E}[u'(\omega(e'))|\underline{e}] \geq \beta(1+r)\mathbb{E}[u'(c')|\underline{e}]$$

   where the first inequality follows from strict concavity of the utility function, second inequality follows from $\beta(1+r) \leq 1$, and the third inequality follows from $c' \geq ra' + \omega(e')$ by proposition 2. Since $u'(a(1+r)+w\underline{e}) > \beta(1+r)\mathbb{E}[u'(c')|\underline{e}]$ holds, $a' = 0$ is optimal. Then $g(a, \underline{e}) = 0$ as we wanted to show. ■

Lemma C.2 *Suppose* $r > 0$ *and* $a \geq 0$*. Then*

1. *There exists a unique* $d \in (-(ra + w\underline{e}), \frac{ra+w\underline{e}}{r})$ *that solves the following expression*

$$u'(ra + w\underline{e} + d) = \mathbb{E}[u'(ra + \omega(e') - rd)|\underline{e}] \tag{11}$$

   and it satisfies $d > 0$*. Moreover* $\Delta \equiv \min\{a, d\}$ *exists, it is unique, and satisfies* $\Delta > 0$ *for all* $a > 0$*.*

2. *Saving policy satisfies* $a - g(a, \underline{e}) \geq \Delta$ *if* $\beta(1+r) \leq 1$ *holds.*

Proof.

1. Define $\underline{x} \equiv -(ra + w\underline{e})$, $\bar{x} \equiv \frac{ra+w\underline{e}}{r}$ and the function $\phi : (\underline{x}, \bar{x}) \to \mathbb{R}$ as $\phi(x) \equiv u'(ra + w\underline{e} + x) - \mathbb{E}[u'(ra + \omega(e') - rx)|\underline{e}]$. Since $u'(.)$ is continuous and strictly decreasing by assumption 1, $\phi(x)$ is strictly decreasing over its domain. By assumption 1, $\lim_{x\to\bar{x}} \phi(x) = -\infty$ holds. Observe that $\phi(0) = u'(ra + w\underline{e}) - \mathbb{E}[u'(ra + \omega(e'))|\underline{e}]$. Since $\omega(e^j) > w\underline{e}$ for at least one accessible state e^j, we have $\phi(0) > 0$. By intermediate value theorem and strict monotonicity of $\phi(.)$, there exists a unique $x^* \in (0, \bar{x})$ that satisfies $\phi(x^*) = 0$. Clearly $d = x^*$ solves equation (11). Stated properties of Δ follow trivially from the properties of d.

2. If $g(a, \underline{e}) = 0$, $a - g(a, \underline{e}) = a \geq \min\{a, d\} = \Delta$ is trivially satisfied. Now, we consider the non-trivial case in which a satisfies $a' = g(a, \underline{e}) > 0$ and show that $a - a' \geq d \geq \Delta$. Suppose, to get a

contradiction, that for some $0 < r \leq \frac{1}{\beta} - 1$, $a - a' < d$ holds. Rearranging expression (11), we get

$$u'((1+r)a+w\underline{e}-(a-d)) = \mathbb{E}[u'(ra'+\omega(e')+r(a-d)-ra')|\underline{e}] \geq \beta(1+r)\mathbb{E}[u'(ra'+\omega(e')+r(a-d)-ra')|\underline{e}]$$

When $a - d < a'$, this inequality implies

$$u'((1+r)a + w\underline{e} - a') > \beta(1+r)\mathbb{E}[u'(ra' + \omega(e'))|e] \geq \beta(1+r)\mathbb{E}[u'(c')|e]$$

where last third inequality follows from $c' \geq ra' + \omega(e')$ due to Proposition 2. But then $a' = g(a, \underline{e}) = 0$, a contradiction.■

Proof of Proposition 3: For the case of $r > 0$, lemmas C.1 and C.2 show that there exists $\underline{a} > 0$ such that $g(a, \underline{e}) = 0$ for all $a \leq \underline{a}$, and for all $a \geq \underline{a} > 0$ there exists $\Delta(a) > 0$, such that $a - g(a, \underline{e}) \geq \Delta(a) > 0$. Since $g(a, \underline{e})$ is continuous, it must be the case that $a > g(a, \underline{e})$ for all $a > 0$.

I proceed with the case of $r \in (-1, 0]$. Suppose, to get a contradiction, that $a' = g(a, \underline{e}) \geq a > 0$. Then, budget constraint implies $c \leq ra + w\underline{e} \leq w\underline{e}$ where the second inequality follows from $r \leq 0$. Since, Euler equation is satisfied with equality by assumption ($a' > 0$), we have $u'(w\underline{e}) \leq u'(c) = \beta(1+r)\mathbb{E}[u'(c')|e'] \leq \beta u'(w\underline{e})$ where the last inequality follows from $c(a', e') \geq w\underline{e}$ by proposition 2, and $\beta(1 + r) \leq \beta$. Since $\beta \in (0, 1)$, we get the desired contradiction $u'(w\underline{e}) < u'(w\underline{e})$.■.

D Proof of Proposition 4

Lemma D.1 *For any* $\Delta \geq 0$*,* $\lim_{c\to\infty} \frac{u'(c+\Delta)}{u'(c)} = 1$.

Proof. Use the following Taylor expansion of the marginal utility function around c:

$$u'(c + \Delta) = u'(c) + u''(c + \tilde{\Delta})\Delta \text{ where } \tilde{\Delta} \in [0, \Delta]$$

Rearranging terms, we get

$$1 \geq \frac{u'(c + \Delta)}{u'(c)} = 1 + \frac{u''(c + \tilde{\Delta})}{u'(c + \tilde{\Delta})} \frac{u'(c + \tilde{\Delta})}{u'(c)} \Delta \geq 1 + \frac{u''(c + \tilde{\Delta})}{u'(c + \tilde{\Delta})} \Delta$$

where the last inequality follows from the fact that last term is negative and $u'(c+\tilde{\Delta}) \leq u'(c)$ by assumption 1. Taking the limit as $c \to \infty$ and using assumption 3, we obtain $\lim_{c\to\infty} \frac{u'(c+\Delta)}{u'(c)} = 1$.■

Proof of Proposition 4: Suppose the claim is not true. Then, for some $e \in E$, there exist a sequence $a_n \to \infty$ such that $a'_n = g(a_n, e) \geq a_n$ for all n. Budget constraint implies $c_n = (1 + r)a_n + we - a'_n \leq ra_n + we$. If $r \leq 0$, $c_n \leq we$ must hold for all n, this is a contradiction since $c_n \to \infty$ as $a_n \to \infty$

which follows from Proposition 2. For the non-trivial case of $r > 0$, by proposition 2, $c'_n \equiv c(a'_n, e') \geq ra'_n + \omega(e') \geq ra'_n + w\underline{e}$ for all $e' \in E$. Since we have $a'_n \geq a_n$, $c'_n \geq ra_n + w\underline{e}$ holds. Euler equation holds with equality at all a_n, therefore these inequalities imply $u'(ra_n + we) \leq \beta(1+r)u'(ra_n + w\underline{e})$. Let $x_n \equiv ra_n + we$ and $\Delta \equiv we - w\underline{e} \geq 0$. Then

$$\frac{u'(x_n + \Delta)}{u'(x_n)} \leq \beta(1+r)$$

Taking the limit as $n \to \infty$, and applying lemma D.1, we obtain $1 \leq \beta(1+r) < 1$, a contradiction. ■

E Proof of Proposition 5

By Proposition 4, the process takes values in a compact set $S = [0, \bar{a}] \times E$. A sufficient condition for existence and uniqueness of a stationary distribution is the ergodicity of the Markov process. I show a stronger result that the Markov process $Q(.,.)$ is *uniformly* ergodic. I invoke theorem 16.0.2 from Meyn and Tweedie (2009) which proves the equivalence of state space S being v_m-small for some $m \in \mathbb{N}_+$ and uniform ergodicity.

A set $C \in \Sigma$ is called a $\mathbf{v_m}$**-small set** if there exists an $m \in \mathbb{N}_+$ and a non-trivial measure v_m on Σ such that for all $s \in C$, $S \in \Sigma$, $Q^m(s, S) \geq v_m(S)$.

Proof of proposition 3 established that there exists $\underline{a} > 0$ such that $g(a, \underline{e}) = 0$ for all $a \leq \underline{a}$ and that $g(a, \underline{e}) < a$ for all $a > 0$. Proposition 4 implies that the assets take values in compact set $[0, \bar{a}]$ for some $\bar{a} \geq 0$. If $\bar{a} \leq \underline{a}$, it is clear that the unique stationary distribution is one in which all agents are borrowing constrained. Suppose $\bar{a} > \underline{a}$. Define $\Delta \equiv \min_{a \in [\underline{a}, \bar{a}]} a - g(a, \underline{e})$>0, which represents the minimal decline in assets in the least productive state of the world. Let $m_1 \equiv \frac{\bar{a}}{\Delta} + 1$. Since $g(a, \underline{e})$ is increasing, conditional on staying in state $\underline{e}$, the household hits the borrowing constraint in at most m_1 periods.

By assumption 4, there exists an integer $m_2 > 0$ such that $[P^{m_2}]_{j1} > 0$ for all j, i.e. state $\underline{e}$ can be reached with strictly positive probability from any initial state e^j. Let $m \equiv m_1 + m_2$. Since $P_{11} > 0$ by assumption 4, starting from any initial state $s = (a_0, e_0) \in S$, there is a strictly positive probability of hitting the borrowing constraint in m periods, i.e $Q^m(s, (0, \underline{e})) \geq q$ for all $s \in S$ for some $q > 0$.[27] This proves that $(0, \underline{e})$ is an accessible atom. Define

$$v_m(C) \equiv \begin{cases} q & (0, \underline{e}) \in C \\ 0 & (0, \underline{e}) \notin C \end{cases}$$

By construction, $Q^m(s, C) \geq v_m(C)$ for all $s \in S$, $C \in \Sigma$. Therefore we have shown that S is a small set and theorem 16.0.2 by Meyn and Tweedie (2009) implies the Markov process on (a, e) is uniformly ergodic. This proves that it has a unique stationary distribution.■

[27] To be more precise, we can pick $q = \left(\min_j [P^{m_2}]_{j1}\right) P_{11}^{m_1}$.

F Proof of Lemma 1

Lemma F.1 *Let $\theta(a,e) \geq 0$ represent the value of the Lagrange multiplier for the borrowing constraint $a' \geq 0$ in state (a,e). Define $\underline{\theta} \equiv u'(w\underline{e}) - \mathbb{E}[u'(\omega(e'))|e]$. When $r > 0$ and $\beta(1+r) \leq 1$, $\theta(0,\underline{e}) \geq \underline{\theta} > 0$.*

Proof. By Proposition 3, $g(0,\underline{e}) = 0$ since $g(\underline{a},\underline{e}) = 0$ for some $\underline{a} > 0$, and $g(.,\underline{e})$ is weakly increasing in a due to proposition 2. Then, first-order necessary condition for optimality reads

$$\theta(0,\underline{e}) = u'(w\underline{e}) - \beta(1+r)\mathbb{E}[u'(c')|\underline{e}] \geq u'(w\underline{e}) - \mathbb{E}[u'(c')|\underline{e}] \geq u'(w\underline{e}) - \mathbb{E}[u'(\omega(e'))|\underline{e}] = \underline{\theta} > 0$$

where the first inequality follows from $\beta(1+r) \leq 1$ and the second inequality follows from $c' \geq ra' + \omega(e')$ due to proposition 2. The last inequality $\underline{\theta} > 0$ follows from the fact that there is at least one state $e^j > \underline{e}$ accessible from $\underline{e}$ by assumption 4.■

Lemma F.2 *For any $\underline{r} \in (0,\bar{r})$ and any $L > 0$, there exists an integer $m > 0$ and $q \in (0,1]$ such that any household with asset level $a \leq L$ reaches state $(0,\underline{e})$ in m periods with probability of at least q, uniformly over all $[\underline{r},\bar{r})$.*

Proof. For any $r > 0$ and for any given asset level a, budget constraint implies, in T periods, assets can reach a maximal value of $(1+r)^T a + \sum_{j=0}^{T-1}(1+r)^j w\bar{e} \leq \frac{a}{\beta^T} + \sum_{j=0}^{T-1}\frac{1}{\beta^j} w\bar{e}$. This upper bound is obtained by assuming that the agent receives the highest value $\bar{e}$ in all T periods and using the fact that $(1+r) \leq \frac{1}{\beta}$. By assumption 4, there is an integer $m_1 > 0$ that satisfies $[P^{m_1}]_{j1} > 0$ for all j. Hence, for any $r > 0$ and any $a \leq L$, the maximal assets by the time an agent reaches the lowest labor productivity $\underline{e}$, is uniformly bounded from above by $\bar{a} \equiv \frac{L}{\beta^{m_1}} + \sum_{j=0}^{m_1-1}\frac{1}{\beta^j} w\bar{e}$.

By Lemma C.1, there exists $\underline{a}(r) > 0$ such that $g(a,\underline{e}) = 0$ for all $a \leq \underline{a}(r)$. Since $\underline{a}(r)$ is continuous in r, we can define $\underline{a} = \min_{r\in[\underline{r},\bar{r}]} \underline{a}(r) > 0$. By Lemma C.2, there exists $\Delta(a,r)$ such that $a - g(a,e) \geq \Delta(a,r) > 0$ for all $a > 0$ and $r > 0$. Define $\Delta \equiv \min_{a\in[\underline{a},\bar{a}], r\in[\underline{r},\bar{r}]} \Delta(a,r) > 0$. Let $m_2 \equiv \frac{\bar{a}}{\Delta} + 1$. Observe that any agent with $a \leq L$ reaches state $(0,\underline{e})$ with strictly positive probability in at most $m \equiv m_1 + m_2$ periods since $P_{11} > 0$ by assumption 4. Let $q \equiv \left(\min_j [P^{m_1}]_{j1}\right)\left(P_{11}^{m_2}\right) > 0$. Since Δ and $\underline{a}$ do not depend on r, this property is uniform over all $r \in [\underline{r},\bar{r})$.■

Proof of Lemma 1: Choose $L > 0$ and $\underline{r} \in (0,\bar{r})$. Let $p(r) \equiv \mu([0,L] \times E; r)$, we want to show that $\lim_{r\uparrow\bar{r}} p(r) = 0$. By Lemma F.2, there exists m and $q \in (0,1]$ such that every household with asset level $a \leq L$ reaches state $(0,\underline{e})$ in m periods with probability of at least q, uniformly over all $r \in [\underline{r},\bar{r})$. This implies transition function satisfies $Q^m((a,e),\{(0,\underline{e})\};r) \geq q$ for all $r \in [\underline{r},\bar{r})$, all $a \leq L$ and $e \in E$. Then for all such r, the stationary distribution satisfies,

$$
\begin{aligned}
\mu(\{(0,\underline{e})\};r) &= \int Q((a,e),\{(0,\underline{e})\};r)d\mu(.;r) = \int Q^m((a,e),\{(0,\underline{e})\};r)d\mu(.;r) \\
&= \int \mathbf{1}_{a\leq L} Q^m((a,e),\{(0,\underline{e})\};r)d\mu(.;r) + \int \mathbf{1}_{a>L} Q^m((a,e),\{(0,\underline{e})\};r)d\mu(.;r) \\
&\geq p(r)q + \int \mathbf{1}_{a>L} Q^m((a,e),\{(0,\underline{e})\};r)d\mu(.;r) \\
&\geq p(r)q
\end{aligned}
$$

By Lemma F.1, there exists $\underline{\theta} \equiv \min_{r\in[\underline{r},\bar{r}]} u'(w\underline{e}) - \mathbb{E}[u'(\omega(e'))|e] > 0$ such that $\theta(0,\underline{e};r) \geq \underline{\theta}$ for all $r \in [\underline{r},\bar{r})$. Hence, we have

$$
\int \theta(a,e;r)d\mu(.;r) \geq \mu(\{(0,\underline{e})\};r)\theta(0,\underline{e};r) \geq p(r)q\underline{\theta} \text{ for all } r \in [\underline{r},\bar{r})
$$

Since $u'(c)$ is bounded, integrating both sides of the Euler equation with respect to the stationary distribution, we obtain

$$
\int u'(c(a,e;r))d\mu(.;r) = \beta(1+r)\int \sum_{e'\in E} P_{ee'}u'(c(g(a,e),e';r))d\mu(.;r) + \int \theta(a,e;r)d\mu(.;r) \text{ for all } r \in [\underline{r},\bar{r})
$$

The integrals for marginal utility are strictly positive and finite since the support of $\mu(.;r)$ is compact for all $r \in [\underline{r},\bar{r})$ and $c \geq w\underline{e}$. Then, stationarity of the distribution allows us to simplify this expression as

$$
\int u'(c(a,e);r)d\mu(.;r) = \frac{\int \theta(a,e;r)d\mu(.;r)}{1-\beta(1+r)} \geq \frac{p(r)q\underline{\theta}}{1-\beta(1+r)}
$$

Since $c \geq w\underline{e}$, we have

$$
(1-\beta(1+r))u'(w\underline{e}) \geq p(r)q\underline{\theta} \geq 0 \text{ for all } r \in [\underline{r},\bar{r})
$$

Taking the limit as $r \uparrow \bar{r}$, we have $\lim_{r\uparrow\bar{r}} p(r) = 0$ as we wanted to show.■

G Proof of Theorem 1

Lemma G.1 *Value function for the sequential problem, i.e. $V(a,e)$ in equation* (1) *satisfies the Bellman equation*

$$
V(a,e) = \max_{a'\in\Gamma(a,e)} u(a(1+r)+we-a') + \beta\mathbb{E}[V(a',e')|e] \tag{12}
$$

where $\Gamma(a,e) \equiv \{a'|a' \in [0, a(1+r)+we]\}$. Hence, the unique solution to this problem is $a' = g(a,e)$.

Proof. Re-write the household's sequential problem (1) as follows:

$$
V(a,e) = \sup_{\{a_{t+1}\}_{t=0}^{\infty}\in\Pi(a,e)} \sum_{t=0}^{\infty} \beta^t \sum_{e_t\in E} [P^t]_{e,e_t} u(a_t(1+r)+we_t-a_{t+1}) \tag{13}
$$

where $\Pi(a, e)$ is the set of feasible allocations for initial state (a, e):

$$\Pi(a, e) \equiv \left\{ \{a_{t+1}\}_{t=0}^{\infty} | a_{t+1} \in [0, a_t(1+r) + we_t], a_{t+1} \text{ is } \mathcal{E}^t\text{-measurable, given } a_0 = a, e_0 = e \right\} \quad (14)$$

Starting from this notation, the proof of the fact that $V(a, e)$ satisfies Bellman equation (12) follows from standard arguments in Stokey, Lucas, and Prescott (1989) and more recently in Miao (2014) (See Lemma 7.1.1 and Theorem 7.1.1, pg 144-145) and the proof is omitted. Uniqueness of the optimal policy and its equivalence to $g(a, e)$ follows from Proposition 1.■.

Lemma G.2 *Let Θ be any compact subset of $\{(r, w, \beta)|\beta \in [0, 1), \beta(1+r) \in [0, 1), w > 0\}$. There exists $\bar{a} \geq 0$ such that $g(a, e; \theta) < a$ for all $a > \bar{a}$, all $e \in E$, and all $\theta \equiv (r, w, \beta) \in \Theta$.*

Proof. Suppose the claim were not true. Then, there exist $e \in E$, and a sequence (a_n, θ_n), where $a_n \to \infty$, $\theta_n \in \Theta$, for which $a'_n \equiv g(a_n, e; \theta_n) \geq a_n$ for all n. Let $\underline{w} \equiv \min_w \Theta > 0$ and $\bar{w} \equiv \max_w \Theta < \infty$. There are two cases to consider:

1. Case of $r_n a_n \to \infty$: Since $a_n \geq 0$, taking a subsequence if necessary, assume without loss of generality that $r_n \geq 0$. Following the same steps as in the proof of proposition 4, it is straightforward to derive $u'(r_n a_n + w_n e) \leq \beta_n(1 + r_n)u'(r_n a_n + w_n \underline{e})$. Since $\underline{w} \leq w_n \leq \bar{w}$ and marginal utility is decreasing, we have $u'(r_n a_n + \bar{w}e) \leq \beta_n(1+r_n)u'(r_n a_n + \underline{we})$. Let $x_n \equiv r_n a_n + \underline{we}$ and $\Delta \equiv \bar{w}e - \underline{we}$. Then we have
$$\beta_n(1 + r_n) \geq \frac{u'(x_n + \Delta)}{u'(x_n)}$$
By assumption, we have $x_n \to \infty$. Taking the limit as $n \to \infty$ and applying lemma D.1, we get $\liminf_{n\to\infty} \beta_n(1 + r_n) \geq 1$, a contradiction.

2. Case of $\liminf_{n\to\infty} r_n a_n = L < \infty$: As in the proof of proposition 4, we can write $c_n \equiv c(a_n, e; \theta_n) \leq r_n a_n + w_n e$, therefore $\limsup_{n\to\infty} c(a_n, e; \theta_n) \leq L + \bar{w}e < \infty$ must hold. However, it is easy to show that $\lim_{n\to\infty} c(a_n, e; \theta_n) = \lim_{n\to\infty} g(a_n, e; \theta_n) = \infty$, which achieves the desired contradiction. The proof of these limits essentially follows the same steps in the proof of Proposition 2 item 3, where I show that $\lim_{a\to\infty} c(a, e; \theta) = \lim_{a\to\infty} g(a, e; \theta) = \infty$ for constant $\theta = (r, w, \beta)$. Therefore, I omit the details.■

Lemma G.3 *Policy function $a' = g(a, e; \theta)$ (and therefore $c = c(a, e; \theta)$) is continuous in $\theta \equiv (r, w, \beta)$ over any compact subset Θ of $\{(r, w, \beta)|\beta \in [0, 1), \beta(1 + r) \in [0, 1), w > 0\}$.*

Proof. By lemma G.2, state space can be chosen to be bounded uniformly over the compact parameter space Θ. Let $\bar{a} \geq 0$ be such an upper bound for assets and $A = [0, \bar{a}]$ represent the space for assets. Next, we show that the value function for the household's sequential problem, $V(a, e; \theta)$ is continuous over the domain $A \times E \times \Theta$.

By Proposition 1, the solution to the household's sequential problem is unique and $V(a,e;\theta)$ is well-defined for all $(a,e,\theta) \in A \times E \times \Theta$. Consider the restricted choice set for the sequential problem

$$\tilde{\Pi}(a,e;\theta) \equiv \left\{ \{a_{t+1}\}_{t=0}^{\infty} | a_{t+1} \in [0, \min\{\bar{a}, a_t(1+r) + we_t\}], a_{t+1} \text{ is } \mathcal{E}^t\text{-measurable, given } a_0 = a, e_0 = e \right\}$$

Clearly $\tilde{\Pi}(a,e;\theta) \subset \Pi(a,e;\theta)$, where $\Pi(a,e;\theta)$ is the unrestricted choice set as defined in equation (14). Since optimal solution features $a_{t+1}^* \in [0,\bar{a}]$ for all $t \geq 0$, initial states $(a,e) \in A \times E$ and all $\theta \in \Theta$ by lemma G.2, we have

$$V(a,e;\theta) = \sup_{\{a_{t+1}\}_{t=0}^{\infty} \in \tilde{\Pi}(a,e;\theta)} \sum_{t=0}^{\infty} \beta^t \sum_{e_t \in E} [P^t]_{e,e_t} u(a_t(1+r) + we_t - a_{t+1})$$

Observe that $\tilde{\Pi}(a,e;\theta)$ is a countable Cartesian product of compact sets, and is therefore compact in the product topology by Tychonoff's Theorem. The constraint set and the objective function are continuous in (a,e,θ), and the solution $\{a_{t+1}^*\}_{t=0}^{\infty}$ exists and is unique for all $(a,e,\theta) \in A \times E \times \Theta$ due to Proposition 1. By Maximum Theorem, $V(a,e;\theta)$ is continuous in all of its arguments.

Next, we proceed with the continuity of the policy functions. By lemma G.1, the value function satisfies the Bellman equation

$$V(a,e;\theta) = \max_{a' \in \Gamma(a,e;\theta)} u(a(1+r) + we - a') + \beta \mathbb{E}[V(a',e';\theta)|e]$$

where $\Gamma(a,e;\theta) \equiv [a' \in [0,\bar{a}] | a' \leq a(1+r) + we\}$. Invoking Maximum Theorem a second time on the "$\max$" problem above and using the fact that the objective is continuous in (a,e,θ), the constraint set is compact, and that the solution is unique, we obtain the desired conclusion that the policy function $g(a,e;\theta)$ is continuous in θ over Θ.■

Proof of Theorem 1: Let $K(r) \equiv F_1^{-1}(r,1)$ represent the demand for capital when the interest rate equals r and let $w(r) \equiv F_2(K(r),1)$ represent the corresponding wage level. For the rest of the proof, to save on notation, I will suppress the dependence of $w(r)$ on r. This function is continuous and strictly decreasing by lemma 2 and all properties stated below apply both to case in which w is constant, and the case in which it depends on r.

Define excess demand function $XD(r) \equiv K(r) - A(r)$. We proceed in 3 steps:

1. *There exists $\bar{\rho} > 0$ such that $XD(\bar{\rho}) < 0$*: Since the state for assets are bounded for all $r < \bar{r}$, $A(r) < \infty$ must hold for all $r < \bar{r}$, moreover $K(r) < \infty$ for all $0 \leq r \leq \bar{r}$ by lemma 2. Proposition 6 implies $\lim_{r \to \bar{r}} A(r) = \infty$. Hence, there exists $\bar{\rho} \in [0,\bar{r})$ such that $A(\bar{\rho}) > K(0)$. Since $K(r)$ is strictly decreasing by lemma 2, $XD(\bar{\rho}) < 0$.

2. *There exists $\underline{\rho} < 0$ such that $XD(\underline{\rho}) > 0$*: By lemma 2, there exists $\underline{r} \in [-\delta, 0)$ such that $\lim_{r \to \underline{r}} K(r) =$

∞. For $r < 0$, budget constraint implies $g(a,e) \le -\frac{w\bar{e}}{r}$ for all $a \le -\frac{w\bar{e}}{r}$. Hence, the support of the stationary distribution/state space is contained in $[0, -\frac{w\bar{e}}{r}] \times E$. Therefore, we have $A(r) \le -\frac{w(r)\bar{e}}{r}$ for all $r \in (0, -\delta)$. For all $r < 0$,

$$XD(r) \ge K(r) + \frac{w(r)\bar{e}}{r} = K(r)\left[1 + \frac{w(r)}{K(r)}\frac{\bar{e}}{r}\right] = K(r)\left[1 + \frac{F_2(K(r),1)}{K(r)}\frac{\bar{e}}{r}\right] \tag{15}$$

Using the CRS assumption for the production function, continuity of $K(r)$, and $\lim_{r\downarrow \underline{r}} K(r) = \infty$ by lemma 2, we obtain $\lim_{r\to\underline{r}} \frac{F_2(K(r),1)}{K(r)} = \lim_{K\to\infty} \frac{F_2(K,1)}{K} = \lim_{K\to\infty}\left\{\frac{F(K,1)}{K} - F_1(K,1)\right\} = 0.$[28] This implies, the right-hand side of (15), and therefore $XD(r)$ diverges to infinity as $r \downarrow \underline{r}$. Since $XD(r) < \infty$ for all $r \in (-\delta, 0)$, there exists $\underline{\rho} < 0$ such that $0 < XD(\underline{\rho}) < \infty$.

3. *There exists $r^* \in [\underline{\rho}, \bar{\rho}]$ such that $XD(r^*) = 0$*: Lemma G.2 implies, there exists a uniform upper bound on the assets for all interest rates in the compact set $[\underline{\rho}, \bar{\rho}]$, i.e. the state space is uniformly compact. Moreover, policy functions are continuous in (r, w) over the set of prices $[\underline{\rho}, \bar{\rho}] \times [w(\bar{\rho}), w(\underline{\rho})]$ by lemma G.3. Then transition function $Q(.;r,w)$ varies continuously with respect to (r,w) in this set. Applying Theorem 12.13 (pg 384) by Stokey, Lucas, and Prescott (1989), the stationary distribution $\mu(.;r)$ is continuous in weak* sense over $[\underline{\rho}, \bar{\rho}]$. Since the support of the distributions are uniformly bounded, weak* continuity implies continuity of the means, i.e. $A(r)$. Then, $XD(r)$ is continuous in r over $[\underline{\rho}, \bar{\rho}]$.[29] We have also established $XD(\bar{\rho}) < 0$ and $XD(\underline{\rho}) > 0$. By intermediate value theorem, there exists an equilibrium interest rate $r^* \in [\underline{\rho}, \bar{\rho}]$ that satisfies $XD(r^*) = 0$.

I have just shown that there exists r^* that clears the capital market. Let $w^* \equiv w(r^*)$. It is easy to check that prices (r^*, w^*), the corresponding value and policy functions, and the stationary distribution $\mu(r^*)$ satisfy all requirements in definition 1.■

H Proof of Proposition 7

Lemma H.1 *Consumption and saving policy are homogeneous of degree 1 in (a,w), i.e., $g(ta,e;r,tw) = tg(a,e;r,w)$, and $c(ta,e;r,tw) = tc(a,e;r,w)$ holds for all $t > 0$, all $a \ge 0$, all $e \in E$, all $w > 0$ and all $r < \frac{1}{\beta} - 1$.*

Proof. Take any $t > 0$. Below, I show that $tc(\frac{\tilde{a}}{t}, e; r, w) = c(\tilde{a}, e; r, tw)$ holds for all $(\tilde{a}, e)$. More specifically, I show that function $tc(\frac{\tilde{a}}{t}, e; r, w)$ is a fixed point of the Coleman operator under prices (r, tw). By construction, policy $c(a,e;r,w)$ is weakly increasing and is less than $(1+r)a + we$ for all (a,e), therefore candidate $tc(\frac{\tilde{a}}{t}, e; r, w)$ is weakly increasing in $\tilde{a}$ and is less than $(1+r)\tilde{a} + twe$ for all $(\tilde{a}, e)$. Hence it belongs in

[28] Note that this follows regardless of whether $\lim_{K\to\infty} F(K,1)$ is finite.

[29] This property is not true in general. Weak*-continuity implies $\int f(a,e)d\mu(.;r_n) \to \int f(a,e)d\mu(.;r^*)$ for all bounded continuous functions $f(a,e)$ for any sequence $r_n \to r^*$. We have $A(r) = \int a d\mu(.;r)$, even though the integrand is not bounded in general, it is bounded over the compact state space S in our case when $r \in [\underline{\rho}, \bar{\rho}]$.

the functional space $\mathcal{C}$ over which Coleman operator is defined when prices equal (r, tw) . For all $(\tilde{a}, e)$, $K(tc(.)) \equiv K(tc(\frac{\tilde{a}}{t}, e; r, w))$ satisfies

$$u'\big(K(tc(.)\big) = \max\left\{\beta(1+r)\mathbb{E}\left\{u'\left(tc\big(\frac{\tilde{a}(1+r)+twe-K(tc(.))}{t}, e; r, w\big)\right)\right\}, u'((1+r)\,\tilde{a}+twe)\right\}$$
$$u'\big(\frac{K(tc(.))}{t}\big) = \max\left\{\beta(1+r)\mathbb{E}\left\{u'\left(c\big(\frac{\tilde{a}}{t}(1+r)+we-\frac{K(tc(.))}{t}, e; r, w\big)\right)\right\}, u'((1+r)\,\frac{\tilde{a}}{t}+we)\right\}$$

where the second expression is obtained by using the CRRA property and dividing both sides of the first expression above by $t^{-\sigma}$. Under prices (r, w) and state $(\frac{\tilde{a}}{t}, e)$, policy function satisfies

$$u'\big(c(\frac{\tilde{a}}{t}, e; r, w)\big) = \max\left\{\beta(1+r)\mathbb{E}\left\{u'\left(c\big(\frac{\tilde{a}}{t}(1+r)+we-c(\frac{\tilde{a}}{t}, e; r, w), e; r, w\big)\right)\right\}, u'((1+r)\,\frac{\tilde{a}}{t}+we)\right\}$$

Comparing the last two expressions, we find that $\frac{K(tc(\frac{\tilde{a}}{t}, e; r, w))}{t} = c(\frac{\tilde{a}}{t}, e; r, w)$ satisfies the first equality (uniquely by Lemma A.1), and hence $K(tc(\frac{\tilde{a}}{t}, e; r, w)) = tc(\frac{\tilde{a}}{t}, e; r, w)$ must hold for all $(\tilde{a}, e)$. This proves $tc(\frac{\tilde{a}}{t}, e; r, w)$ is a fixed point of the Coleman operator under prices (r, tw) and since the fixed point is unique by Proposition 1, $tc(\frac{\tilde{a}}{t}, e; r, w) = c(\tilde{a}, e; r; tw)$ holds for all $\tilde{a}$. The claim in the lemma follows from the change of variables $\tilde{a} = ta$. This property also holds for the saving policy: $tg(a, e; r, w) = (1+r)ta + twe - tc(a, e; r, w) = (1+r)ta + twe - c(ta, e; r, tw) = g(ta, e; r, tw)$.■

The following lemma establishes asymptotic linearity of the policy functions under CRRA utility assumption. This result is not new, for an alternative proof in a slightly different environment, see Benhabib, Bisin, and Zhu (2015).

Lemma H.2 *Saving policy satisfies* $\lim_{a\to\infty} \frac{g(a,e)}{a} = [\beta(1+r)]^{1/\sigma}$ *for all* $e \in E$ *when* $r > 0$.

Proof. Suppose $r > 0$ holds. Define $\bar{\kappa}(e) \equiv \limsup_{a\to\infty} \frac{g(a,e)}{a}$. By proposition 2, when $r > 0$, $c(a, e) \geq ra + w\underline{e}$ and therefore $g(a, e) \leq a + we - w\underline{e}$. This implies $\bar{\kappa}(e) \leq 1$ for all $e \in E$. Let $\bar{\kappa}^* \equiv \max_e \bar{\kappa}(e)$ and $e_M \in \arg\max_e \bar{\kappa}(e)$. By proposition 2, for large enough a, $g(a, e) > 0$ and Euler equality holds. By dividing and multiplying by relevant quantities, the Euler equality for e_M can be written as

$$\left(\frac{c(a, e_M)}{a}\right)^{-\sigma}\left(\frac{g(a, e_M)}{a}\right)^{\sigma} = \beta(1+r)\mathbb{E}\left[\left(\frac{c(g(a, e_M), e')}{g(a, e_M)}\right)^{-\sigma}\middle| e\right] \tag{16}$$

By proposition 2, $r \leq \liminf_{a\to\infty} \frac{c(a,e_M)}{a} = 1 + r - \bar{\kappa}^* \leq 1 + r$ since $ra + w\underline{e} \leq c(a, e) \leq (1+r)a + we$. Similarly, due to proposition 2, $\lim_{a\to\infty} g(a, e) = \infty$, therefore $r \leq \liminf_{a\to\infty} \frac{c(g(a,e_M),e')}{g(a,e_M)} = 1+r-\bar{\kappa}(e') \leq 1 + r$ holds for all $e' \in E$. These limits jointly imply

$$0 < \limsup_{a\to\infty} \beta(1+r)\mathbb{E}\left[\left(\frac{c(g(a, e_M), e')}{g(a, e_M)}\right)^{-\sigma}\middle| e\right] = \beta(1+r)\mathbb{E}[(1+r-\bar{\kappa}(e'))^{-\sigma}|e] < \infty$$

where we used the property that $\limsup\left(\frac{c(g(a,e_M),e')}{g(a,e_M)}\right)^{-\sigma} = \left(\liminf \frac{c(g(a,e_M),e')}{g(a,e_M)}\right)^{-\sigma}$.

Since equality (16) holds for all large a, this limit result implies the left-hand side in (16) also has a positive finite limit superior, moreover,

$$0 < \limsup_{a\to\infty} \left(\frac{c(a, e_M)}{a}\right)^{-\sigma} \left(\frac{g(a, e_M)}{a}\right)^{\sigma} \leq (1 + r - \bar{\kappa}^*)^{-\sigma} \bar{\kappa}^{*\sigma} < \infty$$

where the second inequality follows from the property $\limsup f_1(x) f_2(x) \leq \big(\limsup f_1(x)\big)\big(\limsup f_2(x)\big)$ for any two functions that satisfy $f_1, f_2 \geq 0$. Observe that this result implies $\bar{\kappa}^* > 0$. Hence, we have the following limit inequality

$$(1 + r - \bar{\kappa}^*)^{-\sigma} \bar{\kappa}^{*\sigma} \geq \beta(1 + r)\mathbb{E}[(1 + r - \bar{\kappa}(e'))^{-\sigma}|e]$$

Rearranging this expression, we get

$$\bar{\kappa}^* \leq [\beta(1 + r)]^{1/\sigma} \left\{\mathbb{E}\left[\left(\frac{1 + r - \bar{\kappa}'}{1 + r - \bar{\kappa}^*}\right)^{-\sigma} \middle| e\right]\right\}^{1/\sigma} \leq [\beta(1 + r)]^{1/\sigma} \tag{17}$$

where the second inequality follows from $\bar{\kappa}' \leq \bar{\kappa}^*$.

Now define $\underline{\kappa}(e) = \liminf_{a\to\infty} \frac{g(a,e)}{a}$, let $\underline{\kappa}^* \equiv \min_{e\in E} \underline{\kappa}(e)$, and $e_m \in \arg\min_{e\in E} \underline{\kappa}(e)$. The steps above can be repeated by taking the $\liminf$ of the Euler equality for e_m to get

$$\underline{\kappa}^* \geq [\beta(1 + r)]^{1/\sigma} \left\{\mathbb{E}\left[\left(\frac{1 + r - \underline{\kappa}'}{1 + r - \underline{\kappa}^*}\right)^{-\sigma} \middle| e\right]\right\}^{1/\sigma} \geq [\beta(1 + r)]^{1/\sigma} \tag{18}$$

where the second inequality follows from $\underline{\kappa}' \geq \underline{\kappa}^*$. Then, (17) and (18) jointly imply

$$\underline{\kappa}^* \geq [\beta(1 + r)]^{1/\sigma} \geq \bar{\kappa}^* \geq \underline{\kappa}^*$$

Then $\bar{\kappa}(e) = \underline{\kappa}(e) = \lim_{a\to\infty} \frac{g(a,e)}{a} = [\beta(1 + r)]^{1/\sigma} < 1$ must hold for all $e \in E$.■

Proof of Proposition 7:

1) Fix some $t > 0$ and r. Let $\mathcal{A}$ be the Borel σ-algebra on $\mathbb{R}_+$, let $S = \mathbb{R}_+ \times E$ denote the state space, and define $\Sigma \equiv \mathcal{A} \times \mathcal{E}$ to be the product σ-algebra on S.

Lemma H.1 shows that saving policy satisfies $tg(a, e; w) = g(ta, e; tw)$ for all $t > 0$ and all $(a, e) \in S$. Let $\mu(.; w)$ and $\mu(.; tw)$ be joint stationary distributions of assets and labor efficiency under prices (r, w) and (r, tw) respectively. These distributions exist and are unique by proposition 5. Due to homogeneity of the policy functions, the transition functions satisfy the following property which follows trivially from its definition in expression 5:

$$Q((ta, e), \tilde{C}; tw) = Q((a, e), C; w) \text{ for all } (a, e) \in S, \text{ and all } C \in \Sigma \text{ where } \tilde{C} \equiv tC_A \times C_E \tag{19}$$

Define the following function on Σ: Let $\nu(\tilde{C}) \equiv \mu(C; w)$ for all $C \in \Sigma$, where $\tilde{C} \equiv tC_A \times C_E$. Clearly $\nu : \Sigma \to \mathbb{R}_+$ is a measure on Σ.[30] Now I show that $\nu(.)$ is a stationary distribution under prices (r, tw), i.e. want to show that $\nu(\tilde{C}) = \int_S Q((\tilde{a}, e), \tilde{C}; tw)\nu(d\tilde{a} \times de)$ for all $\tilde{C} \in \Sigma$ defined as above. Observe that by a change of variables $ta = \tilde{a}$, this integral satisfies

$$\int_S Q((\tilde{a}, e), \tilde{C}; tw)\nu(d\tilde{a} \times de) = \int_S Q((ta, e), \tilde{C}; tw)\nu(tda \times de) = \int_S Q((a, e), C; w)\mu(da \times de; w) = \mu(C; w)$$

where the first equality follows from the change of variables, the second follows from the property of the transition function (19), and by definition of ν. The last equality follows from the fact that $\mu(.; w)$ is a stationary distribution under prices (r, w). Now observe that $\nu(\tilde{C}) = \mu(C; w)$ holds by construction, i.e. ν is a stationary distribution under (r, tw). Since the stationary distribution is unique, we have $\mu(.; tw) = \nu(.)$. Now, using the same change of variables, $\tilde{a} = ta$ and the property of the measures, we have

$$A(r, tw) = \int_S \tilde{a}\mu(d\tilde{a} \times de; tw) = \int_S ta\mu(tda \times de; tw) = t\int_S a\mu(da \times de; w) = tA(r, w)$$

This completes the proof.

2) Under CRRA assumption, saving (consumption) policy function is convex (concave) under arbitrary Markov processes and in the presence of borrowing constraints. (See Jensen (2015) Theorem 11 and Remark 10) In fact it can be shown, by following the arguments in Jensen (2015), that Coleman operator maps concave functions in $\mathcal{C}$ to concave functions in $\mathcal{C}$, and hence the fixed point must be concave under pointwise convergence. Since saving policy is increasing, convex and satisfies $g(a, e) \geq 0$, the secant of the policy function at $a = 0$, $f(a) \equiv \frac{g(a,e) - g(0,e)}{a}$ is increasing over all $a > 0$. But then we have

$$f(a) \leq \lim_{a \to \infty} \frac{g(a, e) - g(0, e)}{a} = \kappa(r)$$

where the last equality follows from lemma H.2. Since $f(a) \leq \kappa(r)$ for all $a > 0$, we obtain the inequality $g(a, e) \leq \kappa(r)a + g(0, e)$ holds. Proposition 2 implies $c(0, e) \geq \omega(e)$, therefore $g(0, e) \leq we - \omega(e)$ holds. Then $g(a, e) \leq \kappa(r)a + we - \omega(e)$, as we wanted to show.■.

[30] This is the case because $\phi : \Sigma \to \Sigma$ where $\phi(C) \equiv tC_A \times C_E$ is a bijection.