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Abstract

This paper establishes the link of microstructure and macroeconomic factors to the time-varying conditional correlation of foreign exchange and excess equity returns. By using the proposed DCC model with exogenous variables, capital flows and interest rate differentials are shown to be significant factors in driving this conditional correlation. Furthermore, using this model it provides evidence of the dynamic behavior of global investors as they seek parity in equity returns between home and foreign markets to reduce exchange rate risks.

JEL: C32, F31, G15

Key words: uncovered equity parity, order flow, ADCCX

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The software programs used in this paper were written in Eviews and are available from the author upon request. Comments are welcome.

1. Introduction

Short-run dynamics of nominal exchange rates are difficult to predict using macroeconomic models. Meese and Rogoff (1983a, 1983b) and the survey of the literature by Frankel and Rose (1995) have shown the failure of these models to capture the behavior of exchange rates in short horizons. Again Rogoff (2001) maintained such observation of macroeconomic based exchange rate models. However, the recent shift from macroeconomic to microstructure approach gave rise to more plausible models that can account for a large proportion of the variations in the movement of exchange rates. In microstructure models of exchange rates, Evans and Lyons (2002a, 2002b) showed that order flow can explain 45% to 78% of the variation of the daily returns of the most liquid currencies. It is defined by Evans and Lyons (2002a) as a measure of buying and selling pressure or simply the difference between buyer-initiated and seller-initiated trade.

Related to order flow is the movement of equities across financial markets. Hau and Rey (2004) showed that equity flows have grown from 4% of GDP for 1975 in the United States (US) to 245% of the GDP in 2000 and argued that this movement in equity significantly influences the short-run dynamics of foreign exchange balances. In this interaction between equity and exchange rate, Brooks, et al. (2001) observed that there is a negative correlation between foreign exchange return and *excess* equity return.

Hau and Rey (2006) referred to this phenomenon of negative correlation as uncovered equity parity. They explained that home equity return in excess of foreign equity return corresponds to the depreciation of the home currency. The depreciation is driven by domestic purchases of foreign equities to reduce exchange rate risks. Under complete market assumption this risk can be hedged and eliminated but Levich, et al. (1999) found that only a small fraction of institutional investors actually hedge exchange rate risks. So although the foreign exchange market is very liquid there are limits to the foreign exchange arbitrage trading that investors may conduct in a complete market setting according to Hau and Rey (2004).

Hau and Rey (2006) provided a plausible explanation to how equity and exchange returns relate to each other in integrated financial markets using the idea of portfolio shifts. Asset allocation changes produce capital flows that find their way to the foreign exchange market. They also argued that exchange rates are primarily a function of investment flows resulting from limited forex arbitrage of risk-averse speculators.

Furthermore, Hau and Rey (2004) posited that portfolio rebalancing moves the conditional correlation between equity and foreign exchange returns. Hau and Rey (2006) assumed that this correlation structure between foreign exchange return and excess equity return is constant, although they did consider a structural change in the correlation between two periods.

In exchange rate microstructure theory, Evans and Lyons (2002a) demonstrated that foreign exchange order flow and the exchange rate are not endogenous although both are simultaneously determined. They found that the innovations in the exchange rate are driven largely by order flow but not the other way. This phenomenon supports what Evans and Lyons (2002a) called the pressure hypothesis where the causality goes from order flow to exchange rates. This observed dynamics are consistent with the theoretical models of Glosten and Milgrom (1985) and Kyle (1985) and the empirical investigation of Evans and Lyons (2002b, 2006), Payne (2003) and Froot and Ramadorai (2005) where order flows provide information about payoffs and they therefore drive prices.

Obstfeld and Rogoff (2000) have observed that fundamentals fail to explain the movement of exchange rates. However, Hau and Rey (2006) showed that correlation exists between foreign exchange return and capital flows while Evans and Lyons (2002a, 2006) used regression to show that order flows and interest rate differentials significantly impact the foreign exchange return.

This paper contributes in the current literature by determining whether capital flows and interest rate differentials significantly drive, and in what direction, the time-varying conditional correlation of foreign exchange and excess equity returns. This differs largely from the problem being addressed currently in the literature where the response variable is foreign exchange returns. The literature also typically use regression to measure the impact of order flow on exchange rate returns like in Evans and Lyons (2002a) and Dunne, et al. (2004), while this paper innovates by employing a proposed conditional correlation model with exogenous variables to link the impact of two relevant variables on conditional correlation dynamics.

The dynamic conditional correlation (DCC) model of Engle (2002) and its extensions are widely used in the volatility literature and some of its applications in finance have been made by Manera, et al. (2006) on spot and forward oil price returns, Cappiello, et al. (2006) on international bond and equity returns, Billio, et al. (2006) sectoral asset allocation, Lanza, et al. (2006) on oil forward and future prices, Kuper and Lestano (2007) on stock markets and interest rates, among others.

Although Hafner and Franses (2003), Cappiello, et al. (2006) and Feng (2006) suggested a DCC model with exogenous variables they did not pursue it, but this is a natural extension in the estimation of DCC models where the determinants of time-varying conditional correlation are directly incorporated into the model.

The paper has two main contributions. First, an extension of the DCC model is proposed by including exogenous variables in the evolution of the time-varying conditional correlation. And second, using this DCC model it is shown that the time-varying conditional correlation of the foreign exchange and excess equity returns varies across time and is driven by capital flows and interest rate differentials.

2. Asymmetric DCC Model with Exogenous Variables (ADCCX)

The DCC model of Engle (2002) has the following specification. Let y_t be an $N \times 1$ vector of asset returns and $\Im_{t-1}$ a sigma algebra of information up to time $t-1$, without loss of generality μ_t is assumed to be zero, so

$$
\begin{aligned}
&y_t = \mu_t + \varepsilon_t \\
&\varepsilon_t = H_t^{1/2} u_t \text{ where } u_t \sim N(0,\mathrm{I}) \qquad (1) \\
&\varepsilon_t \mid \Im_{t-1} \sim N(0, H_t).
\end{aligned}
$$

The conditional covariance matrix H_t can be expressed as a function of the DCC,

$$H_t = D_t R_t D_t = \left(\rho_{ij,t}\sqrt{h_{ii,t}h_{jj,t}}\right) \qquad (2)$$

$$R_t = Q_t^{*-1} Q_t Q_t^{*-1}, \text{ where } Q_t^* = diag\left(\sqrt{q_{ii,t}}\right) \qquad (3)$$

where Q_t evolves according to

$$\left(\overline{Q} - \mathrm{A}'\overline{Q}\mathrm{A} - \mathrm{B}'\overline{Q}\mathrm{B}\right) + \mathrm{A}'(\varepsilon_{t-1}^{*}\varepsilon_{t-1}^{*}{}')\mathrm{A} + \mathrm{B}'Q_{t-1}\mathrm{B}. \qquad (4)$$

This model was extended by Cappiello, et al. (2006) to include asymmetric effects, that is Q_t evolves according to

$$\left(\overline{Q} - \mathrm{A}'\overline{Q}\mathrm{A} - \mathrm{B}'\overline{Q}\mathrm{B} - \Gamma'\overline{N}\Gamma\right) + \mathrm{A}'(\varepsilon_{t-1}^{*}\varepsilon_{t-1}^{*}{}')\mathrm{A} + \mathrm{B}'Q_{t-1}\mathrm{B} + \Gamma'(n_{t-1}n_{t-1}{}')\Gamma \tag{5}$$

which is the Asymmetric DCC (ADCC) model.

Here $\varepsilon_t^{*} \sim N\left(0, R_t\right)$ is an $N \times 1$ vector of standardized residuals where $\varepsilon_{i,t}^{*} = \varepsilon_{i,t} h_{ii,t}^{-\frac{1}{2}}$ and $n_t = I\left(\varepsilon_t^{*} < \tau\right) \circ \varepsilon_t^{*}$ captures the asymmetric effects and where τ is typically set to zero. A, B and Γ are $N \times N$ diagonal matrices where $\mathrm{A} = diag\left(\sqrt{\alpha}\right)$, $\mathrm{B} = diag\left(\sqrt{\beta}\right)$ and $\Gamma = diag\left(\sqrt{\eta}\right)$. To ensure positive definiteness of Q_t it is assumed that α, β and η are non-negative coefficients satisfying $\alpha + \beta + \delta\eta < 1$ where δ is the maximum eigenvalue of $\overline{Q}^{-\frac{1}{2}}\left(\overline{N}\right)\overline{Q}^{-\frac{1}{2}}$ which was derived by Cappiello, et al. (2006). Furthermore, $\hat{\overline{Q}} = T^{-1}\sum_{t=1}^{T} \varepsilon_t^{*}\varepsilon_t^{*}{}'$ and $\hat{\overline{N}} = T^{-1}\sum_{t=1}^{T} n_t n_t{}'$ serve as estimators of $\overline{Q}$ and $\overline{N}$, respectively.

In this paper, a model of ADCC which incorporates exogenous variables that drive the time-varying conditional covariance is proposed. Let X_t be a $p \times 1$ vector of exogenous variables, ξ be a $p \times 1$ vector of parameters and K be an $N \times N$ matrix that can either be an identity matrix or matrix of ones. The following specification for the proposed model has the following evolution of Q_t,

$$\begin{aligned}&\left(\overline{Q} - \mathrm{A}'\overline{Q}\mathrm{A} - \mathrm{B}'\overline{Q}\mathrm{B} - \Gamma'\overline{N}\Gamma - \mathrm{K}\xi'\overline{X}\right)\\ &\qquad + \mathrm{A}'(\varepsilon_{t-1}^{*}\varepsilon_{t-1}^{*}{}')\mathrm{A} + \mathrm{B}'Q_{t-1}\mathrm{B} + \Gamma'(n_{t-1}n_{t-1}{}')\Gamma + \mathrm{K}\xi'X_{t-1}\end{aligned} \tag{6}$$

which is called ADCCX, where $\hat{\overline{X}} = T^{-1}\sum_{t=1}^{T} X_t$ is the estimator of $\overline{X}$. It can be easily shown that the ADCCX regresses to a DCCX model if $\eta = 0$; to the ADCC model if $\xi = 0$; and, to the DCC model if $\eta = 0$ and $\xi = 0$.

To ensure the positive definiteness of Q_t, K can be assumed as an identity matrix. It is further specified that $\xi' = \left(\xi_1 \quad \cdots \quad \xi_p\right)'$ where $\xi_k = \sqrt{\xi_k^{(k)}}$ be $\xi_k^{(k)} \in \left(0,1\right)$. This condition on ξ_k, however, might be very restrictive because it implies that the exogenous variables only drive the conditional variances $q_{ii,t}$ but not the conditional covariances $q_{ij,t}$ where $i \neq j$. However, since the conditional correlation $r_{ij,t}$ is equal to $q_{ij,t}\left(q_{ii,t}q_{jj,t}\right)^{-1/2}$, it is still indirectly a function of the exogenous variables. This restriction may be relaxed by setting K as a matrix of ones instead.

Another concern about setting $\xi_k = \sqrt{\xi_k^{(k)}}$ is that it restricts the sign of the parameters to be non-negative. This is very limited and does not allow for the exogenous variable to have a negative impact on the conditional covariance Q_t. A remedy would be to allow ξ_k to take on a positive or negative value when K is an identity matrix provided that the positive definiteness of Q_t, $\forall t$ is not violated.

The maximum likelihood estimation of the ADCCX model is given in the Appendix.

3. Data

The excess equity return, foreign exchange rate return, and capital flow data were sourced from the Princeton University website of Hélène Rey. The data included in this study are from the most liquid and largest equity and foreign exchange markets in Europe: Germany and Great Britain, vis-à-vis the United States (US). The home country refers to the US while the other two are the foreign country. The data consists of monthly observations from January 1980 to December 2001 for a total of 264 observations.

Excess equity return is defined as the difference between the log foreign stock market index return and the log US stock market index return, $dS_t^{f*} - dS_t^h$, where both returns are in their corresponding local currencies. The foreign exchange return dE_t is defined as the log return of E_t where E_t is in foreign currency per US dollar. This means that $-dE_t > 0$ is the foreign currency's appreciation against the dollar. Capital flows is defined as the difference between the net foreign equity purchases by US residents and the net US equity purchases by foreigners normalized by the average flows in the past 12 months, $dK_t^f - dK_t^{h*}$.

Interest rate differential $i_t^{f*} - i_t^h$ is defined as the difference between end-of-the-month yields of the foreign and home interest rates. With Great Britain and the US the spread is the difference between 3-month T-bill yields downloaded from of the Bank of England and the US Federal Reserve websites while between Germany and the US, it is between 1-year T-bond yields, taken from EconStat.com.

4. Results and Discussion

The following DCC models will be evaluated:

$$R_t\left(-dE_t, \left(dS_t^{f*} - dS_t^h\right)\right) = f\left(Q_t\right) \tag{7}$$

where Q_t is given by the ADCCX model in Eq. (6)

$$\begin{aligned} &\overline{Q} - \mathrm{A}'\overline{Q}\mathrm{A} - \mathrm{B}'\overline{Q}\mathrm{B} - \Gamma'\overline{N}\Gamma - \mathrm{K}\xi_1\overline{dK} - \mathrm{K}\xi_2\overline{di} \\ &\quad + \mathrm{A}'(\varepsilon_{t-1}^*\varepsilon_{t-1}^*{}')\mathrm{A} + \mathrm{B}'Q_{t-1}\mathrm{B} + \Gamma'(n_{t-1}n_{t-1}{}')\Gamma + \mathrm{K}\xi_1\left(dK_{t-1}^f - dK_{t-1}^{h*}\right) + \mathrm{K}\xi_2 d\left(i_{t-1}^{f*} - i_{t-1}^h\right), \end{aligned} \tag{8}$$

where $\overline{dK}$ and $\overline{di}$ equal to the mean of $dK_t^f - dK_t^{h*}$ and $d\left(i_t^{f*} - i_t^h\right)$, respectively. Eq. (8) simplifies to ADCC, DCC and DCCX models by setting the appropriate parameters to zero.

In Table 1 the GARCH(1,1) models of foreign exchange returns as well as the excess equity returns are reported. Foreign exchange returns exhibit heteroskedasticity based on the significant coefficients of the GARCH models. The pound and the mark demonstrate persistency in the conditional variance even at the monthly returns. The volatility of excess equity returns is highly persistent for British equities while German equities display large short-run shocks.

Table 1. GARCH(1,1) Models of Foreign Exchange and Excess Equity Returns

Parameters	Foreign Exchange Returns		Excess Equity Returns	
	Germany	Great Britain	Germany	Great Britain
$a_0^{\#}$	0.0010	0.0002	0.0230**	0.0022
	(0.0010)	(0.0002)	(0.0095)	(0.0018)
a_1	0.0214	0.0696**	0.1707**	0.1266**
	(0.0340)	(0.0338)	(0.0833)	(0.0577)
b_1	0.8853***	0.9098***	0.0000	0.7103***
	(0.1119)	(0.0398)	(0.2908)	(0.1607)

$a_0^{\#}$ is multiplied by a factor of 10

Turning now to the DCC models, in Table 2 the parameter estimates of four models arising from Eq. (8) are reported. The ADCC models for both Germany and Britain indicate that there is no asymmetric effect between foreign exchange and excess equity returns since η is not significant. Although not reported here, the estimated parameters of the exogenous variables in the ADCCX models for both markets were not significant. The absence of asymmetric effect between foreign exchange and excess equity returns is the important result in this exercise.

Table 2. DCC and DCCX Models of Foreign Exchange and Excess Equity Returns

Parameters	Germany				Great Britain			
	ADCC	DCC	DCCX1	DCCX2	ADCC	DCC	DCCX1	DCCX2
α	0.0338 (0.0248)	0.0160 (0.0287)	0.0174 (0.0224)	0.0350* (0.0187)	0.0150 (0.0409)	0.0207 (0.0444)	0.0174 (0.0324)	0.0009 (0.0337)
β	0.9349*** (0.0569)	0.9518*** (0.0983)	0.9494*** (0.0450)	0.9254*** (0.0307)	0.8790** (0.0810)	0.8943*** (0.1095)	0.9196*** (0.0461)	0.9285*** (0.0465)
η	0.0105 (0.0524)	–	–	–	0.0986 (0.0710)	–	–	–
ξ_1	–	–	-0.0054 (0.0065)	-0.0151** (0.0065)	–	–	-0.0122 (0.0080)	-0.0163* (0.0088)
ξ_2	–	–	–	0.5203* (0.2781)	–	–	–	0.4834** (0.2135)
AIC	2.0066	1.9932	2.0031	2.0245	2.0133	2.0073	2.0179	1.9937
SIC	2.0473	2.0203	2.0438	2.0789	2.0540	2.0345	2.0587	2.0480
Log L	-260.86	-260.10	-260.40	-262.23	-261.75	-261.96	-262.36	-258.17

Note: K is a matrix of ones.

The conditional correlation of foreign exchange and excess equity returns are highly persistent as shown by the significant parameter estimates of the DCC models and indicate that the correlation between the two is indeed time-varying for both markets.

The sign of the parameter estimates of capital flows, ξ_1, is correct and is significant for both markets in the second DCCX model, DCCX2. For the British market the loglikelihood ratio test between the DCC and the DCCX2 model is significant at the 10% level. This further supports the hypothesis that capital flows together with interest rate differentials significantly account for the time-varying conditional correlation of foreign exchange and excess equity returns. Figures 1 and 2 present the plots of the time-varying conditional correlation for the two markets. The inclusion of capital flows, even in the DCCX1, clearly accentuates the magnitude of correlation. Recall that the negative correlation between foreign exchange and excess equity returns supports the uncovered equity parity assertion of Hau and Rey (2006).

Figure 1. Dynamic Conditional Correlations of the Mark and the German Excess Equity Returns v.v. the US

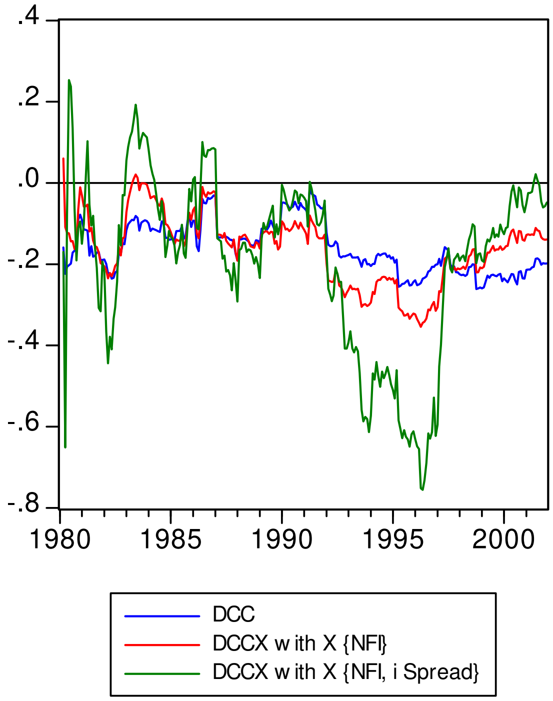
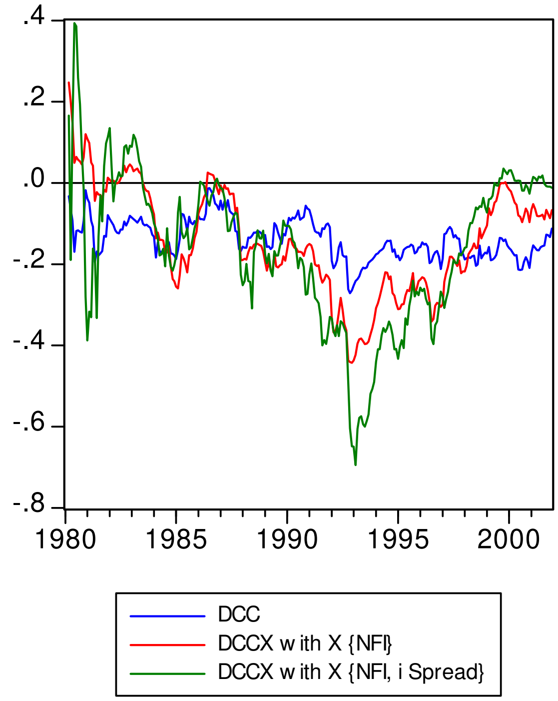


Figure 2. Dynamic Conditional Correlations of the Pound and the British Excess Equity Returns v.v. the US



The outcome of this estimation shows that net capital flows from the home to the foreign market results in foreign currency appreciation that stabilizes the disparity in equity returns between the home and foreign market when there is excess home equity return. This means that capital flows, consistent with the exogenous assumption of Evans and Lyons (2002a, 2006) and Froot and Ramadorai (2005), act to bring equity returns to parity to reduce the exchange rate risk involved when either equity markets have a higher return than the other. It is this portfolio rebalancing explanation of Hau and Rey (2004) that is observed here and the parity is important for global investors because they are expected to minimize the variance in their portfolio holdings in the sense of the classic Markowitz's efficient frontier.

The parameter estimate of ξ_2 of the interest rate differentials is positive and is also correctly signed for both markets and it is significant for the British market. If the foreign interest rate rises it makes the foreign assets more attractive than before and this results in net movement of capital flows towards it resulting in excess foreign equity return. However, the foreign currency depreciates in accordance with uncovered interest parity.

The significance of the capital flows and interest rate differentials implies that the correlation dynamics of foreign exchange and excess equity returns are subject to both microstructure and macroeconomic factors, at least in the sense of capital flows and interest rates, respectively.

5. Conclusion

The extension of the DCC models to include exogenous variables is a natural direction to take in order to identify the factors that drive the time-varying conditional correlation between asset returns. By employing the DCC model, this paper shows that the correlation between foreign exchange and excess equity returns is time-varying. The proposed DCC models with exogenous variables provide a convenient tool for characterizing this time-varying correlation as a function of capital flows and interest rate differentials.

The optimizing behavior of global investors shows that they seek equity parity to minimize the foreign exchange risk in their portfolios. This paper shows that this behavior results in capital flow movements that adjust both the exchange rate and equity returns in both home and foreign financial markets to satisfy uncovered equity parity.

Capital flows contain information about investor decisions, in the microstructure context, and is significant in explaining the time-varying conditional correlation of the foreign exchange and excess equity returns. This confirms that investor behavior is a rich source of information that can account for the short-run dynamics of foreign exchange rate. Furthermore, the interest rate differentials represent macroeconomic information that also significantly drives this time-varying correlation.

The significance of capital flows and interest rate differentials establishes the link of microstructure and macroeconomic factors with the short-run dynamics of foreign exchange and equity returns using DCC models with exogenous variables.

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APPENDIX

Maximum Likelihood Estimation of the ADCCX Model

The likelihood function under the assumption of multivariate normality of y_t is given by

$$L(\theta \mid y_t) = \prod_{t=1}^{T}\left[\frac{1}{\left(\sqrt{2\pi}\right)^N |H_t|^{1/2}} e^{-\frac{1}{2} y_t H_t^{-1} y_t'}\right].$$

Using the two-stage LIML procedure proposed by Engle (2002) the likelihood function is maximized with respect to two sets of parameters in succeeding steps.

The vector θ consists of GARCH parameters for each element of the N-dimensional y_t and the parameters of Q_t, where $y_t = \varepsilon_t$. Engle and Sheppard (2001) have shown the consistency and asymptotic normality of this two-stage procedure. The loglikelihood function is

$$\log L(\theta_1, \theta_2 \mid y_t) = -\frac{1}{2}\sum_{t=1}^{T}\left(N\log(2\pi) + \log|H_t| + y_t H_t^{-1} y_t'\right)$$

$$= -\frac{1}{2}\sum_{t=1}^{T}\left(N\log(2\pi) + \log|R_t| + 2\log|D_t| + y_t' D_t^{-1} R_t^{-1} D_t^{-1} y_t\right)$$

where θ_1 consists of parameters of the MGARCH model, θ_2 consists of parameters of Q_t. Furthermore, $H_t = D_t R_t D_t$ and $D_t = diag\left(h_{11,t}^{1/2} \ldots h_{NN,t}^{1/2}\right)$. Engle and Sheppard (2001) set R_t as the identity matrix in the first stage estimation,

$$\log L(\theta_1 \mid y_t) = -\frac{1}{2}\sum_{t=1}^{T}\left(N\log(2\pi) + \log|I_N| + 2\log|D_t| + y_t' D_t^{-1} I_N^{-1} D_t^{-1} y_t\right)$$

$$\hat{\theta}_1 = \arg\max[\log L(\theta_1 \mid y_t)]$$

which is equivalent to estimation of the univariate GARCH models of y_t.

The second stage estimation involves

$$\log L(\theta_2 \mid \hat{\theta}_1, y_t) = -\frac{1}{2}\sum_{t=1}^{T}\left(N\log(2\pi) + \log|R_t| + 2\log|\hat{D}_t| + y_t' \hat{D}_t^{-1} R_t^{-1} \hat{D}_t^{-1} y_t\right)$$

where $\varepsilon_t^* = \hat{D}_t^{-1} y_t$. And since $R_t = Q_t^{*-1} Q_t Q_t^{*-1}$ where $Q_t^* = diag\left(\sqrt{q_{iit}}\right)$

$$\log L(\theta_2 \mid \hat{\theta}_1, y_t) = -\frac{1}{2}\sum_{t=1}^{T}\left(N\log(2\pi) + \log\left|Q_t^{*-1}Q_tQ_t^{*-1}\right| + 2\log\left|\hat{D}_t\right| + \varepsilon_t^{*}{}'(Q_t^{*-1}Q_tQ_t^{*-1})^{-1}\varepsilon_t^{*}\right).$$

The constant terms $N\log(2\pi)$ and $2\log\left|\hat{D}_t\right|$ are not necessary in the maximization and are dropped from the function so that

$$\log L'(\theta_2 \mid \hat{\theta}_1, y_t) = -\frac{1}{2}\sum_{t=1}^{T}\left(\log\left|Q_t^{*-1}Q_tQ_t^{*-1}\right| + \varepsilon_t^{*}{}'(Q_t^{*-1}Q_tQ_t^{*-1})^{-1}\varepsilon_t^{*}\right)$$

$$\hat{\theta}_2 = \arg\max[\log L'(\theta_2 \mid \hat{\theta}_1, y_t)].$$

An expansion of the second stage loglikelihood function is

$$\log L'(\theta_2 \mid \hat{\theta}_1, y_t) = -\frac{1}{2}\sum_{t=1}^{T}\Big\{\log\left|Q_t^{*-1}\left(\tilde{Q} + \mathrm{A}'(\varepsilon_{t-1}^{*}\varepsilon_{t-1}^{*}{}')\mathrm{A} + \mathrm{B}'Q_{t-1}\mathrm{B} + \Gamma'(n_{t-1}n_{t-1}{}')\Gamma + \mathrm{K}\xi' X_{t-1}\right)Q_t^{*-1}\right|$$
$$+ \varepsilon_t^{*'}\left(Q_t^{*-1}\left(\tilde{Q} + \mathrm{A}'(\varepsilon_{t-1}^{*}\varepsilon_{t-1}^{*}{}')\mathrm{A} + \mathrm{B}'Q_{t-1}\mathrm{B} + \Gamma'(n_{t-1}n_{t-1}{}')\Gamma + \mathrm{K}\xi' X_{t-1}\right)Q_t^{*-1}\right)^{-1}\varepsilon_t^{*}\Big\}$$

where

$\tilde{Q} = \left(\overline{Q} - \mathrm{A}'\overline{Q}\mathrm{A} - \mathrm{B}'\overline{Q}\mathrm{B} - \Gamma'\overline{N}\Gamma - \mathrm{K}\xi'\,\overline{X}\right)$ and

$Q_t = \tilde{Q} + \mathrm{A}'(\varepsilon_{t-1}^{*}\varepsilon_{t-1}^{*}{}')\mathrm{A} + \mathrm{B}'Q_{t-1}\mathrm{B} + \Gamma'(n_{t-1}n_{t-1}{}')\Gamma + \mathrm{K}\xi' X_{t-1}$.

The maximum likelihood estimators of ADCC, DCC and DCCX models can be derived by setting the appropriate parameters to zero.