Social preference under twofold uncertainty

Mongin, Philippe and Pivato, Marcus

CNRS HEC Paris, THEMA, Université de Cergy-Pontoise, France

5 June 2016

Social Preference Under Twofold Uncertainty

Philippe Mongin (CNRS & HEC Paris)
and
Marcus Pivato (Université de Cergy-Pontoise)*

June 5, 2016

Abstract

We investigate the conflict between the *ex ante* and *ex post* criteria of social welfare in a novel axiomatic framework of individual and social decisions, which distinguishes between a subjective and an objective source of uncertainty. This framework permits us to endow the individuals and society not only with *ex ante* and *ex post* preferences, as is classically done, but also with interim preferences of two kinds, and correspondingly, to introduce interim forms of the Pareto principle. After characterizing the *ex ante* and *ex post* criteria, we present a first solution to their conflict that amounts to extending the former as much possible in the direction of the latter. Then, we present a second solution, which goes in the opposite direction, and is our preferred one. This solution combines the *ex post* criterion with an objective interim Pareto principle, which avoids the pitfalls of the *ex ante* Pareto principle, and especially the problem of "spurious unanimity" discussed in the literature. Both solutions translate the assumed Pareto conditions into weighted additive utility representations, and both attribute common individual probability values only to the objective source of uncertainty.

Keywords: Ex ante social welfare; ex post social welfare; objective versus subjective uncertainty; Pareto principle; separability; Harsanyi social aggregation theorem; spurious unanimity.
JEL classification: D70; D81.

1 Introduction

Any normative analysis of collective decisions under uncertainty must confront an old and unresolved problem: the conflict between the *ex ante* and *ex post* criteria of social welfare. This paper proposes a new solution to this problem, based on a distinction between *subjective* and *objective* uncertainty. In our framework, agents may hold different probabilistic

*This research was completed while the first author visited Wissenschaftskolleg zu Berlin. The second author was supported by NSERC grant #262620-2008 and by Labex MME-DII (ANR11-LBX-0023-01).

beliefs about the source of subjective uncertainty, but they must hold the same beliefs about the objective source. Before explaining this resolution, we must state the conflict in its classical form, and explain what difference our twofold uncertainty framework makes.

The *ex ante social welfare criterion* assumes that the individuals form preferences over social uncertain prospects according to some normative decision theory - typically that of subjective expected utility (SEU) - and it applies the Pareto principle to these *ex ante* individual preferences, thus following an *ex ante* version of the principle. In contrast, the *ex post social welfare criterion* assumes that society itself forms preferences over social prospects according to the normative decision theory under consideration, while it endows the individuals only with state-by-state preferences. It then applies the Pareto principle statewise to these *ex post* individual preferences, thus following an *ex post* version of the principle.[1] However, if all agents are subjective expected utility (SEU) maximizers, then the *ex ante* and *ex post* criteria are incompatible. Hence we face a trilemma: we must abandon either SEU theory, or the *ex ante* social welfare criterion, or the *ex post* social welfare criterion.

This clash between the *ex ante* and *ex post* social welfare criteria has long been recognized, although the problem has been formulated in several different ways. The early statements by Starr (1973) and Hammond (1981, 1983) belonged to traditional welfare economics, and envisaged only two extreme solutions to the conflict, i.e., endorsing one of the two criteria and rejecting the other, with an overall preference for the *ex post* criterion. Mongin's (1995) abstract formulation in terms of Savage's (1972) SEU postulates avoided the domain-specific assumptions made by the welfare economists, thus sharpening the conflict, while also bringing out more complex positive solutions. He showed that it was impossible to satisfy both social welfare criteria and the Savage postulates, unless either the individual probabilities or the individual utilities exhibit strong dependencies. This axiomatic approach also facilitated comparison with Harsanyi's (1955) Social Aggregation Theorem. This theorem says that, if both individuals and society form their preferences over social lotteries according to von Neumann-Morgenstern (VNM) theory, and the social preferences satisfy the Pareto principle, then society's preferences can be represented by a weighted ("utilitarian") sum of individual utility representations.[2] As the Pareto principle applies here both *ex ante* (to lotteries) and *ex post* (to final outcomes), Harsanyi's assumptions contain all the ingredients of the two welfare criteria, and his weighted sum formula seems to contradict the claim that the two criteria are incompatible. However, the assumption of a common lottery set amounts to imposing identical probabilities on all individuals, an extreme case of dependency between individual probabilities, which is covered by Mongin's axiomatic treatment. This clarifies the sense in which Harsanyi's theorem contributes only to a limiting case of the initial problem.

The present paper will also exploit the fact that the conflict between the *ex ante* and

[1] Note the difference between a social welfare criterion and the corresponding Pareto principle. There is more to the the *ex ante* (*ex post*) social welfare criterion than just the *ex ante* (*ex post*) Pareto principle, because a criterion also decides where rationality assumptions apply (to the individuals or society).

[2] We do not claim that such weighted sums of VNM utilities have a genuine utilitarian interpretation. Harsanyi took this for granted, but Sen famously denied it, and the debate is still unsettled. See Mongin and Pivato (2016) for a review, and Fleurbaey and Mongin (2016) for a new defence of Harsanyi's position.

ex post social welfare criteria vanishes when probabilities are identical, but in a much more general fashion than Harsanyi. In our twofold uncertainty framework, if the agents have probabilistic beliefs at all, these will be the same for the objective source, but may differ for the subjective source. This amounts to endogenizing Harsanyi's lottery set assumption while narrowing down its scope. At the same time, this allows us to partially reconcile the two welfare criteria, notwithstanding the welfare economists' pessimistic belief that they are mutually exclusive. Our framework complies with standard economic methodology, taking preference relations to be the only primitives. Thus, for both society and the individuals, we obtain probability and utility functions in the conclusions (not the assumptions) of our representation theorems. We distinguish between the two sources of uncertainty by introducing *conditional* preferences, with different properties. That is, for both society and the individuals, there will be preferences *conditional on the objective source* and preferences *conditional on the subjective source*, with each kind obeying distinctive decision-theoretic conditions. This, in turn, leads to two new, interim forms of the Pareto principle, in addition to the classic *ex ante* and *ex post* forms.

Our preferred solution (Theorem 5) mediates between the *ex ante* and *ex post* social welfare criteria as follows. We obtain SEU representations for both the individuals (as in the *ex ante* criterion) and society (as in the *ex post* criterion), which apply to both the objective and subjective sources of uncertainty. The Pareto principle holds in the *ex post* sense, i.e., when all uncertainty is resolved, and also in a limited *ex ante* sense, i.e., when only the objective uncertainty remains to be resolved. We will see that this *objective interim Pareto principle* avoids the pitfalls of the full *ex ante* Pareto principle. Thus, our solution starts from the *ex post* social welfare criterion, but also encompasess part of the *ex ante* criterion. For the sake of comparison, and to acknowledge a subtle argument of Hild, Jeffrey and Risse (2003) and Risse (2003), we also introduce another mixed solution (Theorem 3), which instead starts from the *ex ante* criterion, and moves towards the *ex post* criterion.

Our solutions should be compared with the influential work of Gilboa, Samet and Schmeidler (2004) (hereafter GSS), which has recently attracted significant attention. In the same Savage-based framework as Mongin (1995), GSS assume the *ex post* criterion in full and the *ex ante* criterion in part. They limit the *ex ante* Pareto principle to comparisons of social prospects which do not involve probabilistic disagreements between the individuals. From this, they are able to conclude that (i) society's *ex post* preference is represented by a weighted ("utilitarian") sum of the individuals' utility functions, and (ii) society's probability equals an average of individual probabilities; this is often called the *linear pooling rule* (Genest and Zidek, 1986; Clemen and Winkler, 2007). Importantly, since GSS weaken the *ex ante* Pareto principle, they do *not* conclude that (iii) society's *ex ante* preference can be represented in terms of the individuals' *ex ante* representations, let alone by a weighted sum of them. Thus, they evade the impossibility theorems of Mongin (1995).

The difference between our solution and that of GSS is as follows. In the twofold uncertainty framework, we can derive not only (i), but also weighted utility sums for the *conditional* social preferences. Moreover, and perhaps more importantly, we do *not* obtain (ii), and *we see this as an advantage*. Often, the individuals have different private information; in this case, the linear pooling rule can yield the wrong answer. Society should try to infer this private information from the individual probabilities, rather than aggregate them mechanically. To illustrate this objection, we provide a simple example where the GSS solution (and linear pooling in general) is unappealing (Section 5).

If we differ from GSS on both (i) and (ii), this is because of a difference in framework. We (exogeneously) distinguish between two *sources* of uncertainty, whereas they (endogeneously) distinguish between two *kinds* of uncertainty, one in which the individuals happen to agree in their probabilistic values and the other in which they do not. Our exogenous distinction between objective and subjective uncertainty is related to the distinction made by Anscombe and Aumann (1963), but they represent objective uncertainty in terms of *lotteries* with known probabilities; in contrast, we do not assume any predefined probabilities, even for objective uncertainty.

The paper is organized as an ongoing argument. Section 2 introduces the framework, and the various preference and Pareto conditions. Section 3 axiomatizes the *ex ante* social welfare criterion (Proposition 1) and the *ex post* social welfare criterion (Proposition 2) in their most general form. Section 4 proposes our first solution, based on the *ex ante* criterion (Theorem 3). But Section 5 rejects this solution because of its vulnerability to *spurious unanimity*. It also develops the *informational objection* against the GSS solution and the linear pooling rule. Section 6 proposes our prefered solution, based on the *ex post* criterion (Theorem 5). Finally, Section 7 reviews recent literature.

2 The framework

We assume that states of the world are pairs (s, o), with s representing the subjective component of the uncertainty, and o the objective component. These components vary over finite sets $\mathcal{S}$ and $\mathcal{O}$ with $|\mathcal{S}|, |\mathcal{O}| \geq 2$. Denote by $\Delta_{\mathcal{S}}$ and $\Delta_{\mathcal{O}}$ the sets of probability vectors on $\mathcal{S}$ and $\mathcal{O}$ respectively. We assume that the individuals i belong to a finite set $\mathcal{I}$ with $|\mathcal{I}| \geq 2$, and that each individual i and society face uncertain prospects. In the present framework, these can be *completely uncertain* (when both s and o are unknown), *subjectively uncertain* (o is fixed and s is unknown), or *objectively uncertain* (s is fixed and o is unknown). We think of social prospects in the usual way, as mappings from states to social consequences, but with social consequences directly expressed in terms of *payoff numbers* for the individuals. We leave it for the interpretation to decide whether these numbers represent physical payoffs (levels of consumption in a good) or subjective payoffs (utility values in some metric).

Denoting the payoff numbers by x^i_{so}, we define *completely uncertain* prospects as follows: for an individual $i \in \mathcal{I}$, they are matrices $\mathbf{X}^i = (x^i_{so})_{s\in\mathcal{S},o\in\mathcal{O}} \in \mathbb{R}^{\mathcal{S}\times\mathcal{O}}$, and for society, they are three-dimensional arrays, $\mathbb{X} = (x^i_{so})^{i\in\mathcal{I}}_{s\in\mathcal{S},o\in\mathcal{O}} \in \mathbb{R}^{\mathcal{I}\times\mathcal{S}\times\mathcal{O}}$.[3] Other prospects are obtained by fixing one dimension of the state in these objects. Hence *objectively uncertain* prospects

[3] The order in which s and o, or $\mathcal{S}$ and $\mathcal{O}$, enter the notation is purely conventional. We do not mean to suggest that the s-uncertainty is resolved before the o-uncertainty.

(resp. *subjectively uncertain* prospects) are, for individual i, vectors $\mathbf{x}_s^i = (x_{so}^i)_{o\in\mathcal{O}} \in \mathbb{R}^{\mathcal{O}}$ for some fixed $s \in \mathcal{S}$ (resp. $\mathbf{x}_o^i = (x_{so}^i)_{s\in\mathcal{S}} \in \mathbb{R}^{\mathcal{S}}$ for some fixed $o \in \mathcal{O}$), and for society, matrices $\mathbf{X}_s = (x_{so}^i)_{o\in\mathcal{O}}^{i\in\mathcal{I}} \in \mathbb{R}^{\mathcal{I}\times\mathcal{O}}$ (resp. $\mathbf{X}_o = (x_{so}^i)_{s\in\mathcal{S}}^{i\in\mathcal{I}} \in \mathbb{R}^{\mathcal{I}\times\mathcal{S}}$). Notice that it is the *unresolved* uncertainty that fixes the status of these prospects; thus, an objective prospect is one in which o is unknown and s known, and vice-versa for a subjective prospect. When uncertainty is completely resolved, an individual i faces a scalar x_{so}^i, while society faces a vector $\mathbf{x}_{so} = (x_{so}^i)^{i\in\mathcal{I}} \in \mathbb{R}^{\mathcal{I}}$.

We assume that both the individuals and society assess completely uncertain prospects in terms of *ex ante* preference relations, denoted by $\succsim^i$ for $i \in \mathcal{I}$ and $\succsim$ for society; these are our only preference primitives. Throughout, we take them to be continuous weak orders, thus representable by continuous real-valued utility functions.

The other relations of this paper are *conditionals* induced by either the $\succsim^i$ or $\succsim$. There are six of them to consider: $\succsim_s^i$, $\succsim_o^i$ and $\succsim_{so}^i$ for individual i, and $\succsim_s$, $\succsim_o$ and $\succsim_{so}$ for society. While $\succsim_{so}^i$ and $\succsim_{so}$ make *ex post* comparisons, $\succsim_s^i$, $\succsim_s$, $\succsim_o^i$ and $\succsim_o$ make *interim* comparisons, which are specific to our twofold uncertainty framework. We obtain these conditional preferences by restricting the unconditional preferences to prospects that vary only along the component of interest. For instance, for all i in $\mathcal{I}$ and s in $\mathcal{S}$, the relation $\succsim_s^i$ compares prospects $\mathbf{x}_s^i \in \mathbb{R}^{\mathcal{O}}$ in the same way as $\succsim^i$ compares those larger prospects $\mathbf{X}^i \in \mathbb{R}^{\mathcal{S}\times\mathcal{O}}$ in which all $\mathcal{S}$-components other than s are fixed at some arbitrary values. Likewise, $\succsim_o$ compares prospects $\mathbf{X}_o \in \mathbb{R}^{\mathcal{I}\times\mathcal{S}}$ in the same way as $\succsim$ compares those larger prospects $\mathbb{X} = (x_{so}^i)_{s\in\mathcal{S},o\in\mathcal{O}}^{i\in\mathcal{I}} \in \mathbb{R}^{\mathcal{I}\times\mathcal{S}\times\mathcal{O}}$ in which all $\mathcal{O}$-components other than o are fixed at some arbitrary values.

Importantly, a conditional relation defined in this way is *complete, but not necessarily transitive*, so it is not necessarily an ordering. A special assumption must be added to endow it with this property, *separability*, which says the comparisons made by the unconditional relation are the same, independent of the arbitrary values used to define the conditional. This idea is familiar from decision theory, but we review it in Appendix A.

The individual *ex post* relations $\succsim_{so}^i$ simply compare real numbers according to their natural ordering. This is consistent with our payoff interpretation of these numbers, and means that $\succsim_{so}^i$ is automatically transitive. Formally, for all $(s, o) \in \mathcal{S} \times \mathcal{O}$, all $i \in \mathcal{I}$ and all $x_{so}^i, y_{so}^i \in \mathbb{R}$, we stipulate that

$$x_{so}^i \succsim_{so}^i y_{so}^i \quad \text{if and only if} \quad x_{so}^i \geq y_{so}^i. \tag{1}$$

For all the other conditional relations, we do not assume transitivity in general. If (and only if) a relation *is* transitive, we will call it a *preference*. Thus, when we say below that $\succsim^i$ or $\succsim$ induces an *ex post* or interim *preference*, we mean a (transitive) preference *ordering*, which amounts to assuming that $\succsim^i$ or $\succsim$ is separable in the relevant component. Whenever we assume that a conditional relation is a preference, we will make this assumption *across the uncertainty type*. In other words, we take the ordering property of conditionals to hold either for all (s, o) or for none, for all s or for none, and for all o or for none. Thus, we will simply say, "$\succsim^i$ induces interim preferences $\succsim_s^i$" or "$\succsim$ induces *ex post* preferences $\succsim_{so}$", without adding the implied "for all s" or "for all (s, o)".[4]

[4]The requirement that separability conditions hold across the uncertainty type means that they are

When $\succsim^i$ or $\succsim$ induces conditional preferences of some type, we may, by a separate decision, require that these preferences be *identical across this type*. For example, we may assume not only that $\succsim^i$ induces interim $\succsim^i_s$ preferences, but also that $\succsim^i_s = \succsim^i_{s'}$ for all $s, s' \in \mathcal{S}$; and we may assume not only that $\succsim$ induces *ex post* preferences $\succsim_{so}$, but also that $\succsim_{so} = \succsim_{s'o'}$ for all $s, s' \in \mathcal{S}$ and $o, o' \in \mathcal{O}$. We will then say that the induced preferences are *invariant*. Note that the $\succsim^i_{so}$ preferences are automatically invariant, by statement (1).

Our framework should be compared with Savage's (1972) axiomatization of SEU. When we say that $\succsim^i$ or $\succsim$ induces conditional preferences of some type, we are in effect applying the *Sure-thing Principle* - (P2) in Savage's system - but in a limited form. Similarly, the optional requirement that *ex post* or interim preferences are invariant corresponds to the *event-independent preference* condition - (P3) in Savage's system - but again in limited form.[5] We would recover the entirety of the Sure-thing Principle if we required *all* induced conditionals to be preferences, and we would recover the entirety of the event-independent preference condition if we required *all* these induced preferences to be invariant, but we will refrain from doing that because - as will be explained - this would precipitate the same kind of impossibilities as those obtained for the Savage framework, and we are here are after positive solutions.

Our framework should also be compared with Anscombe and Aumann's (1963) variant of Savage's theory, in which the consequences of prospects take the form of VNM lotteries.[6] Like them, we exogenously distinguish an objective and subjective source of uncertainty. However, unlike them, we locate this distinction within each state and not between the consequences and the states; we do not assume that the objective source already has probabilities attached to it, but derive this property axiomatically; and we are concerned with multiple decision-makers, so that we can associate the objective and subjective sources with shared and idiosyncratic probabilities, respectively.

In the standard uncertainty framework, social preference is subjected to Pareto conditions defined either *ex ante* or *ex post*. But the *twofold* uncertainty framework introduces more options. Now, the *ex ante* condition applies to completely uncertain prospects, the *ex post* condition applies to fully resolved prospects, and two newly defined interim conditions apply to *s-resolved prospects* and *o-resolved prospects*. Formally:

$\succeq$ satisfies the *ex ante* Pareto principle if for all $\mathbb{X}, \mathbb{Y} \in \mathbb{R}^{\mathcal{I}\times\mathcal{S}\times\mathcal{O}}$:

> if $\mathbf{X}^i \succeq^i \mathbf{Y}^i$ for all $i \in \mathcal{I}$, then $\mathbb{X} \succeq \mathbb{Y}$; if, in addition, $\mathbf{X}^i \succ^i \mathbf{Y}^i$ for some $i \in \mathcal{I}$, then $\mathbb{X} \succ \mathbb{Y}$.

$\succeq$ satisfies the *ex post* Pareto principle if for all $(s, o) \in \mathcal{S} \times \mathcal{O}$, and all $\mathbf{x},\ \mathbf{y} \in \mathbb{R}^{\mathcal{I}}$:

> if $x^i \geq y^i$ for all $i \in \mathcal{I}$, then $\mathbf{x} \succeq_{so} \mathbf{y}$; if, in addition, $x^i > y^i$ for some $i \in \mathcal{I}$, then $\mathbf{x} \succ_{so} \mathbf{y}$;

equivalent to *dominance conditions* for the given type. See Appendix A for details.

[5]In a framework with a finite number of states, a *state-independent preference* condition is sufficient to do the work of (P3), which is to determine a probability representation. See, e.g., Karni's (2014) review of SEU theories.

[6]For a recent extension of Anscombe and Aumann's theorem, see Mongin and Pivato (2015).

$\succeq$ satisfies the *objective interim* Pareto principle if for all $s \in \mathcal{S}$:

> if $\mathbf{x}_s^i \succeq_s^i \mathbf{y}_s^i$ for all $i \in \mathcal{I}$, then $\mathbf{X}_s \succeq \mathbf{Y}_s$; if in addition, $\mathbf{x}_s^i \succ_s^i \mathbf{y}_s^i$ for some $i \in \mathcal{I}$, then $\mathbf{X}_s \succ \mathbf{Y}_s$.

The interim Pareto principle introduced here is called *objective*, because it handles prospects in which the only uncertainty concerns the *objective* source . We likewise define the *subjective* interim Pareto principle, which interchanges the roles of the two sources, thus considering subjective prospects with fixed o and uncertain s. Notice that all forms of the Pareto principle except for the *ex ante* one are defined in terms of binary relations rather than preferences (orderings). This is to make the Paretian conditions logically independent of the decision-theoretic assumptions. However, in our conclusions, any time a version of the Pareto principle holds, it applies to preferences, not just relations.

3 The *ex ante* and *ex post* criteria of social welfare

The first result of this section axiomatically characterizes the *ex ante* social welfare criterion. This is done by assuming the *ex ante* Pareto principle plus relevant decision-theoretic conditions on individual preferences. Specifically, we require that each individual have well-defined *invariant* interim preferences for either type of uncertainty. Although this is weaker than Savage's postulates, it turns out to deliver full-fledged SEU representations for the individuals. By contrast, the *ex ante* social preference is simply represented by a function that is increasing with these individual representations.

Proposition 1 *Suppose that for all $i \in \mathcal{I}$, the individual preference $\succeq_i$ induces interim preferences of both kinds, i.e., $\succeq_s^i$ and $\succeq_o^i$, and both kinds are invariant. Suppose also that $\succeq$ satisfies the* ex ante *Pareto principle. Then, for all $i \in \mathcal{I}$, there are strictly positive probability vectors $\mathbf{p}^i \in \Delta_{\mathcal{S}}$ and $\mathbf{q}^i \in \Delta_{\mathcal{O}}$, and an increasing continuous utility function u_i on $\mathbb{R}$, such that the preference ordering $\succeq_i$ admits the following SEU representation:*

$$U_i(\mathbf{X}) \quad := \quad \sum_{s \in \mathcal{S}} \sum_{o \in \mathcal{O}} q_o^i p_s^i u^i(x_{so}), \quad \text{for all } \mathbf{X} \in \mathbb{R}^{\mathcal{S} \times \mathcal{O}}. \tag{2}$$

Moreover, there is a continuous increasing function F on the range of the vector-valued function $(U^i)_{i \in \mathcal{I}}$ such that $\succeq$ is represented by the ex ante social welfare function

$$W_{\text{xa}}(\mathbb{X}) \quad := \quad F\big(\big[U^i(\mathbf{X}^i)\big]_{i \in \mathcal{I}}\big), \quad \text{for all } \mathbb{X} \in \mathbb{R}^{\mathcal{I} \times \mathcal{S} \times \mathcal{O}}. \tag{3}$$

In these representations, for all $i \in \mathcal{I}$, the probability vectors $\mathbf{p}^i$ and $\mathbf{q}^i$ are unique, and u_i is unique up to positive affine transformations, while F is unique up to continuous increasing transformations.

Each SEU representation U^i builds upon *two* probability functions $\mathbf{p}^i$ and $\mathbf{q}^i$, which we can interpret as i's *subjective beliefs* about $\mathcal{S}$ and $\mathcal{O}$, respectively. The multiplicative probability $q_o^i p_s^i$ then means that i believes that $\mathcal{S}$ and $\mathcal{O}$ are *probabilistically independent*. But our symmetric treatment of the two types of conditional preferences does not yet permit us to distinguish $\mathcal{S}$ from $\mathcal{O}$.

The second result of this section axiomatizes the *ex post* social welfare criterion. Thus, we take the *ex post* Pareto principle to be the only unanimity condition, and reserve the decision-theoretic conditions for society. The conclusions reproduce those of Proposition 1 *mutatis mutandis*. They deliver a SEU representation for social preference and make the *ex post* social welfare function increasing in the individual *ex post* utilities.

Proposition 2 *Suppose that the social preference $\succeq$ induces interim preferences of both kinds, i.e., $\succeq_s$ and $\succeq_o$, and both kinds are invariant. Suppose also that $\succeq$ satisfies the* ex post *Pareto principle. Then, social* ex post *preferences $\succeq_{so}$ are well-defined and invariant, and there is a continuous and increasing representation W_{xp} for these preferences. There also exist strictly positive probability vectors $\mathbf{p} \in \Delta_{\mathcal{S}}$ and $\mathbf{q} \in \Delta_{\mathcal{O}}$ such that $\succeq$ has the following SEU representation:*

$$W_{\text{xa}}(\mathbb{X}) \quad := \quad \sum_{s\in\mathcal{S}}\sum_{o\in\mathcal{O}} p_s q_o W_{\text{xp}}(\mathbf{x}_{so}), \quad \textit{for all } \mathbb{X} \in \mathbb{R}^{\mathcal{I}\times\mathcal{S}\times\mathcal{O}}. \tag{4}$$

In this representation, $\mathbf{p}$ *and* $\mathbf{q}$ *are unique, and the* ex post *social welfare function W_{xp} is unique up to positive affine transformations.*

The probabilities $\mathbf{p}$ and $\mathbf{q}$ that appear here - again in multiplicative form - belong to society exclusively. Those of the individuals - if any - are left unspecified, since the only individual data are the orderings $\succeq^i$ and $\succeq^i_{so}$, nothing being said on the other conditionals. As in Proposition 1, our symmetric treatment of $\mathcal{S}$ and $\mathcal{O}$ does not yet make it possible to separate the interpretation of the two sources of uncertainty.

The main results of this paper will exploit the twofold uncertainty framework in order to reconcile the two social welfare criteria. We will consider two intermediary solutions in turn, one based on the *ex ante* criterion (in Section 4), and the other based on the *ex post* criterion (in Section 6, our preferred solution). In both cases , we include as much as possible of the other criterion, pushing the reconciliation to the point where any further step would precipitate an impossibility.

4 An initial reconciliation of *ex ante* and *ex post*

Our first main result starts from the *ex ante* social welfare criterion, and extends it to recover as much of the *ex post* criterion as possible.

Theorem 3 *Suppose the same assumptions as in Proposition 1 hold, and moreover, the social preference $\succeq$ induces interim preferences of both kinds, i.e., $\succeq_s$ and $\succeq_o$, with the* *interim preferences $\succeq_o$ being invariant. Then, for all $i \in \mathcal{I}$, the SEU representations of Proposition 1 for* ex ante *individual preferences hold with $\mathbf{q}_1 = ... = \mathbf{q}_n = \mathbf{q}$, i.e., for all $i \in \mathcal{I}$, we have*

$$U^i(\mathbf{X}) \quad = \quad \sum_{o\in\mathcal{O}}\sum_{s\in\mathcal{S}} q_o p^i_s u^i(x_{so}), \quad \textit{for all } \mathbf{X} \in \mathbb{R}^{\mathcal{S}\times\mathcal{O}}.$$

Furthermore, the ex ante social preference $\succeq$ is now represented by the additive ex ante *social welfare function*

$$W_{\rm xa}(\mathbb{X}) \quad := \quad \sum_{i\in\mathcal{I}} U_i \quad = \quad \sum_{i\in\mathcal{I}}\sum_{s\in\mathcal{S}}\sum_{o\in\mathcal{O}} q_o p^i_s u^i(x^i_{so}), \quad \textit{for all } \mathbb{X} \in\mathbb{R}^{\mathcal{I}\times\mathcal{S}\times\mathcal{O}}, \tag{5}$$

and, for all states $(s,o) \in \mathcal{S}\times\mathcal{O}$, $\succeq$ induces a state-dependent ex post *social preference $\succeq_{so}$ represented by*

$$W_{so}(\mathbb{X}) \quad := \quad \sum_{i\in\mathcal{I}} p^i_s u^i(x^i_{so}), \quad \textit{for all } \mathbf{x} \in \mathbb{R}^{\mathcal{I}}.$$

The interim social preferences $\succeq_s$ and $\succeq_o$ are represented by the relevant sums in the formula for $W_{\rm xa}$. As a consequence, the ex post *and objective interim forms of the Pareto principle also hold.*

In these representations, $\mathbf{q}$ and each $\mathbf{p}_i$ are unique, while the utility functions u_i are unique up to a positive affine transformation with a common multiplier.

Theorem 3 strengthens Proposition 1 in several ways. First of all, the individuals' SEU representations now separate the two sources of uncertainty as they should: each can entertain an idiosyncratic probability vector $\mathbf{p}^i$ on the subjective source, but must accept the common probability vector $\mathbf{q}$ on the objective source. By requiring society, like the individuals, to have *invariant* interim preferences on the objective source, we have not only endowed society with a probability vector $\mathbf{q}$, but also forced the individuals to coordinate their probabilities of this source on $\mathbf{q}$. Unlike them, society has interim preferences on the subjective source that may *not* be invariant. Hence, no social probability exists for this source, a prominent feature of Theorem 3.

Second, Theorem 3 turns the unspecified social welfare function $W_{\rm xa}$ of Proposition 1 into a weighted sum of the individuals' expected utilities, as in Harsanyi's (1955) Social Aggregation Theorem. However, we *derive* the identity of probabilistic beliefs that he merely assumed, and we restrict this identity to the objective source of uncertainty. Additive representations exist for all the social preferences, but with distinctive properties. The representations of the interim social preferences $\succsim_o$:

$$\sum_{i\in\mathcal{I}}\sum_{s\in\mathcal{S}} p^i_s u^i(x^i_{so})$$

are independent of o, but both the representations of the *ex post* social preferences $\succeq_{so}$:

$$\sum_{i\in\mathcal{I}} p^i_s u^i(x^i_{so}),$$

and of the interim social preferences $\succeq_s$:

$$\sum_{i\in\mathcal{I}}\sum_{o\in\mathcal{O}} q_o p^i_s u^i(x^i_{so})$$

depend on the value of s. To assume such *state-dependence* of society's preferences is one way of reconciling the *ex ante* and *ex post* criteria of social welfare (Mongin, 1998; Chambers and Hayashi, 2006). In a standard SEU framework, this solution leaves society without any probabilistic beliefs, and some have objected to it for this reason. Theorem 3 combines social state-dependence for *subjective* uncertainty with a well-defined social probability for *objective* uncertainty.

Third, beside more fully determining the *ex ante* social welfare criterion, Theorem 3 includes as much as possible of the *ex post* social welfare criterion. As the next corollary demonstrates, slightly stronger assumptions would deliver a social probability $\mathbf{p}$ on the subjective source, but also force *the individuals to align their probabilities* $\mathbf{p}_i$ *on* $\mathbf{p}$, an unacceptable conclusion since they may hold different beliefs about $\mathcal{S}$. This is an adaptation of Mongin's (1995) impossibility theorem to the twofold uncertainty framework.

Corollary 4 *Suppose the same assumptions as in Theorem 3 hold, and moreover, the social preference* $\succsim$ *induces invariant interim preferences* $\succsim_s$. *Then, the representations of Theorem 3 hold with a common probability vector* $\mathbf{p}\in\Delta_{\mathcal{S}}$ *such that* $\mathbf{p}_1 = ... = \mathbf{p}_n = \mathbf{p}$.

Theorem 3 is a welcome improvement on the position in welfare economics that merely adopts the *ex ante* criterion and rejects the *ex post* one. Among the more recent participants, Hild, Jeffrey and Risse (2003) and Risse (2003) have made a subtle case for the *ex ante* Pareto principle against the *ex post* one. In effect, they argue that social and individual preferences are always *ex ante*. The distinction between final consequences and uncertain prospects is a matter of convention; a more refined analysis of these consequences and their underlying events would reveal that they define yet *another* class of uncertain prospects. By focusing on this particular class, the *ex post* Pareto principle makes an arbitrary restriction to the *ex ante* principle, while being open to exactly the same difficulties; hence it should be avoided. This troubling argument connects with worries that Savage once expressed on the relevance of his framework.[7] We will return to it in Section 6.

However, we will now raise an objection that even Hild, Jeffrey and Risse concede regarding the *ex ante* Pareto principle, i.e., *spurious unanimity,* an objection introduced by Mongin (1997). The next section illustrates this with an idealized public policy example.

5 Spurious unanimity and complementary ignorance

Imagine the members of society are spread out in two areas, an island and a mainland, with the island being rich, sparsely populated, and beautifully preserved, and the mainland

[7] See Savage's (1972) analysis of "small worlds" and the problem they raise for his SEU theory.

being poor, densely populated, and disfigured by industrialization. Being worried that the island is lacking sufficient public services and the mainland is lacking recreation areas, the Government considers connecting one to the other by a bridge, and being democratically inspired, it organizes a public hearing. Given the relatively high toll that will have to be paid by users of the bridge, it is not clear whether the flow will go from the little populated but rich island, or from the heavily populated but poorer mainland. (But we assume that the bridge is financially feasible, whichever the direction of the main flow of users.) As it happens, the Islanders think that the former consequence is more probable than the latter, while the Mainlanders have the opposite belief. (To connect this assumption with Theorem 3, we take the direction of the flow to be subjectively, not objectively uncertain; formally, it is evaluated by the p_s^i, not the q_0.) It is also the case that both communities are self-concerned, so that the Islanders value the former consequence more than the latter, while the Mainlanders have the opposite preference. Given these data, SEU and even more general decision theories predict that the *two groups will both support the project.* As the well-intended Government takes the Pareto principle very seriously, it will push the project forward. However, this would be a dubious decision to make. The two groups are unanimous in preferring the bridge, but spuriously, since they are in fact *twice opposed* - i.e., in their utility and probability comparisons - and their disagreements just cancel out in the SEU or related calculation. Arguably, the Government should go beyond the individuals' overt preferences and clarify the beliefs and desires underlying these preferences, and if the individuals disagree on these two scores, as is the case here, it should conclude that unanimity does not compel it to build the bridge.[8]

Our diagnosis of the bridge example rests on a normative claim. Individual judgments do not matter to society *by themselves*, but in virtue of the *reasons* that individuals have for holding them; accordingly, society should take these reasons into account before deciding whether unanimous individual judgments are compelling or not; and in particular, if the individuals strongly disagree on each set of reasons separately considered, society does not have to follow suit. The last case - deep disagreement below a surface agreement - is *spurious unanimity*, as defined by Mongin (1997). He applies this analysis to social decision under uncertainty as a particular case; here the judgments are preferences over prospects, and the reasons are beliefs about the states and utilities for the consequences. As the *ex ante* Pareto principle only considers prospects, not consequences or states, it does not discriminate between spurious and nonspurious cases of unanimous preferences. This argument suggests giving up the principle, but not necessarily entirely, since it does not deny that there may be a valid core of *ex ante* unanimity.

Gilboa, Samet and Schmeidler (2004) were the first to argue for a restriction rather than an abandonment of *ex ante* Pareto .[9] They restrict the *ex ante* Pareto principle to comparisons of social prospects in which the events of interest receive the same probability values across all individuals. Formally, they introduce the family $\mathcal{F}$ of all events on whose

[8]Mathematically equivalent examples of spurious unanimity can be constructed in terms of a proposed financial transaction between two speculative traders, a proposed treaty between two countries, or a proposed deal (or duel) between two gentlemen.

[9]From the spurious unanimity objection, Mongin (1997, 1998) moves all the way to the *ex post* solution.

probabilities all individuals agree;[10] then they limit *ex ante* Pareto to comparisons between social prospects that are *measurable* with respect to $\mathcal{F}$. An example is a social prospect which is constant on the cells of a partition, where each cell receives the same probabilities from all individuals. GSS's restriction of *ex ante* Pareto is tailor-made to block the "bridge" example, since it precludes probabilistic disagreement from cancelling utility disagreement. Thus, it seems to purge the *ex ante* Pareto principle of its spurious applications.

However, GSS's solution can go wrong in situations where the individuals have private information. Consider a society of two individuals, Alice and Bob, a partitition of the set of states $\mathcal{S}$ into three events E_1, E_2, E_3, and two prospects, f and g, which we describe in terms of the utility values that each of the three events brings to Alice and Bob. We assume that they share the same utility function, and initially have the same probabilistic beliefs, as shown in the next table.

	E_1	E_2	E_3
Alice's and Bob's utilities for f	1	0	1
Alice's and Bob's utilities for g	0	1	0
Alice's and Bob's probabilities	0.49	0.02	0.49

Initially, Alice and Bob both assign an SEU of 0.98 to f and an SEU of 0.02 to g, so that they both prefer f over g. Thus, GSS's restricted *ex ante* Pareto principle says that society should also prefer f over g.

Suppose now that Alice privately observes the event $E_1 \cup E_2$, while Bob privately observes the event $E_2 \cup E_3$. After Bayesian updating, they have the following probabilities and SEU values:

	$P(E_1)$	$P(E_2)$	$P(E_3)$	$SEU(f)$	$SEU(g)$
Alice	0.96	0.04	0	0.96	0.04
Bob	0	0.04	0.96	0.96	0.04

From this table, we construct the family $\mathcal{F}$ of events on the probabilities of which Alice and Bob agree. Both put $P(E_1 \cup E_3) = 0.96$ and $P(E_2) = 0.04$, so $\mathcal{F} = \{\varnothing, E_1 \cup E_3, E_2, \mathcal{S}\}$. The prospects f and g are constant on $E_1 \cup E_3$ and constant on E_2, hence measurable with respect to $\mathcal{F}$, so GSS's restricted *ex ante* Pareto principle says that that f should be socially preferred to g. However, if the bearer of social preference learns that Alice has observed $E_1 \cup E_2$, and Bob has observed $E_2 \cup E_3$, he should logically conclude that E_2 is the true event. But then g should socially be preferred to f, contrary to the previous conclusion. Alice and Bob unanimously prefer f over g only because each one has information the other one lacks; they are in a state of *complementary ignorance*.

This criticism applies not only to the restricted *ex ante* Pareto principle of GSS, but to the linear pooling rule itself. Regardless of the weights for Alice's and Bob's probabilities in this rule, the common probability value $P(E_2) = 0.04$ is also the social probability value. When presented in this way, the example becomes of interest to the management literature, where linear pooling has acquired somewhat canonical status. [11]

[10] Note that $\mathcal{F}$ might not be an algebra, let alone a σ-agebra of events.

[11] See the related urn example against the linear pooling rule in Mongin (1997). In this example, the social observer infers what each individual has observed just by being told the revised probability values.

At this point, the reader might object that neither the *ex ante* Pareto principle nor the GSS restriction was ever intended to cover the case of changing information. In other words, both are stated under the implicit proviso that the individuals' probabilities are *priors*, unlike the probabilities of the last tables, which are *posteriors*. However, the distinction between "prior" and "posterior" is just a matter of convention. Depending on what one considers to be background knowledge and what one calls new information, what is called a prior in one context would be called a posterior in another. Leaving aside "original position" or "veil of ignorance" constructions, the individual probabilities of interest to society are *always already posteriors*. The only relevant distinction is between those posteriors which can be analyzed so as to reveal information, as in the Alice and Bob example, and those which cannot. The linear pooling rule obliterates this distinction.[12]

As discussed in Section 7, some authors find GSS's Pareto principle *too weak*, arguing that *ex ante* unanimity is sometimes compelling even in cases of probabilistic disagreement. But our example of complementary ignorance suggests, to the contrary, that GSS's Pareto principle is still *too strong*. The next section explores our prefered alternative.

6 A final reconciliation of *ex ante* and *ex post*

At this point, it is clear that any social decision criterion must achieve three goals: first, avoid the impossibility theorems discussed in the introduction; second, avoid the problem of spurious unanimity (unlike the *ex ante*-based solution of Theorem 3), and third, avoid the problem of complementary ignorance (unlike the solution of GSS). Within these constraints, we should maximize the utility information that can be derived from the axioms. Theorem 5 below is our solution to this problem. Reversing the direction taken in Theorem 3, it starts from the *ex post* criterion, and exploits the double uncertainty framework to recover as much as possible of the *ex ante* criterion. What it retains from the latter are, for one thing, the decision-theoretic assumptions relative to the individuals, and for another, the *objective interim Pareto principle*.

Theorem 5 *Suppose the same assumptions as in Proposition 2 hold, and moreover, for all $i \in \mathcal{I}$, the individual preferences $\succeq^i$ induces invariant interim preferences $\succeq^i_s$ and invariant interim preferences $\succeq^i_o$. Suppose also that the objective interim Pareto principle holds. Then, there exist strictly positive probability vectors $\mathbf{p} \in \Delta_{\mathcal{S}}$ and $\mathbf{q} \in \Delta_{\mathcal{O}}$, and for all $i \in \mathcal{I}$, there exist probability vectors $\mathbf{p}^i \in \Delta_{\mathcal{S}}$ and continuous and increasing utility functions u^i, such that the following holds. For all $i \in \mathcal{I}$, the relation $\succeq^i$ has the SEU representation:*

$$U^i(\mathbf{X}) \quad := \quad \sum_{s \in \mathcal{S}} \sum_{o \in \mathcal{O}} p^i_s \, q_o \, u^i(x_{so}), \quad \textit{for all } \mathbf{X} \in \mathbb{R}^{\mathcal{S} \times \mathcal{O}}, \tag{6}$$

while the SEU representation of $\succeq$ in Proposition 2 holds, that is,

$$W_{\mathrm{xa}}(\mathbb{X}) \quad = \quad \sum_{s \in \mathcal{S}} \sum_{o \in \mathcal{O}} p_s \, q_o \, W_{\mathrm{xp}}(\mathbf{x}_{so}), \quad \textit{for all } \mathbb{X} \in \mathbb{R}^{\mathcal{I} \times \mathcal{S} \times \mathcal{O}}. \tag{7}$$

[12] "Original position" or "veil of ignorance" in the style of Harsanyi and Rawls are usually interpreted as referring to pure priors. However, some recent variants allow for private information (Nehring, 2004; Chambers and Hayashi, 2014).

Furthermore, there is a vector of positive weights $\mathbf{r} = (r^i)_{i\in\mathcal{I}}$ *such that the* ex post *social welfare function* W_{xp} *has the additive form*

$$W_{\text{xp}}(\mathbf{x}) \quad := \quad \sum_{i\in\mathcal{I}} r^i\, u^i(x^i), \quad \textit{for all } \mathbf{x}\in\mathbb{R}^{\mathcal{I}}. \tag{8}$$

Finally, for all $s \in \mathcal{S}$ *and* $o \in \mathcal{O}$*, the interim social preferences* $\succeq_s$ *and* $\succeq_o$ *have the representations:*

$$\sum_{o\in\mathcal{O}} q_o\, W_{\text{xp}}(\mathbf{x}_o), \quad \textit{for all } \mathbf{X}\in\mathbb{R}^{\mathcal{O}}, \quad \textit{and} \quad \sum_{s\in\mathcal{S}} p_s\, W_{\mathbf{xp}}(\mathbf{x}_s), \quad \textit{for all } \mathbf{X}\in\mathbb{R}^{\mathcal{S}}. \tag{9}$$

In these representations, the vectors $\mathbf{p}$*,* $\mathbf{q}$ *and* $\mathbf{r}$ *are unique, and the utility functions* $(u^i)_{i\in\mathcal{I}}$ *are unique up to positive affine representations with a common multiplier.*

As with Theorem 3, let us compare the conclusions with those of the base-line statement, here Proposition 2. First, while Proposition 2 said nothing of the decision theory satisfied by the individuals, we now see that they have SEU representations for their preferences $\succeq^i$. These representations break the symmetry between the two sources of uncertainty in the sensible way , by endowing the individuals with identical probabilities on $\mathcal{O}$ and idiosyncratic probabilities on $\mathcal{S}$. Theorem 3 also broke the symmetry, but by using the questionable *ex ante* Pareto principle.

Second, all available Pareto conditions are translated into weighted sum formulas for social preferences. That is, the *ex post* social utility W_{xp}, which Proposition 2 did not determine, turns out to be a weighted sum of *ex post* individual utilities, and one can also check that, for each given s, the objective interim social welfare function

$$\sum_{o\in\mathcal{O}} q_o\, W_{\text{xp}}(\mathbf{x}_o)$$

can be rewritten as a weighted sum of the individuals' objective interim expected utilities:

$$\sum_{i\in\mathcal{I}} r^i \sum_{o\in\mathcal{O}} q_o\, u^i(x_{so}).$$

By contrast, no such translation exists for the subjective interim social utilities and the *ex ante* social utility, which reflects the absence of subjective interim or *ex ante* Pareto hypotheses. Of course, replacing $W_{\text{xp}}(\mathbf{x})$ by its value in the SEU formulas for these two social utilities delivers weighted sums of u^i values, but since society uses its own probability $\mathbf{p}$, not the individual probabilities $\mathbf{p}^i$, what is being added are *not* individual SEU representations.

Third, as with Theorem 3, any further step towards the *ex ante* Pareto criterion precipitates an impossibility. Indeed, adding either the *ex ante* Pareto principle in full or even just the subjective interim Pareto principle collapses all individual probabilities onto the social ones. Formally:

Corollary 6 *Suppose the same assumptions as in Theorem 5 hold. Suppose also that, for all $s \in \mathcal{S}$, $\succsim_s$ satisfies the subjective interim Pareto principle. Then, on top of the previous results, for all $i \in \mathcal{I}$, $\mathbf{p}^i = \mathbf{p}$, and as a consequence, the ex ante Pareto principle holds.*

Thus, Theorem 5 maximizes the utility information that can be derived within the three constraints stated above. By deriving idiosyncratic probabilities on the subjective source, it circumvents the impossibilities, and by imposing common probabilities on the objective source, it becomes immune to spurious unanimity. Perhaps most importantly, regarding the subjective source, it derives *no connection between the social probability and the individual ones*, and in particular eschews the linear pooling rule. (For a quick check that this does not hold, notice that the axioms can be satisfied for any choice of u^i with all $\mathbf{p}^i$ being the same and nonetheless differing from $\mathbf{p}$.)

By comparison with Gilboa, Samet and Schmeidler (2004), Theorem 5 enriches conclusion (i), while discarding (ii). Theorem 5 can also be related to the argument of Hild, Jeffrey and Risse (2003). They end up endorsing the full *ex ante* principle only because their classical SEU framework compels them to choose between it and the rival *ex post* one. In a richer framework in which this dichotomy is superseded, their argument (that so-called *ex post* preferences are *ex ante* preferences in disguise) should rather lead one to reconcile the two welfare criteria, and indeed our objective interim Pareto principle provides one such reconciliation.

7 Related literature

The conflict between the *ex ante* and *ex post* social welfare criteria has attracted renewed attention since Gilboa, Samet and Schmeidler (2004). The general strategy is to weaken either the *ex ante* Pareto principle, or the SEU assumptions (whether on society or the individuals), while preserving the *ex post* Pareto principle. Employing the first strategy , Nehring (2004) and Chambers and Hayashi (2014) have found a new impossibility theorem . They assume that agents have private information, and restrict the *ex ante* Pareto principle to situations where it is common knowledge that one prospect *ex ante* Pareto-dominates another. If society satisfies statewise dominance, even this restricted variant leads to an undesirably strong conclusion: the agents must share a common prior on the common knowledge events. Chambers and Hayashi further show that *ex ante* social welfare is a weighted sum of individual expected utilities, whereas Nehring assumes this. These results refine those of Harsanyi (1955) and Mongin (1995).

Others use both strategies at the same time. For example, Qu (2016) considers the possibility that both society and the individuals conform to the *maximin expected utility* (MEU) theory of Gilboa and Schmeidler (1989), a generalization of SEU where each agent is described by a *set* of probabilistic beliefs. Since Qu operates in the Anscombe and Aumann (1963) framework, which draws an exogenous distinction between objective and subjective uncertainty, he can deploy a Pareto principle for *objective* uncertainty that bears some analogy with ours. He also restricts the *ex ante* Pareto principle to a *Common Taste* version, which regulates comparisons between prospects f and g only when the individuals have unanimous *ex post* preferences over every possible outcome arising from f or g, with comparisons being performed on the certainty equivalents of such prospects (or convex combinations thereof). Qu shows that society satisfies his two Pareto principles if and only if the *ex post* social welfare is a weighted sum of individual utilities and the social set of probabilities $\mathcal{P}$ is a convex combination of the individuals' sets. The first conclusion is identical to conclusion (i) of GSS, and the second generalizes their conclusion (ii). In a variant that replaces MEU by the even more general *Choquet expected utility* (CEU) of Schmeidler (1989), Qu derives (i) again, as well as an appropriate generalization of (ii).

Alon and Gayer (2016) assume Savage's SEU theory for the individuals, and put axioms on society that endow it with a MEU representation. They strengthen GSS's restricted *ex ante* Pareto principle to a *Consensus Pareto* version, which says that if every individual (according to his own probabilistic beliefs) deems that prospect f yields a higher SEU than prospect g *for every individual*, then society should prefer f over g. This excludes the spurious unanimity diagnosed by Mongin (1997), while still respecting unanimity in some situations where individuals have different beliefs. Alon and Gayer show that society satisfies Consensus Pareto if and only if conclusion (i) holds and the social probability set is included in the convex hull of the individual probability measures.

In the Anscombe-Aumann framework, Danan et al. (2016) suppose that society and each individual have partial orders $\succeq$ and $\succeq_i$ that admit representations in the sense of Bewley (2002), i.e., there are sets $\mathcal{P}$ and $\mathcal{P}_i$ of probability distributions such that $f \succeq g$ (resp. $f \succeq_i g$) if and only if f yields at least as high an expected utility as g according to *all* elements in $\mathcal{P}$ (resp. in $\mathcal{P}_i$). The authors refer to these partial orders as *unambiguous preference relations* and show that those of society satisfy an *ex ante* Pareto principle relative to those of the individuals if and only if conclusion (i) holds, and $\mathcal{P}$ is included in the intersection of the $\mathcal{P}_i$. In the particular case of SEU theory, the individuals' unique probability measure is the same for all individuals and society, which also satisfies SEU theory; this recovers Mongin's (1995) impossibility theorem in the Anscombe-Aumann framework. Danan et al. also consider Common Taste Pareto and show that society's unambiguous preference relation satisfies this condition if and only if (i) holds, and $\mathcal{P}$ is included in the convex hull of the unions of the $\mathcal{P}_i$. They also provide a solution to an impossibility theorem of Gajdos, Tallon and Vergnaud (2008), in which a society of ambiguity-sensitive agents is susceptible to a phenomenon of *spurious hedging*, analogous spurious unanimity. [13]

The positive results in the three aforementioned papers have many attractive features, but they are all vulnerable to the problem of complementary ignorance from Section 5. In that example, both individuals satisfy SEU theory, which is a special case of the MEU and Bewley theories (using a singleton set of probabilities) as well as CEU theory (since a probability is a capacity). But in this case, the results of these three papers force the observer to be an SEU maximizer, with probabilistic beliefs given by the linear pooling rule, which, as we have seen, does not properly incorporate private information. Indeed, in our example in Section 5, both agents had the same utility function. Thus, the Consensus

[13]This is explained most clearly in the preprint version (Danan et al., 2015) .

Pareto axiom of Alon and Gayer (2016) and the common taste Pareto axioms of Qu (2016) and Danan et al. (2016) all reduce to *ex ante* Pareto, which yields the wrong answer in that example.

Billot and Vergopoulos (2016) have devised a framework in which neither spurious unanimity nor complementary ignorance can arise. They endow each individual with a personalized state space and a personalized consequence set, and society with a state space and a consequence set that are simply Cartesian products of these spaces. Assuming SEU theory for individuals and society, Billot and Vergopoulos show that the latter satisfies a set of three Pareto conditions if and only if (i) holds and the social probability measure is the *product* of the individual ones, an interesting alternative to (ii). To adopt this framework is tantamount to assuming that individuals face independent risks, as in standard insurance markets, an assumption that works for some public policy applications, but not for all. In the bridge example of Section 5, the risk faced by the Islanders is not independent of the risk faced by the Mainlanders.

Complementing these axiomatic endeavours, other papers usefully investigate the same issues in relation to financial markets. Standard economic theory generally endorses transactions on these markets by assuming the *ex ante* Pareto principle, but uncertainty raises spurious unanimity objections here as it does elsewhere. Thus, Posner and Weyl (2013), Blume et al. (2015) and others identify *purely speculative* transactions with those driven by different beliefs, and argue for public regulation in this case. Defining a new form of Paretian comparison, Gilboa, Samuelson and Schmeidler (2014) say that prospect f *No-Betting Pareto* (NBP) dominates prospect g if there exists some probability measure p such that f yields at least as high an expected utility as g for every agent, according to p. They show that NBP-dominance holds if and only if, for any weighted sum of *ex post* utilities, g does not statewise dominate f. They explore the consequences for financial markets of restricting the *ex ante* Pareto principle to NBP-dominance comparisons, an analysis continued by Gayer et al. (2014). But NBP is still vulnerable to the problem of complementary ignorance.

Instead of Pareto dominance, Brunnermeier, Simsek and Xiong (2014) strengthen the concept of Pareto *inefficiency*, by defining a prospect f to be *belief-neutral inefficient* if, for every probability measure p arising from a convex combination of the individual ones, there is some prospect yielding a higher p-expected utility for every agent than f. They propose to use this criterion to identify speculative transactions, and recommend regulatory scrutiny for these. They also propose a second criterion, which is based on a utilitarian-style social welfare function, and is related to the Bewley preferences considered by Danan et al. (2016). Like the authors cited in the previous paragraph, they argue forcefully and convincingly that unrestricted speculation in financial markets can destroy social welfare.

References

Alon, Shiri, and Gabrielle Gayer. 2016. "Utilitarian preferences with multiple priors." *Econometrica*, 84(3): 1181–1201.

Anscombe, Francis J., and Robert J. Aumann. 1963. "A definition of subjective probability." *Ann. Math. Statist.*, 34: 199–205.

Bewley, Truman. 2002. "Knightian decision theory. Part I." *Decisions in economics and finance*, 25(2): 79–110.

Billot, Antoine, and Vassili Vergopoulos. 2016. "Aggregation of Paretian preferences for independent individual uncertainties." (working paper).

Blume, Lawrence E, Timothy Cogley, David A Easley, Thomas J Sargent, and Viktor Tsyrennikov. 2015. "A case for incomplete markets." Reihe Ökonomie, Institut für Höhere Studien.

Brunnermeier, Markus, Alp Simsek, and Wei Xiong. 2014. "A Welfare Criterion For Models With Distorted Beliefs." *The Quarterly Journal of Economics*, 129(4): 1753–1797.

Chambers, Christopher, and Takashi Hayashi. 2014. "Preference aggregation with incomplete information." *Econometrica*, 82(2): 589–599.

Chambers, Christopher P., and Takashi Hayashi. 2006. "Preference aggregation under uncertainty: Savage vs. Pareto." *Games Econom. Behav.*, 54: 430–440.

Clemen, Robert T., and Robert L. Winkler. 2007. "Aggregating probability distributions." In *Advances in Decision Analysis.* , ed. W. Edwards, R. Miles and D. von Winterfeldt, 154–176. Cambridge, UK:Cambridge University Press.

Danan, Eric, Thibault Gajdos, Brian Hill, and Jean-Marc Tallon. 2015. "Aggregating Tastes, Beliefs, and Attitudes under Uncertainty." (published as Danan et al. 2016).

Danan, Eric, Thibault Gajdos, Brian Hill, and Jean-Marc Tallon. 2016. "Robust Social Decisions." *American Economic Review*, (to appear). (earlier circulated as Danan et al. 2015).

Fleurbaey, Marc, and Philippe Mongin. 2016. "The Utilitarian Relevance of the Social Aggregation Theorem." *American Economic Journal (Microeconomics)*, to appear.

Gajdos, Thibault, Jean-Marc Tallon, and Jean-Christophe Vergnaud. 2008. "Representation and aggregation of preferences under uncertainty." *J. Econom. Theory*, 141: 68–99.

Gayer, Gabrielle, Itzhak Gilboa, Larry Samuelson, and David Schmeidler. 2014. "Pareto Efficiency with Different Beliefs." *Journal of Legal Studies.*

Genest, Christian, and James V. Zidek. 1986. "Combining probability distributions: a critique and an annotated bibliography." *Statist. Sci.*, 1: 114–148.

Gilboa, Itzhak, and David Schmeidler. 1989. "Maxmin expected utility with nonunique prior." *J. Math. Econom.*, 18(2): 141–153.

Gilboa, Itzhak, Dov Samet, and David Schmeidler. 2004. "Utilitarian aggregation of beliefs and tastes." *Journal of Political Economy*, 112: 932–938.

Gilboa, Itzhak, Larry Samuelson, and David Schmeidler. 2014. "No-Betting Pareto Dominance." *Econometrica*, 82: 1405–1442.

Hammond, Peter J. 1981. "Ex ante and ex post welfare optimality under uncertainty." *Economica*, 48: 235–250.

Hammond, Peter J. 1983. "Utilitarianism, Uncertainty and Information." In *Utilitarianism and Beyond.* , ed. A. Sen and B. Williams, 85–102. Cambridge:Cambridge University Press.

Harsanyi, John C. 1955. "Cardinal Welfare, Individualistic Ethics, and Interpersonal comparisons of Utility." *Journal of Political Economy*, 63: 309–321.

Hild, Matthias, Richard Jeffrey, and Mathias Risse. 2003. "Flipping and Ex Post Aggregation." *Social Choice and Welfare*, 20: 267–275.

Karni, Edi. 2014. "Axiomatic Foundations of Expected Utility and Subjective Probability." In *Handbook of the Economics of Risk and Uncertainty.* , ed. Mark J. Machina and W. K. Viscusi, 1–39.

Mongin, Philippe. 1995. "Consistent Bayesian aggregation." *J. Econom. Theory*, 66: 313–351.

Mongin, Philippe. 1997. "Spurious unanimity and the Pareto principle." THEMA, Université de Cergy-Pontoise (working paper). Reissued as LSE-CPNSS working paper (2005).

Mongin, Philippe. 1998. "The paradox of the Bayesian experts and state-dependent utility theory." *J. Math. Econom.*, 29: 331–361.

Mongin, Philippe, and Marcus Pivato. 2015. "Ranking Multidimensional Alternatives and Uncertain Prospects." *Journal of Economic Theory*, 157: 146–171.

Mongin, Philippe, and Marcus Pivato. 2016. "Social Evaluation Under Risk and Uncertainty." In *The Oxford Handbook of Well-Being and Economic Policy.* , ed. Matthew D. Adler and Marc Fleurbaey, Chapter 24. Oxford.

Nehring, Klaus. 2004. "The veil of public ignorance." *Journal of Economic Theory*, 119: 247–270.

Posner, Eric, and Glen Weyl. 2013. "FDA for Financial Innovation: Applying the Insurable Interest Doctrine to Twenty-First-Century Financial Markets." *Northwestern University Law Review*, 107(3): 1307.

Qu, Xiangyu. 2016. "Separate Aggregation of Beliefs and Values under Ambiguity." *Economic Theory*, (to appear).

Risse, Mathias. 2003. "Bayesian Group Aggregation and Two Modes of Aggregation." *Synthese*, 135: 347–377.

Savage, Leonard J. 1972. *The foundations of statistics.* New York:Dover Publications Inc.

Schmeidler, David. 1989. "Subjective probability and expected utility without additivity." *Econometrica*, 57(3): 571–587.

Starr, Ross M. 1973. "Optimal Production and Allocation Under Uncertainty." *Quarterly Journal of Economics*, 87: 81–95.

Appendices

Appendix A states general lemmas and propositions about separability. Appendix B uses these reuslts to prove the results in the paper.

Appendix A: Technical background

We begin by restating the definition of a conditional relation in terms of its master relation, and the separability property that turns a conditional relation into an ordering.

Suppose that a weak preference ordering R is defined on a product set $\mathcal{X} = \prod_{\ell \in \mathcal{L}} \mathcal{X}_\ell$, where $\mathcal{L}$ is a finite set of indexes. Take a subset of indexes $\mathcal{J} \subseteq \mathcal{L}$ and its complement $\mathcal{K} := \mathcal{L} \setminus \mathcal{J}$. Denote the subproduct sets $\prod_{\ell \in \mathcal{J}} \mathcal{X}_\ell$ and $\prod_{\ell \in \mathcal{K}} \mathcal{X}_\ell$ by $\mathcal{X}_\mathcal{J}$ and $\mathcal{X}_\mathcal{K}$, respectively. By definition, the *conditional induced by* R *on* $\mathcal{J}$ is the relation $\mathsf{R}_\mathcal{J}$ on $\mathcal{X}_\mathcal{J}$ thus defined: for all $\xi_\mathcal{J}, \xi'_\mathcal{J} \in \mathcal{X}_\mathcal{J}$,

$$\xi_\mathcal{J} \ \mathsf{R}_\mathcal{J} \ \xi'_\mathcal{J} \ \text{ if and only if } \ \text{for some } \xi_\mathcal{K} \in \mathcal{X}_\mathcal{K}, \ (\xi_\mathcal{J}, \xi_\mathcal{K}) \ \mathsf{R} \ (\xi'_\mathcal{J}, \xi_\mathcal{K}).$$

We denote the conditional $\mathsf{R}_{\{\ell\}}$ by R_ℓ. By a well-known fact, the conditional $\mathsf{R}_\mathcal{J}$ is an ordering if and only if R is *separable in* $\mathcal{J}$, that is: for all $\xi_\mathcal{J}, \xi'_\mathcal{J} \in \mathcal{X}_\mathcal{J}$ and $\xi_\mathcal{K}, \xi'_\mathcal{K} \in \mathcal{X}_\mathcal{K}$,

$$(\xi_\mathcal{J}, \xi_\mathcal{K}) \ \mathsf{R} \ (\xi'_\mathcal{J}, \xi_\mathcal{K}) \text{ if and only if } (\xi_\mathcal{J}, \xi'_\mathcal{K}) \ \mathsf{R} \ (\xi'_\mathcal{J}, \xi'_\mathcal{K}).$$

In this case, we may also say that $\mathcal{J}$ is a R-*separable*. Clearly, separability in $\mathcal{J}$ entails that R is *increasing* with $\mathsf{R}_\mathcal{J}$, that is: for all $\xi_\mathcal{J}, \xi'_\mathcal{J} \in \mathcal{X}_\mathcal{J}$ and $\xi_\mathcal{K} \in \mathcal{X}_\mathcal{K}$,

$$\text{if } \xi_\mathcal{J} \ \mathsf{R}_\mathcal{J} \ \xi'_\mathcal{J}, \text{ then } (\xi_\mathcal{J}, \xi_\mathcal{K}) \ \mathsf{R} \ (\xi'_\mathcal{J}, \xi_\mathcal{K}),$$

and if the $\mathsf{R}_\mathcal{J}$-comparison is in fact strict, so is the resulting R-comparison. Conversely, if $\mathsf{R}_\mathcal{J}$ is some ordering on $\mathcal{X}_\mathcal{J}$, the property that R on $\mathcal{X}$ is increasing with $\mathsf{R}_\mathcal{J}$ entails that R is weakly separable in $\mathcal{J}$.[14]

This apparatus can be applied by taking the $\mathcal{X}_\ell$ sets to be copies of $\mathbb{R}$, and suitably fixing the relation R and the indexing sets $\mathcal{L}$ and subsets $\mathcal{J} \subset \mathcal{L}$. For example, for any $i \in \mathcal{I}$ and $s \in \mathcal{S}$, to translate the statement, "$\succsim^i$ induces a conditional preference $\succsim^i_s$" into a statement about separability, we take $\mathsf{R} = \succsim^i$, $\mathcal{L} = \{(s,o) \mid s \in \mathcal{S}, o \in \mathcal{O}\}$ and $\mathcal{J} = \{(s,o) \mid o \in \mathcal{O}\}$. All the other "conditional preference" statements introduced in Section 2 can be translated in the same way. Recall that we assume throughout that conditional preference orderings exist across the uncertainty type or not at all (i.e., for all s or none, etc). It then follows

[14] For these definitions and basic facts, see Fishburn (1970), Keeny and Raiffa (1976), and Wakker (1989). What is called *separable* here is sometimes called *weakly separable* elsewhere.

from the equivalence between separability and the increasing property just said that our decision-theoretic assumptions could be restated in terms of *dominance*. For example, the statement "$\succsim^i$ induces conditional preferences $\succsim^i_s$" is equivalent to asserting that $\succsim^i$ satisfies dominance with respect to the $(\succsim^i_s)_{s\in\mathcal{S}}$.

Let $\mathcal{X}$ specifically be an open box in $\mathbb{R}^{\mathcal{L}}$, i.e., $\mathcal{X} = \prod_{\ell\in\mathcal{L}} \mathcal{X}_\ell$, where the $\mathcal{X}_\ell$ are open intervals. An ordering $\succeq$ on $\mathcal{X}$ has an *additive representation* if it is represented by a function $U : \mathcal{X}\longrightarrow\mathbb{R}$ of the form

$$U(\mathbf{x}) \quad := \quad \sum_{\ell\in\mathcal{L}} u_\ell(x_\ell), \tag{A1}$$

where $u_\ell : \ \mathcal{X}_\ell\longrightarrow\mathbb{R}$, $\ell \in \mathcal{L}$.

Let us say that $\mathcal{J} \subseteq \mathcal{L}$ is *strictly $\succeq$-essential* if, for all $\mathbf{x} \in \mathcal{X}$, there exist $\mathbf{y}, \mathbf{y}' \in \mathcal{X}$ such that $(y_\ell)_{\ell\in\mathcal{K}} = (y'_\ell)_{\ell\in\mathcal{K}} = (x_\ell)_{\ell\in\mathcal{K}}$, and $\mathbf{y} \succ \mathbf{y}'$. In words, we can create a strict preference by only manipulating the $\mathcal{J}$ coordinates, while keeping the $\mathcal{K}$ coordinates fixed at given values. We now record two classic results due to Gorman (1968) and Debreu (1960), respectively. If every subset $\mathcal{J} \subseteq \mathcal{L}$ is $\succeq$-separable, we say that $\succeq$ is *totally separable*.

Lemma A1 *Let $\succeq$ be a continuous order on an open box $\mathcal{X} \subseteq \mathbb{R}^{\mathcal{L}}$. Let $\mathcal{J}, \mathcal{K} \subseteq \mathcal{L}$ be two $\succeq$-separable subsets of indexes, such that $\mathcal{J} \cap \mathcal{K} \neq \emptyset$. Suppose that $\mathcal{J}$, $\mathcal{K}$, and $\mathcal{J} \cap \mathcal{K}$ are all strictly $\succeq$-essential. Then:*

(a) *$\mathcal{J} \cup \mathcal{K}$ is $\succeq$-separable.*

(b) *$\mathcal{J} \cap \mathcal{K}$ is $\succeq$-separable.*

Lemma A2 *If $\succeq$ is a continuous, totally separable order on an open box $\mathcal{X} \subseteq \mathbb{R}^{\mathcal{L}}$, and $\succeq$ is increasing in every coordinate, then $\succeq$ has an additive utility representation* (A1). *Furthermore, the functions $\{u_\ell\}_{\ell\in\mathcal{L}}$ in this representation are unique up to positive affine transformations (PAT) with a common multiplier.*

We will now adapt these results to our framework. Suppose $\mathcal{L} = \mathcal{I} \times \mathcal{S} \times \mathcal{O}$, and let $\succeq$ be a preference order on $\mathcal{X} = \mathbb{R}^{\mathcal{L}}$. For any $i \in \mathcal{I}$, we will say that $\succeq$ is *separable in i* if $\{i\} \times \mathcal{S} \times \mathcal{O}$ is $\succeq$-separable. Likewise, for any $s \in \mathcal{S}$ (resp. $o \in \mathcal{O}$), say $\succeq$ is *separable in s* (resp. *separable in o*) if $\mathcal{I} \times \{s\} \times \mathcal{O}$ (resp. $\mathcal{I} \times \mathcal{S} \times \{o\}$) is $\succeq$-separable.

Proposition A3 *Take $\mathcal{L} = \mathcal{I} \times \mathcal{S} \times \mathcal{O}$, with $|\mathcal{I}|, |\mathcal{S}|, |\mathcal{O}| \geq 2$, and $\mathcal{X} = \mathbb{R}^{\mathcal{L}}$, viewing elements $\mathbb{X} \in \mathcal{X}$ as arrays $[x^i_{so}]^{i\in\mathcal{I}}_{s\in\mathcal{S},o\in\mathcal{O}}$. If a continuous order $\succeq$ on $\mathcal{X}$ is increasing in every coordinate, and is separable in each $i \in \mathcal{I}$, each $s \in \mathcal{S}$, and each $o \in \mathcal{O}$, then it admits an additive utility representation $U : \mathcal{X}\longrightarrow\mathbb{R}$ of the form*

$$U(\mathbf{x}) \quad := \quad \sum_{i\in\mathcal{I}} \sum_{s\in\mathcal{S}} \sum_{o\in\mathcal{O}} u^i_{so}(x^i_{so}),$$

where each u^i_{so} is a continuous, increasing function from $\mathbb{R}$ to $\mathbb{R}$. Furthermore, the utility functions $\{u^i_{so}\}^{i\in\mathcal{I}}_{s\in\mathcal{S},o\in\mathcal{O}}$ are unique up to PAT with a common multiplier.

Proof. (Sketch)

Since $\succeq$ is increasing in every coordinate, every subset of $\mathcal{L}$ is strictly $\succeq$-essential. For all $i \in \mathcal{I}$, $s \in \mathcal{S}$, and $o \in \mathcal{O}$, the subsets $\{i\} \times \mathcal{S} \times \mathcal{O}$, $\mathcal{I} \times \{s\} \times \mathcal{O}$, and $\mathcal{I} \times \mathcal{S} \times \{o\}$ are $\succeq$-separable, by hypothesis. Thus, Lemma A1 says that the (nonempty) intersections of these sets are $\succeq$-separable, as are their unions. At this point, by further applications of Lemma A1, we can show that every two-element subset of $\mathcal{L}$ is $\succeq$-separable; from there, it can be shown that *every* subset of $\mathcal{L}$ is $\succeq$-separable. In other words, $\succeq$ is totally separable. By hypothesis, $\succeq$ is increasing in every coordinate. Thus, we can apply Lemma A2 to get the additive representation. □

We now specialize the basic sets differently. Take $\mathcal{L} = \mathcal{J} \times \mathcal{K}$ with $|\mathcal{J}|, |\mathcal{K}| \geq 2$, and $\mathcal{X} = \mathbb{R}^{\mathcal{L}}$, viewing elements $\mathbf{X} \in \mathcal{X}$ as matrices $[x_k^j]_{k\in\mathcal{K}}^{j\in\mathcal{J}}$, with $j \in \mathcal{J}$ indexing the rows and $k \in \mathcal{K}$ indexing the columns. Alternatively, we can think of $\mathbf{X}$ as a $\mathcal{J}$-indexed array of row vectors $\mathbf{x}^j := [x_k^j]_{k\in\mathcal{K}} \in \mathbb{R}^{\mathcal{K}}$, or as a $\mathcal{K}$-indexed array of columns vectors $\mathbf{x}_k := [x_k^j]^{j\in\mathcal{J}} \in \mathbb{R}^{\mathcal{J}}$. Now consider a continuous ordering $\succeq$ on $\mathcal{X}$. Here are three axioms that $\succeq$ might satisfy.

Coordinate Monotonicity: For all $\mathbf{X}, \mathbf{Y} \in \mathcal{X}$, if $x_k^j \geq y_k^j$ for all $(j,k) \in \mathcal{J} \times \mathcal{K}$, then $\mathbf{X} \succeq \mathbf{Y}$. If, in addition, $x_k^j > y_k^j$ for some $(j,k) \in \mathcal{J} \times \mathcal{K}$, then $\mathbf{X} \succ \mathbf{Y}$.

Row Preferences: For each column $j \in \mathcal{J}$, $\succeq$ is separable in $\{j\} \times \mathcal{K}$.

Column Preferences: For all rows $k \in \mathcal{K}$, $\succeq$ is separable in $\mathcal{J} \times \{k\}$.

Define $\succeq^j$ and $\succeq_k$ to be the conditional relations of $\succeq$ on j and k, respectively. It follows from Row Preferences that the $\succeq^j$ are orders on $\mathbb{R}^{\mathcal{K}}$, and from Column Preferences that the $\succeq_k$ are orders on $\mathbb{R}^{\mathcal{J}}$. Moreover, $\succeq$ is increasing with respect to each of these conditional relations . The next two axioms force the conditional orders to be invariant.

Invariant Row Preferences: Row Preferences holds, and there is an ordering $\succeq^{\mathcal{J}}$ on $\mathcal{Y}^{\mathcal{K}}$ such that $\succeq^j = \succeq^{\mathcal{J}}$ for all $j \in \mathcal{J}$.

Invariant Column Preferences: Column Preferences holds, and there is an ordering $\succeq_{\mathcal{K}}$ on $\mathcal{Y}^{\mathcal{J}}$ such that $\succeq_k = \succeq_{\mathcal{K}}$ for all $k \in \mathcal{K}$.

These five axioms draw their use from the following proposition, which the proofs in Appendix B will repeatedly use. (Each of these proofs will involve two of the sets $\mathcal{I}$, $\mathcal{S}$, $\mathcal{O}$ taking the place of the abstract indexing sets $\mathcal{J}$ and $\mathcal{K}$.)

Proposition A4 (a) *Suppose a continuous preference order $\succeq$ on $\mathcal{X} = \mathbb{R}^{\mathcal{L}}$ satisfies Coordinate Monotonicity, Row Preferences and Invariant Column Preferences. Then there is a strictly positive probability vector $\mathbf{p} \in \Delta_{\mathcal{K}}$, and for all $j \in \mathcal{J}$, there is an increasing, continuous function $u^j : \mathbb{R} \longrightarrow \mathbb{R}$, such that $\succeq$ is represented by the function $W : \mathcal{X} \longrightarrow \mathbb{R}$ defined by:*

$$W(\mathbf{X}) \; := \; \sum_{k\in\mathcal{K}} \sum_{j\in\mathcal{J}} p_k \, u^j(x_k^j).$$

In this representation, the probability vector $\mathbf{p}$ is unique, and the functions u^j are unique up to PAT with a common multiplier.

(b) *Assume* **Invariant Row Preferences** *instead of* **Row Preferences**, *holding the other conditions the same as in part* (a) . *Then there is an increasing, continuous function* $u : \mathbb{R} \longrightarrow \mathbb{R}$ *and strictly positive probability vectors* $\mathbf{q} \in \Delta_{\mathcal{J}}$ *and* $\mathbf{p} \in \Delta_{\mathcal{K}}$ *such that* $\succeq$ *is represented by the function* $W : \mathcal{X} \longrightarrow \mathbb{R}$ *defined by*

$$W(\mathbf{X}) \;:=\; \sum_{k\in\mathcal{K}} \sum_{j\in\mathcal{J}} q^j\, p_k\, u(x^j_k).$$

In this representation, the probability vectors $\mathbf{q}$ *and* $\mathbf{p}$ *are unique, and the function* u *is unique up to a PAT.*

Proof. See Mongin and Pivato (2015). Part (a) follows from Theorem 1(c,d), and part (b) from Corollary 1(c,d). The axioms of that paper are stated differently, because the domain considered there is not necessarily a Cartesian product. □

Appendix B: Proofs of the results of the paper

Our twofold uncertainty framework may seem to raise the worrying possibility that conditional orderings depend on how they are induced; e.g., that $\succsim_{so}$, as directly induced by $\succsim$, differs from $\succsim_{so}$, as induced by the ordering $\succsim_s$ induced by $\succsim$, or from $\succsim_{so}$, as induced by the ordering $\succsim_o$. However, such a discrepancy cannot occur; the different forms of conditionalization "commute" with one another . We skip the purely formal proof. In the next lemma and elsewhere, we will repeatedly use this *commutativity of conditionalization.*

Lemma B1 *Let* $\succeq$ *be a continuous order on* $\mathbb{R}^{\mathcal{I}\times\mathcal{S}\times\mathcal{O}}$.

(a) *If* $\succeq$ *induces interim preferences* $\succeq_s$ *and* $\succeq_o$, *then it also induces* ex post *preferences* $\succeq_{so}$.

(b) *If, moreover, the interim preferences* $\succeq_o$ *are invariant, then for any given* s, $\succeq_s$ *induces invariant* ex post *preferences* $\succeq_{so}$.

(c) *If, moreover, the interim preferences* $\succeq_o$ *and* $\succeq_s$ *are both invariant, then the* ex post *preferences* $\succeq_{so}$ *are invariant.*

Proof. Let $(s, o) \in \mathcal{S} \times \mathcal{O}$. For all $o \in \mathcal{O}$, let $\mathcal{J}_o := \{(i', s', o);\, i' \in \mathcal{I}$ and $s' \in \mathcal{S}\}$. Then $\mathcal{J}_o$ is a $\succeq$-separable subset of $\mathcal{I}\times\mathcal{S}\times\mathcal{O}$, because, by hypothesis, $\succeq$ induces interim preferences $\succeq_o$. Similarly, for all $s \in \mathcal{S}$, let $\mathcal{K}_s := \{(i', s, o');\, i' \in \mathcal{I}$ and $o' \in \mathcal{O}\}$; this is a $\succeq$-separable subset of $\mathcal{I}\times\mathcal{S}\times\mathcal{O}$, because $\succeq$ induces interim preferences $\succeq_s$. The nonempty intersection $\mathcal{I}_{so} := \mathcal{J}_o \cap \mathcal{K}_s$ is $\succeq$-separable by Lemma A1(b), meaning that $\succeq$ induces *ex post* preferences $\succeq_{so}$.

Adding the assumption that the interim preferences $\succeq_o$ induced by $\succeq$ are invariant, we fix s and consider any pair $o \neq o'$. By commutativity of conditonalization , we can regard the *ex post* preferences $\succeq_{so}$ and $\succeq_{so'}$ as being induced by $\succeq_o$ and $\succeq_{o'}$, respectively. But

$\succeq_o = \succeq_{o'}$, so that $\succeq_{so} = \succeq_{so'}$, and now regarding these *ex post* preferences as being induced by $\succeq_s$, we conclude that this ordering induces invariant *ex post* preferences.

Now we add the assumption that the interim preferences $\succeq_s$ induced by $\succeq$ are invariant, fix o and consider any pair $s \neq s'$. By symmetric reasoning, we conclude that $\succeq_{so} = \succeq_{s'o}$. The two paragraphs together prove that, for all $o, o' \in \mathcal{O}$ and $s, s' \in \mathcal{S}$, $\succeq_{so} = \succeq_{s'o'}$, meaning that $\succeq$ induces invariant *ex post* preferences. □

Proof of Proposition 1. Let $\mathcal{J} := \mathcal{S}$ and $\mathcal{K} := \mathcal{O}$. We will check which of the axioms of Appendix A apply to the ordering $\succeq^i$, for any $i \in \mathcal{I}$. Coordinate Monotonicity holds because $\succeq^i$ induces preference orderings $\succeq^i_{so}$ that coincide with the natural ordering of real numbers. As the $\succeq^i_s$ (resp. the $\succeq^i_o$) are invariant, Invariant Row Preferences (resp. Invariant Column Preferences) holds. Thus, Proposition A4(b) yields the expected utility representation (2) for $\succeq_i$. Since $\succeq$ has a numerical representation that is increasing with the $\succeq^i$ by the *ex ante* Pareto principle, the social representation (3) follows. The uniqueness condition for F is obvious, and the other uniqueness statements follow from Proposition A4(b). □

Proof of Proposition 2. By Lemma B1(c), the assumption that $\succeq$ induces invariant interim preferences of both kinds guarantees that $\succeq$ also induces invariant *ex post* preferences $\succeq_{\mathrm{xp}}$ on $\mathbb{R}^{\mathcal{I}}$. These preferences inherit the continuity of $\succeq$ and the *ex post* Pareto principle makes them increasing in every coordinate. Thus, each of them is represented by a continuous and increasing function $v : \mathbb{R}^{\mathcal{I}} \longrightarrow \mathbb{R}$.

To any $\mathbb{X} \in \mathbb{R}^{\mathcal{I}\times\mathcal{S}\times\mathcal{O}}$, we associate the element $\widetilde{\mathbf{X}} \in \mathbb{R}^{\mathcal{S}\times\mathcal{O}}$ whose (s, o) component is $\widetilde{x}_{so} := v(\mathbf{x}_{so})$. The function $V : \mathbb{R}^{\mathcal{I}\times\mathcal{S}\times\mathcal{O}} \to \mathbb{R}^{\mathcal{S}\times\mathcal{O}}$ defined by $V(\mathbb{X}) := \widetilde{\mathbf{X}}$ is continuous and increasing in each component. By these two properties, the image set $\widetilde{\mathcal{X}} := V(\mathbb{R}^{\mathcal{I}\times\mathcal{S}\times\mathcal{O}})$ is a set of the form $\mathcal{Y}^{\mathcal{S}\times\mathcal{O}}$, where $\mathcal{Y} := v(\mathbb{R}^{\mathcal{I}})$ is an open interval.

Define an ordering $\widetilde{\succeq}$ on $\widetilde{\mathcal{X}}$ by the condition that for all $\widetilde{\mathbf{X}}, \widetilde{\mathbf{Y}} \in \widetilde{\mathcal{X}}$, if $\widetilde{\mathbf{X}} = V(\mathbb{X})$ and $\widetilde{\mathbf{Y}} = V(\mathbb{Y})$, then

$$\widetilde{\mathbf{X}} \mathrel{\widetilde{\succeq}} \widetilde{\mathbf{Y}} \text{ if and only if } \mathbb{X} \succeq \mathbb{Y}. \tag{B1}$$

(To see that $\widetilde{\succeq}$ is mathematically well-defined by (B1), suppose $V(\mathbb{X}) = \widetilde{\mathbf{X}} = V(\mathbb{X}')$ for some $\mathbb{X}, \mathbb{X}' \in \mathcal{X}$. Then for all $(s, o) \in \mathcal{S} \times \mathcal{O}$, we have $v(\mathbf{x}_{so}) = v(\mathbf{x}'_{so})$, and hence $\mathbf{x}_{so} \approx_{\mathrm{xp}} \mathbf{x}'_{so}$. Thus $\mathbb{X} \approx \mathbb{X}'$, because $\succeq$ is increasing relative to $\succeq_{\mathrm{xp}}$.) In terms of the Appendix A, putting $\mathcal{J} := \mathcal{S}$ and $\mathcal{K} := \mathcal{O}$, we conclude that $\widetilde{\succeq}$ is continuous and satisfies Invariant Row Preferences and Invariant Column Preferences, and Coordinate Monotonicity, by using the respective properties that $\succeq$ is continuous, induces invariant interim orderings $\succeq_s$, and induces invariant interim orderings $\succeq_o$, and induces invariant *ex post* orderings $\succeq_{\mathrm{xp}}$. Thus, Proposition A4(b) yields strictly positive probability vectors $\mathbf{p} \in \Delta_{\mathcal{S}}$ and $\mathbf{q} \in \Delta_{\mathcal{O}}$, and a continuous increasing function $u : \mathbb{R} \longrightarrow \mathbb{R}$, such that $\widetilde{\succeq}$ is represented by the function $\widetilde{W} : \widetilde{\mathcal{X}} \longrightarrow \mathbb{R}$ defined by

$$\widetilde{W}(\widetilde{\mathbf{X}}) := \sum_{s\in\mathcal{S}} \sum_{o\in\mathcal{O}} q_o\, p_s\, u(\widetilde{x}_{so}).$$

Now, putting $W_{\mathrm{xa}}(\mathbb{X}) := \widetilde{W} \circ V(\mathbf{X})$ for all $\mathbf{X} \in \mathbb{R}^{\mathcal{I}\times\mathcal{S}\times\mathcal{O}}$, and $W_{\mathbf{xp}}(\mathbf{x}) := u \circ v(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^{\mathcal{I}}$, we obtain the desired representations. The uniqueness properties are those of

Proposition A4(b).[15] $\square$

Proof of Theorem 3. First we show that $\succeq$ is increasing in every coordinate. Let $(i,s,o) \in \mathcal{I}\times\mathcal{S}\times\mathcal{O}$. The representation (2) from Proposition 1 implies that the individual preference order $\succeq^i$ is increasing with respect to the coordinate $x_{s,o}$. Since $\succeq$ increasing with respect to $\succeq^i$ by the *ex ante* Pareto principle, we conclude that $\succeq$ is increasing with respect to the coordinate $x^i_{s,o}$.

As the $\succeq^i$ relations are orderings and the *ex ante* Pareto principle makes $\succeq$ increasing with them, $\succeq$ is separable in each $i \in \mathcal{I}$. As $\succeq$ induces interim preferences of both types, $\succeq$ is also separable in each $s \in \mathcal{S}$ and $o \in \mathcal{O}$. It then follows from Proposition A3 that, for all $(i,s,o) \in \mathcal{I}\times\mathcal{S}\times\mathcal{O}$, there exist continuous and increasing functions $u^i_{so} : \mathbb{R} \longrightarrow \mathbb{R}$ such that $\succeq$ is represented by the function $W_{\text{xa}} : \mathbb{R}^{\mathcal{I}\times\mathcal{S}\times\mathcal{O}} \longrightarrow \mathbb{R}$ defined by

$$W_{\text{xa}}\,(\mathbb{X}) \quad := \quad \sum_{i\in\mathcal{I}}\sum_{s\in\mathcal{S}}\sum_{o\in\mathcal{O}} u^i_{so}(x^i_{so}). \tag{B2}$$

Furthermore, the u^i_{so} are unique up to positive affine transformations (PAT) with a common multiplier. We can fix any $\mathbb{Y} \in \mathbb{R}^{\mathcal{I}\times\mathcal{S}\times\mathcal{O}}$ and add constants to these functions so as to ensure that $u^i_{so}(y^i_{so}) = 0$ for all $(i,s,o) \in \mathcal{I}\times\mathcal{S}\times\mathcal{O}$.[16] For convenience, fix some $\overline{y} \in \mathbb{R}$, and suppose that $y^i_{so} = \overline{y}$ for all $(i,s,o) \in \mathcal{I}\times\mathcal{S}\times\mathcal{O}$.

For all $i \in \mathcal{I}$, equation (B2) implies that the preference ordering $\succeq^i$ can be represented by the function $U^i : \mathbb{R}^{\mathcal{S}\times\mathcal{O}} \longrightarrow \mathbb{R}$ defined by

$$U^i(\mathbf{X}) \quad := \quad \sum_{s\in\mathcal{S}}\sum_{o\in\mathcal{O}} u^i_{so}(x_{so})\ . \tag{B3}$$

From Proposition 1 , there are continuous increasing utility functions $\tilde{u}^i : \mathbb{R} \longrightarrow \mathbb{R}$, and two strictly positive probability vectors $\mathbf{p}^i \in \Delta_{\mathcal{S}}$ and $\mathbf{q}^i \in \Delta_{\mathcal{O}}$, such that $\succeq^i$ is represented by the function $U^i : \mathbb{R}^{\mathcal{S}\times\mathcal{O}} \longrightarrow \mathbb{R}$ defined by

$$U^i(\mathbf{X}) \quad := \quad \sum_{s\in\mathcal{S}}\sum_{o\in\mathcal{O}} q^i_o\, p^i_s\, \tilde{u}^i(x_{so}). \tag{B4}$$

Furthermore, in this representation, $\mathbf{p}^i$ and $\mathbf{q}^i$ are unique, and $\tilde{u}^i$ is unique up to PAT. By adding a constant, we ensure that $\tilde{u}^i(\overline{y}) = 0$.

From the uniqueness property applied to (B3) and (B4), there exist constants $\alpha^i > 0$ and $\beta^i \in \mathbb{R}$ such that :

$$u^i_{so}(x) = \alpha^i\, q^i_o\, p^i_s\, \tilde{u}^i(x) + \beta^i,\ \text{for all } (s,o) \in \mathcal{S}\times\mathcal{O}. \tag{B5}$$

Substituting $x = \overline{y}$ into (B5) leads to $\beta^i = 0$. Then substituting (B5) (for all $i \in \mathcal{I}$) into the representation (B2) yields:

$$W_{\text{xa}}\,(\mathbb{X}) \quad = \quad \sum_{i\in\mathcal{I}}\sum_{s\in\mathcal{S}}\sum_{o\in\mathcal{O}} \alpha^i\, q^i_o\, p^i_s\, \tilde{u}^i(x^i_{so}). \tag{B6}$$

[15] Proposition A4(b) is stated for $\mathbb{R}^{\mathcal{J}\times\mathcal{K}}$, but it carries through to subsets $Y^{\mathcal{J}\times\mathcal{K}} \subseteq \mathbb{R}^{\mathcal{J}\times\mathcal{K}}$, when these are open and take the form of a product of intervals.

[16] To avoid burdening notation, we refer to the original and translated functions by the same symbol. This convention is applied throughout the proofs.

For given $s \in \mathcal{S}$ in this representation, we obtain a representation $V_s : \mathbb{R}^{\mathcal{I}\times\mathcal{O}} \longrightarrow \mathbb{R}$ of the interim preference $\succeq_s$ on $\mathbb{R}^{\mathcal{I}\times\mathcal{O}}$:

$$V_s(\mathbf{X}) \quad := \quad \sum_{i\in\mathcal{I}} \sum_{o\in\mathcal{O}} \alpha^i \, q_o^i \, p_s^i \, \tilde{u}^i(x_o^i). \tag{B7}$$

Let $\mathbf{Y}_s := (\overline{y}, \ldots, \overline{y}) \in \mathbb{R}^{\mathcal{I}\times\mathcal{O}}$; then $V_s(\mathbf{Y}_s) \ = 0$.

Let us now put $\mathcal{J} := \mathcal{I}$ and $\mathcal{K} := \mathcal{O}$, and check which axioms in Appendix A the interim preference $\succeq_s$ satisfies. This is a continuous ordering by the continuity of $\succeq$. By the representation (B7) , $\succeq_s$ is separable in each $\{i\} \times \mathcal{O}$ and each $\mathcal{I} \times \{o\}$, and increasing in every coordinate , and thus satisfies Row Preferences, Column Preferences, and Coordinate Monotonicity . As $\succeq$ induces invariant $\succeq_o$, Lemma B1(b) entails that the induced preferences $\succeq_{so}$ are invariant, meaning that the stronger axiom of Invariant Column Preferences holds. Hence, Proposition A4(a) yields a strictly positive probability vector $\mathbf{r}_s \in \Delta_{\mathcal{O}}$, and for all $i \in \mathcal{I}$, continuous, increasing utility functions $\widehat{u}_s^i : \mathbb{R} \longrightarrow \mathbb{R}$ such that $\succeq_s$ is represented by the function $\widehat{V}_s : \mathbb{R}^{\mathcal{I}\times\mathcal{O}} \longrightarrow \mathbb{R}$ defined by

$$\widehat{V}_s(\mathbf{X}) \quad := \quad \sum_{i\in\mathcal{I}} \sum_{o\in\mathcal{O}} r_{so} \, \widehat{u}_s^i(x_o^i). \tag{B8}$$

In this representation, $\mathbf{r}_s$ is unique and the functions $\widehat{u}_s^i$ are unique up to PAT with a common multiplier. We add constants to ensure that $\widehat{u}_s^i(\overline{y}) = 0$ for all $i \in \mathcal{I}$. It follows that $\widehat{V}_s(\mathbf{Y}_s) \ = 0$.

From the uniqueness property applied to (B7) and (B8), there exist $\gamma_s > 0$ and $\delta_s \in \mathbb{R}$ such that $\widehat{V}_s = \gamma_s \, V_s + \delta_s$. Substituting $\mathbf{Y}_s$ leads to $\delta_s = 0$. Since this holds for all $s \in \mathcal{S}$, we can conclude that

$$\gamma_s \, r_{so} \, \widehat{u}_s^i \quad = \quad \alpha^i \, q_o^i \, p_s^i \, \tilde{u}^i, \text{ for all } (i, s, o) \in \mathcal{I} \times \mathcal{S} \times \mathcal{O} \tag{B9}$$

Let us now fix i and s in these equations. All the coefficients are positive and the increasing functions $\widehat{u}_s^i$ and $\tilde{u}^i$ are nonzero for some $y^* \in \mathbb{R}$. Thus we can derive the relations:

$$\frac{r_{so}}{q_o^i} \quad = \quad \frac{\alpha^i \, p_s^i \, \tilde{u}^i(y^*)}{\gamma_s \, \widehat{u}_s^i(y^*)}, \quad \text{for all } o \in \mathcal{O}. \tag{B10}$$

The right-hand side of (B10) does not depend on o. Thus, the left-hand side must also be independent of o, which means that the vectors $\mathbf{q}^i$ and $\mathbf{r}_s$ are scalar multiples of one another. Thus, since they are probability vectors, we have $\mathbf{q}^i = \mathbf{r}_s$. Since this holds for all i and s, we can drop the indexes. Denote the common probability vector by $\mathbf{q}$. Substituting $\mathbf{q}$ into (B6) and defining $u^i := \alpha^i \, \tilde{u}^i$, we get the formula (5) of the theorem. The other parts readily follow. □

Proof of Theorem 4. For each $i \in \mathcal{I}$, $\succeq^i$ satisfies the assumptions of Proposition 1. Thus, by the argument used to prove this proposition, we conclude that there exist a continuous increasing utility function $u^i : \mathbb{R} \longrightarrow \mathbb{R}$, and strictly positive probability vectors $\mathbf{p}^i \in \Delta_{\mathcal{S}}$ and $\mathbf{q}^i \in \Delta_{\mathcal{O}}$, such that $\succeq^i$ is represented by the function $U^i : \mathbb{R}^{\mathcal{S}\times\mathcal{O}} \longrightarrow \mathbb{R}$ defined by

$$U^i(\mathbf{X}) \quad := \quad \sum_{s\in\mathcal{S}} \sum_{o\in\mathcal{O}} q^i_o \, p^i_s \, u^i(x_{so}),$$

and the u^i are unique up to PAT with a common multiplier. This establishes the SEU representation (6). Fix $\overline{x} \in \mathbb{R}$. By adding constants, we ensure that $u^i(\overline{x}) = 0$ for all $i \in \mathcal{I}$.

Meanwhile, Proposition 2 yields strictly positive probability vectors $\mathbf{p} \in \Delta_{\mathcal{S}}$ and $\mathbf{q} \in \Delta_{\mathcal{O}}$, and a continuous increasing function $W_{\rm xp} : \mathbb{R}^{\mathcal{I}} \longrightarrow \mathbb{R}$, such that $\succeq$ is represented by the function $W_{\rm xa} : \mathcal{X} \longrightarrow \mathbb{R}$ defined by

$$W_{\rm xa}(\mathbb{X}) \quad := \quad \sum_{s\in\mathcal{S}} \sum_{o\in\mathcal{O}} p_s \, q_o \, W_{\rm xp}(\mathbf{x}_{so}),$$

where $\mathbf{p}$ and $\mathbf{q}$ are unique, and $W_{\rm xp}$ is unique up to PAT. This establishes the SEU representation (7). Let $\overline{\mathbf{x}} := (\overline{x}, \ldots, \overline{x})$. By adding a constant, we ensure that $W_{\rm xp}(\overline{\mathbf{x}}) = 0$.

Now let $\mathcal{J} = \mathcal{I}$ and $\mathcal{K} = \mathcal{O}$ and consider how the axioms of Appendix A apply to $\succeq_s$ for any given $s \in \mathcal{S}$, recalling that these interim social preferences are well-defined and invariant (i.e. independent of s) . The objective interim Pareto principle makes $\succeq_s$ separable in each $i \in \mathcal{I}$, so that Row Preferences holds. By Proposition 2, the *ex post* social preferences $\succeq_{so}$ are well-defined and invariant, so that Invariant Column Preferences holds. Then, by Proposition A4(a), there exist a probability vector $\widetilde{\mathbf{q}} \in \Delta_{\mathcal{O}}$, and for all $i \in \mathcal{I}$, continuous increasing functions v^i such that $\succeq_s$ is represented by the function $W : \mathbb{R}^{\mathcal{I}\times\mathcal{O}} \longrightarrow \mathbb{R}$ defined by

$$W(\mathbf{X}) \quad := \quad \sum_{i\in\mathcal{I}} \sum_{o\in\mathcal{O}} \widetilde{q}_o \, v^i(x^i_{so}),$$

where $\widetilde{\mathbf{q}}$ is unique and the v^i are unique up to PAT with a common multiplier. The same representation holds for all $s \in \mathcal{S}$. Adding a constant, we ensure that $v^i(\overline{x}) = 0$ for all $i \in \mathcal{I}$.

We now show that $\mathbf{q} = \widetilde{\mathbf{q}}$. By fixing $s \in \mathcal{S}$ and applying the representation $W_{\rm xa}$ to elements $\mathbb{X}$ whose components for $s' \neq s$ are fixed at some values, we obtain a new representation for $\succeq_s$ and reduce it to the representation just obtained in terms of W by the standard uniqueness property. That is, there exist constants $\alpha > 0$ and β such that

$$\sum_{o\in\mathcal{O}} q_o \, W_{\rm xp}(\mathbf{x}_o) \quad = \quad \alpha \sum_{i\in\mathcal{I}} \sum_{o\in\mathcal{O}} \widetilde{q}_o \, v^i(x^i_o) + \beta \text{ for all } \mathbf{X} \in \mathbb{R}^{\mathcal{I}\times\mathcal{O}}.$$

Substituting $x^i_o = \overline{x}$ for all $i \in \mathcal{I}$ and $o \in \mathcal{O}$ leads to $\beta = 0$. Now fixing o and putting $x^i_{o'} = \overline{x}$ for all $o' \neq o$ leads to the equation:

$$W_{\rm xp}(\mathbf{x}_o) \quad = \quad \frac{\widetilde{q}_o}{q_o} \sum_{i\in\mathcal{I}} \alpha \, v^i(x^i_{so}), \qquad \text{for all } \mathbf{x}_o \in \mathbb{R}^{\mathcal{I}}.$$

Since this holds for all $o \in \mathcal{O}$, the two probability vectors $\mathbf{q}$ and $\widetilde{\mathbf{q}}$ are proportional, hence equal. Hence

$$W_{\text{xp}}(\mathbf{x}_o) \quad = \quad \sum_{i \in \mathcal{I}} \alpha\, v^i(x^i_{so}), \qquad \text{for all } \mathbf{x}_o \in \mathbb{R}^{\mathcal{I}}. \tag{B11}$$

and the invariant conditional preference $\succeq_s$ is represented by the function $\widetilde{W} : \mathbb{R}^{\mathcal{I}\times\mathcal{O}} \longrightarrow \mathbb{R}$ defined by

$$\widetilde{W}(\mathbf{X}) \quad := \quad \sum_{i \in \mathcal{I}} \sum_{o \in \mathcal{O}} q_o\, \alpha\, v^i(x^i_{so}).$$

We now use a similar argument to show that $\mathbf{q} = \mathbf{q}^i$ for all $i \in \mathcal{I}$. Fixing $i \in \mathcal{I}$ and $s \in \mathcal{S}$, we can obtain a representation for $\succeq^i_s$ in two ways, i.e., first from $\widetilde{W}$ by applying this representation to elements of $\mathbb{R}^{\mathcal{I}\times\mathcal{O}}$ whose components for $i' \neq i$ are fixed at some values, and second, from U^i by applying this representation to elements of $\mathbb{R}^{\mathcal{S}\times\mathcal{O}}$ whose components for $s' \neq s$ are fixed at some values. By the standard uniqueness property, there exist $\gamma^i_s > 0$ and δ^i_s such that

$$\sum_{o \in \mathcal{O}} q_o \alpha v^i(x_o) \quad = \quad \gamma^i_s \sum_{o \in \mathcal{O}} q^i_o\, p^i_s\, u^i(x_o) + \delta^i_s, \quad \text{for all } \mathbf{x} \in \mathbb{R}^{\mathcal{O}}. \tag{B12}$$

Substituting $x_o = \overline{x}$ into (B12) leads to $\delta^i_s = 0$. Fix $o \in \mathcal{O}$. Put $x_{o'} = \overline{x}$ for all $o' \neq o$ leads to the equation:

$$\frac{q_o}{q^i_o} \alpha v^i(x) \quad = \quad \gamma^i_s\, p^i_s\, u^i(x) \text{ for all } x \in \mathbb{R}\ . \tag{B13}$$

The right-hand side of (B13) is independent of o. Thus, the probability vectors $\mathbf{q}$ and $\mathbf{q}^i$ are proportional, hence equal, and thus

$$\alpha v^i(x) \quad = \quad \gamma^i_s\, p^i_s\, u^i(x) \text{ for all } x \in \mathbb{R}. \tag{B14}$$

Equation (B14) holds for all $s \in \mathcal{S}$. Hence, for all $i \in \mathcal{I}$, the product $r^i := \gamma^i_s\, p^i_s$ is independent of s; note that $r^i > 0$. Equation (B14) now says $\alpha v^i = r^i\, u^i$. Substituting this into the representation (B11) yields the representation (8) for $\succeq_{\text{xp}}$. This completes the proof. $\square$