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Abstract

This paper extends the idea in "Analysis of average value of a Fourier series using z-transform" by the author. The main difference is that a three-pole filter is used instead of a two-pole filter. This paper reaches qualitatively the same conclusion.

1 Introduction

The core argument has been presented in [3], including econometrics and graph theory examples. Rather than repeating the argument, I will present the parts that differ from the referenced paper.

2 Three-pole filter

The filter is:

$$H(z) = \frac{1}{(z-p_1)(z-p_2)(z-p_3)} \tag{1}$$

In practice, it is better to have z^3 in the numerator - since the ordinary assumption is to have filter output before time $t = 0$ to be zero, along with filter input being zero before $t = 0$. However, this can easily be taken care of, even with Equation 1. The above form allows us to focus on what is important.

Definition 2.1 $(\mu_1, \mu_2, \gamma_1, \gamma_2, \gamma_3)$**.** $\mu_1(t) = e^{iut}$, $\mu_2(t) = 1$. $\gamma_1 = 1 - p_1$, $\gamma_2 = 1 - p_2$, $\gamma_3 = 1 - p_3$.

Let $\gamma_2 \gg |u|$, $\gamma_1, \gamma_3 \ll |u|$, $\gamma_3 \ll \gamma_1$.
The focus will be on $1/{\gamma_2}^2$ terms.

$$H(z)\mu_1(z) = \frac{z}{(z-p_1)(z-p_2)(z-p_3)(z-e^{iu})} \tag{2}$$

Taking the inverse z-transform residue for pole p_1 and subtracting away $t = 0$ part at $t = 1$:

$$\frac{-\gamma_1}{(\gamma_2 - \gamma_1)(\gamma_1 - \gamma_3)(\gamma_1 + iu - \frac{u^2}{2} + ..)} \tag{3}$$

The dominant term in the denominator is $\gamma_2\gamma_1 iu$. Thus, the initial term to be expanded is:

$$\frac{-1}{\gamma_2 iu}$$

The idea of expanding factors is based on the following principle. Suppose that $d = c/(a + b + e + f + ...)$, and a is the dominant term. The first term will be c/a.

$$d - \frac{c}{a} = \frac{(-c/a)(a + b + e + ...)}{(a + b + e + f..)}$$

Notice that in the numerator, $b, e, f..$ can be noticed, and in the denominator, the dominant term a appears again. Thus $b/a, e/a, f/a$ are referred to as expanding factors.

A sequence of multiplying expanding factors can be used to obtain constant terms - first let us see the expanding factors that are dependent strictly on $1/u$ or are constant terms. First for $1/u$-dependent expanding factors:

- $$\frac{\gamma_2{\gamma_1}^2}{\gamma_2\gamma_1 iu} = \frac{\gamma_1}{iu} \ \ [a, 1, 1]$$
- $$\frac{-\gamma_2\gamma_3\gamma_1}{\gamma_2\gamma_1 iu} = \frac{-\gamma_3}{iu} \ \ [a, 1, 2]$$
- $$\frac{-{\gamma_1}^3}{\gamma_2\gamma_1 iu} = \frac{-{\gamma_1}^2}{\gamma_2 iu} \ \ [a, 1, 3]$$
- $$\frac{{\gamma_1}^2\gamma_3}{\gamma_2\gamma_1 iu} = \frac{\gamma_1\gamma_3}{\gamma_2 iu} \ \ [a, 1, 4]$$

Multiplying the additive inverse of the initial term by $[a, 1, 1]$:

$$\frac{1}{\gamma_2 iu}\frac{\gamma_1}{iu} = \frac{-\gamma_1}{\gamma_2 u^2}$$

I will make this term to be smaller than $1/{\gamma_2}^2$ by assuming:

- $$\gamma_2 \ll 1$$
- $$\gamma_1 \ll u^2$$

Other expanding terms labelled with $[a, 1,_]$ are all smaller than $[a, 1, 1]$, and thus will be ignored.
Now the constant expanding factors.

- $$\frac{-\gamma_2\gamma_3 iu}{\gamma_2\gamma_1 iu} = \frac{-\gamma_3}{\gamma_1} \quad [a, 2, 1]$$
- $$\frac{-{\gamma_1}^2 iu}{\gamma_2\gamma_1 iu} = \frac{-\gamma_1}{\gamma_2} \quad [a, 2, 2]$$
- $$\frac{\gamma_1\gamma_3 iu}{\gamma_2\gamma_1 iu} = \frac{\gamma_3}{\gamma_2} \quad [a, 2, 3]$$

I will assume:

- $$\frac{\gamma_3}{\gamma_1} \ll \frac{\gamma_1}{\gamma_2}$$

Multiplying the additive inverse of the initial term by $[a, 2, 2]$:

$$\frac{1}{\gamma_2 iu}\frac{-\gamma_1}{\gamma_2} = \frac{-\gamma_1}{{\gamma_2}^2 iu}$$

Since $\gamma_1 \ll |u|$, the product is smaller than $1/{\gamma_2}^2$. Other expanding factors will be ignored.
Thus to eliminate u in the denominator of the additive inverse of the initial term, only a single expanding factor dependent on u^1 and no higher powers can lead to $1/{\gamma_2}^2$ terms. The expanding factor with the greatst magnitude dependent on single power of u, u^1, is:

$$\frac{\gamma_2\gamma_1\frac{-u^2}{2}}{\gamma_2\gamma_1 iu} = \frac{-u}{2i}$$

Multiplying the additive inverse of theinitial term,

$$\frac{1}{\gamma_2 iu}\frac{-u}{2i} = \frac{1}{2\gamma_2}$$

Since $\gamma_2 \ll 1$, this term is smaller than $1/{\gamma_2}^2$.
Now to pole p_2. The inverse z-transform residue calculation at $t = 1$ after subtracting the $t = 0$ part results in:

$$\frac{-\gamma_2}{(\gamma_1 - \gamma_2)(\gamma_2 - \gamma_3)(\gamma_2 + iu - \frac{u^2}{2} - ...)} \tag{4}$$

The initial term is:

$$\frac{1}{{\gamma_2}^2} \tag{5}$$

I will stop here, since this is the term being searched for.
Now to pole p_3. The inverse z-transform residue calculation at $t = 1$ after subtracting the $t = 0$ part results in:

$$\frac{-\gamma_3}{(\gamma_1 - \gamma_3)(\gamma_3 - \gamma_2)(\gamma_3 + iu - \frac{u^2}{2} - ..)} \tag{6}$$

The initial term is $\gamma_3/(\gamma_1\gamma_2 iu)$. By the previously assumed conditions, the initial term is smaller than $1/{\gamma_2}^2$. Thus, pole p_3 will be ignored.
Now to pole $p_4 = e^{iu}$. The residue calculation after adjustments:

$$\frac{iu - \frac{u^2}{2} - ..}{(\gamma_1 + iu - \frac{u^2}{2} - ..)(\gamma_2 + iu - \frac{u^2}{2} - ..)(\gamma_3 + iu - \frac{u^2}{2} - ..)} \tag{7}$$

The initial terms are:

- $$\frac{iu}{-\gamma_2 u^2} = \frac{-i}{\gamma_2 u}$$
- $$\frac{-\frac{u^2}{2}}{-\gamma_2 u^2} = \frac{1}{2\gamma_2}$$
- Smaller terms continuing indefinitely

Since the second initial term is smaller than $1/{\gamma_2}^2$, it will be ignored.
To obtain a constant term from the initial term $-i/(\gamma_2 u)$, u in the denominator has to be eliminated. First, let us consider the expanding factors dependent on $1/u^2$:

- $$\frac{\gamma_1\gamma_2\gamma_3}{-\gamma_2 u^2} = \frac{-\gamma_1\gamma_3}{u^2} \quad [d, 1, 1]$$

Multiplying the additive inverse of the initial term by $[d, 1, 1]$:

$$\frac{i}{\gamma_2 u}\frac{-\gamma_1\gamma_3}{u^2} = \frac{-\gamma_1\gamma_3 i}{\gamma_2 u^3}$$

since γ_3 can be made small arbitrarily, this term will be ignored. The expanding factors dependent on $1/u$:

- $$\frac{\gamma_1\gamma_2 iu}{-\gamma_2 u^2} = \frac{-\gamma_1 i}{u}$$
- $$\frac{\gamma_1\gamma_3 iu}{-\gamma_2 u^2} = \frac{-\gamma_1\gamma_3 i}{\gamma_2 u}$$

- $$\frac{\gamma_2\gamma_3 iu}{-\gamma_2 u^2} = \frac{-\gamma_3 i}{u}$$

Note that γ_1's lower bound in magnitude is zero - there is no limit, as long as relation to γ_3 and u^2 is satisfied. Thus, one can ignore these expanding factors, because multiplying the additive inverse of the initial term by these expanding factors can be made much smaller than $1/{\gamma_2}^2$.
Now the constant expanding factors:

- $$\frac{\gamma_1\gamma_2\frac{-u^2}{2}}{-\gamma_2 u^2} = \frac{\gamma_1}{2}$$
- $$\frac{\gamma_1\gamma_3\frac{-u^2}{2}}{-\gamma_2 u^2} = \frac{\gamma_1\gamma_3}{2\gamma_2}$$
- $$\frac{\gamma_2\gamma_3\frac{-u^2}{2}}{-\gamma_2 u^2} = \frac{\gamma_3}{2}$$
- $$\frac{\gamma_1(iu)(iu)}{-\gamma_2 u^2} = \frac{\gamma_1}{\gamma_2}$$
- $$\frac{\gamma_3(iu)(iu)}{-\gamma_2 u^2} = \frac{\gamma_3}{\gamma_2}$$

As before, since γ_1 and γ_3 only has zero as magnitude lower bound, we will ignore these expanding factors.
These considerations show that to obtain the $1/{\gamma_2}^2$ terms, the only way is multiplying by an expanding factor dependent on u. The only expanding factor that works is:

$$\frac{(iu)^3}{-\gamma_2 u^2} = \frac{iu}{\gamma_2}$$

Multiplying the additive inverse of the initial term by the above expanding factor:

$$\frac{i}{\gamma_2 u}\frac{iu}{\gamma_2} = \frac{-1}{{\gamma_2}^2} \tag{8}$$

Adding up equation 5 and 8,

$$0 \tag{9}$$

Now let us consider the case when the input is $\mu_2(t) = 1$.

$$\mu_2(z)H(z) = \frac{z}{(z-p_1)(z-p_2)(z-p_3)(z-1)} \tag{10}$$

For pole p_1 the inverse z-transform calculation after adjustments:

$$\frac{-1}{(\gamma_2-\gamma_1)(\gamma_1-\gamma_3)} \tag{11}$$

The initial term is $-1/(\gamma_2\gamma_1)$ Expanding factors are:

- $$\frac{-\gamma_2\gamma_3}{\gamma_2\gamma_1}=\frac{-\gamma_3}{\gamma_1}$$
- $$\frac{-\gamma_1{}^2}{\gamma_2\gamma_1}=\frac{-\gamma_1}{\gamma_2}$$
- $$\frac{\gamma_1\gamma_3}{\gamma_2\gamma_1}=\frac{\gamma_3}{\gamma_2}$$

All of these expanding factors can be made arbitrarily small by adjusting γ_3 and γ_1. Thus, pole p_1 will be ignored.
For pole p_2, the inverse z-transform calculation after adjustments:

$$\frac{-1}{(\gamma_1-\gamma_2)(\gamma_2-\gamma_3)} \tag{12}$$

The initial term is:

$$\frac{1}{\gamma_2{}^2} \tag{13}$$

I will stop here, since this is the term being searched for.
For pole p_3, the inverse z-transform calculation after adjustments:

$$\frac{-1}{(\gamma_1-\gamma_3)(\gamma_3-\gamma_2)} \tag{14}$$

The initial term is $1/(\gamma_2\gamma_1)$.
The expanding factors are:

- $$\frac{\gamma_2\gamma_3}{-\gamma_2\gamma_1}=\frac{-\gamma_3}{\gamma_1}$$
- $$\frac{\gamma_3\gamma_1}{-\gamma_2\gamma_1}=\frac{-\gamma_3}{\gamma_2}$$
- $$\frac{\gamma_3{}^2}{\gamma_2\gamma_1}$$

These expanding factors can be made arbitrarily small by adjusting γ_3. Thus, pole p_3 will be ignored.
Finally, pole 1. But notice that $1^t = 1$, and subtracting away $t = 0$ case would obviously result in zero. Thus pole 1 can be ignored.
Now one can compare two results: Equation 9 and 13. The first one results in 0, while the second one results in $1/{\gamma_2}^2$.
Thus, by exploiting this difference, under some restrictions in addition to the Fourier series restriction, one can get the linear trend information.

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