Numeraire Invariance and application to Option Pricing and Hedging

Jamshidian, Farshid

University of Twente, AtomPro Structured Products

14 February 2008

Numeraire Invariance and application to Option Pricing and Hedging

FARSHID JAMSHIDIAN

Keywords: Numeraire invariance, hedging, self-financing trading strategy, predictable representation, unique pricing, arbitrage-free, martingale, homogeneous payoff, Markovian, Itô's formula, SDE, PDE, geometric Brownian motion, exponential Poisson process.

1. Introduction

Numeraire invariance is a well-known technique in option pricing and hedging theory. It takes a convenient asset as the *numeraire*, as if it were the medium of exchange, and expresses all other asset and option prices in units of this numeraire. Since the price of the numeraire relative to itself is identically 1 at all times, this reduces pricing and hedging to a market with zero-interest rates. A somewhat controversial implication is that the modelling focus should be more on the asset price *ratios* rather than on the asset price processes themselves.

The idea of numeraire invariance is already implicit in Merton (1973), and since then many authors have contributed to its development. After a brief survey of its origins, we state and prove the numeraire invariance principle for general semimartingale price processes, following essentially Duffie [3]. We then present its application to unique pricing in arbitrage-free models and discuss nondegeneracy and unique hedging.

Next, using numeraire invariance, we show that if the underlying asset *ratios* follow a diffusion, then a payoff that is a *homogeneous function* of the asset payoffs can *always* be replicated (subject to mild growth conditions) and hence also uniquely priced. The deltas (hedge ratios) are given by the partial derivatives of the either the "projective option price function," or equivalently, of the "homogenous option price function," either of which is the solution of a PDE. We illustrate the classical multivariate lognormal case from this angel.

To illustrate replication under the presence of jumps, we work out a little-known *exponential Poisson model*, first for the exchange option, and then for a multivariate generalization with an arbitrary homogenous payoff function. Here, the option price function satisfies a partial *difference* equation, and the deltas are given by partial *differences*. We mention a connection to martingale representation, from which the explicit formulae are actually drawn.

In the final section, we first highlight the role played by homogeneity, emphasizing that if the covariation matrix of the underlying assets is nondegenerate, then nonhomogeneous payoffs *cannot* be replicated. We then extend the discussion to assets with dividends. Finally, we derive the ubiquitous bivariate lognormal exchange option formula by a change of measure.

We will confine the discussion to European options with expiration denoted T.

---

†Part-time Professor of Applied Mathematics, FELAB, University of Twente.

‡Cofounder, AtomPro Structured Products, http://www.atomprostructuredproducts.nl/index.html.

††Version 14-Feb-2008. For possible future updates visit wwwhome.math.utwente.nl/˜ jamshidianf.

This paper is a short version of the paper "Exchange Options" (2007) and is prepared for the proceedings of the *Actuarial and Financial Mathematics Conference*, Brussels, 2008.

2. A brief survey

2.1. Merton's extension of Black-Scholes. Let be given a zero-dividend asset with price process $A = (A_t)$. Let $C = (C_t)$ denote the price process of a call option on A with strike price K and expiration T, which we wish to find. So, the option payoff is

$$C_T = (A_T - K)^+.$$

To ***replicate*** C, another asset is needed. Black-Scholes (1973) take as the second asset a money market of the form e^{rt}. Merton's idea is to take the T-maturity zero-coupon bond B with principal K, i.e., $B_T = K$. The payoff can now be expressed in terms of both assets:

$$C_T = (A_T - B_T)^+.$$

The payoff's ***homogeneity*** allows one to factor out B:

$$F_T = (X_T - 1)^+,$$

where

$$X := \frac{A}{B}, \quad F := \frac{C}{B},$$

are the ***forward prices*** of the asset and the option. Merton (1973) argues that it is sufficient to replicate the forward option by trading the forward asset, i.e., to find a δ such that

$$dF_t = \delta_t dX_t.$$

The *same* δ should then serve as the hedge ratio with respect to asset A.

Assuming $F_t = f(t, X_t)$ for some f, by ***Itô's formula*** the equation $dF = \delta dX$ is equivalent to the following formula for δ_t and **PDE** for $f(t,x)$ with terminal condition $f(T,x) = (x-1)^+$:

$$\delta_t = \frac{\partial f}{\partial x}(t, X_t),$$

$$\frac{\partial f}{\partial t} + \frac{1}{2}\sigma_t^2 x^2 \frac{\partial^2 f}{\partial x^2} = 0,$$

where σ_t is the ***forward-price volatility*** (assumed deterministic by Merton):

$$d[X]_t = \sigma_t^2 X_t^2 dt.$$

(The first (second) equation follows by equating the martingale (drift) terms of the two equations for dF.) Thus by "factoring out" asset B, the problem with a stochastic interest rate reduces to a call option struck at 1 in the Black-Scholes model with *zero interest rate.*

More generally, when asset A pays dividends at a constant rate y, the above applies with the forward asset price $X_t = e^{-y(T-t)} A_t / B_t$.

2.2. Margrabe's extension to exchange options. Margrabe (1978) showed that Merton's argument extends to an option to exchange any two assets A and B. His idea was to replicate the exchange option price process C according to the SDE

$$dC_t = \delta_t^A dA_t + \delta_t^B dB_t.$$

Assuming $C_t = c(t, A_t, B_t)$ for some function $c(t,a,b)$, he noted that by *Itô's formula* this equation is implied by the system of equations

$$\delta_t^A = \frac{\partial c}{\partial a}(t, A_t, B_t), \qquad \delta_t^B = \frac{\partial c}{\partial b}(t, A_t, B_t),$$

$$\frac{\partial c}{\partial t} + \frac{1}{2}\sigma_A^2 a^2 \frac{\partial^2 c}{\partial a^2} + \frac{1}{2}\sigma_B^2 b^2 \frac{\partial^2 c}{\partial b^2} + \sigma_A \sigma_B \rho\, ab \frac{\partial^2 c}{\partial a \partial b} = 0,$$

where σ_A and σ_B are the volatilities of A and B and ρ is their correlation, assumed constants. The converse is also true if $|\rho| \neq 1$. (Note however, this nondegeneracy condition excludes the Black-Scholes and 1-factor short-rate diffusion models).

Margrabe stated that if the option payoff is ***homogenous*** of degree 1 in (a, b) (such as $(a-b)^+$ as in the case of an exchange option), then the PDE above should have a homogenous solution $c(t, a, b)$. But then, *Euler's formula* for homogenous functions implies $c = a\partial c/\partial a + b\partial c/\partial b$. Thus if we choose $\delta^A = \partial c/\partial a$ and $\delta^B = \partial c/\partial b$ as above, we get

$$C_t = \delta_t^A A_t + \delta_t^B B_t.$$

Together with the equation $dC = \delta^A dA + \delta^B dB$, this means these deltas are ***self financing***.

Merton had made similar observations and provided the homogenous solution $c(t, a, b)$ of the above PDE by reducing it to the 1-dimensional PDE of Sec. 2.1 via the transformation

$$f(t, x) = c(t, a, b)/b = c(t, x, 1), \quad x = a/b,$$

with volatility σ in the 1-dimensional PDE given by that of ***asset ratio*** A/B:

$$\sigma^2 = \sigma_A^2 + \sigma_B^2 - 2\sigma_A\sigma_B\rho.$$

Coining the term ***numeraire***, Margrabe presented (acknowledging Stephen Ross) a financial interpretation of Merton's algebraic reduction. He proposed to measure the asset and option prices in terms of asset B, as in a barter economy where B serves as the medium of exchange. This provided the intuition behind Merton's reduction to zero interest rates.

Note, the exchange option is replicated here by dynamic trading in only assets A and B.

2.3. **Equivalent martingale measures.** Harrison and Kreps (1979) and Harrison and Pliska (1981) pioneered the application of martingale theory to option pricing. They showed that ***no-arbitrage*** in the sense of no *free lunches* is essentially equivalent to the existence of an equivalent measure under which discounted prices are martingales. (See [2] for the general theory.) Options can thus be priced by computing the ***discounted payoff expectation***.

For discounting, they utilized the finite variation money market numeraire $\exp(\int_0^t r_s ds)$, where r_t is the instantaneous interest rate. This included the Black-Scholes and short-rate models, but did not address Merton's and Margrabe's approach where the numeraire had infinite variation. With the advent of the ***forward measure***, it was clear that the discounting could also be done with a zero-coupon bond, and this often simplified the calculation as discounting was in effect performed outside the expectation (e.g., [7] and [4]).

Another useful numeraire, "the annuity", was used by Neuberger (1990) to price interest-rate swaptions. It serves as the industry standard to this date for quoting swaption volatilities. Eventually, El-Karoui, Geman and Rochet (1995) showed that one can ***change numeraire*** to any asset B and associate to it an equivalent probability measure under which A/B is a martingale for all other assets A. In some problems (such as certain Asian options or the passport option), it is advantageous to take the underlying asset itself as the numeraire.

3. The principle of numeraire invariance

We fix a stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ with a finite time horizon $t \in [0, T]$. We denote the stochastic integral of a locally bounded predictable integrand $\theta = (\theta^1, \cdots, \theta^n)$ against a (vector) semimartingale $X = (X^1, \cdots, X^n)$ by

$$\theta \cdot X = \sum_{i=1}^n \int_0^\cdot \theta_t^i dX_t^i.$$

In what follows, A will denote a vector semimartingale:

$$A = (A^1, \cdots, A^m). \qquad (m \geq 2)$$

Each A^i represents the observable price process of a traded (or replicable) *zero-dividend* asset. When $A^m, A^m_- > 0$, we will set

$$X^i := \frac{A^i}{A^m},$$

and

$$X := (X^1, \cdots, X^n), \qquad n := m - 1.$$

3.1. **Self-financing trading strategies (SFTS).** A SFTS δ for a semimartingale $A = (A^1, \cdots, A^m)$ is a locally bounded predictable process $\delta = (\delta^1, \cdots, \delta^m)$ such that

$$\sum_{i=1}^m \delta^i A^i = \sum_{i=1}^m \delta^i_0 A^i_0 + \delta \cdot A. \tag{3.1}$$

This is equivalent to saying that $C = C_0 + \delta \cdot A$, i.e.,

$$dC = \sum_{i=1}^m \delta^i dA^i, \tag{3.2}$$

where C is the SFTS *price process* defined by

$$C := \sum_{i=1}^m \delta^i A^i. \tag{3.3}$$

Clearly C is then a semimartingale, $\Delta C = \sum_i \delta^i \Delta A_i$, and thus

$$C_- = \sum_{i=1}^m \delta^i A^i_-.$$

The hedge ratio δ^i_t is interpreted as the number of shares invested in asset A^i at time t.

3.2. **Numeraire invariance.** *Let δ be a SFTS for A and S be any (scalar) semimartingale. Then δ is also a SFTS for $SA = (SA^1, \cdots, SA^m)$, i.e., (with $C := \sum_1^m \delta^i A^i$) ,*

$$d(SC) = \sum_{i=1}^m \delta^i d(SA^i). \tag{3.4}$$

Proof. By Itô's product rule, then substituting for dC and C_- and regrouping, followed by Itô's product rule again,

$$\begin{aligned} d(SC) &= S_- dC + C_- dS + d[S, C] \\ &= S_- \sum_{i=1}^m \delta^i dA^i + \sum_{i=1}^m \delta^i A^i_- dS + \sum_{i=1}^m \delta^i d[S, A^i] \\ &= \sum_{i=1}^m \delta^i (S_- dA^i + A^i_- dS + d[S, A^i]) = \sum_{i=1}^m \delta^i d(SA^i). \qquad \square \end{aligned}$$

To our best knowledge, this result first appeared in the 1992 edition of Duffie [3], where it is called the ***numeraire invariance theorem***. Duffie gives the same proof, but assumes that the A^i are (continuous) Itô processes. The only difference in the general case here is the use of left limits, primarily, substituting $C_- = \sum_1^m \delta^i A^i_-$ instead of $C = \sum_1^m \delta^i A^i$.

Interpreting S as an exchange rate, numeraire invariance means that the self-financing property is independent of the choice of base currency, which is intuitively obvious.

If $S, S_- > 0$, then $1/S$ is also a semimartingale. The result applied to $1/S$ implies that:

δ is a SFTS for A if and only if it is one for SA. Thus, *if (3.3) holds then (3.2) and (3.4) are equivalent.*

3.3. **Taking an asset as numeraire.** Assume now $A^m, A^m_- > 0$, and apply the result to $S = 1/A^m$. It follows that

δ is a SFTS for A if and only if it is a SFTS for $A/A^m = (X, 1)$, i.e., if and only if $F := C/A^m$ satisfies $F = F_0 + \delta' \cdot X$ where $\delta' := (\delta^1, \cdots, \delta^n)$. Clearly then

$$\delta^m = F - \sum_{i=1}^{n} \delta^i X^i = F_- - \sum_{i=1}^{n} \delta^i X^i_-. \quad (F := \frac{C}{A^m})$$

Conversely, given $\delta' = (\delta^1, \cdots, \delta^n)$ and an F_0, then with δ^m as above, $\delta = (\delta', \delta^m)$ is a SFTS for $(X, 1)$ with price process $F := F_0 + \delta' \cdot X$. Hence by numeraire invariance δ is a SFTS for A with price process $C = A^m F$. Numeraire invariance thus reduces dimensionality by one:

In order to find a SFTS δ with a given time-T payoff C_T, it is sufficient to find a process δ' and an F_0 such that $F_T = C_T/A^m_T$, where $F = F_0 + \delta' \cdot X$, or equivalently to find a process F such that $F_T = C_T/A^m_T$ and $dF = \sum_1^n \delta^i dX^i$ for some $\delta^1, \cdots, \delta^n$.

Since $\delta^m = F - \sum_{i=1}^n \delta^i X^i$, the m-th delta δ^m is like F determined by δ' and F_0. As such, one interprets the m-th asset as the ***numeraire asset*** chosen to finance an otherwise arbitrary trading strategy δ' in the other assets, post an initial investment of $C_0 = A^m_0 F_0$.

3.4. **Application to unique pricing.** One calls A ***arbitrage free*** if there exists a *state price density*, i.e., semimartingale S such that $S, S_- > 0$ and SA^i are martingales for all i.

The (bounded) ***law of one price*** then holds: *If A is arbitrage free and δ is a **bounded** SFTS for A then SC is a martingale where $C := \sum_{i=1}^m \delta^i A^i$; consequently $C = 0$ if $C_T = 0$.*

Proof. By *numeraire invariance*, $d(SC) = \sum_{i=1}^m \delta^i d(SA^i)$. Thus SC is a local martingale. But since δ is bounded, SC is dominated by a martingale. So SC is a martingale. □

By a simple and well-known argument: *If $A^m, A^m_- > 0$, then A is arbitrage free if and only if there exists an equivalent probability measure $\mathbb{Q}$ such that A^i/A^m are $\mathbb{Q}$-martingales, all i.*

The equivalent martingale measure $\mathbb{Q}$ is related to S by

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \frac{S_T A^m_T}{\mathbb{E}[S_0 A^m_0]}.$$

*If δ is a **bounded** SFTS, then C/A^m is a $\mathbb{Q}$-martingale, where $C := \sum_{i=1}^m \delta^i A^i$; hence*

$$C_t = A^m_t \mathbb{E}^{\mathbb{Q}}[\frac{C_T}{A^m_T} \,|\, \mathcal{F}_t].$$

Proof. By *numeraire invariance*, $d(C/A^m) = \sum_{i=1}^m \delta^i d(A^i/A^m)$. So C/A^m is a local martingale. Since δ is bounded, C/A^m is dominated by a martingale. So C/A^m is a martingale. □

3.5. **Unique hedging.** Let A be arbitrage-free and δ be a bounded SFTS for A. Then, as before, $X^i := A^i/A^m$ and $F := C/A^m$ are $\mathbb{Q}$-martingales, and $dF = \sum_{i=1}^n \delta^i dX^i$ by numeraire invariance. Assume that X^i are $\mathbb{Q}$-locally square-integrable (e.g., continuous). Then, $d\langle F\rangle^{\mathbb{Q}} = \sum_{ij=1}^n \delta^i\delta^j d\langle X^i, X^j\rangle^{\mathbb{Q}}$. (Here, $\langle X^i, X^j\rangle^{\mathbb{Q}}$ is the $\mathbb{Q}$-compensator of $[X^i, X^j]$; so it equals the latter in the continuous case.) Clearly, $\langle F\rangle^{\mathbb{Q}} = 0$ if $F_T = 0$. Thus: *If $\langle X^i\rangle^{\mathbb{Q}}$ are absolutely continuous and the $n\times n$ matrix $(\frac{d}{dt}\langle X^i, X^j\rangle^{\mathbb{Q}})$ is* ***nonsingular****, then given any random variable R, there exists* ***at most one bounded SFTS*** *δ for A with $\sum_{i=1}^m \delta_T^i A_T^i = R$.*

When there are "redundant assets", the matrix is singular, and replication is *not* unique.

4. Application to diffusion processes

4.1. **Pricing and hedging.** Let $A = (A^1, \cdots, A^m)$ be a semimartingale with $A, A_- > 0$ such that the price ratios $X^i := A^i/A^m$ follow the SDE system

$$dX_t^i = X_t^i \sum_{j=1}^k \varphi_{ij}(t, X_t)(dZ_t^j + \phi^j dt), \qquad (i = 1, \cdots, n := m-1)$$

where Z^j are independent Brownian motions, $\varphi_{ij}(t,x)$ are *bounded* continuous functions, and $\mathbb{E}\, e^{\frac{1}{2}\sum_j \int_0^T (\phi_t^j)^2 dt} < \infty$. (Note, we allow A^i be discontinuous.) Define the martingale

$$M := e^{-\sum_{j=1}^k (\int \phi^j dZ^j + \frac{1}{2}\int (\phi^i)^2 dt)},$$

and the measure $\mathbb{Q}$ by $d\mathbb{Q} = M_T d\mathbb{P}$. Then $W^j := Z^j + \int \phi^j dt$ are $\mathbb{Q}$-Brownian motions and are $\mathbb{Q}$-independent since $[W^j, W^k] = 0$ for $j \neq k$. The X^i are $\mathbb{Q}$-martingales since

$$dX_t^i = X_t^i \sum_{j=1}^k \varphi_{ij}(t, X_t) dW_t^j, \tag{4.1}$$

and $\varphi_{ij}(t,x)$ are bounded. *Thus A is* ***arbitrage-free****.*

For each $s \le T$ and $x \in \mathbb{R}_+^n$, there is a unique continuous positive $\mathbb{Q}$-square-integrable martingale $X^{s,x} = (X_t^{s,x})$ on $[s,T]$ with $X_s^{s,x} = x$ satisfying this SDE, and we have $X = X^{0,X_0}$.

Now, let $h(a)$, $a \in \mathbb{R}_+^m > 0$, be a homogenous Borel function of linear growth. Define

$$g(x) := h(x, 1), \quad x \in \mathbb{R}_+^n.$$

Define the function $f(t,x)$ satisfying $f(T,x) = g(x)$ by,

$$f(t,x) := \mathbb{E}^{\mathbb{Q}}\, g(X_T^{t,x}). \tag{4.2}$$

(Intuitively, $f(t,x) = \mathbb{E}[g(X_T) \mid X_t = x]$.) Then the Markov property holds, i.e., we have,

$$F_t := f(t, X_t) = \mathbb{E}^{\mathbb{Q}}(g(X_T) \mid \mathcal{F}_t). \tag{4.3}$$

Thus $F = (f(t, X_t))$ is a $\mathbb{Q}$-martingale, and since X^i are too, assuming that $f(t,x)$ is $C^{1,2}$, Itô's formula yields (setting the martingale and drift parts equal),

$$dF_t = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(t, X_t) dX_t^i, \tag{4.4}$$

and

$$\frac{\partial f}{\partial t}(t, X_t)dt + \frac{1}{2}\sum_{i,j=1}^n \frac{\partial^2 f}{\partial x_i \partial x_j}(t, X_t) d[X^i, X^j]_t = 0. \tag{4.5}$$

By (4.4) and ***numeraire invariance***, δ is a SFTS for A, where

$$\delta_t^i := \frac{\partial f}{\partial x_i}(t, X_t),\ i \le n, \qquad \delta^m := F - \sum_{i=1}^{n} \delta^i X_i. \tag{4.6}$$

Clearly, the price process of this SFTS is $C = A^m F$ (by the definition of δ^m). Moreover, $C_T = h(A_T)$ since $F_T = g(X_T)$ and $h(a)$ is homogenous.

By (4.5), on the support X, $f(t, x)$ satisfies the PDE

$$\frac{\partial f}{\partial t} + \frac{1}{2} \sum_{i,j=1}^{n} x_i x_j \sigma_{ij}(t, x) \frac{\partial^2 f}{\partial x_i \partial x_j} = 0, \tag{4.7}$$

where

$$\sigma_{ij}(t, x) := \sum_{l=1}^{k} \varphi_{il}(t, x) \varphi_{jl}(t, x).$$

By the invariance of Itô's formula under the change of coordinates, the change of variable $L^i = \frac{X^i}{X^{i+1}} - 1$ $(i < n)$, $L^n = X^n - 1$, transforms (4.7) into the *Libor market model* PDE.

4.2. **The homogenous solution.** The option price process and the deltas are already found, but let us also discuss the homogenous option price function defined by

$$c(t, a) := a_m f(t, \frac{a_1}{a_m}, \cdots, \frac{a_n}{a_m}).$$

Then $C_t = c(t, A_t)$. Agreeably, $\delta_t^i = \frac{\partial c}{\partial a_i}(t, A_t)$ by (4.6). (For $i = m$ use Euler's formula for $c(t, a)$). By the continuity of X and (4.6), $\delta_t^i = \frac{\partial c}{\partial a_i}(t, A_{t\_})$ too. Therefore by Itô's formula,

$$\frac{\partial c}{\partial t}(t, A_{t\_})dt + \frac{1}{2} \sum_{i,j=1}^{m} \frac{\partial^2 c}{\partial a_i \partial a_j}(t, A_{t\_}) d[A^i, A^j]_t^c = 0. \tag{4.8}$$

(The sum of jumps term in Itô's formula drops out since $\Delta C = \sum \delta^i \Delta A^i$.) This yields the PDE $\frac{\partial c}{\partial t} + \frac{1}{2} \sum_{i,j} a_i a_j \sigma_{ij}^A(t, a) \frac{\partial^2 c}{\partial a_i \partial a_j} = 0$ for the special case $d[A^i, A^j]_t = A_t^i A_t^j \sigma_{ij}^A(t, A_t)dt$ for some functions $\sigma_{ij}^A(t, a)$. The quotient-space PDE (4.7) is more fundamental for it holds in general (even when A is not a diffusion or is discontinuous) and has one lower dimension.

4.3. **Deterministic volatility case.** Assume φ_{ij}, and hence σ_{ij}, are independent of x. Then we simply have $X_T^{t,x} = x X_T / X_t$. Hence by (4.2),

$$f(t, x) := \mathbb{E}^{\mathbb{Q}}[g(x_1 \frac{X_T^1}{X_t^1}, \cdots, x_n \frac{X_T^n}{X_t^n})]. \tag{4.9}$$

Conditioned on $\mathcal{F}_t$ and unconditionally, X_T / X_t is $\mathbb{Q}$-*multivariately lognormally distributed*, with mean $(1, \cdots, 1)$ and log-covariances $\int_t^T \sigma_{ij}(s)ds$. Let $P(t, T, z)$, denote its distribution function. Then by (4.9), we obtain

$$f(t, x) = \int_{\mathbb{R}_+^n} g(x_1 z_1, \cdots, x_n z_n) P(t, T, dz). \tag{4.10}$$

If $\partial g / \partial x_i$ and $g(x) - \sum x_i \partial g / \partial x_i$ are bounded, then so is δ, since

$$\frac{\partial f}{\partial x_i}(t, x) = \mathbb{E}^{\mathbb{Q}}[\frac{X_T^i}{X_t^i} \frac{\partial g}{\partial x_i}(x_1 \frac{X_T^1}{X_t^1}, \cdots, x_n \frac{X_T^n}{X_t^n})].$$

5. Application to exponential Poisson model

5.1. **Option to exchange two assets.** Let A and B denote the asset price processes. Assume $A = BX$, where

$$X_t = X_0 e^{\beta P_t - (e^\beta - 1)\lambda t} \tag{5.1}$$

for some constants $\beta \neq 0$, $\lambda > 0$ and semimartingale P such that $[P] = P$ and $P_0 = 0$ (so, $P_t = \sum_{s \leq t} 1_{\Delta P_s \neq 0}$), e.g., a Poisson (or Cox) process. Equivalently, by Itô's formula, X follows

$$dX_t = X_{t-}(e^\beta - 1) d(P_t - \lambda t). \tag{5.2}$$

Define the function $f(t, x)$, $x > 0$ by

$$f(t,x) := \sum_{n=0}^{\infty} (x e^{\beta n - (e^\beta - 1)\lambda(T-t)} - 1)^+ \frac{\lambda^n}{n!}(T-t)^n e^{-\lambda(T-t)}, \tag{5.3}$$

Clearly $f(T, x) = (x-1)^+$. Define $u(t,p) := f(t, X_0 e^{\beta p - (e^\beta - 1)\lambda t})$. One directly verifies that

$$\frac{\partial u}{\partial t}(t,p) + \lambda(u(t, p+1) - u(t,p)) = 0,$$

Using this, one can show that

$$dF = \delta^A dX, \qquad F_t := f(t, X_t). \tag{5.4}$$

where,

$$\delta^A_t := \delta_A(t, X_{t-}), \quad \delta_A(t,x) := \frac{f(t, e^\beta x) - f(t,x)}{(e^\beta - 1)x}.$$

Thus by ***numeraire invariance*** (δ^A, δ^B) is a SFTS for A with price process $C = BF$, where

$$\delta^B := F_- - \delta^A X_- = F - \delta^A X.$$

Further, $C_T = (A_T - B_T)^+$ since $F_T = (X_T - 1)^+$.

Also, this is a *bounded* SFTS. In fact, $0 \leq \delta^A \leq 1$ and $-1 \leq \delta^B \leq 0$.

5.2. **Multivariate exponential Poisson model.** Let $A > 0$ be an m-dimensional semimartingale with $A_- > 0$. Set $X := (A^i/A^m)_{i=1}^n$, $n := m - 1$. Assume

$$X^i_t := X^i_0 \exp\left(\sum_{j=1}^{k} (\beta_{ij} P^j_t - (e^{\beta_{ij}} - 1)\lambda_j t)\right),$$

$(1 \leq k \leq n)$ or equivalently,

$$dX^i_t = X^i_{t-} \sum_{j=1}^{k} (e^{\beta_{ij}} - 1)(dP^j_t - \lambda_j dt),$$

where, β_{ij} are constants with the $n \times k$ matrix $(e^{\beta_{ij}} - 1)$ of full rank, $\lambda_j > 0$ are constants, and P^j are semimartingales such that $[P^j] = P^j$, $P^j_0 = 0$ and $[P^j, P^l] = 0$ for $j \neq l$.

Let $h(a)$, $a \in \mathbb{R}^m_+$ be a given payoff function, assumed ***homogenous*** of degree 1 and of linear growth in a. Define

$$g(x) := h(x, 1), \quad x \in \mathbb{R}^n_+, \ \ n := m - 1.$$

Define

$$f(t,x) := \sum_{q_1,\cdots,q_n=0}^{\infty} g(x_1 e^{\sum_{j=1}^n (\beta_{1j} q_j - (e^{\beta_{1j}}-1)\lambda_j (T-t))}, \cdots,$$
$$x_n e^{\sum_{j=1}^n (\beta_{nj} q_j - (e^{\beta_{nj}}-1)\lambda_j (T-t))}) \prod_{i=1}^n \frac{\lambda_i^{q_i}}{q_i!} (T-t)^{q_i} e^{-\lambda_i (T-t)}.$$

Let $\alpha = (\alpha_{ij})$ be any $n \times k$ matrix such that for $1 \le j, l \le k$, $\sum_{i=1}^n (e^{\beta_{il}} - 1)\alpha_{ij} = 1$ if $j = l$ and 0 otherwise. Define

$$\delta_t^i := \delta_i(t, X_{t-}), \qquad (1 \le i \le n) \tag{5.5}$$

where

$$\delta_i(t,x) := \frac{1}{x_i} \sum_{j=1}^k \alpha_{ij} (f(t, e^{\beta_{1j}} x_1, \cdots, e^{\beta_{nj}} x_n) - f(t,x)).$$

Then one can show

$$dF = \sum_{i=1}^n \delta^i dX^i, \qquad F_t := f(t, X_t). \tag{5.6}$$

Hence by ***numeraire invariance***, $\delta = (\delta^1, \cdots, \delta^n, \delta^m)$ is a SFTS for A, where $\delta^m := F - \sum_{i=1}^n \delta^i X^i$. Its price process $C = \sum_1^m \delta^i A^i = C_0 + \delta \cdot A$ is clearly given by $A^m F$:

$$C_t = A_t^m f(t, X_t). \tag{5.7}$$

Further, $C_T = h(A_T)$ because $h(a)$ is homogenous of degree 1 and $f(T,x) = g(x) := h(x,1)$.

Moreover, δ^i are bounded if $\gamma_i(x)$ are bounded, where $\gamma_m(x) := g(x) - \sum_{i=1}^n \gamma_i(x) x_i$ and

$$\gamma_i(x) := \frac{1}{x_i} \sum_{j=1}^k \alpha_{ij} (g(e^{\beta_{1j}} x_1, \cdots, e^{\beta_{nj}} x_n) - g(x)). \qquad (i \le n)$$

5.3. **Relation to Poisson predictable representation.** Let $P = (P^1, \cdots, P^k)$ be a vector of independent Poisson processes P^i with intensities $\lambda_i > 0$. Let $v(p)$, $p \in \mathbb{R}^k$, be a function of exponential linear growth. Then, one has the following representation:

$$v(P_T) = \sum_{q_1,\cdots,q_k=0}^{\infty} v(q_1, \cdots, q_k) \prod_{i=1}^k \frac{\lambda_i^{q_i}}{q_i!} T^{q_i} e^{-\lambda_i T} + \sum_{i=1}^k \int_0^T \Delta_i u(t, P_{t-}) d(P_t^i - \lambda_i t),$$

where $\Delta_i u(t,p) := u(t, p_1, \cdots, p_i + 1, \cdots p_n) - u(t,p))$ and

$$u(t,p) := \sum_{q_1,\cdots,q_k=0}^{\infty} v(p+q) \prod_{i=1}^k \frac{\lambda_i^{q_i}}{q_i!} (T-t)^{q_i} e^{-\lambda_i (T-t)}.$$

Also, $u(t,p)$ satisfies the partial difference equation and the SDE

$$\frac{\partial u}{\partial t}(t,p) + \sum_{i=1}^k \lambda_i \Delta_i u(t,p) = 0;$$

$$du(t, P_t) = \sum_{i=1}^k \Delta_i u(t, P_-) d(P^i - \lambda_i t).$$

6. Miscellaneous considerations

6.1. **The role of homogeneity.** Let A be continuous semimartingale and δ be a SFTS for A. Assume $C_t = c(t, A_t)$ for a $C^{1,2}$ function $c(t,a)$. Since $dC = \sum \delta^i dA^i$, by Itô's formula,

$$\frac{\partial c}{\partial t}(t, A_t)dt + \frac{1}{2}\sum_{i,j=1}^{m} \frac{\partial^2 c}{\partial a_i \partial a_j}(t, A_t)d[A^i, A^j]_t = \sum_{i=1}^{m}(\delta_t^i - \frac{\partial c}{\partial a_i}(t, A_t))dA_t^i. \tag{6.1}$$

In general, $\sum_{i,j}(\delta^i - \frac{\partial c}{\partial a_i})(\delta^j - \frac{\partial c}{\partial a_j})d[A^i, A^j] = 0$ since the (left so) right hand side of (6.1) has finite variation and hence zero quadratic variation. Thus, if $[A^i]$ are absolutely continuous and the $m \times m$ matrix $(\frac{d}{dt}[A^i, A^j])$ is *nonsingular*, then $\delta_t^i = \frac{\partial c}{\partial a_i}(t, A_t)$, and so by (6.1),

$$\frac{\partial c}{\partial t}(t, A_t)dt + \frac{1}{2}\sum_{i,j=1}^{m} \frac{\partial^2 c}{\partial a_i \partial a_j}(t, A_t)d[A^i, A^j]_t = 0. \tag{6.2}$$

Moreover, since $C = \sum_i \delta^i A^i$, we then have $c(t, A_t) = \sum_i \frac{\partial c}{\partial a_i}(t, A_t)A_t^i$. So, if the support of A_t is a cone, then it follows that $c(t,a)$ is necessarily *homogenous of degree 1* in a on that cone. Consequently, only homogenous payoffs can be so replicated in this nonsingular case.

In some singular cases, e.g., the Black-Scholes or Markovian short-rate models, there also exist infinitely many *nonhomogenous* functions $c(t,a)$ satisfying $C_t = c(t, A_t)$. This is simply because for each t the support of A_t is a proper surface in $\mathbb{R}^m$ in these models, and obviously there exist infinitely many distinct functions on $\mathbb{R}^m$ that coincide on any surface.

Assume $M^i := e^{-\int_0^{\cdot} r_t dt} A^i$ are local martingales under an equivalent measure for some predictable process r. Then $dA^i = rA^i dt + e^{\int r dt} dM^i$. Thus, using $C = \sum_i \delta^i A^i$ and (6.1),

$$\frac{\partial c}{\partial t}(t, A_t)dt + \frac{1}{2}\sum_{i,j=1}^{m} \frac{\partial^2 c}{\partial a_i \partial a_j}(t, A_t)d[A^i, A^j]_t = r_t(C_t - \sum_{i=1}^{m} \frac{\partial c}{\partial a_i}(t, A_t)A_t^i)dt. \tag{6.3}$$

This "PDE" is valid also for nonhomogeneous functions. It is the type of PDE encountered in the Black-Scholes or Markovian short-rate models. Of course, if we choose $c(t,a)$ to be homogenous - which we can thanks to numeraire invariance - then it simplifies to (6.2).

6.2. **Extension to dividends.** Consider m assets with positive price processes $\hat{A}^i$ and continuous dividend yields y_t^i. When there exist traded or replicable zero-dividend assets A^i such that $A_T^i = \hat{A}_T^i$, the problem reduces to pricing and hedging (European) options on the A^i.

All that is required is that the $2m$ assets A^i and $\tilde{A}^i$ be arbitrage free, where

$$\tilde{A}_t^i := e^{\int_0^t y_s^i ds} \hat{A}_t^i$$

is the price of the zero-dividend asset that initially buys one share of $\hat{A}$ and thereon continually reinvests all dividends in $\hat{A}$ itself. (When y^i is deterministic, this requires $A_t^i = e^{-\int_t^T y_s^i ds} \hat{A}_t^i$.)

For instance, consider an exchange option ($m = 2$). Say $\hat{A}$ and $\hat{B}$ are the yen/dollar and yen/Euro exchange rates viewed as yen-denominated dividend assets. Then A is the yen-value of the U.S. T-maturity zero-coupon bond and $\tilde{A}$ is the yen-value of the U.S. money market asset. This exchange option is equivalent to a Euro-denominated call struck at 1 on the Euro/dollar exchange rate $\hat{A}/\hat{B}$. The ratio A/B is the *forward* Euro/dollar exchange rate. If it has deterministic volatility, we are as in a setting of [7] with results similar to next section.

6.3. **Change of numeraire.** For the exchange option, one has to calculate $\mathbb{E}\,(X-Y)^+$ for certain integrable random variables X and $Y>0$. Such expectations often become more tractable by a change of measure as in [4]. Define the equivalent probability measure $\mathbb{Q}$ by

$$\frac{d\mathbb{Q}}{d\mathbb{P}} := \frac{Y}{\mathbb{E}(Y)}.$$

Clearly,

$$\mathbb{E}^{\mathbb{Q}}(\frac{X}{Y}) = \frac{\mathbb{E}(X)}{\mathbb{E}(Y)}. \tag{6.4}$$

Replacing X by $(X-Y)^+$ in (6.4) and using the homogeneity to factor out Y,

$$\mathbb{E}\,(X-Y)^+ = \mathbb{E}(Y)\mathbb{E}^{\mathbb{Q}}(\frac{X}{Y}-1)^+. \tag{6.5}$$

If X/Y is $\mathbb{Q}$-lognormally distributed then (6.4) and (6.5) readily yield,

$$\mathbb{E}\,(X-Y)^+ = \mathbb{E}(X)N(\frac{\log(\mathbb{E}X/\mathbb{E}Y)}{\sqrt{\nu^{\mathbb{Q}}}} + \frac{\sqrt{\nu^{\mathbb{Q}}}}{2}) - \mathbb{E}(Y)N(\frac{\log(\mathbb{E}X/\mathbb{E}Y)}{\sqrt{\nu^{\mathbb{Q}}}} - \frac{\sqrt{\nu^{\mathbb{Q}}}}{2}), \tag{6.6}$$

where $\nu^{\mathbb{Q}} := \mathrm{var}^{\mathbb{Q}}[\log(X/Y)]$ and $N(\cdot)$ denotes standard the normal distribution function.

If X and Y are bivariately lognormally distributed, as in Merton's and Margrabe's models, then it is not difficult to show that X/Y is lognormally distributed in both $\mathbb{P}$ and $\mathbb{Q}$ with the same log-variance $\nu^{\mathbb{Q}} = \nu := \mathrm{var}[\log(X/Y)]$. Then $\nu^{\mathbb{Q}}$ can be replaced with ν in (6.6).

References

[1] Black, F., M. Scholes, M.: The Pricing of Options and Corporate Liabilities. *Journal of Political Economics*, 81, 637-59, (1973).

[2] Delbaen, F. Schachermayer, W.: *The Mathematics of Arbitrage*, Springer (2006).

[3] Duffie, D.: *Dynamic Asset Pricing Theory*, third edition, Princeton University Press (2001).

[4] El-Karoui, N., Geman, H., Rochet, J.C.: Change of numeraire, change of probability measure, and option pricing, *Journal of Applied Probability* 32, 443-458 (1995).

[5] Harrison, M.J., Kreps , D.M.: Martingales and arbitrage in multiperiod securities markets. *J. Econ. Theory* 20, 381-408 (1979).

[6] Harrison, M.J., Pliska, S.: Martingales and stochastic integrals in the theory of continuous trading. *Stoc Proc Appl*, 11, 215-260 (1981).

[7] Jamshidian, F: Options and Futures Evaluation with Deterministic Volatilities. *Mathematical Finance 3*(2), 149-159 (1993).

[8] Margrabe, W.: The Value of an Option to Exchange One Asset for Another. *Journal of Finance* 33, 177-86 (1978).

[9] Merton, R: Theory of Rational Option Pricing. *Bell Journal of Economics* 4(1), 141-183 (1973).

[10] Neuberger, A.: Pricing Swap Options Using the Forward Swap Market. IFA Preprint (1990).