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Abstract

Strategic games are considered where each player's total utility is the sum of local utilities obtained from the use of certain "facilities." All players using a facility obtain the same utility therefrom, which may depend on the identities of users and on their behavior. If a regularity condition is satisfied by every facility, then the game admits an exact potential; both congestion games and games with structured utilities are included in the class and satisfy that condition. Under additional assumptions the potential attains its maximum, which is a Nash equilibrium of the game.
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Key words: Potential game; Congestion game; Game with structured utilities; Game of social interactions; Additive aggregation

1 Introduction

When Monderer and Shapley (1996) introduced the notion of a *potential game*, the main example they had in mind were Rosenthal's (1973) *congestion games*. Their Theorems 3.1 and 3.2 showed that a finite game admits an exact potential if and only if it can be represented as a congestion game (the sufficiency part was implicit in Rosenthal's reasoning). An alternative, more transparent proof was given in Voorneveld et al. (1999, Theorem 3.3).

Kukushkin (2007) introduced *games with structured utilities*, in a sense, "dual" to congestion games; the players there do not choose *which* facilities to use, only *how* to use facilities from a fixed list. The idea of such a structure of utility functions can be traced back to Germeier and Vatel' (1974), although the local utilities in that paper were aggregated with the minimum function. Theorem 5 from Kukushkin (2007) showed that a strategic game admits an exact potential if and only if it can be represented as a game with structured utilities.

Thus, two different classes of potential games were considered in Kukushkin (2007); one universal for finite potential games, the other for all of them. The possibility to combine both constructions was not discussed, actually, was overlooked altogether.

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Le Breton and Weber (2011) described a class of strategic games which possess Nash equilibria because they admit an exact potential which, in its turn, attains its maximum. There was some similarity with both constructions, and with their potentials as well; however, those games need not belong to either class.

The motivation for this paper was purely technical, one could say, aesthetical. Our main objective is to figure out how those two constructions generating potential games could be combined into a single, unified construction. We define a general class of strategic games where the players are able to choose both which facilities to use and how to use them. Then we formulate conditions ensuring that such a game admits an exact potential; naturally, they are satisfied for both congestion games and games with structured utilities, as well as games of Le Breton and Weber. Since those conditions are formulated independently for every facility, a necessity result becomes obtainable: if a facility does not satisfy them, adding it to a potential game may destroy that property.

Our basic construction is described in the following section. In Section 3, the key definitions of a regular facility and a regular game are given; Theorem 1 asserts the presence of an exact potential in every regular game. Theorem 2 in Section 4 shows kind of necessity of regularity for this property.

In Section 5, the question of when the potential attains its maximum is addressed; since the strategy sets in our games may be infinite, this question is not trivial. We formulate a list of assumptions ensuring the upper semiconinuity of the potential, and hence the existence of a Nash equilibrium (Theorem 3). The proof of the theorem is in Section 6.

Section 7 demonstrates that the Le Breton–Weber construction is, indeed, a particular case of ours. In Section 8, we show that every game from the class considered by Harks et al. (2011) can be naturally represented as one from our class. Section 9 summarizes the message of the paper.

2 Basic definitions

A *strategic game* Γ is defined by a finite set N of *players*, and, for each $i \in N$, a set X_i of *strategies* and a real-valued *utility function* u_i on the set $X_N := \prod_{i \in N} X_i$ of *strategy profiles*. We denote $\mathcal{N} := 2^N \setminus \{\emptyset\}$ and $X_I := \prod_{i \in I} X_i$ for each $I \in \mathcal{N}$. Given $i, j \in N$, we use notation X_{-i} instead of $X_{N \setminus \{i\}}$ and X_{-ij} instead of $X_{N \setminus \{i,j\}}$.

A function $P : X_N \to \mathbb{R}$ is an *exact potential* of Γ (Monderer and Shapley, 1996) if

$$u_i(y_N) - u_i(x_N) = P(y_N) - P(x_N) \tag{1}$$

whenever $i \in N$, $y_N, x_N \in X_N$, and $y_{-i} = x_{-i}$. If $x_N^0 \in X_N$ maximizes P over X_N, then, obviously, x_N^0 is a Nash equilibrium.

A *game with* (*additive*) *common local utilities* (a *CLU game*) may have an arbitrary finite set N of players and arbitrary sets of strategies X_i ($i \in N$), whereas the utilities are defined by the following construction. First of all, there is a set A of *facilities*; we denote $\mathcal{B}$ the set of all (nonempty) finite subsets of A. For each $i \in N$, there is a mapping $\mathrm{B}_i \colon X_i \to \mathcal{B}$ describing what facilities player i

uses having chosen x_i. Every strategy profile x_N determines *local utilities* at all facilities $\alpha \in \mathrm{A}$; each player's *total utility* is the sum of local utilities over chosen facilities. The exact definitions need plenty of notations.

For every $\alpha \in \mathrm{A}$, we denote $I_\alpha^- := \{i \in N \mid \forall x_i \in X_i\, [\alpha \in \mathrm{B}_i(x_i)]\}$ and $I_\alpha^+ := \{i \in N \mid \exists x_i \in X_i\, [\alpha \in \mathrm{B}_i(x_i)]\}$; without restricting generality, we may assume $I_\alpha^+ \neq \emptyset$. For each $i \in I_\alpha^+$, we denote $X_i^\alpha := \{x_i \in X_i \mid \alpha \in \mathrm{B}_i(x_i)\}$; if $i \in I_\alpha^-$, then $X_i^\alpha = X_i$. Then we set $\mathcal{I}_\alpha := \{I \in \mathcal{N} \mid I_\alpha^- \subseteq I \subseteq I_\alpha^+\}$ and $\Xi_\alpha := \{\langle I, x_I\rangle \mid I \in \mathcal{I}_\alpha \;\&\; x_I \in X_I^\alpha\}$. The *local utility function* at $\alpha \in \mathrm{A}$ is $\varphi_\alpha \colon \Xi_\alpha \to \mathbb{R}$. For every $\alpha \in \mathrm{A}$ and $x_N \in X_N$, we denote $I(\alpha, x_N) := \{i \in N \mid \alpha \in \mathrm{B}_i(x_i)\}$; obviously, $I_\alpha^- \subseteq I(\alpha, x_N) \subseteq I_\alpha^+$. The *total utility function* of each player i is

$$u_i(x_N) := \sum_{\alpha \in \mathrm{B}_i(x_i)} \varphi_\alpha(I(\alpha, x_N), x_{I(\alpha,x_N)}). \tag{2}$$

3 Regularity

We call a facility $\alpha \in \mathrm{A}$ *regular* if these two conditions are satisfied:

1) whenever $i \notin J \subset N$, $I_\alpha^+ \neq J \cup \{i\} \in \mathcal{I}_\alpha$, $x_i, y_i \in X_i^\alpha$, and $x_J \in X_J^\alpha$, there holds

$$\varphi_\alpha(J \cup \{i\}, (x_J, x_i)) = \varphi_\alpha(J \cup \{i\}, (x_J, y_i)); \tag{3a}$$

2) whenever $J \subset N$ and $i, j \in N \setminus J$ are such that $i \neq j$, $J \cup \{i\} \in \mathcal{I}_\alpha \ni J \cup \{j\}$, and $x_{J\cup\{i,j\}} \in X_{J\cup\{i,j\}}^\alpha$, there holds

$$\varphi_\alpha(J \cup \{i\}, x_{J\cup\{i\}}) = \varphi_\alpha(J \cup \{j\}, x_{J\cup\{j\}}) \tag{3b}$$

($J = \emptyset$ is allowed in both conditions, in which case the term x_J should be just ignored).

For every $\alpha \in \mathrm{A}$, we denote $n^-(\alpha) := \min_{I \in \mathcal{I}_\alpha} \#I = \max\{1, \#I_\alpha^-\}$.

Proposition 1. *A facility $\alpha \in \mathrm{A}$ is regular if and only if there is a real-valued function $\psi_\alpha(\cdot)$ defined for integer m between $n^-(\alpha)$ and $\#I_\alpha^+ - 1$ such that*

$$\varphi_\alpha(I, x_I) = \psi_\alpha(\#I) \tag{4}$$

whenever $I \in \mathcal{I}_\alpha$, $I \neq I_\alpha^+$, and $x_I \in X_I^\alpha$.

In other words: whenever a regular facility α is not used by all potential users, neither the identities of the users, nor their strategies matter, only the number of users.

Proof. The implication "if" is obvious. Let $\alpha \in \mathrm{A}$ be a regular facility and m be an integer between $n^-(\alpha)$ and $\#I_\alpha^+ - 1$. First of all, the existence of such m implies that $\#I_\alpha^+ > n^-(\alpha)$. Further, whenever $I \in \mathcal{I}_\alpha$, $I \neq I_\alpha^+$, and $x_I, y_I \in X_I^\alpha$, we can, picking, one by one, $i \in I$ and replacing x_i with y_i, obtain, by (3a), that $\varphi_\alpha(I, x_I) = \varphi_\alpha(I, y_I)$, i.e., the choice of strategies does not matter indeed.

Let us show the irrelevance of the identities of users. If $I_\alpha^- \neq \emptyset$ and $m = \#I_\alpha^-$, we define $\psi(m) := \varphi_\alpha(I_\alpha^-, x_{I_\alpha^-})$, which does not depend on $x_{I_\alpha^-} \in X^\alpha_{I_\alpha^-}$ by the argument of the preceding paragraph. There is no other $I \in \mathcal{I}_\alpha$ with the same $\#I$. If $I_\alpha^- = \emptyset$, we set $\psi(1) := \varphi_\alpha(\{i\}, x_i)$, which does not depend on $i \in I_\alpha^+$ by (3b) with $J = \emptyset$, or on $x_i \in X_i^\alpha$ by the argument of the preceding paragraph again.

Finally, supposing that $I, J \in \mathcal{I}_\alpha$, $n^-(\alpha) < \#I = \#J < \#I_\alpha^+$, $x_I \in X_I^\alpha$ and $y_J \in X_J^\alpha$, we have to prove that $\varphi_\alpha(I, x_I) = \varphi_\alpha(J, y_J)$. Obviously, there is a one-to-one correspondence between $J \setminus I := \{j_1, \ldots, j_k\}$ and $I \setminus J := \{i_1, \ldots, i_k\}$. Consecutively applying (3b), we obtain:

$$\varphi_\alpha(I, x_I) = \varphi_\alpha((I \cap J) \cup \{j_1, i_2, \ldots, i_k\}, (x_{(I\cap J)\cup\{i_2,\ldots,i_k\}}, y_{j_1})) = \\ \varphi_\alpha((I \cap J) \cup \{j_1, j_2, i_3, \ldots, i_k\}, (x_{(I\cap J)\cup\{i_3,\ldots,i_k\}}, y_{\{j_1,j_2\}})) = \cdots = \varphi_\alpha(J, y_J).$$

Now we can set $\psi(m) := \varphi_\alpha(I, x_I)$ for an arbitrary $I \in \mathcal{I}_\alpha$ with $\#I = m$ and an arbitrary $x_I \in X_I^\alpha$, and have (4) satisfied. $\square$

We call a CLU game *regular* if so is every facility. It is instructive to check that both congestion games and games with structured utilities are regular. In the first case, (4) holds for all $I \in \mathcal{I}_\alpha$, even for $I = I_\alpha^+$; in the second case, conversely, $I_\alpha^- = I_\alpha^+$ for each facility α and hence (4) is not required at all.

Theorem 1. *Every regular CLU game admits an exact potential.*

Proof. Given $x_N \in X_N$, we denote $\mathrm{A}(x_N) := \{\alpha \in \mathrm{A} \mid I(\alpha, x_N) \neq \emptyset\}$ and $\mathrm{A}^+(x_N) := \{\alpha \in \mathrm{A} \mid \#I(\alpha, x_N) > n^-(\alpha)\}$ $[\subseteq \mathrm{A}(x_N)]$; since N and each $\mathrm{B}_i(x_i)$ are finite, $\mathrm{A}(x_N)$ is finite too. Now we define our potential function in this way:

$$P(x_N) := \sum_{\alpha \in \mathrm{A}(x_N)} \varphi_\alpha(I(\alpha, x_N), x_{I(\alpha,x_N)}) + \sum_{\alpha \in \mathrm{A}^+(x_N)} \sum_{m=n^-(\alpha)}^{\#I(\alpha,x_N)-1} \psi_\alpha(m). \tag{5}$$

Given $i \in N$ and $x_{-i} \in X_{-i}$, we denote $I_{-i}(\alpha, x_{-i}) := \{j \in N \setminus \{i\} \mid \alpha \in \mathrm{B}_j(x_j)\}$, $\mathrm{A}_{-i}(x_{-i}) := \{\alpha \in \mathrm{A} \mid I_{-i}(\alpha, x_{-i}) \neq \emptyset\}$ and $\mathrm{A}^+_{-i}(x_{-i}) := \{\alpha \in \mathrm{A} \mid \#I_{-i}(\alpha, x_{-i}) > n^-(\alpha)\}$ $[\subseteq \mathrm{A}_{-i}(x_{-i})]$. Then we define these auxiliary functions $Q_{-i} \colon X_{-i} \to \mathbb{R}$ $(i \in N)$:

$$Q_{-i}(x_{-i}) := \sum_{\alpha \in \mathrm{A}_{-i}(x_{-i})} \varphi_\alpha(I_{-i}(\alpha, x_{-i}), x_{I_{-i}(\alpha,x_{-i})}) + \sum_{\alpha \in \mathrm{A}^+_{-i}(x_N)} \sum_{m=n^-(\alpha)}^{\#I_{-i}(\alpha,x_{-i})-1} \psi_\alpha(m). \tag{6}$$

Once we show that

$$P(x_N) = u_i(x_N) + Q_{-i}(x_{-i}) \tag{7}$$

for all $i \in N$ and $x_N \in X_N$, Theorem 2.1 of Voorneveld et al. (1999) will imply that P is an exact potential.

Whenever $\alpha \notin \mathrm{B}_i(x_i)$, we have $I_{-i}(\alpha, x_{-i}) = I(\alpha, x_N)$; therefore, this α brings to $Q_{-i}(x_{-i})$ the same contribution as to $P(x_N)$, while no contribution at all to $u_i(x_N)$. For every $\alpha \in \mathrm{B}_i(x_i)$, we have $I_{-i}(\alpha, x_{-i}) = I(\alpha, x_N) \setminus \{i\}$ and hence $\#I_{-i}(\alpha, x_{-i}) = \#I(\alpha, x_N) - 1$. If $I(\alpha, x_N) = \{i\}$, then this α brings to $u_i(x_N)$ the same contribution, $\varphi_\alpha(\{i\}, x_i)$, as to $P(x_N)$, while no contribution at all to $Q_{-i}(x_{-i})$. If $I(\alpha, x_N) = \{i, j\}$, then this α contributes $\varphi_\alpha(\{i, j\}, (x_i, x_j))$ to $u_i(x_N)$, contributes $\varphi_\alpha(\{j\}, x_j)$ to $Q_{-i}(x_{-i})$, and contributes $\varphi_\alpha(\{i, j\}, (x_i, x_j)) + \psi_\alpha(1)$ to $P(x_N)$. Since $\varphi_\alpha(\{j\}, x_j) = \psi_\alpha(1)$ by Proposition 1, total contributions coincide again. Finally, if $\#I(\alpha, x_N) > 2$, we argue virtually in the same way as in the previous case of $\#I(\alpha, x_N) = 2$. Equality (7) being satisfied, Theorem 1 is proven. □

4 Necessity of regularity

Let a finite set N of players be fixed. An *autonomous facility* α is defined by two subsets $I_\alpha^- \subseteq I_\alpha^+ \in \mathcal{N}$ [I_α^- may be empty], a set X_i^α of relevant strategies for each $i \in I_\alpha^+$, and a local utility function $\varphi_\alpha \colon \Xi^\alpha \to \mathbb{R}$, where $\mathcal{I}_\alpha := \{I \in \mathcal{N} \mid I_\alpha^- \subseteq I \subseteq I_\alpha^+\}$ and $\Xi^\alpha := \{\langle I, x_I^\alpha \rangle \mid I \in \mathcal{I}_\alpha \ \&\ x_I^\alpha \in X_I^\alpha\}$, exactly as in Section 2. We call an autonomous facility α *regular* if it satisfies the same conditions (3).

Let α be an autonomous facility, and let Γ be a CLU game with the same set N, a finite set A such that $\alpha \notin \mathrm{A}$, and $X_i \cap X_i^\alpha = \emptyset$ for each $i \in N$. An *extension of* Γ *with* α is a strategic game Γ^* satisfying these conditions: $N^* = N$; $\mathrm{A}^* = \mathrm{A} \cup \{\alpha\}$; for each $i \in N$, $X_i^* = X_i \cup X_i^\alpha$ if $i \in I_\alpha^+$ and $X_i^* := X_i$ otherwise, $\mathrm{B}_i^*(x_i) = \mathrm{B}_i(x_i)$ for each $x_i \in X_i$, and, for each $x_i^\alpha \in X_i^\alpha$, there is $\sigma_i(x_i^\alpha) \in X_i$ such that $\mathrm{B}_i^*(x_i^\alpha) = \{\alpha\} \cup \mathrm{B}_i(\sigma_i(x_i^\alpha))$; whenever $I \in \mathcal{I}_\alpha$ and $x_I^\alpha \in X_I^\alpha$, there holds $\varphi_\alpha^*(I, x_I^\alpha) = \varphi_\alpha(I, x_I^\alpha)$; whenever $\beta \in \mathrm{A}$, $I \in \mathcal{I}_\beta$, $x_I \in X_I^{*\beta}$, and $J = \{i \in I \mid x_i \in X_i^\alpha\}$, there holds $\varphi_\beta^*(I, x_I) = \varphi_\beta(I, (x_{I \setminus J}, \sigma_J(x_J)))$.

Theorem 2. *For every autonomous facility α the following statements are equivalent:*

1. *α is regular.*
2. *Whenever Γ^* is an extension with α of a regular CLU game Γ, Γ^* admits an exact potential.*
3. *Whenever Γ^* is an extension with α of a congestion game Γ, Γ^* admits an exact potential.*
4. *Whenever Γ^* is an extension with α of a game with structured utilities Γ, Γ^* admits an exact potential.*

Proof. The implication Statement 1 ⇒ Statement 2 immediately follows from Theorem 1; the implications Statement 2 ⇒ Statement 3 and Statement 2 ⇒ Statement 4 are trivial. We only have to show the implications Statement 3 ⇒ Statement 1 and Statement 4 ⇒ Statement 1; so let Statement 3 hold.

Claim 2.1. *Let $i, j \in I_\alpha^+$, $i \notin I \in \mathcal{I}_\alpha$, $j \in I$, $x_I^\alpha \in X_I^\alpha$ and $y_j^\alpha \in X_j^\alpha$. Then $\varphi_\alpha(I, x_I^\alpha) = \varphi_\alpha(I, (x_{I \setminus \{j\}}^\alpha, y_j^\alpha))$.*

Proof of Claim 2.1. Let us consider a congestion game Γ with the same set N of players, a singleton set of facilities $\mathrm{A} := \{\beta\}$, and a singleton set of strategies $X_h := \{x_h^\beta\}$ with $\mathrm{B}_h(x_h^\beta) := \{\beta\}$ for each $h \in N$, and an arbitrary constant (in lieu of a function) $\varphi_\beta(N, x_N^\beta) = \psi_\beta(\#N)$. We define an extension Γ^* of Γ with α by: $N^* := N$; $\mathrm{A}^* := \{\alpha, \beta\}$; $X_h^* := \{x_h^\beta\}$ for each $h \in N \setminus I_\alpha^+$; $X_h^* := \{x_h^\beta\} \cup X_h^\alpha$ for each $h \in I_\alpha^+$; $\mathrm{B}_h^*(x_h^\beta) = \{\beta\}$; $\mathrm{B}_h^*(x_h^\alpha) = \mathrm{A}^*$ for each $x_h^\alpha \in X_h^\alpha$; $\varphi_\alpha^*(J, x_J^\alpha) = \varphi_\alpha(J, x_J^\alpha)$ for every $J \in \mathcal{I}_\alpha$ and $x_J^\alpha \in X_J^\alpha$; $\varphi_\beta^*(N, x_N^\beta) = \varphi_\beta(N, x_N^\beta)$.

Since we assumed Statement 3 to hold, Γ^* admits an exact potential; hence so does every subgame. As was noted by Monderer and Shapley (1996, Theorem 2.8), it is enough to consider 2×2 subgames. We leave players i and j with two strategies each: $\{x_i^\alpha, x_i^\beta\}$ and $\{x_j^\alpha, y_j^\alpha\}$, respectively, fixing strategies for all other players: x_h^α for $h \in I$ and x_h^β for $h \notin I$. Introducing auxiliary notations, $u^\beta := \varphi_\beta(N, x_N^\beta)$, $v_x^\alpha := \varphi_\alpha(I \cup \{i\}, x_{I\cup\{i\}}^\alpha)$, $v_y^\alpha := \varphi_\alpha(I \cup \{i\}, (x_{I\cup\{i\}\setminus\{j\}}^\alpha, y_j^\alpha))$, $u_x^\alpha := \varphi_\alpha(I, x_I^\alpha)$, and $u_y^\alpha := \varphi_\alpha(I, (x_{I\setminus\{j\}}^\alpha, y_j^\alpha))$, we obtain the following matrix of the resulting subgame:

$$\begin{array}{cc} & \begin{array}{cc} \qquad x_j^\alpha \qquad & \qquad y_j^\alpha \end{array} \\ \begin{array}{c} x_i^\alpha \\ x_i^\beta \end{array} & \begin{bmatrix} \langle v_x^\alpha + u^\beta, v_x^\alpha + u^\beta \rangle & \langle v_y^\alpha + u^\beta, v_y^\alpha + u^\beta \rangle \\ \langle u^\beta, u_x^\alpha + u^\beta \rangle & \langle u^\beta, u_y^\alpha + u^\beta \rangle \end{bmatrix} \end{array}.$$

Straightforward calculations show that $P(x_i^\beta, y_j^\alpha, x_{-ij}) - P(x_i^\beta, x_j^\alpha, x_{-ij}) = [P(x_i^\alpha, x_j^\alpha, x_{-ij}) - P(x_i^\beta, x_j^\alpha, x_{-ij})] + [P(x_i^\alpha, y_j^\alpha, x_{-ij}) - P(x_i^\beta, x_j^\alpha, x_{-ij})] + [P(x_i^\beta, y_j^\alpha, x_{-ij}) - P(x_i^\alpha, y_j^\alpha, x_{-ij})] = v_x^\alpha + (v_y^\alpha - v_x^\alpha) - v_y^\alpha = 0$. Therefore, $\varphi_\alpha(I, x_I^\alpha) = u_j(x_i^\beta, y_j^\alpha, x_{-ij}) = u_j(x_i^\beta, x_j^\alpha, x_{-ij}) = \varphi_\alpha(I, (x_{I\setminus\{j\}}^\alpha, y_j^\alpha))$. In other words, (3a) is established. $\square$

Claim 2.2. *Let $i, j \in I \in \mathcal{I}_\alpha$, $I_\alpha^- \subseteq I\setminus\{i, j\}$, and $x_I^\alpha \in X_I^\alpha$. Then $\varphi_\alpha(I\setminus\{i\}, x_{I\setminus\{i\}}^\alpha) = \varphi_\alpha(I\setminus\{j\}, x_{I\setminus\{j\}}^\alpha)$.*

Proof of Claim 2.2. We consider the same congestion game Γ used in the proof of Claim 2.1 and the same extension Γ^* of Γ with α. This time, we consider a 2×2 subgame where players i and j have two strategies each: $\{x_i^\alpha, x_i^\beta\}$ and $\{x_j^\alpha, x_j^\beta\}$, respectively, while the strategies of all other players are fixed: x_h^α for $h \in I$ and x_h^β for $h \notin I$.

Again, this subgame must admit an exact potential. Introducing auxiliary notations, $u^\beta := \varphi_\beta(N, x_N^\beta)$, $u^\alpha := \varphi_\alpha(I, x_I^\alpha)$, $v_i^\alpha := \varphi_\alpha(I \setminus \{i\}, x_{I\setminus\{i\}}^\alpha)$ and $v_j^\alpha := \varphi_\alpha(I \setminus \{j\}, x_{I\setminus\{j\}}^\alpha)$, we obtain the following matrix:

$$\begin{array}{cc} & \begin{array}{cc} \qquad x_j^\alpha \qquad & \qquad x_j^\beta \end{array} \\ \begin{array}{c} x_i^\alpha \\ x_i^\beta \end{array} & \begin{bmatrix} \langle u^\alpha + u^\beta, u^\alpha + u^\beta \rangle & \langle v_j^\alpha + u^\beta, u^\beta \rangle \\ \langle u^\beta, v_i^\alpha + u^\beta \rangle & \langle u^\beta, u^\beta \rangle \end{bmatrix} \end{array}.$$

Now we have $0 = [P(x_i^\alpha, x_j^\alpha, x_{-ij}) - P(x_i^\beta, x_j^\alpha, x_{-ij})] + [P(x_i^\alpha, x_j^\beta, x_{-ij}) - P(x_i^\alpha, x_j^\alpha, x_{-ij})] + [P(x_i^\beta, x_j^\beta, x_{-ij}) - P(x_i^\alpha, x_j^\beta, x_{-ij})] + [P(x_i^\beta, x_j^\alpha, x_{-ij}) - P(x_i^\beta, x_j^\beta, x_{-ij})] = u^\alpha - u^\alpha - v_j^\alpha + v_i^\alpha = v_i^\alpha - v_j^\alpha$. Therefore, $\varphi_\alpha(I \setminus \{i\}, x_{I\setminus\{i\}}^\alpha) = v_i^\alpha = v_j^\alpha = \varphi_\alpha(I \setminus \{j\}, x_{I\setminus\{j\}}^\alpha)$. In other words, (3b) is established. $\square$

The proof of the implication Statement 4 $\Rightarrow$ Statement 1 is now straightforward: the congestion game Γ used in the proofs of Claims 2.2 and 2.1 can as well be perceived as a game with structured utilities. Theorem 2 is proven. $\square$

Theorem 2 takes it for granted that the players add up their local utilities. Actually, the necessity (in a sense) of addition was showed in Kukushkin (2007): If the players may aggregate local utilities with arbitrary (continuous and strictly increasing) functions, then the existence of an exact potential is ensured regardless of other characteristics of the game only if the players sum up local utilities; that statement remains valid when attention is restricted to congestion games (Theorem 2 of Kukushkin, 2007), or to games with structured utilities (Theorem 4).

Strictly speaking, those theorems do not exclude the possibility that the aggregation of local utilities with some other, *non-strictly* increasing functions might also ensure the existence of an exact potential, but there is no reason to expect anything interesting here. On the other hand, the minimum aggregation, as envisaged by Germeier and Vatel' (1974), ensures the acyclicity of coalitional improvements and hence the existence of a strong Nash equilibrium (Harks et al., 2013; Kukushkin, 2014).

5 The existence of Nash equilibrium

To ensure that the potential P attains a maximum, some additional assumptions are needed. The simplest approach would be to have P upper semicontinuous and X_N compact. A certain degree of subtlety is required, however, as was shown by Le Breton and Weber (2011) even in a particular case.

Assumption 1. *The set of facilities* A *and each strategy set* X_i *are metric spaces; each mapping* B_i *is continuous in the Hausdorff metric on the target; for every* $\alpha \in \mathrm{A}$ *and* $I \in \mathcal{I}_\alpha$, *the function* $\varphi_\alpha(I, \cdot)\colon X_I \to \mathbb{R}$ *is upper semicontinuous.*

Henceforth, we assume each set X_I ($I \in \mathcal{N}$) to be endowed with the maximum metrics. For each $i \in N$ and $m \in \mathbb{N}$, we denote $X_i^m := \{x_i \in X_i \mid \#\mathrm{B}_i(x_i) = m\}$.

Assumption 2. *For each* $i \in N$ *and* $m \in \mathbb{N}$, *either* $X_i^m = \emptyset$ *or* X_i^m *is a compact subset of* X_i.

Assumption 3. *For each* $i \in N$, $X_i^m \neq \emptyset$ *only for a finite number of* $m \in \mathbb{N}$.

Assumptions 1 – 3 have technical implications useful in the following.

Lemma 1. *Let* $i \in N$, $x_i^k \in X_i$ *for all* $k \in \mathbb{N}$, *and* $x_i^k \to x_i^\omega \in X_i$; *let open neighborhoods* O_α *of* $\alpha \in \mathrm{B}_i(x_i^\omega)$ *be such that* $O_\alpha \cap O_\beta = \emptyset$ *whenever* $\alpha \neq \beta$. *Then* $\#\mathrm{B}_i(x_i^k) = \#\mathrm{B}_i(x_i^\omega)$ *and* $\#(\mathrm{B}_i(x_i^k) \cap O_\alpha) = 1$ *for all* $\alpha \in \mathrm{B}_i(x_i^\omega)$ *and all* $k \in \mathbb{N}$ *large enough.*

Proof. By Assumption 3, there is a finite number of possible values of $\#\mathrm{B}_i(x_i^k)$; therefore, we must have $x_i^k \in X_i^m$ for some $m \in \mathbb{N}$, $m \neq 0$, and an infinite number of $k \in \mathbb{N}$. Since X_i^m is compact by

Assumption 2, and hence closed in X_i, we have $x_i^\omega \in X_i^m$ too. It follows immediately that such an m must be unique, i.e., $x_i^k \in X_i^m$ for all $k \in \mathbb{N}$ large enough.

By Assumption 1, we have $\mathrm{B}_i(x_i^k) \to \mathrm{B}_i(x_i^\omega)$. Therefore, for each $k \in \mathbb{N}$ large enough and for each $\alpha \in \mathrm{B}_i(x_i^\omega)$, there is $\alpha^k \in \mathrm{B}_i(x_i^k) \cap O_\alpha$. Since $O_\alpha \cap O_\beta = \emptyset$ whenever $\alpha \neq \beta$, and $\#\mathrm{B}_i(x_i^k) = \#\mathrm{B}_i(x_i^\omega)$, we must have $\#(\mathrm{B}_i(x_i^k) \cap O_\alpha) = 1$ indeed. □

For every $\alpha \in \mathrm{A}$, we denote

$$I_\alpha^\circ := \left\{ i \in I_\alpha^+ \mid \exists O \left[(O \text{ is an open subset of A}) \ \& \ \alpha \in O \ \& \ \forall \beta \in O \left[i \in I_\beta^+ \Rightarrow \beta = \alpha \right] \right] \right\}; \tag{8}$$

loosely speaking, I_α° is the set of players in whose strategy sets α is topologically isolated.

Lemma 2. *For every $\alpha \in \mathrm{A}$, there holds $I_\alpha^- \subseteq I_\alpha^\circ$.*

Proof. Supposing the contrary, let $i \in I_\alpha^- \setminus I_\alpha^\circ$. By the negation of (8), there is a sequence $\alpha^k \to \alpha$ and a sequence $x_i^k \in X_i$ such that $\alpha^k \in \mathrm{B}_i(x_i^k)$ and $\alpha^k \neq \alpha$ for each $k \in \mathbb{N}$. Since X_i is compact by Assumptions 2 and 3, we may assume that $x_i^k \to x_i^\omega \in X_i$. Since $\mathrm{B}_i(x_i^\omega)$ is finite, there is an open neighborhood O_β of each $\beta \in \mathrm{B}_i(x_i^\omega)$ such that $O_\gamma \cap O_\beta = \emptyset$ whenever $\gamma \neq \beta \in \mathrm{B}_i(x_i^\omega)$. Moreover, we may assume that $\alpha^k \in O_\alpha$ for each $k \in \mathbb{N}$.

Now we have $\alpha^k \in \mathrm{B}_i(x_i^k)$ by the choice of α^k and $\alpha \in \mathrm{B}_i(x_i^k)$ since $i \in I_\alpha^-$. Since $\alpha, \alpha^k \in O_\alpha$, we have $\#(\mathrm{B}_i(x_i^k) \cap O_\alpha) \geq 2$, which contradicts Lemma 1. □

Our final assumption combines some sorts of upper semicontinuity (of φ_α "in α") and monotonicity (of φ_α "in I").

Assumption 4. *For every $\alpha \in \mathrm{A}$, $I \in \mathcal{I}_\alpha$, and $\varepsilon > 0$, there is $\delta > 0$ such that:*

$$\varphi_\alpha(I, x_I) > \varphi_\beta(J, y_J) - \varepsilon \tag{9}$$

whenever $\beta \in \mathrm{A} \setminus \{\alpha\}$, $J \in \mathcal{I}_\beta$, $x_I \in X_I^\alpha$, $y_J \in X_J^\beta$, $J \subseteq I \setminus I_\alpha^\circ$, and the distances between α and β in A as well as between x_J and y_J in X_J are less than δ.

If A is finite as, e.g., in a game with structured utilities or in a congestion game, then Assumption 4 holds vacuously since $I_\alpha^\circ = I_\alpha^+$ and hence no $J \in \mathcal{N}$ could satisfy the conditions.

Theorem 3. *Every regular CLU game satisfying Assumptions 1, 2, 3, and 4 possesses a (pure strategy) Nash equilibrium.*

The proof is deferred to Section 6.

It is impossible to argue that the assumptions imposed in Theorem 3 are *necessary* in a proper sense. After all, neither upper semicontinuity, nor compactness are necessary for a function to attain its maximum. Nonetheless, dropping any one of them makes the theorem wrong. There is no need to discuss Assumption 1, but for the three others, appropriate counterexamples follow. In Examples 1 and 2, even one-player games suffice.

Example 1. Let us consider a "congestion game with an infinite set of facilities," where $N := \{1\}$, $\mathrm{A} := [0,1]$, $X_1 := \{\{0\}\} \cup \{\{1/2^m, 1/2^{m+1}\}\}_{m\in\mathbb{N}}$, $\mathrm{B}_1(x_1) := \{x_1\}$ for every $x_1 \in X_1$, and $\psi_\alpha(1) := 1-\alpha$ for every $\alpha \in [0,1]$. All assumptions of Theorem 3 except Assumption 2 are satisfied, X_1 is compact (in the Hausdorff metrics), but X_1^2 is not. And there is no Nash equilibrium, i.e., maximum of u_1: $\sup_{x_1\in X_1^2} u_1(x_1) = 2$, whereas $u_1(x_1) < 2$ for every $x_1 \in X_1$.

Example 2. Let us consider a "congestion game with an infinite set of facilities," where $N := \{1\}$, $\mathrm{A} := [0,1]$, $X_1 := \{\{0\}\} \cup \{\{1/2^m - k/[(m+1)2^{m+1}]\}_{k=0,\ldots,m}\}_{m\in\mathbb{N}}$, $\mathrm{B}_1(x_1) := \{x_1\}$ for every $x_1 \in X_1$, and $\psi_\alpha(1) := 1-\alpha$ for every $\alpha \in [0,1]$. All assumptions of Theorem 3 except Assumption 3 are satisfied, X_1 is compact, as well as each X_1^m ($m \in \mathbb{N}$), which is actually a singleton. And again, there is no Nash equilibrium, i.e., maximum of u_1, since $\sup_{x_1\in X_1} u_1(x_1) = +\infty$.

Example 3. Let us consider a "congestion game with an infinite set of facilities," where $N := \{1,2\}$, $\mathrm{A} := X_1 := X_2 := [0,1]$, $\mathrm{B}_i(x_i) := \{x_i\}$ for every $x_i \in X_i$, $\psi_\alpha(2) := 1-\alpha$ and $\psi_\alpha(1) := 2-\alpha$ for every $\alpha \in [0,1]$. All assumptions of Theorem 3 except Assumption 4 are satisfied, but there is no Nash equilibrium. Let $(x_1, x_2) \in X_N$ and $i \in N$. If $x_i \neq 0$, then player i is better off slightly decreasing x_i. On the other hand, $(\{0\},\{0\})$ is not an equilibrium either, because each player will be better off choosing any $x_i \in]0,1[$.

6 Proof of Theorem 3

As was hinted at the start of Section 5, our strategy is to show that P defined by (5) is upper semicontinuous on a compact X_N. Then any strategy profile which maximizes P will be a Nash equilibrium.

The compactness of X_N immediately follows from Assumptions 2 and 3. Let $x_N^k \to x_N^\omega \in X_N$; we have to show that

$$P(x_N^\omega) \geq \limsup_{k\to\infty} P(x_N^k).$$

Since $\mathrm{A}(x_N^\omega)$ is finite, there is an open neighborhood O_α of each $\alpha \in \mathrm{A}(x_N^\omega)$ such that $O_\alpha \cap O_\beta = \emptyset$ whenever $\alpha \neq \beta \in \mathrm{A}(x_N^\omega)$. Moreover, those neighborhoods can be picked in such a way that each O_α is included in the neighborhood O from (8) for each $i \in I_\alpha^\circ$. Now Lemma 1 applies; therefore, we may, without restricting generality, assume that $\mathrm{B}_i(x_i^k) \cap O_\alpha = \{\beta_i^k\}$ for each $\alpha \in \mathrm{A}(x_N^\omega)$, $i \in I(\alpha, x_N^\omega)$, and $k \in \mathbb{N}$ [we should have written $\beta_i^k(\alpha)$, but β_i^k related to different α's will never be considered simultaneously]. Note that $\beta_i^k \to \alpha$ for each $i \in N$. Since there is a finite number of possible values of $I(\beta_i^k, x_N^k)$, we may, without restricting generality, assume that, given $i \in I(\alpha, x_N^\omega)$, the set $I(\beta_i^k, x_N^k)$ is the same for all k. Similarly, we may assume that $I(\alpha, x_N^\omega)$ is partitioned into $\bar{I}(\alpha, x_N^\omega) := \{i \in I(\alpha, x_N^\omega) \mid \forall k \in \mathbb{N}\, [\beta_i^k = \alpha]\}$ $[= \{i \in I(\alpha, x_N^\omega) \mid \forall k \in \mathbb{N}\, [\alpha \in \mathrm{B}_i(x_i^k)]\}]$ and $\tilde{I}(\alpha, x_N^\omega) := \{i \in I(\alpha, x_N^\omega) \mid \forall k \in \mathbb{N}\, [\beta_i^k \neq \alpha]\}$ $[= \{i \in I(\alpha, x_N^\omega) \mid \forall k \in \mathbb{N}\, [\alpha \notin \mathrm{B}_i(x_i^k)]\}]$. By definition (8), we have $\tilde{I}(\alpha, x_N^\omega) \subseteq I(\alpha, x_N^\omega) \setminus I^\circ(\alpha, x_N^\omega)$.

Arguing quite similarly to the proof of Lemma 2, we may assume that $I_{\beta_i^k}^- = \emptyset$ whenever $\beta_i^k \neq \alpha$.

Now we are ready to analyze and compare the right-hand side of (5) for x_N^k,

$$\sum_{\alpha\in \mathrm{A}(x_N^k)} \varphi_\alpha(I(\alpha,x_N^k),x^k_{I(\alpha,x_N^k)}) + \sum_{\alpha\in \mathrm{A}^+(x_N^k)} \sum_{m=n^-(\alpha)}^{\#I(\alpha,x_N^k)-1} \psi_\alpha(m), \tag{10a}$$

and for x_N^ω,

$$\sum_{\alpha\in \mathrm{A}(x_N^\omega)} \varphi_\alpha(I(\alpha,x_N^\omega),x_{I(\alpha,x_N^\omega)}) + \sum_{\alpha\in \mathrm{A}^+(x_N^\omega)} \sum_{m=n^-(\alpha)}^{\#I(\alpha,x_N^\omega)-1} \psi_\alpha(m). \tag{10b}$$

If $\tilde{I}(\alpha,x_N^\omega)=\emptyset$ and hence $I(\alpha,x_N^\omega)=I(\alpha,x_N^k)=: I$ for each k, then this α contributes

$$\varphi_\alpha(I,x_I^k)+\Big[\sum_{m=n^-(\alpha)}^{\#I-1}\psi_\alpha(m)\Big]$$

to (10a) [the term in square brackets disappears if $\#I=n^-(\alpha)$] and

$$\varphi_\alpha(I,x_I^\omega)+\Big[\sum_{m=n^-(\alpha)}^{\#I-1}\psi_\alpha(m)\Big]$$

to (10b); since $\varphi_\alpha(I,\cdot)$ is upper semicontinuous by Assumption 1, there is no problem with this α.

If $\tilde{I}(\alpha,x_N^\omega)\neq\emptyset$, the analysis is more complicated. For brevity, we denote $I:=I(\alpha,x_N^\omega)$, $\bar{I}:=\bar{I}(\alpha,x_N^\omega)$, $\tilde{I}:=\tilde{I}(\alpha,x_N^\omega)$, and $I(i):=I(\beta_i^k,x_N^k)$ for each $i\in\tilde{I}$; as was noted above, $I(i)$ does not depend on k. Now the contribution of this α to (10a) is

$$\varphi_\alpha(\bar{I},x^k_{\bar{I}})+\Big[\sum_{m=n^-(\alpha)}^{\#\bar{I}-1}\psi_\alpha(m)\Big]+\sum_{i\in\tilde{I}}\frac{1}{\#I(i)}\Big\{\varphi_{\beta_i^k}(I(i),x^k_{I(i)})+\Big[\sum_{m=1}^{\#I(i)-1}\psi_{\beta_i^k}(m)\Big]\Big\}. \tag{11a}$$

[The terms in square brackets disappear if, respectively, $\#\bar{I}=n^-(\alpha)$ or $\#I(i)=1$; we divide the rightmost sum in (11a) by $\#I(i)$ to compensate for multiple counting of the same terms.]

The contribution of the same α to (10b) is

$$\varphi_\alpha(I,x_I^\omega)+\Big[\sum_{m=n^-(\alpha)}^{\#\bar{I}-1}\psi_\alpha(m)+\sum_{\#\bar{I}}^{\#I-1}\psi_\alpha(m)\Big]. \tag{11b}$$

Taking into account Assumption 4 and the fact that $\tilde{I}(\alpha,x_N^\omega)\subseteq I(\alpha,x_N^\omega)\setminus I^\circ(\alpha,x_N^\omega)$, we see that the upper limit of (11a) cannot be greater than (11b).

The upper semicontinuity of P is proven, and so is the theorem.

7 Le Breton–Weber construction

To show that the games considered by Le Breton and Weber (2011) are regular CLU games, we reproduce their construction, in somewhat streamlined notations. All strategy sets X_i are compact subsets of a Euclidean space $\mathbb{R}^T$; $X = \bigcup_{i \in N} X_i$. Given a strategy profile $x_N \in X_N$ and $x \in X$, we denote $n(x, x_N)$ the number of players with $x_i = x$. The payoff $U_i(x_N)$ of player i is the sum of three terms ("taste component," "local social interaction component," and "global social interaction component"):

$$U_i(x_N) = V_i(x_i) + \sum_{j \in N \setminus \{i\}} W_i^j(x_i, x_j) + H(x_i, n(x_i, x_N)). \tag{12}$$

Three substantial assumptions are made: (1) each function V_i, W_i^j, and $H(\cdot, m)$ ($m \in \mathbb{N}$) is upper semi-continuous; (2) $W_i^j(x_i, x_j) = W_j^i(x_j, x_i)$ for every $i, j \in N$, every $x_i \in X_i$ and every $x_j \in X_j$; (3) $H(x, \cdot)$ is increasing for all $x \in X$. Under those assumptions, Le Breton and Weber (2011) showed that the following function is an upper semi-continuous exact potential:

$$\Psi(x_N) = \sum_{i \in N} V_i(x_i) + 1/2 \sum_{i \in N} \sum_{j \in N \setminus \{i\}} W_i^j(x_i, x_j) + \sum_{x \in X \,:\, n(x, x_N) > 0} \; \sum_{m=1}^{n(x, x_N)} H(x, m). \tag{13}$$

Denoting $\mathcal{N}_2$ the set of all unordered pairs in N, i.e., subsets of cardinality 2, we define $\mathrm{A} := N \cup \mathcal{N}_2 \cup X$; $\mathrm{B}_i(x_i) := \{i\} \cup \{\{i, j\}\}_{j \in N \setminus \{i\}} \cup \{x_i\}$; $\varphi_i(x_i) := V_i(x_i)$; $\varphi_{\{i,j\}}(x_i, x_j) := W_i^j(x_i, x_j)$; $\psi_x(m) := H(x, m)$. It is easy to check now that utility functions (2) coincide with utility functions (12), while exact potential (5) coincides with potential (13). The assumptions of Le Breton and Weber (2011) imply our Assumptions 1–4 (actually, the latter were developed as a generalization of the former).

8 Player-specific local utilities

Congestion games with player-specific local utilities are a natural generalization of Rosenthal's (1973) model. Typically, one cannot expect the existence of an equilibrium, to say nothing of an exact potential, in such games, even though there are results on the existence of a Nash equilibrium (Milchtaich, 1996) or even a strong Nash one (Konishi, et al., 1997) in some particular cases. This big and important topic is mostly left out here. The only objective of this section is to show that both classes of congestion games with player-specific local utilities shown by Harks et al. (2011) to admit exact potentials can be obtained from our construction.

We consider even a more general model, which simultaneously includes both "weighted congestion games with facility-dependent demands" and "weighted congestion games with elastic demands" of Harks et al. (2011). There is a finite set N of *players* and an arbitrary set A of *facilities*; we denote X the set of $x = \langle x^\alpha \rangle_{\alpha \in \mathrm{A}} \in \mathbb{R}^{\mathrm{A}}$ such that $\mathrm{B}(x) := \{\alpha \in \mathrm{A} \mid x^\alpha \neq 0\}$ is finite. For each player $i \in N$, there is a *strategy set* $X_i \subseteq X$ and a function $F_i \colon X_i \to \mathbb{R}$; for every $\alpha \in \mathrm{A}$, there is a constant $c_\alpha \in \mathbb{R}$. Given a strategy profile $x_N \in X_N$, the *local utility* obtained by player i from a facility α is

$$\varphi_i^\alpha(x_N^\alpha) := x_i^\alpha \cdot \sum_{j \in N} c_\alpha \cdot x_j^\alpha. \tag{14}$$

The *total utility function* is

$$u_i(x_N) := F_i(x_i) + \sum_{\alpha \in \mathrm{B}(x_i)} \varphi_i^\alpha(x_N^\alpha). \tag{15}$$

We may say that the players belonging to the set $I(\alpha, x_N) := \{i \in N \mid x_i^\alpha \neq 0 \, [\equiv \alpha \in \mathrm{B}(x_i)]\}$ have chosen facility α. Then we notice that $\varphi_i^\alpha(x_N^\alpha) = 0$ whenever $i \notin I(\alpha, x_N)$; similarly, $\varphi_i^\alpha(x_N^\alpha)$ only depends on $x_{I(\alpha,x_N)}^\alpha$, so we could write $\varphi_i^\alpha(x_{I(\alpha,x_N)}^\alpha)$ in the left hand side of (14) and the right hand side of (15). Now we see that (15) can be viewed as a generalization of (2) where different players may extract different local utilities from the same facility.

As a straightforward example of such a game, assume that A is the set of edges of a network. A strategy of each player is a flow through the network with given source and destination nodes, satisfying Kirchhoff's law at each intermediate node. The cost of pushing a unit of flow through an edge α is affine in the total load: $-b_\alpha - c_\alpha \sum_{i \in N} x_i^\alpha$. The function F_i is the gain obtained from the flow minus the costs incurred independently of the behavior of other players, i.e., plus $\sum_{\alpha \in \mathrm{A}} b_\alpha \cdot x_i^\alpha$. Under this interpretation, (15) is an adequate description of the payoff to player i. The natural assumptions are $b_\alpha, c_\alpha \leq 0$ and $x_i^\alpha \geq 0$ for all i and α. Upper restrictions on x_i^α can be added; moreover, there may be arbitrary restrictions on x_i^α as well, e.g., they may be all integer.

Imposing some superfluous restrictions (e.g., A was finite throughout; the choice of a flow satisfying Kirchhoff's law was excluded; etc.), Harks et al. (2011) showed that such a game admits an exact potential. Therefore, it can be represented as a game with structured utilities (Kukushkin, 2007, Theorem 5), or, if all strategy sets X_i are finite, as a congestion game (Monderer and Shapley, 1996, Theorem 3.2). In either case, however, rather artificial constructions may be needed.

Given a game Γ with utilities defined by (14) and (15), we define, in a simple and natural way, a regular CLU game Γ^* which is isomorphic to Γ. There are the same players with the same strategy sets, $N^* := N$ and $X_i^* := X_i$ for each $i \in N$. The set of facilities is modified: $\mathrm{A}^* := N \cup (\mathrm{A} \times N) \cup (\mathrm{A} \times \mathcal{N}_2)$, where, as in Section 7, $\mathcal{N}_2$ is the set of all unordered pairs in N, i.e., subsets of cardinality 2. Given $i \in N$ and $x_i \in X_i$, we define $\mathrm{B}_i^*(x_i) := \{i\} \cup \{\{(\alpha, i)\} \cup \{(\alpha, \{i, j\})\}_{j \in N \setminus \{i\}}\}_{\alpha \in \mathrm{B}(x_i)}$; thus, $I_i^+ = I_{(\alpha,i)}^+ = \{i\}$ and $I_{(\alpha,\{i,j\})}^+ = \{i, j\}$ for all $\alpha \in \mathrm{A}$ and $i, j \in N$, $i \neq j$. The local utilities are defined in this way: $\varphi_i^*(\{i\}, x_i) := F_i(x_i)$; $\varphi_{(\alpha,i)}^*(\{i\}, x_i^\alpha) := x_i^\alpha \cdot c_\alpha \cdot x_i^\alpha$; $\varphi_{(\alpha,\{i,j\})}^*(I, x_I^\alpha) := c_\alpha \cdot x_i^\alpha \cdot x_j^\alpha$ if $I = \{i, j\}$, while $\varphi_{(\alpha,\{i,j\})}^*(I, x_I^\alpha) := 0$ if $I \neq \{i, j\}$.

It is easy to check now that utility functions (2) coincide with utility functions (15). Moreover, the regularity conditions (4) are obvious. Thus, the existence of an exact potential immediately follows from Theorem 1.

It is worthwhile to note that, if A is finite, we could set $\mathrm{B}_i^*(x_i) := \{i\} \cup \{\{(\alpha, i)\} \cup \{(\alpha, \{i, j\})\}_{j \in N \setminus \{i\}}\}_{\alpha \in \mathrm{A}}$ for all $i \in N$ and $x_i \in X_i$, in which case Γ^* would be a game with structured utilities. For an infinite A, such a representation is inadmissible without a revision of our basic definitions.

If all strategy sets X_i are finite, then we have the existence of a Nash equilibrium as well. If A is finite, then it will be enough to assume that each X_i is compact (the utility functions are continuous anyway). Otherwise, we may assume that A is a metric space; then each X_i is also a metric space and Assumption 1 holds. We have to impose Assumptions 2 and 3 as they are; as to Assumption 4, it holds if $D_i^\alpha \subset \mathbb{R}_+$ for all $i \in N$ and $\alpha \in \mathrm{A}$, while $b_\alpha, c_\alpha > 0$ for all $\alpha \in \mathrm{A}$. In other words, it is natural in this case to restrict attention to *positive* externalities (Le Breton and Weber, 2011, have already come to the same conclusion).

Harks et al. (2011) also showed the necessity of the affine combinations in local utilities for the guaranteed existence of a potential; it does not seem possible to derive the fact from our Theorem 2. On the other hand, if we drop the idea that the costs should be the same for all users of a given facility, then polylinear combinations with symmetric coefficients would be acceptable as well. For instance, consider local utility functions of the form

$$\varphi_i^\alpha(x_N^\alpha) := x_i^\alpha \cdot \Big[\sum_{j \in N} c_\alpha^{ij} \cdot x_j^\alpha + \sum_{j,k \in N \setminus \{i\},\, j \neq k} d_\alpha^{ijk} \cdot x_j^\alpha \cdot x_k^\alpha\Big],$$

where coefficients c_α^{ij} and d_α^{ijk} are invariant w.r.t. permutations of the indices from N for every $\alpha \in \mathrm{A}$. Virtually the same argument as above shows that such a game is isomorphic to a regular CLU game. One may doubt that such cost functions could adequately describe any real-world interrelationships, but an interesting point is that they also emerge in the study of Cournot tâtonnement in aggregative games with monotone best responses (Kukushkin, 2005).

9 Conclusion

Let us summarize our main findings. It is, in principle, possible to allow the players in a congestion game to choose some additional parameters beside the facilities they use (e.g., type of vehicle, load, etc.) without destroying the presence of an exact potential. The "only" restriction is that those additional parameters should not affect the local utility unless *all* players able to use the facility actually show up. Games with structured utilities fit here since each player uses the same list of facilities under every strategy.

This generalization allowed us to include the constructions considered by Le Breton and Weber (2011) and Harks et al. (2011) into the same general scheme. It seems quite possible that other examples could be found as well, but, so far, I have been unable to produce anything specific.

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