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14 February 2008

Online at <https://mpra.ub.uni-muenchen.de/7160/>  
MPRA Paper No. 7160, posted 15 Feb 2008 00:16 UTC

# INFORMATION REVELATION AND RANDOM ENTRY IN SEQUENTIAL ASCENDING AUCTIONS

MAHER SAID<sup>†</sup>

**ABSTRACT.** We examine a model in which multiple buyers with single-unit demand are faced with an infinite sequence of auctions. New buyers arrive on the market probabilistically, and are each endowed with a constant private value. Moreover, objects also arrive on the market at random times, so the number of competitors and the degree of informational asymmetry among them may vary across from one auction to the next. We demonstrate by way of a simple example the inefficiency of the second-price sealed-bid auction in this setting, and therefore assume that each object is sold via ascending auction.

We then characterize an efficient and fully revealing equilibrium for the game in which the objects are sold via ascending auctions. We show that each buyer's bids and payoffs depend only upon their rank amongst their competitors and the (revealed) values of those with lower values. Furthermore, strategies are memoryless—bids depend only upon the information revealed in the current auction, and not on any information that may have been revealed in earlier periods. We then demonstrate that the sequential ascending auction serves as an indirect mechanism that is equivalent—in our setting—to the dynamic marginal contribution mechanism introduced by Bergemann and Välimäki (2007) and generalized in Cavallo et al. (2007).

**KEYWORDS:** Sequential auctions, Ascending auctions, Random arrivals, Information revelation, Dynamic Vickrey-Clarke-Groves mechanism, Marginal contribution

**JEL CLASSIFICATION:** C73, D44, D83.

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*Date:* February 14, 2008.

I would like to thank my advisor, Dirk Bergemann, for his constructive advice and comments, as well as Alessandro Bonatti and Rahul Deb for their many helpful discussions. All errors are, of course, my own. This work was generously supported by a National Science Foundation Graduate Research Fellowship and a Cowles Foundation Anderson Fellowship.

## 1. INTRODUCTION

Many markets, most notably internet markets such as eBay, sell multiple objects via sequential auctions in which one object is sold at a time. In this paper, we examine a model of such markets in which new buyers arrive on the market at random times. Each bidder has an independently drawn private value for purchasing an object. In contrast to much of the literature that makes use of sealed-bid auctions, we focus on the ascending auction. Although these two auctions are in many respects equivalent in a static setting with private values, this equivalence does not hold in a dynamic environment, primarily due to the information revelation inherent in the ascending auction format. The difference between the two formats is further exacerbated in the sequential auction setting when we allow for the entry of new buyers. In particular, the entry of a new buyer introduces an additional informational asymmetry. We show, however, that this asymmetry is easily resolved when the ascending auction is used. In equilibrium, each buyer's bids and payoffs depend only on the buyer's rank amongst their competitors and the (revealed) values of those opponents with lower values. Furthermore, these strategies have the remarkable property of being memoryless—in each auction conducted, bids are independent of any information that may have been revealed in previous periods, despite the fact that *all* private information is revealed in each and every auction.

We feel that this model serves as a useful abstraction of online auction sites such as eBay or uBid, especially when considering the extensive market on these sites for individual units of brand-new homogenous goods. Typically, a variety of auctions for identical items are open simultaneously, but may be ordered by their closing time. Thus, abstracting away from intra-auction dynamics, a sequential auction model yields a good approximation.<sup>1</sup> With this in mind, many authors (see Sailer (2006) or Zeithammer (2006), for instance) make use of the second-price sealed-bid auction, citing evidence from Roth and Ockenfels (2002) and Bajari and Hortacısu (2003) about the prevalence of “sniping” (last-second bidding) in online auctions in defense of their modeling choice. However, as shown by Cai et al. (2007), pure-strategy symmetric equilibria *do not* exist in sequential sealed-bid auctions when buyer values are fixed across time and bids are made publicly observable after each auction. As most online auctions bear a close resemblance to English auctions in regards to intra-auction dynamics as well as the visibility of submitted bids (both during an auction and after an auction has closed), we believe that the ascending auction is better-suited than the sealed-bid second-price auction for modeling online auction markets.<sup>2</sup>

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<sup>1</sup>Nekipelov (2007) examines within-auction dynamics with an eye towards entry behavior in a model of eBay auctions.

<sup>2</sup>Of course, either choice is a compromise, abstracting away from important real-world issues for the sake of tractability.

What is more, we feel that the sequential ascending auction is important for another, independent, reason. Bergemann and Välimäki (2007) demonstrate the suitability of sequential ascending auctions as a simple way to provide for the truthful implementation of the socially efficient allocation in some dynamic environments. In particular, they provide an example in which sequential ascending auctions are equivalent to their dynamic marginal contribution mechanism. This equivalence fails to hold, however, in a more complex setting with multiple-unit demand. Cavallo et al. (2007) generalize this dynamic version of the classic Vickrey-Clarke-Groves mechanism to settings in which agents may be “inaccessible” for periods of time. The present work complements these papers, as we show that the sequential ascending auction serves as an (easily implemented and understood) indirect mechanism that is equivalent to the dynamic marginal contribution mechanism, and is therefore an incentive compatible mechanism for inducing socially efficient choices.

The present work is closely related to several papers in the sequential auctions literature. Milgrom and Weber (2000) examine the properties of a variety of auction formats for the (simultaneous or sequential) sale of multiple objects with a fixed set of buyers and objects.<sup>3</sup> In regards to the ascending auction with private values, they show that, in equilibrium, buyers bid exactly their values. However, they allow for neither discounting nor the entry of new buyers, features that play a large role our model. The vast majority of the literature following that work has chosen to focus on sealed-bid auctions; for example, the previously mentioned Sailer (2006) and Zeithammer (2006) conduct empirical studies of eBay auctions making use of sequential second-price sealed-bid auctions and assumptions of an effectively static environment. Kittsteiner et al. (2004) examine the role of discounting in sequential sealed-bid auctions, and prove a revenue equivalence result for auctions in which the only information revealed is the valuations of bidders who have already left the market, while Jeitschko (1998) considers a model of first-price sealed-bid auctions in which winner’s bids are revealed, allowing the remaining buyers to update their beliefs about their opponents’ valuations. On the other hand, Cai et al. (2007) demonstrate the nonexistence of pure-strategy symmetric equilibria in sealed-bid sequential auction models in which all bids are revealed. The only paper that we are aware of that examines sequential ascending auctions is that of Caillaud and Mezzetti (2004), who examine reserve prices in a model with only two auctions.

Certain elements within the bargaining with incomplete information literature are also related to our model. Inderst (2008) considers a bargaining model in which a seller is randomly visited by heterogeneous buyers. If the seller is currently engaged in bargaining with one agent when another arrives, she may choose to switch from one buyer to

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<sup>3</sup>Note that this work was originally completed in 1982 and was in wide circulation in working paper form until its eventual publication in 2000.

the other. However, this switch is permanent, implying that the arrival of a new buyer either “restarts” the game or is completely irrelevant. Fuchs and Skrzypacz (2007) take a different approach: they consider an incomplete information bargaining problem between a buyer and a seller, and allow for the possibility of the arrival of “events” which end the game and yield a particular expected payoff to each agent. Their interpretation is that these events may be viewed as triggers for some sort of multi-lateral mechanism involving new entrants (a second-price auction, for example) for which the expected payoffs are a reduced-form representation. Thus, while both works are primarily concerned with characterizing the endogenous option value that results from the potential arrival of additional participants to the market, they do this in a framework of bilateral bargaining which fails to capture the dynamic nature of competition among several current and potential market participants. On the other hand, Nekipelov (2007) studies the role of entry *during* an online ascending auction, while Said (2008) examines the role of buyer entry *between* periods in a model of sequential second-price auctions in which objects are stochastically equivalent. By way of comparison, the present work incorporates buyer entry between ascending auctions in a more standard private values framework, and further demonstrates the relationship between the endogenous option value arising from participating in future auctions with the marginal contribution to social welfare.

The paper is organized as follows. We present our model in Section 2, and then provide a simple example demonstrating the superiority of the ascending auction format over the second-price sealed-bid auction in a dynamic setting with buyer entry in section 3. Section 4 solves for the equilibrium in our model with buyer entry and demonstrates some of its desirable properties. In section 5, we discuss the relationship between our model and the dynamic Vickrey-Clarke-Groves mechanism and generalize our setting to allow for the random arrival of objects. Finally, section 6 concludes.

## 2. THE MODEL

We consider a market in which time is discrete; periods are indexed by  $t \in \mathbb{N}$ . There is a finite number  $n_t$  of risk-neutral buyers with single-unit demand in the market in any given period  $t$ . Each buyer  $i \in \{1, \dots, n_t\}$  has a valuation  $v_i \in \mathbb{R}_+$ , where  $v_i$  is drawn from the distribution  $F$  with corresponding density  $f$ . We assume that valuations are private information, and are independently and identically distributed across buyers. Moreover, additional buyers may arrive on the market in each period. We will assume that at most one buyer arrives at a time, and that this arrival occurs with some exogenously given probability  $q \in [0, 1]$ . Finally, we assume that buyers discount the future exponentially with discount factor  $\delta \in (0, 1)$ .

In each period, there is exactly one object available for sale via an ascending auction. The auction begins with the price at zero and all bidders participating in the auction. Each bidder may choose to any price at which to drop out of the auction. This exit decision is irreversible (in the current period), and is observable by all agents currently present in the market. Finally, the auction ends whenever exactly 1 active bidder remains, and the price paid by this winning bidder is the price at which the last exit occurred. Note that we assume that the number of active bidders is commonly known throughout the auction. With this in mind, each bidder's strategy *within* a given period is not a single bid, but rather a sequence of functions, each of which is contingent on the (observed) exit prices of the bidders who have already dropped out of the current auction.

Throughout, we will denote by  $\hat{v}$  the ordered vector of realized buyer valuations, where

$$\hat{v}_1 > \hat{v}_2 > \dots > \hat{v}_{n_t}.$$

Furthermore, we will denote by  $V_{j,k}(\hat{v})$  the expected payoff of the buyer with the  $j$ -th highest of  $k$  values. For example, if there are three bidders present, with  $v_2 > v_3 > v_1$ , then  $\hat{v} = \{v_2, v_3, v_1\}$ , bidder 1's payoff is  $V_{3,3}(\hat{v})$ , bidder 2's payoff is  $V_{1,3}(\hat{v})$ , and bidder 3's payoff is  $V_{2,3}(\hat{v})$ .

### 3. A MOTIVATING EXAMPLE

Suppose that there are two buyers on the market with values  $v_1, v_2 \in [0, 1]$ , where, without loss of generality, we assume that  $v_1 > v_2$ . In addition, a third potential buyer with value  $v_3 \sim F$ , where  $F$  is the uniform distribution on  $[0, 1]$  may enter the market with probability  $q \in (0, 1)$ . Each of these buyers wishes to purchase exactly one unit of some object which is being sold via a sequence of three auctions. All buyers discount time with discount factor  $\delta \in (0, 1)$ . Furthermore, we make the assumption that  $v_1$  and  $v_2$  are commonly known amongst all buyers, which may be viewed as the result of information being revealed via bidding behavior in some (unmodeled) previous periods. The new entrant's value, however, is her own private information. We will consider two versions of this example; first, we assume that objects are sold via second-price auctions in which the buyers' bids are revealed after each round, and then we will assume that objects are sold via ascending auctions.

We begin with the second-price auction. Note that in any round in which there is only one bidder present, that bidder receives the object at a price of zero, regardless of her bid. Therefore, if there are two bidders present in the second period, each bidder  $i$  has an option value of  $\delta v_i$  from losing. Thus, regardless of the information that each bidder has about the other, it is weakly dominant for each bidder to submit a bid of their true value less their option value—the optimal bid for each bidder  $i$  is  $(1 - \delta)v_i$ . Thus, denoting the payoff of a bidder in the second round when there are two bidders present as  $U(v_i, v_j)$ ,

we have

$$(3.1) \quad U(v_i, v_j) = \begin{cases} v_i - (1 - \delta)v_j & \text{if } v_i > v_j, \\ \delta v_i & \text{if } v_i \leq v_j. \end{cases}$$

Using this expression, we may write the payoff of a lone bidder with value  $v_i$  as  $U(v_i, 0)$ .

Now consider the third bidder (when present). Under the assumption that bidder 1 bids a greater amount than bidder 2 (that is, that  $b_1 > b_2$ ), the third bidder faces a choice between winning the auction and receiving a payoff of  $v_3 - b_1$  or losing the auction and facing bidder 2 in the next period, yielding a payoff of  $\delta U(v_3, v_2)$ . Thus, bidder 3 prefers to win if, and only if,  $v_3 - b_1 \geq \delta U(v_3, v_2)$ , or, equivalently,  $b_1 \leq v_3 - \delta U(v_3, v_2)$ . She can then win the auction if, and only if, it is optimal for her to do so by bidding

$$(3.2) \quad b_3(v_3) = v_3 - \delta U(v_3, v_2) = \begin{cases} (1 - \delta)v_3 + \delta(1 - \delta)v_2 & \text{if } v_3 > v_2, \\ (1 - \delta^2)v_3 & \text{if } v_3 \leq v_2. \end{cases}$$

Note that  $b_3$  is strictly increasing in  $v_3$ , and hence fully identifies bidder 3's valuation in the next period when bids are revealed. For convenience, we will denote by  $u_1$  and  $u_2$  the values of bidder 3 that submit bids equal to those of bidders 1 and 2, respectively; that is,

$$u_1 = b_3^{-1}(b_1) \text{ and } u_2 = b_3^{-1}(b_2).$$

Now consider the case of bidder 2's bid in the first period of the game. If she submits a winning bid in the first period, she receives a payoff of  $v_2 - b^*$ , where  $b^*$  is the highest competing bid that she faces. On the other hand, if she loses the first-round auction, she receives a payoff of  $\delta \mathbb{E}[U(v_2, v^*)]$ , where

$$v^* = \begin{cases} 0 & \text{with probability } 1 - q, \\ v_3 & \text{with probability } qF(u_1), \\ v_1 & \text{with probability } q(1 - F(u_1)). \end{cases}$$

Thus, bidder 2 prefers to win if, and only if,  $v_2 - b^* \geq \delta \mathbb{E}[U(v_2, v^*)]$ . She may then guarantee that she wins only when it is desirable to do so by bidding

$$(3.3) \quad \begin{aligned} b_2 &= v_2 - \delta \mathbb{E}[U(v_2, v^*)] \\ &= v_2 - \delta \left[ \begin{aligned} &(1 - q)v_2 + \delta q(1 - F(u_1))v_2 \\ &+ q \int_0^{v_2} (v_2 - (1 - \delta)v') dF(v') + q \int_{v_2}^{u_1} \delta v_3 dF(v') \end{aligned} \right] \\ &= (1 - \delta)(1 + \delta q)v_2 - (1 - \delta)\delta q \frac{v_2^2}{2}. \end{aligned}$$

Finally, let us consider buyer 1's bidding behavior in the first period of the game. Note first that  $u_2 < v_2 < v_1$ , implying that if bidder 1 loses today, she will definitely win the auction in the next period. To see this, note that if bidder 3 enters and wins the first round, bidder 1 faces  $v_2 < v_1$  in the next period. On the other hand, if bidder 2 is the high bidder

in the first round, then bidder 1 is either alone or faces  $v_3 < u_2 < v_1$  in the second round. Thus, when the high opponent bid is  $b^*$ , winning yields bidder 1 a payoff of  $v_1 - b^*$ , while losing yields a payoff of  $\delta U(v_1, v^*)$ , where

$$v^* = \begin{cases} 0 & \text{with probability } 1 - q, \\ v_3 & \text{with probability } qF(u_2), \\ v_2 & \text{with probability } q(1 - F(u_2)). \end{cases}$$

Thus, similar to the cases of bidders 2 and 3, bidder 1 may guarantee that she wins only when it is desirable for her to do so by bidding

$$b_1 = v_1 - \delta \mathbb{E}[U(v_1, v^*)] = v_1 - \delta \left[ (1 - q)v_1 + \delta q(1 - F(u_2))(v_1 - (1 - \delta)v_2) + q \int_0^{u_2} (v_1 - (1 - \delta)v') dF(v') \right].$$

Recall that  $u_2 = b_3^{-1}(b_2) < v_2$ , implying that  $u_2 = b_2/(1 - \delta^2)$ . Combining this with the assumption that  $F(x) = x$  implies that

$$u_2 = \frac{b_2}{1 - \delta^2} = \frac{1 + \delta q}{1 + \delta} v_2 - \frac{\delta q}{1 + \delta} \frac{v_2^2}{2}.$$

Thus, we may conclude that

$$(3.4) \quad b_1 = (1 - \delta)v_1 + (1 - \delta)\delta q v_2 - \frac{\delta(1 - \delta)(1 + \delta q)(1 + \delta(2 - q))q v_2^2}{2(1 + \delta)^2} + \frac{\delta^3(1 - \delta)(1 - q)q^2 v_2^3}{2(1 + \delta)^2} + \frac{\delta^3(1 - \delta)q^3 v_2^4}{8(1 + \delta)^2}.$$

For clarity, Figure 1 plots the bids of all three buyers for fixed parameter values.<sup>4</sup> The key features to note are that  $u_1 < v_1$  and  $u_2 < v_2$ ; use of the second-price auction in this context may lead to inefficient outcomes, as “low” values of bidder 3 may outbid bidders 1 and 2 despite their having higher values. This result is driven by two main features of our setting: first, agents discount the future and hence the order in which objects are allocated matters; and second, there is a fundamental asymmetry in information—not only is bidder 3’s value private information, but the very presence of bidder 3 on the market is information that is asymmetrically distributed amongst the buyers. Thus, in addition to the nonexistence of symmetric equilibria in sequential second-price sealed-bid auctions as demonstrated by Cai et al. (2007), allowing for random entry may induce inefficient outcomes—even in the asymmetric equilibria.

We now demonstrate that the ascending auction does not share the inefficiency of the second-price auction in this setting. Note that when there are only two bidders present, the losing bidder is guaranteed a payoff of  $\delta v_i$  in the next period. Therefore, bidders are willing to remain active in an auction until the price reaches  $(1 - \delta)v_i$ . Thus, the expected

<sup>4</sup>The qualitative features of the equilibrium do not depend on these parameter values.



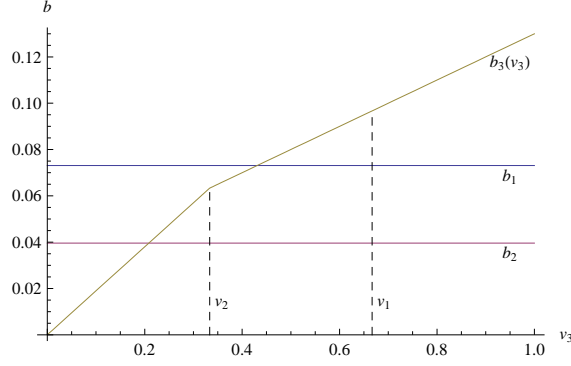


Figure 1: Bids when  $v_1 = \frac{2}{3}$ ,  $v_2 = \frac{1}{3}$ ,  $v_3 \sim U[0, 1]$ ,  $\delta = \frac{9}{10}$ , and  $q = \frac{1}{4}$ .

payoff of a bidder when she has only one opponent present on the market is given by  $U(v_i, v_j)$  from Equation 3.1.

When there are three bidders present, matters are slightly different. In particular, the very nature of an ascending auction immediately reveals to all bidders the number of participants. Thus, bidder 3 is unable to keep private her presence on the market. This implies that the first bidder to drop out of the auction knows that they have the lowest value among three bidders, and hence will receive an expected payoff of  $\delta^2 v_i$ . Thus, each of the three bidders remains active until the price reaches

$$(1 - \delta^2)v_i.$$

Denoting by  $\hat{v}_3$  the lowest of the three values, the two remaining bidders now know that they are guaranteed a payoff of  $U(v_i, \hat{v}_3)$  in the following period, and are hence willing to remain active until they are indifferent between winning at the current price and winning the object in the following period; that is, until the price reaches

$$(1 - \delta)v_i + \delta(1 - \delta)\hat{v}_3.$$

Notice that these cutoff prices are strictly increasing in each bidder's value, and hence are both efficient and fully revealing.<sup>5</sup> Thus, we have established that the ascending auction does not suffer from the same shortcomings as the second-price auction in this relatively simple setting. Thus, we will focus exclusively on the ascending auction from this point forward.

<sup>5</sup>In addition, it is straightforward to verify that these strategies do, in fact, constitute an equilibrium. Conditional on participation, no bidder wishes to deviate from these strategies. Furthermore, no bidder wishes to postpone their participation to a future period.

#### 4. EQUILIBRIUM ANALYSIS

**4.1. Preliminaries and Equilibrium Strategies.** One of the most remarkable features of the equilibrium that we construct in this model is that buyer's bids and payoffs do not depend upon the valuations of higher-ranked bidders (neither in expectation nor realization), even if that information is publicly available. Recall that  $\hat{v}$  is the *ordered* vector of realized buyer valuations, where

$$\hat{v}_1 > \cdots > \hat{v}_n,$$

and that we denote by  $V_{k,n}(\hat{v})$  the expected payoff of the buyer with the  $k$ -th highest of  $n$  values. To show the property described above, we will show that (abusing notation slightly) we may write

$$V_{k,n}(\hat{v}_1, \dots, \hat{v}_n) = V_{k,n}(\hat{v}_k, \dots, \hat{v}_n).$$

A formal statement of this result may be found in the subsequent section; in the meantime, we will describe the equilibrium taking this property as a given.

So, suppose that an auction is in progress with  $n$  bidders, where  $\hat{v} \in \mathbb{R}_+^n$  denotes the (ordered) vector of realized buyer valuations. When all bidders are still active, a bidder with valuation  $v_i$  who drops out of the bidding learns (and reveals) that, in equilibrium, she has the lowest value; that is, that  $\hat{v}_n = v_i$ . Therefore, her expected payoff in the next period is  $V_{n-1,n-1}(v_i)$ , as at the beginning of the next period, there will be  $n - 1$  bidders remaining (the current period's  $n$  bidders less the winning buyer) and she will have the lowest value. Therefore, each bidder  $i$  should remain in the auction until the current price  $p$  is such that

$$v_i - p = \delta V_{n-1,n-1}(v_i).$$

At this price, bidder  $i$  is indifferent between purchasing the object today and waiting until the next period when  $i$  will be the lowest-valued buyer. Thus, when no one has dropped out, bidder  $i$  will remain in the auction until the price reaches

$$(4.1) \quad \beta_{n,n}(v_i) := v_i - \delta V_{n-1,n-1}(v_i).$$

Once someone drops out of the auction, the remaining  $n - 1$  bidders learn the realization of  $\hat{v}_n$  and that they are not the lowest-valued competitor.<sup>6</sup> Therefore, the next bidder (with value  $v_i$ ) to drop out reveals herself to be the second-lowest of the  $n$  bidders; therefore, her expected payoff in the next period is  $V_{n-2,n-1}(v_i, \hat{v}_n)$ , as she will be the second-lowest of the  $n - 1$  buyers remaining in the following period. Thus, each bidder  $i$  who has not already dropped out should remain in the auction until the current price  $p$  is such that she is indifferent between purchasing the object in the present period and waiting until

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<sup>6</sup>This of course requires  $\beta_{n,n}$  to be a strictly increasing function, something that we will verify in short order.

the next period

$$v_i - p = \delta V_{n-2,n-1}(v_i, \hat{v}_n).$$

Thus, when no one has dropped out, bidder  $i$  remains in the auction until the price reaches

$$(4.2) \quad \beta_{n-1,n}(v_i, \hat{v}_n) := v_i - \delta V_{n-2,n-1}(v_i, \hat{v}_n).$$

Proceeding inductively, we define for each  $k = 2, \dots, n$  the bidding function

$$(4.3) \quad \beta_{k,n}(v_i, \hat{v}_{k+1}, \dots, \hat{v}_n) := v_i - \delta V_{k-1,n-1}(v_i, \hat{v}_{k+1}, \dots, \hat{v}_n).$$

These bidding functions define the drop-out points for a bidder with value  $v_i$  when there are  $k$  buyers still active in the auction. Notice that this implies that the final price in this auction will be

$$\beta_{2,n}(\hat{v}_2, \dots, \hat{v}_n) = \hat{v}_2 - \delta V_{1,n-1}(\hat{v}_2, \dots, \hat{v}_n).$$

Keep in mind, however, that we must verify that these bid functions are strictly monotone (so that values are revealed), and also that these bidding strategies in fact form an equilibrium. This requires a characterization of the expected payoff functions  $V_{j,k}$ .

**4.2. The Payoff Functions.** As a preview of our results, consider first the case of a lone buyer present on the market at the beginning of a period with valuation  $v_1$ , and that a second buyer may arrive with probability  $q$ . Once the price clock starts rising, it is immediately revealed whether there are one or two bidders present. Thus, there is no asymmetric information regarding the number of active bidders.

Note that if the second bidder does not arrive, the lone bidder receives the object for free. In the case of two bidders present, however, each bidder  $i = 1, 2$  will stay in the auction until the price rises to  $\beta_{2,2}(v_i) = v_i - \delta V_{1,1}(v_i)$ . Thus,

$$V_{1,1}(v_1) = (1 - q)v_1 + q \left[ \int_0^{v_1} (v_1 - \beta_{2,2}(v')) dF(v') + \int_{v_i}^{\infty} \delta V_{1,1}(v_1) dF(v') \right].$$

The first term in this expression is bidder 1's payoff if she is alone on the market. The second term is her expected payoff if a second bidder arrives, and is the sum of her expected winnings if the second bidder has a lower value than her and her expected continuation payoff if she loses the auction. Differentiation of this expression with respect to  $v_1$  and then substituting for  $\beta_{2,2}(v_1)$  yields

$$\begin{aligned} V'_{1,1}(v_1) &= (1 - q) + q(f(v_1)v_1 + F(v_1) - f(v_1)\beta_{2,2}(v_1)) \\ &\quad - \delta q((1 - F(v_1))V'_{1,1}(v_1) - f(v_1)V_{1,1}(v_1)) \\ &= \frac{1 - q(1 - F(v_1))}{1 - \delta q(1 - F(v_1))}. \end{aligned}$$

Note that we may rewrite this expression as

$$\sum_{t=0}^{\infty} (\delta q(1 - F(v_1)))^t [1 - q(1 - F(v_1))],$$

which is the summation of the expected per-period gain from a marginal increase in  $v_1$ , discounted by the probability of that gain being realized in any given period.

Furthermore, note that  $V_{1,1}(0) = 0$ , implying that

$$(4.4) \quad V_{1,1}(v_i) = V_{1,1}(0) + \int_0^{v_i} V_{1,1}(v') dv' = \int_0^{v_i} \frac{1 - q(1 - F(v'))}{1 - \delta q(1 - F(v'))} dv'.$$

Note that the integrand above is strictly positive for all  $v' \in \mathbb{R}_+$ . Hence,  $V_{1,1}$  is strictly increasing, as is  $\beta_{2,2}$ .

We will now proceed to characterize  $V_{j,k}$  inductively for all  $k \in \mathbb{N}$  and all  $j \in \{1, \dots, k\}$  via a series of propositions.

**Proposition 1** (Existence and uniqueness of  $V_{j,k}$ ).

Fix any  $k > 1$ , and suppose that the expected payoff to a buyer when a period starts with  $k - 1$  bidders present depends only on the rank of that bidder and the values of those with values lower than her; that is, given (known) values  $\hat{v} \in \mathbb{R}_+^{k-1}$ , the  $j$ -th highest of the  $k - 1$  bidders receives expected payoff  $V_{j,k-1}(\hat{v}_j, \dots, \hat{v}_{k-1})$ . Then the expected payoff of the  $j$ -th highest of  $k$  bidders, for all  $j = 1, \dots, k$ , is given by  $V_{j,k}(\hat{v}_j, \dots, \hat{v}_k)$ . Furthermore, given  $\{V_{j,k-1}\}_{j=1}^k$ , the functions  $\{V_{j,k}\}_{j=1}^k$  are uniquely determined.

*Proof.* The proof may be found in the appendix. ■

Thus, the strategies in Equation 4.3 lead to well-defined and unique value functions for the buyers. In addition, following these strategies implies that these expected payoffs are not dependent upon history—they do not depend upon the values or prices paid in previous periods—but rather depend only upon the values of those ranked below a bidder.

We may also use the indifference inherent in the definition of our conjectured equilibrium strategy in order to illustrate the link between the various payoff functions. In particular, we have the following

**Proposition 2** (Relationship between  $V_{l,k}$  and  $V_{1,k}$ ).

Fix any  $k \in \mathbb{N}$ . Then for all  $l = 1, \dots, k$ , the expected payoff to the  $l$ -th ranked of  $k$  buyer is equal to that of the highest-ranked buyer when she is tied with  $l - 1$  of her opponents; that is,

$$V_{l,k}(\hat{v}_l, \dots, \hat{v}_k) = V_{1,k}(\hat{v}_l, \dots, \hat{v}_l, \hat{v}_{l+1}, \dots, \hat{v}_k).$$

*Proof.* The proof may be found in the appendix. ■

As mentioned above, this result makes heavy use of the indifference conditions built into the bidding strategies described in Equation 4.3, and in particular the indifference of the buyer with the second-highest value. This bidder drops out at a price at which she is indifferent between winning immediately or waiting one period. Unsurprisingly, when the top two buyers have the same value, they must receive the same payoff, regardless of the tie-breaking rule used to determine which one of the two should receive the object when they drop out simultaneously. The intuition behind the relationship between lower-ranked buyers' payoff functions is analogous. Moreover, Proposition 2 implies that knowledge of the functions  $\{V_{1,k}\}_{k=1}^{\infty}$  is sufficient to determine the remaining value functions. Thus, define the functions  $\mu : \mathbb{R}_+ \rightarrow [1 - q, 1]$  and  $\eta : \mathbb{R}_+ \rightarrow [1 - \delta q, 1]$  by

$$(4.5) \quad \begin{aligned} \mu(v) &= 1 - q(1 - F(v)), \\ \eta(v) &= 1 - \delta q(1 - F(v)). \end{aligned}$$

We have the following

**Theorem 1** (Characterization of  $V_{l,k}$ ).

For all  $k \in \mathbb{N}$  and all  $l = 1, \dots, k$ ,

$$(4.6) \quad V_{l,k}(\hat{v}) = \delta^{l-1} \int_0^{\hat{v}_l} \frac{\mu^l(v')}{\eta^l(v')} dv' - (1 - \delta) \sum_{j=l+1}^k \delta^{j-2} \int_0^{\hat{v}_j} \frac{\mu^{j-1}(v')}{\eta^j(v')} dv'.$$

*Proof.* Note that we may write  $V_{1,k}(\hat{v})$  as

$$(4.7) \quad \begin{aligned} V_{1,k}(\hat{v}) &= (1 - q) [\hat{v}_1 - \beta_{2,k}(\hat{v}_2, \dots, \hat{v}_k)] \\ &+ q \left[ \sum_{j=0}^{k-1} \int_{\hat{v}_{k-j+1}}^{\hat{v}_{k-j}} \left( \hat{v}_1 - \beta_{2,k+1}(\hat{v}_{-1}(v')) \right) dF(v') + \int_{\hat{v}_1}^{\infty} \delta V_{1,k}(\hat{v}) dF(v') \right]. \end{aligned}$$

We will denote by  $V_{1,k}^{(j)}$  the partial derivative of  $V_{1,k}$  with respect to its  $j$ -th argument. Differentiation with respect to  $\hat{v}_1$  then implies that

$$V_{1,k}^{(1)}(\hat{v}) = 1 - q(1 - F(\hat{v}_1)) + \delta q(1 - F(\hat{v}_1))V_{1,k}^{(1)}(\hat{v}) = \frac{\mu(\hat{v}_1)}{\eta(\hat{v}_1)}.$$

Notice that this result is independent of  $k$ . Furthermore, note that this implies that  $V_{1,k}^j(\hat{v})$  does not depend on  $\hat{v}_1$  for any  $j \neq 1$ ; equivalently,

$$V_{1,k}^{(1,j)}(\hat{v}) = 0 \text{ for all } \hat{v} \in \mathbb{R}_+^k \text{ and } j \neq 1.$$

Differentiating Equation 4.7 with respect to  $\hat{v}_2$  now leads to

$$\begin{aligned} V_{1,k}^{(2)}(\hat{v}) &= -(1 - q(1 - F(\hat{v}_2))) + \delta(1 - q)V_{1,k-1}^{(1)}(\hat{v}_2, \dots, \hat{v}_k) + \delta q(1 - F(\hat{v}_1))V_{1,k}^{(2)}(\hat{v}) \\ &\quad + \delta q \sum_{j=0}^{k-2} \int_{\hat{v}_{k-j+1}}^{\hat{v}_{k-j}} V_{1,k}^{(1)}(\hat{v}_{-1}(v')) dF(v') + \delta q \int_{\hat{v}_2}^{\hat{v}_1} V_{1,k}^{(2)}(v', \hat{v}_2, \dots, \hat{v}_k) dF(v'). \end{aligned}$$

Note first that

$$\int_{\hat{v}_2}^{\hat{v}_1} V_{1,k}^{(2)}(v', \hat{v}_2, \dots, \hat{v}_k) dF(v') = (F(\hat{v}_1) - F(\hat{v}_2)) V_{1,k}^{(2)}(\hat{v})$$

since  $V_{1,k}^{(1,2)} = 0$ . Moreover,

$$\sum_{j=0}^{k-2} \int_{\hat{v}_{k-j+1}}^{\hat{v}_{k-j}} V_{1,k}^{(1)}(\hat{v}_{-1}(v')) dF(v') = \int_0^{\hat{v}_2} \frac{\mu(\hat{v}_2)}{\eta(\hat{v}_2)} dF(v') = F(\hat{v}_2) \frac{\mu(\hat{v}_2)}{\eta(\hat{v}_2)}.$$

Thus, we have

$$V_{1,k}^{(2)}(\hat{v}) = \frac{-\mu(\hat{v}_2) + \delta\mu(\hat{v}_2)\frac{\mu(\hat{v}_2)}{\eta(\hat{v}_2)}}{\eta(\hat{v}_2)} = -\frac{\mu(\hat{v}_2)(\eta(\hat{v}_2) - \delta\mu(\hat{v}_2))}{\eta^2(\hat{v}_2)} = -(1 - \delta)\frac{\mu(\hat{v}_2)}{\eta^2(\hat{v}_2)}.$$

Note that, as with  $V_{1,k}^{(1)}$ ,  $V_{1,k}^{(2)}$  depends only on the second argument of  $V_{1,k}$ . Thus,

$$V_{1,k}^{(2,j)}(\hat{v}) = 0 \text{ for all } \hat{v} \in \mathbb{R}_+^{k-1} \text{ and } j \neq 2.$$

Proceeding inductively, fix any  $l \in \{3, \dots, k\}$  for arbitrary  $k \in \mathbb{N}$ , and suppose that

$$V_{1,k}^{(j)}(\hat{v}) = -(1 - \delta)\delta^{j-2}\frac{\mu^{j-1}(\hat{v}_j)}{\eta^j(\hat{v}_j)}$$

for all  $j = 2, \dots, l-1$ . Differentiating Equation 4.7 with respect to  $\hat{v}_l$  yields

$$\begin{aligned} V_{1,k}^{(l)}(\hat{v}) &= \delta(1 - q)V_{1,k-1}^{(l-1)}(\hat{v}_2, \dots, \hat{v}_k) + \delta q \sum_{j=0}^{k-l} \int_{\hat{v}_{k-j+1}}^{\hat{v}_{k-j}} V_{1,k}^{(l-1)}(\hat{v}_{-1}(v')) dF(v') \\ &\quad + \delta q \sum_{j=k-l+1}^{k-1} \int_{\hat{v}_{k-j+1}}^{\hat{v}_{k-j}} V_{1,k}^{(l)}(\hat{v}_{-1}(v')) dF(v') + \delta q(1 - F(\hat{v}_1))V_{1,k}^{(l)}(\hat{v}). \end{aligned}$$

Since  $V_{1,k}^{(l-1)}$  does not depend on any of its arguments but the  $(l-1)$ -th,

$$\sum_{j=0}^{k-l} \int_{\hat{v}_{k-j+1}}^{\hat{v}_{k-j}} V_{1,k}^{(l-1)}(\hat{v}_{-1}(v')) dF(v') = F(\hat{v}_l)V_{1,k-1}^{(l-1)}(\hat{v}_{-1}).$$

In addition,  $V_{1,k}^{(l,j)} = 0$  for all  $j < l$  implies that

$$\sum_{j=k-l+1}^{k-1} \int_{\hat{v}_{k-j+1}}^{\hat{v}_{k-j}} V_{1,k}^{(l)}(\hat{v}_{-1}(v')) dF(v') = (F(\hat{v}_1) - F(\hat{v}_l)) V_{1,k}^{(l)}(\hat{v}).$$

Thus, we have

$$\begin{aligned} V_{1,k}^{(l)}(\hat{v}) &= \delta(1-q)V_{1,k-1}^{(l-1)}(\hat{v}_{-1}) + \delta q F(\hat{v}_l) V_{1,k-1}^{(l-1)}(\hat{v}_{-1}) + \delta q(1-F(\hat{v}_l))V_{1,k}^{(l)}(\hat{v}) \\ &= \frac{\delta\mu(\hat{v}_k) \left[ -(1-\delta)\delta^{l-3} \frac{\mu^{l-2}(\hat{v}_l)}{\eta^{l-1}(\hat{v}_l)} \right]}{\eta^l(\hat{v}_l)} = -(1-\delta)\delta^{l-2} \frac{\mu^{l-1}(\hat{v}_l)}{\eta^l(\hat{v}_l)}. \end{aligned}$$

By induction, the above expression holds for all  $l = 2, \dots, k$ , where  $k \in \mathbb{N}$  is arbitrary.

We may then apply the boundary condition  $V_{1,k}(0) = 0$  in order to show that

$$\begin{aligned} (4.8) \quad V_{1,k}(\hat{v}) &= V_{1,k}(0) + \sum_{j=1}^k \int_0^{\hat{v}_j} V_{1,k}^{(j)}(v') dv' \\ &= \int_0^{\hat{v}_1} \frac{\mu(v')}{\eta(v')} dv' - (1-\delta) \sum_{j=2}^k \delta^{j-2} \int_0^{\hat{v}_j} \frac{\mu^{j-1}(v')}{\eta^j(v')} dv'. \end{aligned}$$

Applying Proposition 2 and some arithmetic manipulation to the above expression then yields the desired result that

$$(4.9) \quad V_{l,k}(\hat{v}) = \delta^{l-1} \int_0^{\hat{v}_l} \frac{\mu^l(v')}{\eta^l(v')} dv' - (1-\delta) \sum_{j=l+1}^k \delta^{j-2} \int_0^{\hat{v}_j} \frac{\mu^{j-1}(v')}{\eta^j(v')} dv'. \quad \blacksquare$$

To better understand this result, let us consider two ‘‘corner’’ cases. In particular, notice that if  $q = 0$  (that is, if no new buyers ever arrive on the market), then

$$V_{l,k}(\hat{v}) = \sum_{j=l}^k \delta^{j-1} (\hat{v}_j - \hat{v}_{j+1}) \text{ for all } l = 1, \dots, k \text{ and any } k \in \mathbb{N},$$

where we define  $\hat{v}_{k+1} = 0$ . Thus, the expected payoff to a buyer in this case is the discounted difference between consecutively ranked valuations. Note that this is also exactly the externality imposed by the  $l$ -th buyer on all those ranked below her when there is no entry, as she postpones each one’s receipt of an object by exactly one period. On the other hand, if  $\delta = 1$  and buyers are ‘‘infinitely patient,’’ then for any  $q$ , we have

$$V_{l,k}(\hat{v}) = \hat{v}_l \text{ for all } l = 1, \dots, k \text{ and any } k \in \mathbb{N}.$$

In this case, buyers care only about their eventual receipt of an object, but not about the timing of that event. Therefore, their bids are all equal to zero, and any random assignment of objects leaves the buyers equally well off.

**4.3. Equilibrium.** With the characterization derived in Theorem 1, we may now reformulate the bidding strategies from Equation 4.3 as

$$(4.10) \quad \beta_{l,k}(v_i, \hat{v}_{l+1}, \dots, \hat{v}_k) = v_i - \delta^{l-1} \int_0^{v_i} \frac{\mu^{l-1}(v')}{\eta^{l-1}(v')} dv' + (1-\delta) \sum_{j=l+1}^k \delta^{j-2} \int_0^{\hat{v}_j} \frac{\mu^{j-2}(v')}{\eta^{j-1}(v')} dv'.$$

This expression allows us to demonstrate the properties of bids in the following

**Proposition 3** (Information revelation and consistency of  $\beta_{l,k}$ ).

*The buyers' bids  $\beta_{l,k}$ , where  $k \in \mathbb{N}$  and  $l \in \{1, \dots, k\}$ , are strictly increasing in each buyers' own valuation. Furthermore, when the buyers use these bidding functions, the exit of a lower-ranked bidder does not induce the immediate exit of any higher-ranked bidders.*

*Proof.* See the appendix. ■

Note that this proposition verifies our previous assumption that buyers' values are revealed after each round—since the bidding functions are strictly increasing in each buyers' own private valuation, the price at which they drop out of the auction is an invertible function, thereby allowing the inference of their value by their competitors. Furthermore, since the bidding functions are “consistent,” a higher-ranked bidder remains in the auction instead of immediately exiting after a lower-ranked bidder drops out, thereby allowing the other buyers to (eventually) deduce their value.

Finally, it remains to be shown that the bidding strategies described in Equation 4.10 are in fact an equilibrium of this model. We have the following

**Theorem 2** (Equilibrium verification).

*Suppose that in each period, buyers bid according to the cutoff strategies*

$$\beta_{l,k}(v_i, \hat{v}_{l+1}, \dots, \hat{v}_k) = v_i - \delta^{l-1} \int_0^{v_i} \frac{\mu^{l-1}(v')}{\eta^{l-1}(v')} dv' + (1 - \delta) \sum_{j=l+1}^k \delta^{j-2} \int_0^{\hat{v}_j} \frac{\mu^{j-2}(v')}{\eta^{j-1}(v')} dv',$$

*where  $k \in \mathbb{N}$  is the number of bidders present on the market, and  $l \in \{2, \dots, k\}$  is the number of active bidders remaining in the current period. This strategy profile forms a Markov perfect equilibrium of this game.*

*Proof.* Consider any period with  $k \in \mathbb{N}$  buyers on the market, and fix an arbitrary bidder  $i$ . Suppose that all bidders other than  $i$  are using the conjectured strategy. We must show that bidder  $i$  has no incentive to make a one-shot deviation from the collection of bidding functions  $\{\beta_{l,k}\}_{l=2}^k$ .

Note first that if  $v_i \neq \hat{v}_1$ , dropping out of the auction early has no bearing on expected future payoffs due to the memorylessness of the bidding strategies—in each period, the process of information revelation is repeated, and hence a one-shot deviation to an early exit will not affect the bidding behavior in future periods. On the other hand, suppose that  $v_i = \hat{v}_1$ ; that is, bidder  $i$  has the highest realized valuation among those bidders present on the market. Following the conjectured equilibrium leads to a payoff of  $\hat{v}_1 - \beta_{2,k}(\hat{v}_2, \dots, \hat{v}_k)$ , while deviating and exiting at a lower price leads to the second-ranked bidder winning



and a payoff to  $i$  of  $\delta V_{1,k-1}(\hat{v}_1, \hat{v}_3, \dots, \hat{v}_k)$ . Letting  $\hat{w} = (\hat{v}_3, \dots, \hat{v}_k)$ , we then have

$$\begin{aligned} \hat{v}_1 - \beta_{2,k}(\hat{v}_2, \hat{w}) - \delta V_{1,k-1}(\hat{v}_1, \hat{w}) &= \hat{v}_1 - \hat{v}_2 + \delta (V_{1,k-1}(\hat{v}_2, \hat{w}) - V_{1,k-1}(\hat{v}_1, \hat{w})) \\ &= \hat{v}_1 - \hat{v}_2 + \delta \left( \int_0^{\hat{v}_2} \frac{\mu(v')}{\eta(v')} dv' - \int_0^{\hat{v}_1} \frac{\mu(v')}{\eta(v')} dv' \right) \\ &= \int_{\hat{v}_2}^{\hat{v}_1} \frac{\eta(v') - \delta \mu(v')}{\eta(v')} dv' = \int_{\hat{v}_2}^{\hat{v}_1} \frac{(1-\delta)}{\eta(v')} dv'. \end{aligned}$$

Since  $\hat{v}_1 > \hat{v}_2$  and  $\eta(v') > 0$  for all  $v' \in \mathbb{R}_+$ , the above expression is strictly positive. Thus, deviating and exiting the auction early leads to a strict decrease in utility if the realized values are such that bidder  $i$  has the highest value, and does not affect payoffs otherwise.

On the other hand, bidder  $i$  also has available to her the option of remaining in the auction beyond the cutoffs specified in the conjectured equilibrium. If the realized values are such that  $v_i = \hat{v}_1$ , delaying exit will have no effect, as the other bidders will drop out of the auction earlier than  $i$ . If, on the other hand,  $v_i = \hat{v}_l$  for some  $l > 1$ , then delaying exit may have an effect on  $i$ 's payoffs. To be precise, if  $i$  exits before the eventual winner, her payoff will remain unchanged (again, because behavior in future periods does not depend upon the information revealed in the current period). Thus, in order to influence her payoff,  $i$  must remain present in the auction until all other bidders have dropped out, thereby winning the auction. Winning the auction yields a payoff of  $\hat{v}_l - \beta_{2,k}(\hat{v}_1, \dots, \hat{v}_{l-1}, \hat{v}_{l+1}, \dots, \hat{v}_k)$ , while following the strategy in Equation 4.10 leads to an expected payoff of  $\delta V_{l-1,k-1}$ . Letting  $\hat{w} = (\hat{v}_{l+1}, \dots, \hat{v}_k)$  and  $\hat{v}_{-l} = (\hat{v}_1, \dots, \hat{v}_{l-1}, \hat{w})$ , we have

$$\begin{aligned} \hat{v}_l - \beta_{2,k}(\hat{v}_{-l}) - \delta V_{l-1,k-1}(\hat{v}_l, \hat{w}) &= \hat{v}_l - \hat{v}_1 + \delta (V_{1,k}(\hat{v}_1, \dots, \hat{v}_{l-1}, \hat{w}) - V_{l-1,k-1}(\hat{v}_l, \hat{w})) \\ &= \hat{v}_l - \hat{v}_1 + \delta (V_{1,k}(\hat{v}_1, \dots, \hat{v}_{l-1}, \hat{w}) - V_{1,k-1}(\hat{v}_l, \dots, \hat{v}_l, \hat{w})) \\ &= \hat{v}_l - \hat{v}_1 + \delta \left( \int_{\hat{v}_l}^{\hat{v}_1} \frac{\mu(v')}{\eta(v')} dv' \right. \\ &\quad \left. - (1-\delta) \sum_{j=2}^{l-1} \delta^{j-2} \int_{\hat{v}_l}^{\hat{v}_j} \frac{\mu^{j-1}(v')}{\eta^j(v')} dv' \right) \\ &= \int_{\hat{v}_l}^{\hat{v}_1} \frac{\delta \mu(v') - \eta(v')}{\eta(v')} dv' - (1-\delta) \sum_{j=2}^{l-1} \delta^{j-1} \int_{\hat{v}_l}^{\hat{v}_j} \frac{\mu^{j-1}(v')}{\eta^j(v')} dv' \\ &= -(1-\delta) \sum_{j=1}^{l-1} \delta^{j-1} \int_{\hat{v}_l}^{\hat{v}_j} \frac{\mu^{j-1}(v')}{\eta^j(v')} dv' < 0, \end{aligned}$$

where the second line follows from Proposition 2, the third from Theorem 1, and the final inequality from the fact that  $\mu(v), \eta(v) > 0$  for all  $v \in \mathbb{R}_+$ . Hence, deviating and exiting the auction later than prescribed has no effect if  $i$  has the highest value, but may lead to a strict decrease in utility if the realized values are such that  $v_i < \hat{v}_1$ .

Thus, we may conclude that bidder  $i$  has no incentive to make a one-shot deviation from the collection of bidding functions  $\{\beta_{i,k}\}_{k=2}^k$  regardless of the realized values. Furthermore, the choice of  $k$  throughout was arbitrary, implying that bidding according to Equation 4.10 forms a Markov perfect equilibrium. ■

## 5. DYNAMIC VICKREY-CLARKE-GROVES MECHANISM

Bergemann and Välimäki (2007) develop the dynamic contribution mechanism (also referred to as the dynamic Vickrey-Clarke-Groves mechanism), a direct mechanism that implements the socially efficient allocation in a dynamic private value environment in which agents receive private information over time. In particular, the mechanism that they propose, agents receive in each period their marginal contribution to the social welfare in a dynamic generalization of the standard Vickrey-Clarke-Groves Mechanism. In this mechanism, the truth-telling strategy is incentive compatible and forms an ex-post equilibrium. Moreover, the authors show that the sequential ascending auction yields an identical implementation in the case of a scheduling problem with a fixed set of independent tasks. Cavallo et al. (2007) pushes their model one step further, demonstrating that dynamic VCG truthfully implements the socially efficient allocation in more general dynamic settings. In this section, we show that the equilibrium in the sequential ascending auction discussed above is equivalent to the truth-telling equilibrium of the dynamic VCG mechanism. In addition, we use the result of Cavallo et al. (2007) to characterize equilibrium in the sequential ascending auction when objects are no longer available with certainty in every period, and hence there may be (effectively) multiple new entrants participating in a given auction.

**5.1. Constant Availability of Objects.** We first consider the model examined above in which exactly one object is available for sale in every period. It is immediately obvious that the socially efficient policy is to allocate each object to the buyer with the highest valuation present on the market.<sup>7</sup>

Let us define  $W_0$  to be the expected value to the social planner when there are no buyers present on the market. Then, letting  $\bar{v}$  denote the expected value of the distribution  $F$ , we have

$$W_0 = q \int_0^\infty (v' + \delta W_0) dF(v') + (1 - q)\delta W_0 = \frac{q\bar{v}}{1 - \delta}.$$

Denoting by  $W_n(\hat{v})$  the expected value to the social planner when there are  $n$  buyers with values  $\hat{v}_1 > \dots > \hat{v}_n$ , we may recursively solve for the planner's value function. In particular, we have the following

<sup>7</sup>Since a new object arrives in every period and future entrants' values are independent of the current state, there is no benefit to not allocating the object in any particular period. Furthermore, allocating an object to a lower-valued buyer is inefficient due to the fact that the common discount factor  $\delta$  is smaller than one.

**Proposition 4** (Planner's payoff function).

The social planner's payoff at the beginning of a period in which there are  $n$  buyers present on the market with values  $\hat{v}_1 > \dots > \hat{v}_n$  is given by

$$(5.1) \quad W_n(\hat{v}) = W_0 + \sum_{j=1}^n \delta^{j-1} \int_0^{\hat{v}_j} \left( \frac{\mu(v')}{\eta(v')} \right)^j dv'.$$

*Proof.* The proof is similar to that of Theorem 1, and may therefore be found in the appendix. ■

Effectively, this theorem yields an analogue to the social planner's payoff in the case of a fixed number of buyers without any entry. With  $n$  buyers whose values are given by  $v_1 > \dots > v_n$ , the efficient allocation yields a value to the planner given by

$$\sum_{j=1}^n \delta^{j-1} v_j.$$

In our setting, however, potential entrants can and do rearrange the efficient allocation, postponing the time at which buyers with lower valuations receive an object. Thus, their contribution to social welfare must take this effect into account. So, consider the buyer with the highest valuation  $\hat{v}_1$ . If we increase his valuation by some infinitesimal amount, the planner gains an equal amount with probability  $1 - q(1 - F(\hat{v}_1))$ , the probability that no new entrant arrives with a higher value. On the other hand, with the complementary probability  $q(1 - F(\hat{v}_1))$ , assignment of the object to our buyer (and the realization of the planner's gain) is postponed. Thus, the benefit from the increase in  $\hat{v}_1$  is

$$\begin{aligned} & (1 - q(1 - F(\hat{v}_1))) + q(1 - F(\hat{v}_1))\delta [(1 - q(1 - F(\hat{v}_1))) + q(1 - F(\hat{v}_1))\delta \dots] \\ &= \sum_{m=0}^{\infty} (\delta q(1 - F(\hat{v}_1)))^m (1 - q(1 - F(\hat{v}_1))) = \frac{1 - q(1 - F(\hat{v}_1))}{1 - \delta q(1 - F(\hat{v}_1))}. \end{aligned}$$

Integrating this ratio therefore captures the total contribution (relative to assigning the object to a buyer with value 0) of the high-value buyer. Analogous reasoning follows for the remaining buyers.

We may then use this result to provide an interpretation for the buyer payoff functions characterized in Theorem 1. In particular, each buyer's expected payoff at any point in time is simply her marginal contribution to the social welfare. This is stated formally as

**Theorem 3** (Relationship between  $V$  and  $W$ ).

For any  $n \in \mathbb{N}$  and any  $k \in \{1, \dots, n\}$ , the expected payoff of the  $k$ -th ranked buyer is equal to her marginal contribution to the social welfare; that is,

$$(5.2) \quad V_{k,n}(\hat{v}) = W_n(\hat{v}) - W_{n-1}(\hat{v}_{-k}).$$

*Proof.* The proof proceeds via straightforward arithmetic:

$$\begin{aligned}
 W_n(\hat{v}) - W_{n-1}(\hat{v}_{-k}) &= \left[ W_0 + \sum_{j=1}^n \delta^{j-1} \int_0^{\hat{v}_j} \frac{\mu^j(v')}{\eta^j(v')} dv' \right] \\
 &\quad - \left[ W_0 + \sum_{j=1}^{k-1} \delta^{j-1} \int_0^{\hat{v}_j} \frac{\mu^j(v')}{\eta^j(v')} dv' + \sum_{j=k+1}^n \delta^{j-2} \int_0^{\hat{v}_j} \frac{\mu^{j-1}(v')}{\eta^{j-1}(v')} dv' \right] \\
 &= \delta^{k-1} \int_0^{\hat{v}_k} \frac{\mu^k(v')}{\eta^k(v')} dv' + \sum_{j=k+1}^n \delta^{j-2} \int_0^{\hat{v}_j} \left( \delta \frac{\mu^j(v')}{\eta^j(v')} - \frac{\mu^{j-1}(v')}{\eta^{j-1}(v')} \right) dv' \\
 &= \delta^{k-1} \int_0^{\hat{v}_k} \frac{\mu^k(v')}{\eta^k(v')} dv' - (1 - \delta) \sum_{j=k+1}^n \delta^{j-2} \int_0^{\hat{v}_j} \frac{\mu^{j-1}(v')}{\eta^j(v')} dv' \\
 &= V_{k,n}(\hat{v}). \quad \blacksquare
 \end{aligned}$$

Note that we may rewrite this marginal contribution to social welfare in terms of the revised scheduling of future assignments. In particular, the removal of the  $k$ -th highest buyer from the market does not affect the order in which those with higher values receive objects. However, all those who would receive objects *after* the  $k$ -th highest buyer now receive objects one period prior to when they otherwise would have. Thus, we may write

$$\begin{aligned}
 V_{k,n}(\hat{v}) &= \sum_{j=k}^n \delta^{j-1} \int_0^{\hat{v}_j} \frac{\mu^j(v')}{\eta^j(v')} dv' - \sum_{j=k+1}^n \delta^{j-2} \int_0^{\hat{v}_j} \frac{\mu^{j-1}(v')}{\eta^{j-1}(v')} dv' \\
 (5.3) \quad &= \sum_{j=k}^n \delta^{j-1} \left( \int_0^{\hat{v}_j} \frac{\mu^j(v')}{\eta^j(v')} dv' - \int_0^{\hat{v}_{j+1}} \frac{\mu^j(v')}{\eta^j(v')} dv' \right) = \sum_{j=k}^n \delta^{j-1} \int_{\hat{v}_{j+1}}^{\hat{v}_j} \frac{\mu^j(v')}{\eta^j(v')} dv',
 \end{aligned}$$

where  $\hat{v}_{n+1}$  is defined to be equal to zero. Thus, the marginal contribution of a buyer, and hence their expected payoff in the equilibrium of the sequential ascending auction game, is exactly the difference in the scheduling of object assignments to those bidders who have lower values.

Moreover, this demonstrates the equivalence between the dynamic marginal contribution mechanism and the sequential ascending auction in this setting. Not only are continuation payoffs identical in the two settings, but the timing of payments and object allocations are also the same.

**5.2. Random Arrival of Objects.** We now consider a generalization of the setting of the previous sections. Instead of a single object arriving with certainty in every period, we now allow the arrival of objects to be probabilistic. In particular, at most one object arrives on the market in each period with probability  $p \in (0, 1)$ . Thus, the number of buyers present on the market may increase between auctions, as it is possible for multiple new

buyers to arrive before another object becomes available for sale.<sup>8</sup> Notice that the socially efficient policy remains unchanged from the previous section—when available, objects should be allocated to the highest-valued buyer currently present on the market.

Once again, we denote by  $W_n(\hat{v})$  the expected value to the social planner when there are  $n$  buyers present at the beginning of a period with (ordered) values  $\hat{v}_1 > \dots > \hat{v}_n$ .  $W_n$  must satisfy the relationship given by

$$(5.4) \quad \begin{aligned} W_n(\hat{v}) = & pq \left[ \sum_{j=0}^{n-1} \int_{\hat{v}_{n-j+1}}^{\hat{v}_{n-j}} (\hat{v}_1 + \delta W_n(\hat{v}_{-1}(v'))) dF(v') + \int_{\hat{v}_1}^{\infty} (v' + \delta W_n(\hat{v})) dF(v') \right] \\ & + p(1-q) [\hat{v}_1 + \delta W_{n-1}(\hat{v}_{-1})] + (1-p)q \left[ \sum_{j=0}^n \delta \int_{\hat{v}_{n-j+1}}^{\hat{v}_{n-j}} W_{n+1}(\hat{v}(v')) dF(v') \right] \\ & + (1-p)(1-q) [\delta W_n(\hat{v})]. \end{aligned}$$

Define  $\lambda : \mathbb{R}_+ \rightarrow [0, 1]$  by

$$\lambda(v) = \frac{1 - \delta [pq(1 - F(v)) + (1-p)(1 - q(1 - F(v)))]}{2\delta(1-p)q(1 - F(v))} - \frac{\sqrt{1 - 2\delta [pq(1 - F(v)) + (1-p)(1 - q(1 - F(v)))] + \delta^2 (1-p-q(1 - F(v)))^2}}{2\delta(1-p)q(1 - F(v))}.$$

Then we have the following

**Proposition 5** (Planner's payoffs with random object arrivals).

*The social planner's expected payoff at the beginning of a period in which there are  $n$  buyers present on the market with values  $\hat{v}_1 > \dots > \hat{v}_n$  and objects arrive randomly is given by*

$$(5.5) \quad W_n(\hat{v}) = W_0 + \delta^{-1} \sum_{j=1}^n \int_0^{\hat{v}_j} \lambda^j(v') dv',$$

where  $W_0$  is a constant equal to the planner's payoff when no buyers are present on the market.

*Proof.* The proof may be found in the appendix. ■

Thus, as in the case in which objects arrive deterministically in every period, the planner's expected payoff is an additively separable sum of contributions from each buyer present on the market, where the magnitude of each contribution is inversely proportional to the buyer's rank amongst her competitors (and hence the amount of time before the contribution is realized). Although  $\lambda$  is not as straightforward to interpret as the ratio  $\frac{\mu}{\eta}$ , it reflects the anticipated marginal benefit from increasing the value of the highest-ranked buyer, taking into account the fact that multiple new entrants may arrive before

<sup>8</sup>Note that this assumption not only leads to a more general competitive environment amongst buyers, but also leads to a greater expected delay before the assignment of an object to any given market participant.

the next object arrives, thereby delaying the realization of the increase to the social welfare from the increase in  $\hat{v}_1$ .

Notice that our setting satisfies the conditions discussed by Cavallo et al. (2007) for the incentive compatible and truthful implementation of the socially efficient policy by the dynamic marginal contribution mechanism.<sup>9</sup> Thus, using the social planner's payoffs from Proposition 5, we may construct contingent transfers such that truthful revelation of private values is an ex post equilibrium in this dynamic game. In particular, at every point in time, each buyer receives a transfer equal to their "flow marginal contribution" in that period. This is defined as the social welfare excluding the player in question when making the allocation that is efficient given her existence, less the total welfare of all other players when acting as though the player in question is not present on the market.

To make this clearer, consider a buyer  $i$  with value  $v_i$  who is present on the market, and suppose that there are  $n$  other buyers present on the market with (ordered) values  $\hat{v}_1 > \dots > \hat{v}_n$ . Let us suppose first that  $v_i > \hat{v}_1$ . Thus, if an object is available, the efficient policy involves assigning the object to buyer  $i$ , yielding her a flow payoff of  $v_i$ . Furthermore,  $i$  should receive a transfer equal to her marginal contribution to the social welfare, less the current-period utility she receives; this transfer may be written as

$$(5.6) \quad \underbrace{(0 + \delta W_n(\hat{v}))}_{\text{Others' welfare with } i} - \underbrace{(\hat{v}_1 + \delta W_{n-1}(\hat{v}_{-1}))}_{\text{Social welfare excluding } i} = -\hat{v}_1 + \delta (W_n(\hat{v}) - W_{n-1}(\hat{v}_{-1})).$$

On the other hand, if an object is not available,  $i$  cannot be allocated an object in the current period. Therefore,  $i$  receives a payoff of

$$\underbrace{(0 + \delta W_n(\hat{v}))}_{\text{Others' welfare with } i} - \underbrace{(0 + \delta W_n(\hat{v}))}_{\text{Social welfare excluding } i} = 0.$$

Suppose on the other hand that  $i$  does *not* have the highest value. If an object is available, she will not receive it; however, she will receive a transfer of

$$\underbrace{(\hat{v}_1 + \delta W_{n-1}(\hat{v}_{-1}))}_{\text{Others' welfare with } i} - \underbrace{(\hat{v}_1 + \delta W_{n-1}(\hat{v}_{-1}))}_{\text{Social welfare excluding } i} = 0.$$

If no object is available,  $i$  still does not receive an object, and is given a transfer of

$$\underbrace{(0 + \delta W_n(\hat{v}))}_{\text{Others' welfare with } i} - \underbrace{(0 + \delta W_n(\hat{v}))}_{\text{Social welfare excluding } i} = 0.$$

In future periods,  $i$  (again) receives no transfers until she is eventually the highest-ranked buyer, as her flow marginal contribution remains zero until that point.

<sup>9</sup>In particular, their Theorem 6 applies to the present model.

Thus, a buyer's expected continuation payoff at every point in time in this dynamic Vickrey-Clarke-Groves mechanism is given by her marginal contribution to the social welfare. Therefore, by choosing bidding functions that yield continuation payoffs equal to these marginal contributions, we will have constructed an equilibrium of the sequential ascending auction game. So, letting  $V_{k,n}$  denote the expected payoff to a buyer with the  $k$ -th highest of the  $n$  values present at the beginning of a period, we define

$$(5.7) \quad V_{k,n}(\hat{v}) := W_n(\hat{v}) - W_{n-1}(\hat{v}_{-k}) = \delta^{-1} \sum_{j=k}^n \int_{\hat{v}_{j+1}}^{\hat{v}_j} \lambda^j(v') dv'$$

In addition, denoting by  $\beta_{k,n}$  the drop-out point of a bidder when there are  $k$  active bidders in an auction with  $n$  buyers, define

$$\beta_{k,n}(v_i, \hat{v}_{k+1}, \dots, \hat{v}_n) := v_i - \delta V_{k-1,n-1}(v_i, \hat{v}_{k+1}, \dots, \hat{v}_n).$$

We have the following

**Theorem 4** (Auction equilibrium with random object arrivals).

*Suppose that in each period in which an object is available, buyers bid according to the cutoff strategies*

$$(5.8) \quad \beta_{k,n}(v_i, \hat{v}_{k+1}, \dots, \hat{v}_n) = v_i - \int_{\hat{v}_{k+1}}^{v_i} \lambda^{k-1}(v') dv' - \sum_{j=k+1}^n \int_{\hat{v}_{j+1}}^{\hat{v}_j} \lambda^{j-1}(v') dv',$$

where  $n \in \mathbb{N}$  is the number of buyers present on the market, and  $k \in \{2, \dots, n\}$  is the number of active bidders remaining in the current period, and  $\hat{v}_{n+1} = 0$ . Then this strategy profile forms an efficient and fully revealing Markov perfect equilibrium of this game.<sup>10</sup>

*Proof.* Note first that  $\beta_{k,n}$  is strictly increasing in  $v_i$  for all  $k$  and  $n$ ; in particular, since  $\lambda(v') \in (0, 1)$  for all  $v' \in \mathbb{R}_+$ ,

$$\frac{\partial}{\partial v_i} \beta_{k,n}(v_i, \hat{v}_{k+1}, \dots, \hat{v}_n) = 1 - \lambda^{k-1}(v_i) > 0.$$

Furthermore, notice that

$$\begin{aligned} \beta_{k,n}(v_i, \hat{v}_{k+1}, \dots, \hat{v}_n) - \beta_{k+1,n}(\hat{v}_{k+1}, \dots, \hat{v}_n) &= v_i - \hat{v}_{k+1} - \int_{\hat{v}_{k+1}}^{v_i} \lambda^{k-1}(v') dv' \\ &= \int_{\hat{v}_{k+1}}^{v_i} dv' - \int_{\hat{v}_{k+1}}^{v_i} \lambda^{k-1}(v') dv' \\ &= \int_{\hat{v}_{k+1}}^{v_i} (1 - \lambda^{k-1}(v')) dv' > 0 \end{aligned}$$

<sup>10</sup>Notice that we have constructed equilibrium strategies in the sequential auction game by using the payoffs of the dynamic marginal contribution mechanism. A simple adaptation of this argument provides an alternate proof of Theorem 2.

whenever  $v_i > \hat{v}_{k+1}$ . Thus, a lower-ranked bidder's exit from an auction does not induce the immediate exit of a higher-ranked bidder, implying that following the bidding strategies in Equation 5.8 is efficient and fully revealing of all private information.

Note that in the sequential ascending auction game, buyers do not make any payments unless they win an auction. Moreover, when they do win, they make a payment equal to the drop-out point of their last remaining opponent in that auction. Following the bidding strategies defined in Equation 5.8 implies that a bidder with value  $v_i > \hat{v}_1 > \dots > \hat{v}_n$ , when engaged in an auction, will win the auction and make a payment of

$$\beta_{2,n+1}(\hat{v}) = \hat{v}_1 - \delta V_{1,n}(\hat{v}) = \hat{v}_1 - \delta (W_n(\hat{v}) - W_{n-1}(\hat{v}_{-1})),$$

where the second equality comes from the definition of  $V_{k,n}$  in Equation 5.7. Notice that this payment is exactly the dynamic Vickrey-Clarke-Groves transfer described in Equation 5.6. Thus, following the bidding strategies of Equation 5.8 yields exactly the payoffs of truthful reporting in the direct mechanism.

Finally, suppose that some player  $i$  with value  $v_i$  has an incentive to deviate from the bidding strategy. Since the cutoff bids are fully revealing, this is equivalent to  $i$  bidding as though her value were  $v'_i \neq v_i$ . Since the bidding strategies yield the same payoffs as the dynamic Vickrey-Clarke-Groves mechanism, this implies that player  $i$  has an incentive to misreport her type to the social planner. However, truth-telling is incentive compatible in every period of that direct mechanism, a contradiction. Thus, no player has an incentive to deviate—all players bidding according to the cutoff values given by Equation 5.8 is a Markov perfect equilibrium of the sequential ascending auction game. ■

Not only does Theorem 4 characterize an efficient, fully revealing, and symmetric equilibrium of the sequential ascending auction game, it also generalizes the equivalence result of the previous subsection; in a setting with random arrivals of both buyers and objects, the sequential ascending auction serves as a straightforward and intuitive indirect mechanism that is equivalent to the dynamic marginal contribution mechanism.

## 6. DISCUSSION

This paper solves for a Markov perfect equilibrium in a model of online auctions. In particular, we show that in sequential ascending auctions, objects are allocated efficiently in a manner that employs the truthful revelation of private information. Moreover, the bidding strategy employed by buyers in this equilibrium has the remarkable property of being robust to the random entry of new buyers whose valuations are private information—in each period, *all* private information is revealed anew, and hence there is no incentive for new entrants to attempt to manipulate the outcome of future



periods by altering the information that they (truthfully) reveal upon their entry. Furthermore, we show that the sequential ascending auction in this setting is equivalent to the dynamic marginal contribution mechanism developed and characterized by Bergemann and Välimäki (2007) and Cavallo et al. (2007).

There are several interesting avenues for future research in this area. For example, it would be desirable to have a fully developed model of seller behavior and competition in “overlapping” auctions, as well as introducing some notion of endogenous arrival and entry deterrence. Another important question regards the usefulness of sequential ascending auctions as an indirect mechanism that implements socially efficient policies when agents are not constrained to have single-unit demand. Bergemann and Välimäki (2007) provide an example that demonstrates the failure of the sequential ascending auction in implementing the efficient policy in one such setting; it would be useful to understand how this example may be generalized. These questions are, however, left for future work.

## APPENDIX

In order to prove Proposition 1, we will make heavy use of the following result regarding closed subsets of the set of continuous real-valued functions on a Euclidean space. So, let  $\mathbf{C}(\mathbb{R}_+^n)$  denote the set of all continuous real-valued functions on  $\mathbb{R}_+^n$  endowed with the sup-metric topology. Furthermore, let  $\mathbf{C}_k(\mathbb{R}_+^n)$  denote the set of all continuous real-valued functions on  $\mathbb{R}_+^n$  that *do not* depend on their first  $k \leq n$  arguments. We have the following

**Lemma 1** ( $\mathbf{C}_k(\mathbb{R}_+^n)$  is closed).

For any  $k \leq n$ ,  $\mathbf{C}_k(\mathbb{R}_+^n)$  is a closed subset of  $\mathbf{C}(\mathbb{R}_+^n)$ .

*Proof.* Fix any convergent sequence  $\{f_m\}_{m=1}^\infty$  in  $\mathbf{C}_k(\mathbb{R}_+^n)$ , and let  $f^* \in \mathbf{C}(\mathbb{R}_+^n)$  denote the limit of this sequence. Suppose that there exist distinct  $x, y \in \mathbb{R}_+^n$  such that  $x_i = y_i$  for  $i = k + 1, k + 2, \dots, n$ , but

$$\epsilon := |f^*(x) - f^*(y)| > 0.$$

Since uniform convergence implies point-wise convergence and  $f_m$  converges to  $f^*$ , there exists  $M_x \in \mathbb{N}$  such that  $|f_m(x) - f^*(x)| < \frac{\epsilon}{2}$  for all  $m > M_x$ . Similarly, there exists  $M_y \in \mathbb{N}$  such that  $|f_m(y) - f^*(y)| < \frac{\epsilon}{2}$  for all  $m > M_y$ . Therefore, for any  $m > \max\{M_x, M_y\}$ ,

$$\begin{aligned} \epsilon &= |f^*(x) - f^*(y)| \leq |f^*(x) - f_m(x)| + |f_m(x) - f_m(y)| + |f_m(y) - f^*(y)| \\ &< \frac{\epsilon}{2} + 0 + \frac{\epsilon}{2} = \epsilon, \end{aligned}$$

a contradiction. Note that the first inequality above follows from the triangle inequality, and the second is due to the fact that  $f_m \in \mathbf{C}_k(\mathbb{R}_+^n)$  implies  $f_m(x) = f_m(y)$ . Thus, we must have  $f^*(x) = f^*(y)$ ; that is,  $f^* \in \mathbf{C}_k(\mathbb{R}_+^n)$ .  $\blacksquare$

*Proof of Proposition 1.* Let  $\hat{v} \in \mathbb{R}_+^k$  denote the ordered vector of values of those bidders present at the beginning of the period, and suppose that they are commonly known. Furthermore, suppose that all buyers use the bidding strategies described in Equation 4.3. If there are no entrants, then the highest-valued buyer (without loss of generality, bidder 1) wins the object, and pays the price

$$\beta_{2,k}(\hat{v}_2, \dots, \hat{v}_k) = \hat{v}_2 - \delta V_{1,k-1}(\hat{v}_2, \dots, \hat{v}_k).$$

On the other hand, if a new entrant enters with value  $v'$ , bidder 1 may no longer win the object. Furthermore, even if she does win, the price she pays will depend upon the realization of  $v'$ . In particular, we may write the expected payoff of bidder 1 as

$$\begin{aligned} (A.1) \quad V_{1,k}(\hat{v}) &= (1 - q) [\hat{v}_1 - \beta_{2,k}(\hat{v}_2, \dots, \hat{v}_k)] \\ &+ q \left[ \sum_{j=0}^{k-1} \int_{\hat{v}_{k-j+1}}^{\hat{v}_{k-j}} \left( \hat{v}_1 - \beta_{2,k+1}(\hat{v}_{-1}(v')) \right) dF(v') + \int_{\hat{v}_1}^{\infty} \delta V_{1,k}(\hat{v}) dF(v') \right], \end{aligned}$$

where  $\hat{v}(v')$  is the ordered vector of values including the new entrant, and we define  $\hat{v}_{k+1} := 0$ . The first term (multiplied by  $1 - q$ ) is bidder 1's payoff when no entrant arrives, while the second term is the (probability-weighted) sum of the payoffs for each possible realized ranking of the entrant.

Substituting the definition of  $\beta_{2,k}$  and  $\beta_{2,k+1}$  from Equation 4.3 and simplifying, we see that  $V_{1,k}$  is defined as the fixed point of the operator  $T_{1,k} : \mathbf{C}(\mathbb{R}_+^k) \rightarrow \mathbf{C}(\mathbb{R}_+^k)$  defined by

$$(A.2) \quad \begin{aligned} [T_{1,k}(W)](\hat{v}) &:= (1 - q) [\hat{v}_1 - \hat{v}_2 + \delta V_{1,k-1}(\hat{v}_2, \dots, \hat{v}_k)] \\ &+ q \left[ \int_0^{\hat{v}_1} \hat{v}_1 dF(v') - \int_0^{\hat{v}_2} \hat{v}_2 dF(v') - \int_{\hat{v}_2}^{\hat{v}_1} v' dF(v') \right. \\ &\quad \left. + \delta \sum_{j=0}^{k-1} \int_{\hat{v}_{k-j+1}}^{\hat{v}_{k-j}} W(\hat{v}_{-1}(v')) dF(v') + \int_{\hat{v}_1}^{\infty} \delta W(\hat{v}) dF(v') \right]. \end{aligned}$$

Fix any  $W, W' \in \mathbf{C}(\mathbb{R}_+^k)$  such that  $W \geq W'$ . Then

$$[T_{1,k}(W) - T_{1,k}(W')](\hat{v}) = \delta q \left[ \begin{aligned} &\sum_{j=0}^{k-1} \int_{\hat{v}_{k-j+1}}^{\hat{v}_{k-j}} [W - W'](\hat{v}_{-1}(v')) dF(v') \\ &+ (1 - F(\hat{v}_1))[W - W'](\hat{v}) \end{aligned} \right] \geq 0.$$

In addition, for any  $W \in \mathbf{C}(\mathbb{R}_+^k)$  and any  $\alpha \in \mathbb{R}_{++}$ ,

$$[T_{1,k}(W + \alpha)](\hat{v}) = [T_{1,k}(W)](\hat{v}) + \delta q \alpha.$$

Thus,  $T_{1,k}$  satisfies the monotonicity and discounting conditions of Blackwell's Contraction Lemma, and hence we may apply the Banach Fixed Point Theorem to show that  $V_{1,k}$  is the unique fixed point of  $T_{1,k}$ .<sup>11</sup>

Now consider  $V_{2,n}$ . Suppose (again without loss of generality) that bidder 1 has the second-highest of the  $k$  values; that is, that  $\hat{v}_2 = v_1$ . If there are no new entrants, then bidder 1 loses the auction, but has the highest value in the next period. On the other hand, if a new entrant arrives, bidder 1 will still lose the auction. However, in the next period, her ranking depends on the realization of the new entrant's value. Thus, we may write her payoff as the fixed point of the operator  $T_{2,k} : \mathbf{C}(\mathbb{R}_+^k) \rightarrow \mathbf{C}(\mathbb{R}_+^k)$  defined by

$$(A.3) \quad \begin{aligned} [T_{2,k}(W)](\hat{v}) &= \delta(1 - q)V_{1,k-1}(\hat{v}_2, \dots, \hat{v}_k) + q \left[ \sum_{j=0}^{k-2} \int_{\hat{v}_{k-j+1}}^{\hat{v}_{k-j}} \delta V_{1,k}(\hat{v}_{-1}(v')) dF(v') \right. \\ &\quad \left. + \int_{\hat{v}_2}^{\hat{v}_1} \delta W(\hat{v}_{-1}(v')) dF(v') + \int_{\hat{v}_1}^{\infty} \delta W(\hat{v}) dF(v') \right]. \end{aligned}$$

Applying exactly the same technique and steps as with  $T_{1,n}$ , we see that  $T_{2,n}$  is a contraction mapping on  $\mathbf{C}(\mathbb{R}_+^k)$ . Notice that  $T_{2,k}$  in fact maps elements of  $\mathbf{C}_1(\mathbb{R}_+^k)$  into  $\mathbf{C}_1(\mathbb{R}_+^k)$

<sup>11</sup>See Section C.6 of Ok (2007) for precise statements of the results we are applying.

itself; thus, applying Lemma 1, the unique fixed point of  $T_{2,n}$  does not depend upon its first argument. We may therefore, with a slight abuse of notation, write  $V_{2,k}(\hat{v}) = V_{2,k}(\hat{v}_2, \dots, \hat{v}_k)$ .

Now consider any arbitrary  $l$  such that  $1 < l \leq k$ , and suppose that  $V_{l-1,k} \in \mathbf{C}_{l-2}(\mathbb{R}_+^k)$ . Then  $V_{l,k}$  is given by a fixed point of the operator  $T_{l,k} : \mathbf{C}(\mathbb{R}_+^k) \rightarrow \mathbf{C}(\mathbb{R}_+^k)$ , where  $T_{l,k}$  is defined by

$$(A.4) \quad [T_{l,k}(W)](\hat{v}) = \delta(1-q)V_{l-1,k-1}(\hat{v}_l, \dots, \hat{v}_k) + q \left[ \sum_{j=0}^{k-l} \int_{\hat{v}_{k-j+1}}^{\hat{v}_{k-j}} \delta V_{l-1,k}(\hat{v}_{-1}(v')) dF(v') \right. \\ \left. + \sum_{j=k-l+1}^{k-1} \int_{\hat{v}_{k-j+1}}^{\hat{v}_{k-j}} \delta W(\hat{v}_{-1}(v')) dF(v') + \int_{\hat{v}_1}^{\infty} \delta W(\hat{v}) dF(v') \right].$$

We may again apply Blackwell's Contraction Lemma and the Banach Fixed Point Theorem to show that  $V_{l,k}$  is the unique fixed point of  $T_{l,k}$ . Furthermore, it is straightforward to show that  $T_{l,k}$  maps elements of  $\mathbf{C}_{l-1}(\mathbb{R}_+^k)$  back into  $\mathbf{C}_{l-1}(\mathbb{R}_+^k)$ . Therefore, again using Lemma 1, we may write  $V_{l,k}(\hat{v}) = V_{l,k}(\hat{v}_l, \dots, \hat{v}_k)$ .

Thus, by induction, the bidding strategies in Equation 4.3 lead to unique value functions  $\left\{ \{V_{j,k}\}_{j=1}^k \right\}_{k=1}^{\infty}$  such that, for all  $k$  and all  $j \leq k$ ,  $V_{j,k} \in \mathbf{C}_{j-1}(\mathbb{R}_+^k)$ . ■

*Proof of Proposition 2.* Recall from Equation A.2 in the proof of Proposition 1 that  $V_{1,k}$  is defined as the unique fixed point of

$$[T_{1,k}(W)](\hat{v}) := (1-q)[\hat{v}_1 - \hat{v}_2 + \delta V_{1,k-1}(\hat{v}_2, \dots, \hat{v}_k)] \\ + q \left[ \int_0^{\hat{v}_1} \hat{v}_1 dF(v') - \int_0^{\hat{v}_2} \hat{v}_2 dF(v') - \int_{\hat{v}_2}^{\hat{v}_1} v' dF(v') \right. \\ \left. + \delta \sum_{j=0}^{k-1} \int_{\hat{v}_{k-j+1}}^{\hat{v}_{k-j}} W(\hat{v}_{-1}(v')) dF(v') + \int_{\hat{v}_1}^{\infty} \delta W(\hat{v}) dF(v') \right].$$

Let  $\hat{w} := \hat{v}_{-1}$ . Then

$$V_{1,k}(\hat{w}) = [T_{1,k}(V_{1,k})](\hat{w}) = \delta(1-q)V_{1,k-1}(\hat{v}_2, \dots, \hat{v}_k) \\ + \delta q \left[ \sum_{j=0}^{k-2} \int_{\hat{v}_{k-j+1}}^{\hat{v}_{k-j}} V_{1,k}(\hat{w}(v')) dF(v') + \int_{\hat{v}_2}^{\infty} \delta V_{1,k}(\hat{w}) dF(v') \right].$$

However, this is identical to the definition of  $T_{2,k}$  given in Equation A.3, implying that, for all  $k \in \mathbb{N}$ ,  $V_{1,k}(\hat{v}_2, \hat{v}_2, \dots, \hat{v}_k) = V_{2,k}(\hat{v}_2, \dots, \hat{v}_k)$ .

Fix  $l > 1$ , and suppose that  $V_{l-1,k}(\hat{v}_l, \hat{v}_l, \dots, \hat{v}_k) = V_{l,k}(\hat{v}_l, \dots, \hat{v}_k)$  for all  $k \geq l$ . Redefine  $\hat{w} := (l \hat{+} 1, l \hat{+} 1, \dots, \hat{v}_k)$ , and consider  $V_{l,k}(\hat{w})$ . Recalling from Equation A.4 the definition

of  $T_{l,k}$ , we have

$$\begin{aligned}
V_{l,k}(\hat{w}) &= [T_{l,k}(V_{l,k})](\hat{w}) = \delta(1-q)V_{l-1,k-1}(\hat{v}_{l+1}, \hat{v}_{l+1}, \dots, \hat{v}_k) \\
&\quad + q \left[ \sum_{j=0}^{k-l} \int_{\hat{v}_{k-j+1}}^{\hat{v}_{k-j}} \delta V_{l-1,k}(\hat{w}(v')) dF(v') \right. \\
&\quad \left. + \sum_{j=k-l+1}^{k-1} \int_{\hat{v}_{k-j+1}}^{\hat{v}_{k-j}} \delta V_{l,k}(\hat{w}(v')) dF(v') + \int_{\hat{v}_1}^{\infty} \delta V_{l,k}(\hat{w}) dF(v') \right] \\
&= \delta(1-q)V_{l,k-1}(\hat{v}_{l+1}, \dots, \hat{v}_k) \\
&\quad + q \left[ \sum_{j=0}^{k-l-1} \int_{\hat{v}_{k-j+1}}^{\hat{v}_{k-j}} \delta V_{l,k}(\hat{w}(v')) dF(v') + \int_{\hat{v}_{l+1}}^{\infty} \delta V_{l,k}(\hat{w}) dF(v') \right].
\end{aligned}$$

The above is a reformulation of the expression for  $T_{l+1,k}$ , taking into account that the fixed point of this operator lies in  $\mathbf{C}_l(\mathbb{R}_+^k)$ . Since  $V_{l,k}(\hat{w})$  is a fixed point of the operator, the uniqueness result from Proposition 1 implies that  $V_{l+1,k}(\hat{v}_{l+1}, \dots, \hat{v}_k) = V_{l,k}(\hat{w})$  for all  $k \geq l+1$ .

Thus, by induction on  $l$ , we have established that, for arbitrary  $k \in \mathbb{N}$  and for all  $l = 1, \dots, k$ ,  $V_{l,k}(\hat{v}_l, \dots, \hat{v}_k) = V_{1,k}(\hat{v}_l, \dots, \hat{v}_l, \hat{v}_{l+1}, \dots, \hat{v}_k)$ . ■

*Proof of Proposition 3.* To prove the first part of this proposition, it suffices to simply differentiate the bid function  $\beta_{l,k}$  with respect to the bidder's value  $v_i$ . In particular, we have, for all  $k \in \mathbb{N}$  and  $l \in \{1, \dots, k\}$ ,

$$\frac{\partial}{\partial v_i} \beta_{l,k}(v_i, \hat{v}_{l+1}, \dots, \hat{v}_k) = 1 - \left( \delta \frac{\mu(v_i)}{\eta(v_i)} \right)^{l-1}.$$

However,  $\delta \in (0, 1)$  implies  $\frac{\mu(v)}{\eta(v)} \in (0, 1)$ , and hence  $\frac{\partial}{\partial v_i} \beta_{l,k}(v_i, \hat{v}_{l+1}, \dots, \hat{v}_k) > 0$ .

As for the second part of this proposition, let  $\hat{w} = (\hat{v}_{l+1}, \dots, \hat{v}_k)$  and note that

$$\begin{aligned}
\beta_{l,k}(v_i, \hat{w}) - \beta_{l+1,k}(\hat{w}) &= v_i - \delta V_{l-1,k-1}(v_i, \hat{w}) - \hat{v}_{l+1} + \delta V_{l,k-1}(\hat{w}) \\
&= (v_i - \hat{v}_{l+1}) - \delta (V_{1,k-1}(v_i, \dots, v_i, \hat{w}) - V_{1,k-1}(\hat{v}_{l+1}, \dots, \hat{v}_{l+1}, \hat{w})) \\
&= (v_i - \hat{v}_{l+1}) - \delta \left[ \int_{\hat{v}_{l+1}}^{v_i} \frac{\mu(v')}{\eta(v')} dv' - (1-\delta) \sum_{j=2}^l \delta^{j-2} \int_{\hat{v}_{l+1}}^{v_i} \frac{\mu^{j-1}(v')}{\eta^j(v')} dv' \right] \\
&= \int_{\hat{v}_{l+1}}^{v_i} \frac{\eta(v') - \delta \mu(v')}{\eta(v')} dv' + (1-\delta) \sum_{j=2}^l \delta^{j-1} \int_{\hat{v}_{l+1}}^{v_i} \frac{\mu^{j-1}(v')}{\eta^j(v')} dv' \\
&= (1-\delta) \sum_{j=1}^l \delta^{j-1} \int_{\hat{v}_{l+1}}^{v_i} \frac{\mu^{j-1}(v')}{\eta^j(v')} dv'.
\end{aligned}$$

Since  $\mu(v), \eta(v) > 0$  for all  $v \in \mathbb{R}_+$ , this expression is positive if, and only if,  $v_i > \hat{v}_{i+1}$ . Thus, the exit of a lower-ranked bidder does not induce the immediate exit of a higher-ranked bidder who is using the bidding strategy given in Equation 4.10. ■

*Proof of Proposition 4.* We begin by showing that  $W_1$  has the desired form and then proceed inductively. Note that  $W_1$  is a fixed point of the operator  $\hat{T}_1 : \mathbf{C}(\mathbb{R}_+) \rightarrow \mathbf{C}(\mathbb{R}_+)$  defined by

$$(A.5) \quad [\hat{T}(g)](x) = (1 - q)(x + \delta W_0) + q \left[ \int_0^x (x + \delta g(y)) dF(y) + \int_x^\infty (y + \delta g(x)) dF(y) \right].$$

This operator is clearly a self-map from  $\mathbf{C}(\mathbb{R}_+)$  into itself. Furthermore, it is straightforward to see that  $\hat{T}_1$  is a contraction mapping. Fix any  $g, g' \in \mathbf{C}(\mathbb{R}_+)$  such that  $g' > g$ . Then

$$[\hat{T}_1(g' - g)](x) = \delta q \left[ \int_0^x (g'(y) - g(y)) dF(y) + (1 - F(x))(g'(x) - g(x)) \right] > 0.$$

Furthermore, for any  $g \in \mathbf{C}(\mathbb{R}_+)$  and any  $\alpha \in \mathbb{R}_{++}$ ,

$$[\hat{T}_1(g + \alpha)](x) = [\hat{T}_1(g)](x) + \delta q \alpha.$$

Since  $\delta q < 1$ , we may apply Blackwell's Contraction Lemma and the Banach Fixed Point Theorem, implying that  $\hat{T}_1$  has a unique fixed point  $W_1$  such that

$$W_1(\hat{v}_1) = (1 - q)(\hat{v}_1 + \delta W_0) + q \left[ \int_0^{\hat{v}_1} (\hat{v}_1 + \delta W_1(v')) dF(v') + \int_{\hat{v}_1}^\infty (v' + \delta W_1(\hat{v}_1)) dF(v') \right].$$

Differentiating this expression with respect to  $\hat{v}_1$  yields

$$W_1'(\hat{v}_1) = (1 - q) + qF(\hat{v}_1) + \delta q(1 - F(\hat{v}_1))W_1'(\hat{v}_1) = \frac{\mu(\hat{v}_1)}{\eta(\hat{v}_1)}.$$

Finally, note that  $W_1(0) = W_0$ , since a buyer with value zero adds nothing to the social welfare. Therefore, by the Fundamental Theorem of Calculus, we have

$$(A.6) \quad W_1(\hat{v}_1) = W_0 + \int_0^{\hat{v}_1} \frac{\mu(v')}{\eta(v')} dv'.$$

Now consider  $W_n(\hat{v})$  for arbitrary  $n > 1$ , and suppose that  $W_{n-1}$  takes the desired form.<sup>12</sup>  $W_n$  is defined to be a fixed point of the operator  $\hat{T}_n : \mathbf{C}(\mathbb{R}_+^n) \rightarrow \mathbf{C}(\mathbb{R}_+^n)$  defined by

$$(A.7) \quad \begin{aligned} [\hat{T}_n(g)](x) &= (1 - q)(x_1 + \delta W_{n-1}(x_{-1})) \\ &+ q \left[ \sum_{j=0}^{n-1} \int_{x_{n-j+1}}^{x_{n-j}} (x_1 + \delta g(x_2, \dots, x_{n-j}, y, x_{n-j+1}, \dots, x_n)) dF(y) \right. \\ &\quad \left. + \int_{x_1}^\infty (y + \delta g(x)) dF(y) \right]. \end{aligned}$$

<sup>12</sup>Notice that this implies that all of the cross-derivatives of  $W_{n-1}$  are identically zero.

Note that for any  $g, g' \in \mathbf{C}(\mathbb{R}_+^n)$  such that  $g' > g$ , we have

$$\begin{aligned} [\hat{T}_n(g' - g)](x) &= \delta q \left[ \sum_{j=0}^{n-1} \int_{x_{n-j+1}}^{x_{n-j}} [g' - g](x_2, \dots, x_{n-j}, y, x_{n-j+1}, \dots, x_n) dF(y) \right. \\ &\quad \left. + (1 - F(x_1))(g'(x) - g(x)) \right]. \end{aligned}$$

Furthermore, for any  $g \in \mathbf{C}(\mathbb{R}_+^n)$  and any  $\alpha \in \mathbb{R}_{++}$ ,

$$[\hat{T}_n(g + \alpha)](x) = [\hat{T}_n(g)](x) + \delta q \alpha.$$

Since  $\delta q < 1$ , Blackwell's monotonicity and discounting conditions are satisfied. Thus, Blackwell's Contraction Lemma and the Banach Fixed Point Theorem imply that  $\hat{T}_n$  has a unique fixed point  $W_n$  such that

$$\begin{aligned} (A.8) \quad W_n(\hat{v}) &= (1 - q)(\hat{v}_1 + \delta W_{n-1}(\hat{v}_{-1})) \\ &\quad + q \left[ \sum_{j=0}^{n-1} \int_{\hat{v}_{n-j+1}}^{\hat{v}_{n-j}} (\hat{v}_1 + \delta W_n(\hat{v}_2, \dots, \hat{v}_{n-j}, v', \hat{v}_{n-j+1}, \dots, \hat{v}_n)) dF(v') \right. \\ &\quad \left. + \int_{\hat{v}_1}^{\infty} (v' + \delta W_n(\hat{v})) dF(v') \right]. \end{aligned}$$

Differentiating this expression implicitly with respect to  $\hat{v}_1$  yields

$$W_n^{(1)}(\hat{v}) = (1 - q) + qF(\hat{v}_1) + \delta q(1 - F(\hat{v}_1))W_n^{(1)}(\hat{v}) = \frac{\mu(\hat{v}_1)}{\eta(\hat{v}_1)}.$$

Note that this expression is independent of  $n$  and of  $\hat{v}_j$  for  $j \neq 1$ , implying that  $W_n^{(1,j)}$  is identically zero for all  $j \neq 1$ .

Similarly, implicit differentiation with respect to  $\hat{v}_2$  yields

$$\begin{aligned} W_n^{(2)}(\hat{v}) &= (1 - q)\delta W_{n-1}^{(1)}(\hat{v}_{-1}) + \delta q \left[ \sum_{j=0}^{n-2} \int_{\hat{v}_{n-j+1}}^{\hat{v}_{n-j}} W_n^{(1)}(\hat{v}_{-1}(v')) dF(v') \right. \\ &\quad \left. + \int_{\hat{v}_2}^{\hat{v}_1} W_n^{(2)}(v', \hat{v}) dF(v') + (1 - F(\hat{v}_1))W_n^{(2)}(\hat{v}) \right], \end{aligned}$$

where  $\hat{v}(\hat{v})$  is the re-ordering of  $\hat{v}$  and  $v'$ . Since  $W_n^{1,j}$  is identically zero,

$$\sum_{j=0}^{n-2} \int_{\hat{v}_{n-j+1}}^{\hat{v}_{n-j}} W_n^{(1)}(\hat{v}_{-1}(v')) dF(v') = F(\hat{v}_2) \frac{\mu(\hat{v}_2)}{\eta(\hat{v}_2)}.$$

Furthermore,  $W_n^{(2,1)} = 0$  implies that

$$\int_{\hat{v}_2}^{\hat{v}_1} W_n^{(2)}(v', \hat{v}) dF(v') + (1 - F(\hat{v}_1))W_n^{(2)}(\hat{v}) = (1 - F(\hat{v}_2))W_n^{(2)}(\hat{v}).$$

Finally, our inductive hypothesis implies that

$$W_{n-1}^{(1)}(\hat{v}_{-1}) = \frac{\mu(\hat{v}_2)}{\eta(\hat{v}_2)}.$$

Thus, we may conclude that

$$(A.9) \quad W_n^{(2)} = \delta(1-q) \frac{\mu(\hat{v}_2)}{\eta(\hat{v}_2)} + \delta q \left[ F(\hat{v}_2) \frac{\mu(\hat{v}_2)}{\eta(\hat{v}_2)} + (1-F(\hat{v}_2)) W_n^{(2)}(\hat{v}) \right] = \delta \left( \frac{\mu(\hat{v}_2)}{\eta(\hat{v}_2)} \right)^2.$$

Once again, note that this expression is independent of  $n$  and of  $\hat{v}_j$  for  $j \neq 2$ , implying that  $W_n^{2,j}$  is identically zero for all  $j \neq 2$ .

Proceeding inductively, consider the derivative of  $W_n$  with respect to its  $k$ -th argument, where  $k \leq n$ . We have

$$\begin{aligned} W_n^{(k)}(\hat{v}) &= (1-q) \delta W_{n-1}^{(k-1)}(\hat{v}_{-1}) + \delta q \left[ \sum_{j=0}^{n-k} \int_{\hat{v}_{n-j+1}}^{\hat{v}_{n-j}} W_n^{(1)}(\hat{v}_{-1}(v')) dF(v') \right. \\ &\quad \left. + \sum_{j=n-k+1}^{n-1} \int_{\hat{v}_{n-j+1}}^{\hat{v}_{n-j}} W_n^{(k)}(\hat{v}_1(v')) dF(v') + (1-F(\hat{v}_1)) W_n^{(k)}(\hat{v}) \right]. \end{aligned}$$

Applying the same simplifications as above, we have

$$\begin{aligned} W_n^{(k)} &= \delta(1-q) W_{n-1}^{(k-1)}(\hat{v}_{-1}) + \delta q \left[ F(\hat{v}_k) W_{n-1}^{(k-1)}(\hat{v}_{-1}) + (1-F(\hat{v}_k)) W_n^{(k)}(\hat{v}) \right] \\ &= \frac{\delta \mu(\hat{v}_k)}{\eta(\hat{v}_k)} W_{n-1}^{(k-1)}(\hat{v}_{-1}) \\ (A.10) \quad &= \delta^{k-1} \left( \frac{\mu(\hat{v}_k)}{\eta(\hat{v}_k)} \right)^k. \end{aligned}$$

Finally, note that  $W_n(0, \dots, 0) = W_0$  since, as with one agent with value zero, assigning an object to a “null” agent yields no increase in social welfare. Thus, we may conclude that, as desired,

$$(A.11) \quad W_n(\hat{v}) = W_0 + \sum_{j=1}^n \delta^{j-1} \int_0^{\hat{v}_j} \left( \frac{\mu(v')}{\eta(v')} \right)^j dv'. \quad \blacksquare$$

In order to prove Proposition 5, we will need to make use of the following result.

**Lemma 2** ( $W$  is Additively Separable).

*The social planner’s expected payoff function in the case of random object arrivals is an additively separable function of the (ordered) values of the the buyers present on the market.*

*Proof.* Recall that the socially optimal policy in this setting is for the planner to assign an object—whenever it is available—to the highest-valued buyer on the market. Since the arrival process of new buyers and the realized valuations of these new entrants are independent of the number of agents present on the market, a marginal increase in the



value of the highest-valued buyer present on the market does not affect the planner's expectations of future realized values, nor does it impact the anticipated plan of object assignments. Therefore, for any values  $\hat{v}_1 > \dots > \hat{v}_n > \hat{v}_{n+1}$ , we must have

$$\frac{\partial}{\partial \hat{v}_1} W_n(\hat{v}_1, \dots, \hat{v}_n) = \frac{\partial}{\partial \hat{v}_1} W_{n+1}(\hat{v}_1, \dots, \hat{v}_{n+1}).$$

Similarly, for any  $k = 2, \dots, n$ , a marginal increase in  $\hat{v}_k$ , the value of the  $k$ -th highest-ranked buyer, affects neither future arrivals and valuations nor the planner's optimal plan of action. Therefore, we must have

$$\frac{\partial}{\partial \hat{v}_k} W_n(\hat{v}_1, \dots, \hat{v}_n) = \frac{\partial}{\partial \hat{v}_k} W_{n+1}(\hat{v}_1, \dots, \hat{v}_{n+1}).$$

Thus, for any  $n \in \mathbb{N}$  and any  $k = 1, \dots, n$ ,

$$(A.12) \quad W_n^{(k)}(\hat{v}_1, \dots, \hat{v}_n) = W_{n+1}^{(k)}(\hat{v}_1, \dots, \hat{v}_n, \hat{v}_{n+1}),$$

where  $\hat{v}_1 > \dots > \hat{v}_n > \hat{v}_{n+1}$ .

Now consider any  $n \in \mathbb{N}$  and ordered values  $\hat{v}_1 > \dots > \hat{v}_n$ . Equation A.12 implies that

$$W_n^{(1)}(\hat{v}_1, \dots, \hat{v}_n) = W_{n-1}^{(1)}(\hat{v}_1, \dots, \hat{v}_{n-1}) = W_{n-2}^{(1)}(\hat{v}_1, \dots, \hat{v}_{n-2}) = \dots = W_1^{(1)}(\hat{v}_1).$$

Therefore, it must be the case that

$$W_2^{(1,2)}(\hat{v}_1, \hat{v}_2) = \frac{\partial}{\partial \hat{v}_2} W_2^{(1)}(\hat{v}_1, \hat{v}_2) = \frac{\partial}{\partial \hat{v}_2} W_1^{(1)}(\hat{v}_1) = 0.$$

Straightforward induction therefore yields

$$W_k^{(1,k)}(\hat{v}_1, \dots, \hat{v}_k) = \frac{\partial}{\partial \hat{v}_k} W_k^{(1)}(\hat{v}_1, \dots, \hat{v}_k) = \frac{\partial}{\partial \hat{v}_k} W_{k-1}^{(1)}(\hat{v}_1, \dots, \hat{v}_{k-1}) = 0$$

for all  $k = 2, \dots, n$ . Hence, it must be the case that, for any  $k \neq 1$ ,  $W_k^{(1)}(\hat{v}_1, \dots, \hat{v}_k)$  is a function of  $\hat{v}_1$  alone.

Since Equation A.12 implies that

$$W_n^{(2)}(\hat{v}_1, \dots, \hat{v}_n) = W_{n-1}^{(2)}(\hat{v}_1, \dots, \hat{v}_{n-1}) = W_{n-2}^{(2)}(\hat{v}_1, \dots, \hat{v}_{n-2}) = \dots = W_2^{(2)}(\hat{v}_1, \hat{v}_2),$$

we may perform a similar exercise as above. In particular,

$$W_3^{(2,3)}(\hat{v}_1, \hat{v}_2, \hat{v}_3) = \frac{\partial}{\partial \hat{v}_3} W_3^{(2)}(\hat{v}_1, \hat{v}_2, \hat{v}_3) = \frac{\partial}{\partial \hat{v}_3} W_2^{(2)}(\hat{v}_1, \hat{v}_2) = 0.$$

The same inductive reasoning again leads to the conclusion that

$$W_k^{(2,k)}(\hat{v}_1, \dots, \hat{v}_k) = \frac{\partial}{\partial \hat{v}_k} W_k^{(2)}(\hat{v}_1, \dots, \hat{v}_k) = \frac{\partial}{\partial \hat{v}_k} W_{k-1}^{(2)}(\hat{v}_1, \dots, \hat{v}_{k-1}) = 0$$

for all  $k = 3, \dots, n$ . Combining this with the fact that  $W_k^{(1)}$  is independent of its second argument yields the conclusion that  $W_k^{(2)}(\hat{v}_1, \dots, \hat{v}_k)$  is a function of  $\hat{v}_2$  alone. Proceeding

inductively in this manner, we may conclude that

$$(A.13) \quad W_n^{(j,k)}(\hat{v}_1, \dots, \hat{v}_n) = 0 \text{ for all } j \neq k,$$

where  $j, k \in \{1, \dots, n\}$ .

Thus, Equation A.13 implies that, for any  $n \in \mathbb{N}$  and any  $k = 1, \dots, n$ , we may write

$$W_n^{(k)}(\hat{v}_1, \dots, \hat{v}_n) = g_{k,n}(\hat{v}_k)$$

for some real-valued function  $g_{k,n}$ . Combining this with Equation A.12 immediately implies that

$$g_{k,n} = g_{k,m} \text{ for all } m \geq n.$$

Therefore, there exists a sequence of real-valued functions  $\{g_k\}_{k=1}^\infty$  such that

$$W_n^{(k)}(\hat{v}_1, \dots, \hat{v}_n) = g_k(\hat{v}_k)$$

for every  $k = 1, \dots, n$  and arbitrary  $n \in \mathbb{N}$ . Therefore, applying the Fundamental Theorem of Calculus allows us to conclude that, for arbitrary  $n$ ,

$$(A.14) \quad W_n(\hat{v}_1, \dots, \hat{v}_n) = W_n(0, \dots, 0) + \sum_{k=1}^n \int_0^{\hat{v}_k} g_k(v') dv';$$

that is,  $W_n$  is additively separable in its arguments. ■

*Proof of Proposition 5.* Note that we may rewrite Equation 5.4 as

$$\begin{aligned} W_n(\hat{v}) &= pq \left[ F(\hat{v}_1)\hat{v}_1 + \int_{\hat{v}_1}^\infty v' dF(v') + \delta \sum_{j=0}^{n-1} \int_{\hat{v}_{n-j+1}}^{\hat{v}_{n-j}} W_n(\hat{v}_{-1}(v')) dF(v') + \delta(1 - F(\hat{v}_1))W_n(\hat{v}) \right] \\ &\quad + p(1 - q) [\hat{v}_1 + \delta W_{n-1}(\hat{v}_{-1})] + (1 - p)q \left[ \sum_{j=0}^n \delta \int_{\hat{v}_{n-j+1}}^{\hat{v}_{n-j}} W_{n+1}(\hat{v}(v')) dF(v') \right] \\ &\quad + (1 - p)(1 - q) [\delta W_n(\hat{v})]. \end{aligned}$$

Differentiating this expression with respect to  $\hat{v}_1$  yields

$$\begin{aligned} W_n^{(1)}(\hat{v}) &= p[1 - q(1 - F(\hat{v}_1))] + \delta[pq(1 - F(\hat{v}_1)) + (1 - p)(1 - q)]W_n^{(1)}(\hat{v}) \\ &\quad + \delta(1 - p)q \sum_{j=0}^{n-1} \int_{\hat{v}_{n-j+1}}^{\hat{v}_{n-j}} W_{n+1}^{(1)}(\hat{v}(v')) dF(v') + \delta(1 - p)q \int_{\hat{v}_1}^\infty W_{n+1}^{(2)}(v', \hat{v}) dF(v'). \end{aligned}$$

Similarly, differentiating  $W_n(\hat{v})$  with respect to  $\hat{v}_k$  for  $k > 1$  yields

$$\begin{aligned}
W_n^{(k)}(\hat{v}) &= \delta pq \left[ \sum_{j=0}^{n-k-1} \int_{\hat{v}_{n-j+1}}^{\hat{v}_{n-j}} W_n^{(k-1)}(\hat{v}_{-1}(v')) dF(v') + \sum_{j=n-k}^{n-1} \int_{\hat{v}_{n-j+1}}^{\hat{v}_{n-j}} W_n^{(k)}(\hat{v}_{-1}(v')) dF(v') \right] \\
&\quad + \delta pq(1 - F(\hat{v}_1))W_n^{(k)}(\hat{v}) + \delta(1 - p)q \left[ \sum_{j=0}^{n-k} \int_{\hat{v}_{n-j+1}}^{\hat{v}_{n-j}} W_{n+1}^{(k)}(\hat{v}(v')) dF(v') \right] \\
&\quad + \delta(1 - p)q \left[ \sum_{j=n-k+1}^n \int_{\hat{v}_{n-j+1}}^{\hat{v}_{n-j}} W_{n+1}^{(k+1)}(\hat{v}(v')) dF(v') \right] + \delta p(1 - q)W_{n-1}^{(k-1)}(\hat{v}_{-1}) \\
&\quad + \delta(1 - p)(1 - q)W_n^{(k)}(\hat{v}).
\end{aligned}$$

Applying Lemma 2 allows us to denote  $W_n^{(m)}(\hat{v})$  by  $g_m(\hat{v}_m)$ , where  $m, n \in \mathbb{N}$  and  $m \leq n$ . Thus, we may rewrite the two expressions above as

$$\begin{aligned}
(A.15) \quad g_1(v) &= \delta [pq(1 - F(v)) + (1 - p)(1 - q(1 - F(v)))] g_1(v) \\
&\quad + \delta(1 - p)q(1 - F(v))g_2(v) + p[1 - q(1 - F(v))]
\end{aligned}$$

and, for  $k > 1$ ,

$$\begin{aligned}
(A.16) \quad g_k(v) &= \delta [pq(1 - F(v)) + (1 - p)(1 - q(1 - F(v)))] g_k(v) \\
&\quad + \delta(1 - p)q(1 - F(v))g_{k+1}(v) + \delta p[1 - q(1 - F(v))] g_{k-1}(v).
\end{aligned}$$

Note, however, that Equation A.16 also holds for  $k = 1$  if we define

$$g_0(v) := \frac{1}{\delta} \text{ for all } v \in \mathbb{R}_+.$$

Thus, the partial derivatives of  $W_n$  are determined by a second-order difference equation. Defining

$$y_m(v) = \begin{bmatrix} g_{m+1}(v) \\ g_m(v) \end{bmatrix} \text{ and } A(v) = \begin{bmatrix} a(v) & b(v) \\ 1 & 0 \end{bmatrix},$$

where

$$a(v) = \frac{1 - \delta [pq(1 - F(v)) + (1 - p)(1 - q(1 - F(v)))]}{\delta(1 - p)q(1 - F(v))} \text{ and } b(v) = -\frac{\delta p[1 - q(1 - F(v))]}{\delta(1 - p)q(1 - F(v))},$$

we may rewrite this difference equation as the first-order system

$$y_{m+1}(v) = A(v)y_m(v).$$

Induction immediately yields the solution

$$(A.17) \quad y_m(v) = [A(v)]^m y_0(v).$$

We denote by  $\lambda_1(v)$  and  $\lambda_2(v)$  the eigenvalues of  $A(v)$ ; in fact, these can be shown to take the form

$$\lambda_1(v) = \frac{a(v) - \sqrt{a^2(v) + 4b(v)}}{2} \text{ and } \lambda_2(v) = \frac{a(v) + \sqrt{a^2(v) + 4b(v)}}{2}.$$

Furthermore, since  $p, q, \delta \in (0, 1)$  and  $F(v) \in [0, 1]$  for all  $v \in \mathbb{R}_+$ , arithmetic manipulation yields

$$\lambda_2(v) > 1 > \lambda_1(v) > 0 \text{ for all } v \in \mathbb{R}_+.$$

Finally, note that  $A(v)$  is diagonalizable, and, moreover,

$$[A(v)]^m = \frac{1}{\lambda_2(v) - \lambda_1(v)} \begin{bmatrix} \lambda_2^{m+1}(v) - \lambda_1^{m+1}(v) & \lambda_1^{m+1}(v)\lambda_2(v) - \lambda_1(v)\lambda_2^{m+1}(v) \\ \lambda_2^m(v) - \lambda_1^m(v) & \lambda_1^m(v)\lambda_2(v) - \lambda_1(v)\lambda_2^m(v) \end{bmatrix}.$$

Thus, Equation A.17 may be rewritten as

$$\begin{aligned} g_m(v) &= \frac{1}{\lambda_2(v) - \lambda_1(v)} [(\lambda_2^m(v) - \lambda_1^m(v))g_1(v) + (\lambda_1^m(v)\lambda_2(v) - \lambda_1(v)\lambda_2^m(v))g_0(v)] \\ \text{(A.18)} \quad &= \frac{\lambda_2^m(v)}{\lambda_2(v) - \lambda_1(v)} (g_1(v) - \lambda_1(v)g_0(v)) + \frac{\lambda_1^m(v)}{\lambda_2(v) - \lambda_1(v)} (\lambda_2(v)g_0(v) - g_1(v)). \end{aligned}$$

Since  $\lambda_2(v) > 1$ , the first term in Equation A.18 is divergent unless

$$g_1(v) = \lambda_1(v)g_0(v).$$

Such a divergence would, of course, be contradictory; the marginal impact of an increase of the  $m$ -th highest value on the social welfare must be bounded above by  $\delta^{m-1}$ , as the benefit of this increase is not realized for *at least*  $m$  periods. Thus,

$$\text{(A.19)} \quad g_m(v) = \lambda_1^m(v)g_0(v).$$

Recalling that  $W_n^m(\hat{v}) = g_m(\hat{v}_m)$  and that  $g_0(v) = \delta^{-1}$ , application of Lemma 2 allows us to conclude that

$$\text{(A.20)} \quad W_n(\hat{v}) = W_n(0) + \delta^{-1} \sum_{j=1}^n \int_0^{\hat{v}_j} \lambda_1^j(v') dv'.$$

Finally, note that any agent with a “null” value adds nothing to the social welfare, and hence the planner’s payoff when they are not present is given by  $W_n(0) = W_0$ , where

$$\begin{aligned} W_0 &= pq \left[ \int_0^\infty (v' + \delta W_0) dF(v') \right] + p(1-q) [\delta W_0] \\ &\quad + (1-p)q \left[ \int_0^\infty \delta W_1(v') dF(v') \right] + (1-p)(1-q) [\delta W_0] \\ &= \frac{q}{1-\delta} \left[ p \int_0^\infty v' dF(v') + (1-p) \int_0^\infty \left( \int_0^{v''} \lambda_1(v') dv' \right) dF(v'') \right]. \quad \blacksquare \end{aligned}$$

## REFERENCES

- BAJARI, P. AND A. HORTAÇSU (2003): "The Winner's Curse, Reserve Prices, and Endogenous Entry: Empirical Insights from eBay Auctions," *The RAND Journal of Economics*, 34, 329–355.
- BERGEMANN, D. AND J. VÄLIMÄKI (2007): "Dynamic Marginal Contribution Mechanism," Cowles Foundation Discussion Paper 1616, Yale University.
- CAI, G., P. R. WURMAN, AND X. CHAO (2007): "The Non-Existence of Equilibrium in Sequential Auctions when Bids are Revealed," *Journal of Electronic Commerce Research*, 8, 141–156.
- CAILLAUD, B. AND C. MEZZETTI (2004): "Equilibrium Reserve Prices in Sequential Ascending Auctions," *Journal of Economic Theory*, 117, 78–95.
- CAVALLO, R., D. C. PARKES, AND S. SINGH (2007): "Efficient Online Mechanisms for Persistent, Periodically Inaccessible Self-Interested Agents," Unpublished manuscript, Harvard University.
- FUCHS, W. AND A. SKRZYPACZ (2007): "Bargaining with Arrival of New Traders," Unpublished manuscript, Stanford University.
- INDERST, R. (2008): "Dynamic Bilateral Bargaining under Private Information with a Sequence of Potential Buyers," *Review of Economic Dynamics*, 11, 220–236.
- JEITSCHKO, T. (1998): "Learning in Sequential Auctions," *Southern Economic Journal*, 65, 98–112.
- KITTSTEINER, T., J. NIKUTTA, AND E. WINTER (2004): "Declining Valuations in Sequential Auctions," *International Journal of Game Theory*, 33, 89–106.
- MILGROM, P. R. AND R. J. WEBER (2000): "A Theory of Auctions and Competitive Bidding, II," in *The Economic Theory of Auctions*, ed. by P. Klemperer, Cheltenham, UK: Edward Elgar Publishing, vol. 2, 179–194.
- NEKIPELOV, D. (2007): "Entry Deterrence and Learning Prevention on eBay," Unpublished manuscript, Duke University.
- OK, E. A. (2007): *Real Analysis with Economic Applications*, Princeton University Press.
- ROTH, A. AND A. OCKENFELS (2002): "Last-Minute Bidding and the Rules for Ending Second-Price Auctions: Evidence from eBay and Amazon Auctions on the Internet," *American Economic Review*, 94, 1093–1103.
- SAID, M. (2008): "Sequential Auctions with Random Arrivals," Unpublished manuscript, Yale University.
- SAILER, K. (2006): "Searching the eBay Marketplace," CESifo Working Paper 1848, University of Munich.
- ZEITHAMMER, R. (2006): "Forward-Looking Bidding in Online Auctions," *Journal of Marketing Research*, 43, 462–476.