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Abstract

The introduction of the bootstrap methods by Efron (1979) enables many empirical researches, which would otherwise be difficult if not totally impossible. Nowadays, bootstrapping has become an important aspect in research. This paper reviews various ways of bootstrapping data for cross-sectional and time series samples. Various ways of bootstrapping confidence intervals for estimators, an important application of bootstrap methods, are also discussed in this paper. Several other applications of bootstrap methods are briefly mentioned preceding to the concluding remarks this paper.

1. Introduction

Statistical inference plays a dominant role in empirical research. Researchers use to draw general conclusions and recommend policy suggestions based on the inference they made from their statistical estimates of available sample data, which are usually limited. Thus, getting accurate statistical estimates as well as reliable statistical inference inevitably is an obligatory requirement for fine empirical research. Specifically, in the field of economic research, many test-statistics for hypothesis testing intention, including the commonly exploited t-statistics and F-statistics, imposed normality assumption. This strong assumption is easily violated in reality as most economic data series are found to possess non-normal empirical distribution functions (EPF), let alone their unknown population probability distribution functions (PDF). As such, be oblivious to its importance, statistical inference can be a formidable wall of empirical research. By the introduction of the bootstrap method (Efron, 1979) and the dissemination of a series of related study (Efron, 1981; Efron, 1985; Efron, 1987; Efron, 1990; Efron and Tibshirani, 1985; Efron and Tibshirani, 1986, just to name a few), Bradley Efron not only bulldozes this wall, but also opens up alternative road to explore many of the otherwise inaccessible area of research[1].

Bootstrap is a computer-intensive method of statistical inference that can answer many real statistical questions. Briefly, bootstrap enables researchers to estimate the precision of sample statistics (such as medians, variances, percentiles) by drawing randomly with replacement from a set of available data. Bootstrap method is very useful in circumstances where the asymptotic distributions of the test statistics of interest are unknown or statistically too complicated to derive. Besides, in cases where normality assumption have been violated (as mentioned above) thereby invalidate the use of conventional standard errors, confidence intervals as well as *t*-statistics, bootstrap method offers us an alternative procedure in estimating these statistics of interest by resampling with replacement the original finite sample, which has been branded as bootstrapping nowadays. Practically, bootstrapping is similar to that of a Monte Carlo simulation, with an essential difference in the generation of the random variables (stochastic errors). The

[1] Indeed, the contributions of Bradley Efron are so significance that it is not unreasonable for one to predict that he will one day be awarded the Nobel Prize.

latter generates the random variables from a given distribution (assumed known in priori) such as the Normal, chi-squared, Student-t and F-distributions, whereas the bootstrap drawn the random variables from the empirical distribution function. Essentially, it is this simple method but powerful so-called *plug-in principle* (Efron and Tibshirani, 1993) – the use of empirical distribution function as an estimate of their actual distribution – that enables the estimation of reliable parameters from finite samples. See also Davidson and MacKinnon (2004, Section 4.6) for more discussion that is reasonably accessible on the differences between Monte Carlo simulation and bootstrapping.

Bootstrap methods are not only useful in cases whereby certain assumptions (on the distribution of disturbance terms, for instance) are clearly not met or even unknown, but also crucial for the applications of new test statistics (for instance, nonlinear unit roots tests, panel unit root tests) in which the complete set of critical values is unavailable yet. As bootstrapping is almost unavoidable in today's research that involves statistics, it is important to understand how bootstrap works. In the light of this, this paper reviews the various ways of bootstrapping sample data in the literature. From this bootstrap sample, one may proceed to compute the statistics of interest, and whenever necessarily (such as in the case of computing bootstrap confidence intervals, critical values and the like), the bootstrap procedures discussed below may be replicated as many times as needed. Various ways of bootstrapping confidence intervals will be discussed in this paper, as illustration on the usefulness of bootstrap methods. To keep the idea simple and to ensure a smooth flow of discussion, this paper focuses on the illustration of computing the bootstrap linear regression coefficients.

The remainder of this paper is structured as follows. Section 2 describes the regression model that spins the bootstrap sample data. Two ways of bootstrapping cross-sectional data are respectively discussed in Sections 3 and 4. Time series model that generates time series data is discussed in Section 5. This is followed by Section 6, which deals with bootstrapping residuals method for time series data. Two alternatives, namely bootstrapping block method for time series data and bootstrapping moving block method for time series data are mentioned in Sections 7 and 8 respectively. One common application of bootstrapped sample is to compute bootstrap confidence intervals. Various ways of bootstrapping confidence intervals are discussed in Section 9. The final section contained concluding remarks.

2. The regression model

Consider the regression model:

$$y_i = f(x_{ji}, \beta_j) + \varepsilon_i \tag{1}$$

where y_i is the dependent variable, $f(x_{ji}, \beta_j)$ may be either linear or a nonlinear function of x_{ij} and β_j, where x_{ji} ($j = 1,...p$ and $i = 1,..., N$ where N is in turn the sample size) in turn is the ith observation of the jth explanatory variable which is fixed in repeating sampling and β_j is the unknown parameters to be estimated. ε_i is the error terms, which is traditionally assumed to be identically and independently distributed with commonly distribution G with zero mean and finite variance (σ^2).

One form of linear regression model is given by

$$f(x_{ji}, \beta_j) = \sum_{j=1}^{p} \beta_j x_{ji} + \varepsilon_i \text{ , } i = 1,\ldots, N \tag{2}$$

which sometimes may be more conveniently expressed in the following matrix form:

$$Y = X\beta + \varepsilon \tag{3}$$

where $Y = (Y_1,\ldots,Y_N)^{'}$ is a $N \times 1$ column vector of dependent variable, X denotes $N \times p$ matrix of explanatory variables, $\beta = (\beta_1,\ldots,\beta_p)^{'}$ is a $p \times 1$ column vector of unknown parameters and $\boldsymbol{\varepsilon} = (\varepsilon_1,\ldots,\varepsilon_N)^{'}$ is a $N \times 1$ column vector of N error terms.

The conventional least square (OLS) estimate $\hat{\beta}$ for β is given by

$$\hat{\beta} = \left(X^{'}X\right)^{-1} X^{'}Y \tag{4}$$

It is a textbook case that under the classical assumptions $\hat{\beta}$ has mean β and variance-covariance matrix $\sigma^2\left(X^{'}X\right)^{-1}$. Further, it can be shown that (see Freedman, 1981, for example) the asymptotical distribution of $\sqrt{N}(\hat{\beta} - \beta)$ is normal with mean zero and variance covariance given by $N\sigma^2\left(X^{'}X\right)^{-1}$.

In practice, however, both G and σ^2 are unknown. In such case, Efron (1979) proposes the use of empirical distribution function $\hat{G}$ of the centered residuals of Equation (1) in place of the unknown true G[2]. As such, $\hat{G}$ will have the desire expectation zero. Note that it is critical to center the residuals otherwise bootstrap may fail (Freedman, 1981, Davidson and MacKinnon, 2004). Besides, it is also common to rescale the empirical residuals by a factor of $\sqrt{N/(N-k)}$ to give the desired homogeneous variance (Davison and Hinkley, 1997, Section 6.2)[3, 4].

3. Bootstrapping residuals method for cross-sectional data

Efron (1979) originally proposes to bootstrap a random sample of size n drawn with replacement from the centered residuals such that places a probability of $1/N$ on each of the residuals by construction. Given $\hat{G}$, we can generate another sample of Y, known as the bootstrap sample, denoted by Y^* for the estimation of β. The resulting estimated β is known as bootstrap β, denoted by $\hat{\beta}^*$. This bootstrapping procedure is known as bootstrap residuals method and its algorithm is given as follows[5]:

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[2] This is especially applicable if no intercept is included in the regression model (Berkowit and Killian, 2000).

[3] If there is reason to suggest that the residuals variance is proportional to $x \in X$, then the i th empirical residuals may be rescaled by the square root of the i th observation of x (Hinkley, 1998).

[4] See also Masarotto (1990), Li and Maddala (1996), Kim (1999), Berkowitz and Killian (2000), Lam and Veall (2002), Alonso *et al*. (2003) and Christopher and Zhou (2003) for additional information on autoregression models.

[5] In cases when $f(x_{ji}, \beta_j)$ is called upon, all the bootstrap procedures discussed in this section are just valid provided the OLS estimation process is substituted by Nonlinear Least Squares (NLS) or Maximum Likelihood (ML) method.

Algorithm 1 (Bootstrapping Residuals Method for Cross-sectional Data)

1. From the observed sample of Y and X, obtain $\hat{\beta}$ as in Equation (4).
2. Compute empirical residuals $\hat{\varepsilon} = Y - f(X, \hat{\beta})$.
3. Transform the residuals $\hat{\varepsilon}$ of Step 2 into centered residuals $\tilde{\varepsilon}$. Practically, compute $\tilde{\varepsilon}_i = \hat{\varepsilon}_i - \frac{1}{N}\sum_{i=1}^{N}\hat{\varepsilon}_i$, where $\tilde{\varepsilon}_i$ and $\hat{\varepsilon}_i$ are the ith element of vectors $\tilde{\varepsilon}$ and $\hat{\varepsilon}$ respectively.
4. Randomly draw a sample ε^*, known as bootstrap residuals from $\tilde{\varepsilon}$ with replacement.
5. Generate Y^* from the equation $Y^* = f(x_{ji}, \hat{\beta}_j) + \varepsilon^* = X\hat{\beta} + \varepsilon^*$.
6. Regress Y^* on X to obtain OLS $\hat{\beta}^*$ given by $\hat{\beta}^* = (X'X)^{-1} X'Y^*$.

It has been shown that the distribution of the bootstrap will give the same asymptotical results as classical methods (Freedman, 1981). In particular, $\sqrt{N}(\hat{\beta}^* - \beta)$ which can be computed directly from the sample data approximates the distribution of $\sqrt{N}(\hat{\beta}^* - \beta)$, provided N is large and $\sigma^2 p \bullet trace(X^T X)^{-1}$ is small[6].

4. Bootstrap cases method

In Algorithm 1, Y^* values are bootstrapped from the preliminary estimated model (4). The idea is to fit a suitable model to the data, to recenter and resample the model's residuals and to generate new series by incorporating the resultant residuals into the fitted model. Thus, this method is also known as model-based bootstrap method. One may bootstrap cases of Y and X values using the bootstrapping cases method, instead of just the Y values as in the bootstrapping residuals method. This method does not rely on the prior information of the relationship between Y and X variables as well as assumption of

[6] The trace of a square matrix is the sum of its diagonal elements.

residuals. The procedures are much more simplified and time-saving. This bootstrap cases method involves resampling of cases of (Y, X) with replacement from the original series, and then computing the β^* from the resampled (Y^*, X^*) data. The algorithm is given as:

Algorithm 2 (Bootstrap Cases Method)

1. From the observed sample of Y and X, bootstrap N cases of (Y^*, X^*)[7]. As such, we have $(y_1^*, x_{11}^*, x_{21}^*, ..., x_{p1}^*)$, $(y_2^*, x_{12}^*, x_{22}^*, ..., x_{p2}^*)$, …, $(y_N^*, x_{1N}^*, x_{2N}^*, ..., x_{pN}^*)$.
2. Regress Y^* on X^* to obtain OLS $\hat{\beta}^*$ given by $\hat{\beta}^* = \left(X^{*'}X^*\right)^{-1} X^{*'}Y^*$.

As in the case of model-based bootstrap, the OLS $\hat{\beta}^*$ obtained in Algorithm 2 also approximate the properties of the true β (Davison and Hinkley, 1997, Section 8.2.2).

5. Time series model

Both Algorithms 1 and 2 are valid in the context of cross-sectional data only. However, it is widely known that most economic research involved the use of time series data. In this regards, the model-based bootstrap may be slightly modified to suit the requirements of time series, whereas the bootstrapping cases method is helpless as it destroys the serial dependency property of the time series. To circumvent this problem, bootstrapping block method has been introduced as an alternative the model-based bootstrap method.

[7] If there is only one X variable, we have pairs of (Y, X) and the method is accordingly regarded as bootstrap pairs method.

Suppose the series Y is determined endogeneously by its own lagged values, given by the following autoregressive representation:

$$y_t = f(x_{jt}, \beta_j) = y_0 + \sum_{i=1}^{p} \beta_i y_{t-i} + \varepsilon_{42t} \qquad i = 1,..., p,\ t = 1,...,T \tag{5}$$

where y_0 is an intercept parameter, which take the value of zero if the series y_t ($t = 1,...,T$ where T is the sample size) has a zero mean, otherwise it will be a non-zero constant. β_i's ($i = 1,..., p$ where p is the optimal autoregressive lag length) are known as autoregressive parameters.

Model (5) is called AR(p) model, which stands for autoregressive model of order p[8]. It may be compactly rewritten in the matrix form as:

$$A(L)(Y - \bar{Y}) = \varepsilon \tag{6}$$

where $\bar{Y} = \frac{1}{T}\sum_{t=1}^{T} y_t$ is the mean value of Y and $A(L) = (1 - \beta_1 L - \beta_2 L^2 - ... - \beta_p L^p)$ is an invertible polynomial[9] in the lag operator with L^i ($i = 1,..., p$) is the backshift operator such that $B^i\ y_t = y_{t-i}$.

[8] See Berkowitz and Killian (2000) for the extension of bootstrap algorithm for AR(p) model into the general autoregressive moving average ARMA(p, q) models of the form $A(L)(Y - \bar{Y}) = B(L)\varepsilon$, where $A(L)$ and $B(L)$ are invertible polynomials in the lag operator.

6. Bootstrapping residuals method for time series data

There are two methods of bootstrapping Y in the time series context. The simplest is analogous to the model-based bootstrap method as in Algorithm 1. This method is described in Algorithm 3 below:

Algorithm 3 (Bootstrapping Residuals Method for Time Series Data)

1. From the observed sample of Y, determine the order of the AR(p) model.
2. Estimate $\hat{y}_0$ and $\hat{\beta}_i$ ($i = 1,\ldots, p$) as in Equation (5).
3. Compute empirical residuals $\hat{\varepsilon} = Y - f(X, \hat{\beta})$.[10]
4. Transform the residuals $\hat{\varepsilon}_t$ of Step 2 into centered residuals $\tilde{\varepsilon}_t$ using the formula $\tilde{\varepsilon}_t = \hat{\varepsilon}_t - \frac{1}{T}\sum_{t=1}^{T}\hat{\varepsilon}_t$.
5. Randomly draw a sample ε_t^*, known as bootstrap residuals from $\tilde{\varepsilon}_t$ with replacement.
6. Initialized $y_1^* = y_2^* =,\ldots,= y_p^* = 0$ and generate Y^* from the relationship $Y^* = y_t^* = f(x_{ji}, \hat{\beta}_j) + \varepsilon_t^* = \hat{y}_0 + \sum_{i=1}^{p}\hat{\beta}_i y_{t-i} + \varepsilon_t^*$, for $t = p+1,\ldots,T$.
7. Using Y^*, reestimate Equation (5) to obtain OLS $\hat{\beta}^*$.

The series so generated may not be stationary due to initialization effect. To overcome this problem, it is advisable to generate extra so-called ‘burn-in’ observations (Davison and Hinkley, 1997). It is recommended that we should generate $m+T$ observations and keep only the last T observations for latter use by discarding the first m observations.

[9] An invertible polynomial satisfies the condition $A(L) \neq 0$ for all $|L| \leq 1$ (Brockwell and Davis, 1996, p. 84).

[10] One may rescale the empirical residuals by a factor of $\sqrt{(T-p)/(T-p-d)}$, where d stands for the number of estimated coefficients to give the desired variance (Freedman and Peters, 1984; Stine, 1987).

For instance, for T ranging from 30 to 1000, Lam and Veall (2002) use m = 100. Mean while Field and Zhou (2003) choose m = 100 for T ranging from 20 to 1000.

7. Bootstrapping block method for time series data

The second method bootstrapping a time series data is the bootstrapping block method. In this method, the data is divided into b non-overlapping blocks of length l, such that $T = bl$, yielding blocks $z_1, z_2, ..., z_b$, where $z_1 = (y_1, ..., y_l)'$, $z_2 = (y_{l+1}, ..., y_{2l})'$, …, $z_b = (y_{(b-1)l+1}, ..., y_T)'$. The idea is to take a bootstrap sample with equal probabilities $1/b$ from the z_i $(i = 1, ..., b)$, and then pool these end-to-end to form a new series.

Algorithm 4 (Bootstrapping Block Method for Time Series Data)

1. Divide observed sample of $Y = (y_1, ..., y_T)'$ into blocks $z_1 = (y_1, ..., y_l)'$, $z_2 = (y_{l+1}, ..., y_{2l})'$, …, $z_b = (y_{(b-1)l+1}, ..., y_T)'$. This yields b non-overlapping blocks of length l, such that $T = bl$[11].
2. Bootstrap $Y^* = (z_1^*, z_2^*, ..., z_b^*)'$ from $Y = (z_1, z_2, ..., z_b)'$.
3. Using Y^*, estimate Equation (5) to obtain OLS $\hat{\beta}^*$.

Bose (1998) shows that model-based method improves the asymptotic properties of the estimate of the distribution of OLS estimates in the autoregressive model, whereas Smith and Field (1993) show, through a simulation study, that the performance of the block bootstrap is quite dependent on the window size. The latter finding is in contrast to Künsch (1989) and Bühlman (1994) who have proven that the block bootstrap provides a valid approximation to the unknown distribution of the statistics given by smooth

[11] However, it is likely that T/l is not an integer. In such case the last block will be shorter than l.

functions of normalized sample mean. In this respect, Davison and Hinkley (1997, Section 8.2.3) argue that if the length of the blocks in the bootstrapping block method is long enough, the form of dependency in the original data will be preserved faithfully.

Thus the estimated OLS $\hat{\beta}^*$ will have approximately the same distribution as those obtained from the model-based bootstrap method. One may choose $l > p$ in order to optimally preserve the dependency.

8. Bootstrapping moving block method for time series data

Another way to preserve the original dependency is to divide the original sample into overlapping blocks. The resulting blocks will be something like $z_1 = (y_1,..., y_l)^{'}$, $z_2 = (y_2,..., y_{l+1})^{'}$, …, $z_{N-l+1} = (y_{N-1+1},...., y_T)^{'}$. Due to the way the sample is blocked, this method is referred to moving blocks bootstrap. Algorithm 4 is still applicable with the above modification incorporated in the first step. This modification ensures that each original observation has a more equal chance of appearing in the simulated series. It also has the advantage of removing the minor problem with the non-overlapping procedure that the last block is shorter than the rest if T/l is not an integer. The moving blocks bootstrap method is given as:

Algorithm 5 (Bootstrapping Moving Block Method for Time Series Data)

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1. Divide observed sample of $Y =$ $(y_1,..., y_T)^{'}$ into blocks $z_1 = (y_1,..., y_l)^{'}$, $z_2 = (y_2,..., y_{l+1})^{'}$, …, $z_{N-l+1} = (y_{N-1+1},...., y_T)^{'}$. This yields $b =$ $N - l + 1$ overlapping blocks of length l.
2. Bootstrap $Y^* = (z^*_1, z^*_2,..., z^*_b)^{'}$ from $Y = (z_1, z_2,..., z_b)^{'}$.

3. Using Y^*, estimate Equation (5) to obtain OLS $\hat{\beta}^*$.

9. Bootstrapping of confidence intervals

Practically, bootstrap methods have a variety of applications in various research fields, including economics. Besides being used to bootstrap regression models and OLS estimators as described above, it is most commonly applied in the bootstrapping of confidence intervals. Three ways of bootstrapping confidence intervals, namely (1) studentized-*t* confidence intervals, (2) bootstrap-*t* confidence intervals, and (3) bootstrap percentile confidence intervals are discussed as follows.

Suppose $\hat{\theta}$ is a consistent estimator of $\theta \sim N(\theta, \sigma^2(\theta))$. Then the studentized estimator is given as:

$$t = (\hat{\theta} - \theta) / \hat{\sigma}(\hat{\theta}) \tag{7}$$

where $\hat{\sigma}$ is the standard deviation of the estimator.

It is possible to approximate this studentized-*t* by:

$$t^* = (\hat{\theta}^* - \theta) / \hat{\sigma}^*(\hat{\theta}^*) \tag{8}$$

where parameters with * are obtained through bootstrap method.

As such, the $1-2\alpha$ studentized-*t* confidence intervals (based on *t* table) for θ may be represented by:

$$(\hat{\theta}-t_{1-\alpha}\hat{\sigma},\hat{\theta}+t_{\alpha}\hat{\sigma}) \tag{10}$$

where t_{λ} denotes the (100λ) th percentile of the *t* distribution (tabulated in most statistics or econometrics textbook) on $N-1$ degrees of freedom.

Lam and Veall (2002) illustrates via a simulation study the confidence intervals as in (10) is inaccurate when the disturbances are non-normally distributed and that this inaccuracy may increase rather than diminish as sample size grows[12]. This finding is similar to that of Godfrey and Orme (2000), who have demonstrated that non-normal distributions can adversely affect the performance of prediction error in the case of regression models. These empirical findings are in line with Efron and Tibshirani (1993), who point out that studentized-*t* confidence intervals fail to account for skewness in the underlying population or other error that can result when θ is not in the sample mean. The latter propose the bootstrap-*t* confidence intervals and bootstrap percentile confidence intervals, which adjust for these errors. The $1-2\alpha$ bootstrap-*t* confidence intervals are defined as:

$$(\hat{\theta}-t^{*}_{1-\alpha}\hat{\sigma}^{*},\hat{\theta}+t^{*}_{\alpha}\hat{\sigma}^{*}) \tag{11}$$

[12] Another noteworthy issue addressed in Lam and Veall (2002) is that autoregressive forecast model is important and in many cases more realistic than approaches that require known values for independent variables.

where t^*_λ is the λth value of the ranked (in the ascending order) t^*s values, which are obtained by repeatedly computing Equation (7) for B times. In order to yield robust confidence intervals, B may be as large as 999 or more (MacKinnon, 1999; Davidson and MacKinnon, 2004). Davidson and MacKinnon (2004, Section 4.6) mentioned that B has to be sufficiently large; otherwise the power of tests (the ability of the tests to reject the null hypothesis when it is false) will be negatively affected. Another problem that may arise due to insufficiently large sample is that the sequence of random numbers used to generate the bootstrap samples may influence the outcome of the tests. Thus, B=999 is recommended by Davidson and MacKinnon (2004) as a rule of thumb. The authors also suggested that if bootstrapping is inexpensive and the outcome is at all ambiguous, it is desirable to use a large number like 9999. Otherwise, it may be safe to use a value as small as 99.

In large sample, the coverage of bootstrap-*t* confidence intervals tends to be closer than the desired level (for example 10%) than the coverage of the studentized confidence intervals based on *t* table. The only shortcoming is that unlike the latter, which applies to all samples, the former applies only to the given sample. However, with the advancement in computer technology, it is possible to compute bootstrap-*t* confidence intervals for each sample one encounters (Efron and Tibshirani, 1993).

The bootstrap approximation in Equation (11) is at least approximately asymptotically pivotal (Efron and Tibshirani, 1993). A statistics is said to be asymptotically pivotal if its limiting distribution does not depend on any nuisance parameters (unknowns). In this regard, many statistics of interest based on AR(p) or ARMA(p, q) models are asymptotically normal and can be studentized to make them asymptotically pivotal; see, for example, Berkowitz and Killian (2000). Among others, Li and Maddala (1996) note that bootstrapping asymptotical pivotal statistics produces more accurate finite-sample confidence.

A faster way of obtaining bootstrap confidence intervals is to directly replicate θ^*, the bootstrap version of θ for B times. From the ranked (in the ascending order) θ^*, the $1-2\alpha$ bootstrap confidence intervals is constructed as:

$$(\theta^*_{\alpha}, \theta^*_{1-\alpha}) \tag{12}$$

where θ^*_{λ} is the (100λ)th empirical percentile of the θ^* values.

The intervals obtained this way are known as bootstrap percentile confidence intervals. It has the advantage of by-passing the process of computing t^* and thus is time-saving. Efron and Tibshirani (1993) show that if the distribution of θ^* is roughly normal, then the bootstrap percentile confidence intervals will converge to the studentized confidence intervals. However, as economic data are non-normal most of the time, the discrepancy between the two confidence intervals will be substantial in practice. Under such circumstance, it is obvious that the former is preferable as it follows the empirical distribution faithfully. In this respect, it is worth pointing out that the two above-mentioned bootstrap intervals may be obtained without having to make normal theory assumptions. In fact, one does not even have to know the asymptotic distribution of the statistics of interest!

Empirically, Lam and Veall (2002) find that bootstrap prediction intervals based on either the percentile principle or the percentile-*t* principle performs substantially better than the studentized confidence intervals. This finding is consistent with Kim (1999) who uses the percentile-*t* bootstrap method in studying prediction intervals for vector autoregression models. Apart from this, the small-sample effectiveness of bootstrap method in constructing hypothesis tests and confidence intervals for the parameters of cointegrating regressions with autocorrelated errors has also been demonstrated in Psaradakis (2001). See Davidson and MacKinnon (2004, Section 5.3) for overview on bootstrap confidence intervals, and Efron and Tibshirani (1993, Chapters 12 and 13) and Davison and Hinkley (1997, Chapter 5) for a comprehensive discussion regarding the theoretical aspects and algorithms of constructing these intervals.

10. Concluding remarks

Bootstrap is a computer-intensive method of statistical inference that can answer many real statistical questions. This paper reviews the various ways of bootstrapping data for cross-sectional and time series samples. From this bootstrap sample, one may proceed to compute the statistics of interest, for instance, OLS estimators, *t*-test statistic and *F*-test statistic. Perhaps, more importantly, the bootstrap sample may also be utilized to compute bootstrap confidence intervals, critical values, marginal significance values, and etc. Chong *et al*. (2008) and Liew and Yusuf (2008a), for instance, constructed the bootstrap percentile confidence intervals for the estimated intercept and coefficient of trend term in their study on income convergence. Other than constructing empirical confidence intervals, bootstrap method has also been popularly applied in evaluating the performance of existing test statistics such as unit root test in the works of Harris (1992), Angelis *et al.* (1997), Ohtani (2000), Psaradakis (2002) and van Giersbergen (2003). Besides, bootstrap methods are also commonly applied nowadays in newly proposed test statistics including linearity tests (Andrson *et al*., 1999; Chong *et al*., 2008), nonlinear unit root tests (Sarno, 2001; Sekioua, 2003; Chong *et al*., 2008; Liew *et al*. 2008) and nonlinear cointegration tests (Escribano and Mira, 2001; and Dufrénot *et al*., 2006). Another usefulness of the bootstrap methods is to compute the marginal significance values of the test statistics of interest (Henry, Olekalns and Summers, 2001; Basher and Haug, 2003; Sekioua, 2003; Liew, 2004, 2008; Liew and Yusuf, 2007, 2008; Liew *et al*. 2008)[13].

To conclude this section, bootstrap methods first introduced by Bradley Efron in 1979 is very useful in empirical research, especially when the test statistics of interest possess unknown asymptotical sample distribution. Note that modern researches, especially those involved in newly developed statistical testing procedures, rely heavily on bootstrap method for robustness of their findings. In fact, the usefulness of bootstrap has become

[13] Just to mention a few examples. Researches, especially those in the field of economics, finance and econometrics, that apply bootstrap methods are ample in recent literature.

very influential in research so much so that authors for textbooks on qualitative method like Davidson and MacKinnon (2004) have to discuss bootstrap in depth. It is noteworthy that economic researchers nowadays need to equip themselves with computer programming technology so as to keep pace with the competitive and yet evolving research environment. While bootstrapping may be familiar to some, it may be unfamiliar to many others. Interested readers may refer to Efron and Tibshirani (1993) and Davison and Hinkley (1997), among others, for comprehensive and reasonably accessible discussion on bootstrap methods and their applications.

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