Live and Let Live: Sustainable Heterogeneity Will Generally Prevail

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Abstract

As is well known, the most patient household (i.e., the household possessing the lowest rate of time preference) will eventually own all capital in an economy if it behaves unilaterally without considering the optimality of the other households. This paper shows that choosing to engage in unilateral behavior is not always better for the most patient household than choosing multilateral behavior because unilateral behavior results in fewer educational opportunities for most people and constrains innovation in technologically advanced societies. Therefore, the rate of growth on the path when unilateral behavior is taken will be generally lower than that on the path when multilateral behavior is taken.

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1 INTRODUCTION

Becker (1980) showed that the most patient household eventually will own all capital in an economy. Heterogeneity in the rate of time preference (RTP) therefore results in an extremely dire state. A similar result is obtained for the case of heterogeneity in the degree of risk aversion (DRA) (Harashima, 2010). However, Harashima (2010) showed that if relatively more advantaged households behave multilaterally considering the optimality of less advantaged households, a state where all optimality conditions of all heterogeneous households are satisfied can be achieved; that is, the situation that Becker (1980) predicted can be averted. Harashima called this state "sustainable heterogeneity."

Nevertheless, if relatively more advantaged households behave unilaterally (i.e., they do not consider the optimality of less advantaged households), sustainable heterogeneity will not be achieved and the economy will fall into the dire state that Becker (1980) predicted. Which behavior (multilateral or unilateral) is better for the most advantaged household? Harashima (2010) showed that if the political resistance of less advantaged households against the unilateral behavior of the most advantaged household is strong enough, the most advantaged household can be forced to behave multilaterally. If the resistance is insufficient, however, it appears that the most advantaged household will choose to behave unilaterally because a higher rate of growth seems to be guaranteed, but is this actually true?

In this paper, the choice of the most advantaged household is examined based on a non-scale endogenous growth model. I show that choosing the unilateral behavior is not always better because choosing unilateral behavior has an important negative side effect. Overall innovation activities (generation of new technologies) are severely constrained as a result of the unilateral behavior of the most advantaged household. Unilateral behavior severely restricts the opportunities of less advantaged households to receive higher education, and education is indispensable to generate innovations (Becker, 1964; Weisbroad, 1966; Lynch, 1991). Because of this negative side effect, economies' capacity to generate new technologies will be severely constrained, and the rate of growth will be notably lower. As a result, unilateral behavior is not always in the best interest of the most advantaged household. Furthermore, at the present time, it is highly likely that multilateral behavior is better for the most advantaged households as well as all other households.

2 MULTILATERAL AND UNILATERAL PATHS

2.1 The multilateral path (sustainable heterogeneity)

The model constructed in Harashima (2010) is used to examine the choice of the most advantaged household (details of the model are shown in Appendix B). First, suppose that there are H ($\in N$) economies that are identical except for RTP, DRA, and productivity. Each economy is interpreted as representing a group of identical households. The population in each economy is identical and constant. The economies are fully open to each other, and goods, services, and capital are freely transacted among them, but labor is immobilized in each economy. Note households also provide laborers whose abilities are one of factors that determine productivity of each economy.

The model indicates that if and only if

$$\lim_{t\to\infty}\frac{\dot{c}_{i,t}}{c_{i,t}}=\left(\frac{\sum_{q=1}^{H}\varepsilon_q\omega_q}{\sum_{q=1}^{H}\omega_q}\right)^{-1}\left\{\left[\frac{\varpi\alpha\sum_{q=1}^{H}\omega_q}{Hmv(1-\alpha)}\right]^{\alpha}-\frac{\sum_{q=1}^{H}\theta_q\omega_q}{\sum_{q=1}^{H}\omega_q}\right\} \tag{1}$$

for any economy i (= 1, 2, … , H), all the optimality conditions of all heterogeneous economies are satisfied at steady state, and

$$\lim_{t\to\infty}\frac{\dot{c}_{i,t}}{c_{i,t}}=\lim_{t\to\infty}\frac{\dot{k}_{i,t}}{k_{i,t}}=\lim_{t\to\infty}\frac{\dot{y}_{i,t}}{y_{i,t}}=\lim_{t\to\infty}\frac{\dot{A}_t}{A_t}=\lim_{t\to\infty}\frac{\dot{\tau}_{i,j,t}}{\tau_{i,j,t}}=\lim_{t\to\infty}\frac{\frac{d\int_0^t \tau_{i,j,s}ds}{dt}}{\int_0^t \tau_{i,j,s}ds}$$

for any i and j ($i \neq j$), where $c_{i,t}$, $k_{i,t}$, and $y_{i,t}$ are per capita consumption, capital, and output of economy i in period t, respectively; θ_i, ε_i, and ω_i are RTP, DRA, and productivity of economy i, respectively; A_t is technology in period t; and α, m, v, and ϖ are constants. In addition, $\tau_{i,j,t}$ is the current account balance of economy i with economy j, where i = 1, 2, … , H, j = 1, 2, … , H, and $i \neq j$. Equation (1) is identical to equation (B29) in Appendix B. I call the state satisfying this condition (Equation [1]) "sustainable heterogeneity" and the path in which sustainable heterogeneity is achieved the "multilateral path."

2.2 Unilateral path

Sustainable heterogeneity is not naturally achieved—it depends on the behavior of relatively more advantaged economies (see Section B3 in Appendix B). If they behave unilaterally without considering the optimality of relatively less advantaged economies, sustainable heterogeneity will not be achieved.

For simplicity, consider a two-economy model (i.e., H = 2), and the two economies are identical except for their RTPs ($\theta_1 < \theta_2$). All of the optimality conditions of economy 1 can be satisfied only if either

$$\lim_{t\to\infty}\frac{\dot{\tau}_t}{\tau_t}=\lim_{t\to\infty}\frac{\frac{d\left(\int_0^t \tau_s ds\right)}{dt}}{\int_0^t \tau_s ds}=\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}} \tag{2}$$

or

$$\lim_{t\to\infty}\frac{\dot{\tau}_t}{\tau_t}=\lim_{t\to\infty}\frac{\frac{d\left(\int_0^t \tau_s ds\right)}{dt}}{\int_0^t \tau_s ds}=\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{1-\alpha} \tag{3}$$

(see Section B3.1 in Appendix B). Equations (2) and (3) are identical to equations (B32) and (B33) in Appendix B, respectively. On the other hand, economy 2 can achieve optimality only when economy 1 chooses the path satisfying equation (2). Equation (2) corresponds to the case of sustainable heterogeneity (i.e., the multilateral path). If economy 1 chooses the path of equation (3) (i.e., it behaves unilaterally without considering the optimality of economy 2), economy 2 cannot achieve optimality. Economy 2 will eventually lose ownership of all capital, and the state predicted by Becker (1980) will be realized. I call the path satisfying equation (3) the "unilateral path." The same result is obtained for heterogeneity in DRA ($\varepsilon_1 < \varepsilon_2$) (see Section B3.2 in Appendix B), but heterogeneity in productivity can be sustainable for economy 2 even if economy 1 behaves unilaterally (see Section B3.3 in Appendix B).

2.3 *Comparison of growth rates*

The optimal growth rate of economy 1 is

$$\frac{\dot{c}_{1,t}}{c_{1,t}} = \varepsilon^{-1}\left[\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha} + \left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{1-\alpha}\frac{\partial\left(\int_0^t \tau_s ds\right)}{\partial k_{1,t}} - \frac{\partial \tau_t}{\partial k_{1,t}} - \theta_1\right] . \tag{4}$$

Equation (4) is identical to equation (B14) in Appendix B. Therefore, if economy 1 chooses the unilateral path (i.e., behaves according to equation [3]), the growth rate of economy 1 is

$$\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}} = \varepsilon^{-1}\left[\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha} - \theta_1\right] . \tag{5}$$

On the other hand, on the multilateral path (i.e., if economy 1 chooses the path that leads to sustainable heterogeneity), the growth rate of economy 1 is

$$\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}} = \lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}} = \varepsilon^{-1}\left[\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha} - \frac{\theta_1+\theta_2}{2}\right] . \tag{6}$$

Equation (6) is identical to equation (B25) in Appendix B.

Equations (5) and (6) indicate that, for economy 1, the growth rate on the unilateral path is higher than that on the multilateral path because $\theta_1 < \frac{\theta_1+\theta_2}{2}$. This result means that behaving unilaterally is always better for economy 1 from the point of view of the growth rate. If the political resistance of economy 2 can be easily overcome, economy 1 will always choose the unilateral path.

Note that the same result is obtained in a multi-economy model of heterogeneity with multiple elements (see Section B2.6 in Appendix B). The most advantaged economy will still always choose the unilateral path if political resistance in less advantaged economies can be easily subdued, because equation (1) holds when sustainable heterogeneity is achieved.

However, the unilateral path includes a significant negative side effect that is not considered in the previous discussion, which is examined in Section 3.

3 A BETTER PATH FOR THE ADVANTAGED HOUSEHOLD

3.1 *A non-scale endogenous growth model*

Here I use an endogenous growth model that is the base model of that presented in Section 2 to examine the negative side effect (details of the model are presented in Appendix A). The advantage of using this endogenous growth model is that it is free from both scale effects and the influence of population growth.

3.1.1 The model

Let Y_t (≥ 0) be output, K_t (≥ 0) be capital input, L_t (≥ 0) be labor input, A_t be technology, C_t (≥ 0)

be consumption, and $n_t \left(= \frac{\dot{L}_t}{L_t} \right)$ be population growth rate in period *t*. In addition, $y_t = \frac{Y_t}{L_t}$, $k_t = \frac{K_t}{L_t}$, and $c_t = \frac{C_t}{L_t}$. Output Y_t is the sum of consumption C_t, the increase in capital K_t, and the increase in technology A_t such that

$$Y_t = C_t + \dot{K}_t + v\dot{A}_t \quad .$$

Thus,

$$\dot{k}_t = y_t - c_t - \frac{v\dot{A}_t}{L_t} - n_t k_t \quad ,$$

where $v(>0)$ is a constant, and a unit of capital K_t and v^{-1} units of technology A_t are equivalent; that is, they are produced using the same quantities of inputs (capital, labor, and technology). This means that technologies are produced with capital, labor, and technology in the same way as consumer goods and services and capital. For any period,

$$m = \frac{M_t}{L_t} \quad ,$$

where M_t is the number of firms (which are assumed to be identical) and m (> 0) is a constant. In addition, for any period,

$$\frac{\partial Y_t}{\partial K_t} = \frac{\varpi}{M_t^{1-\rho}} \frac{\partial Y_t}{\partial (vA_t)} \quad ; \tag{7}$$

thus,

$$\frac{\partial y_t}{\partial k_t} = \frac{\varpi L_t^{\rho}}{m^{1-\rho} v} \frac{\partial y_t}{\partial A_t} \tag{8}$$

is always kept, where $\varpi(>1)$ and $\rho(0 \le \rho < 1)$ are constants. The parameter ρ describes the effect of uncompensated knowledge spillovers, and the parameter ϖ indicates the effect of patent protection. ρ will naturally decrease to zero as a result of firms' profit-seeking behavior; thus, it is assumed that $\rho = 0$ (see Section A1.3 in Appendix A).

The optimization problem of households is to maximize expected utility

$$E \int_0^{\infty} u(c_t) \exp(-\theta t) dt$$

subject to

$$\dot{k}_t = \frac{mL_t(1-\alpha)}{mL_t(1-\alpha) + \varpi\alpha} \left[\left(\frac{\varpi\alpha}{mv} \right)^{\alpha} (1-\alpha)^{-\alpha} k_t - c_t - n_t k_t \right] \quad . \tag{9}$$

where $u(\bullet)$ is a constant relative risk aversion (CRRA) utility function, and E is the expectation operator. Equation (9) is identical to equation (A8) in Appendix A.

3.1.2 The rate of growth

The rate of growth on the balanced growth path is

$$\frac{\dot{c}_t}{c_t} = \varepsilon^{-1}\left\{\frac{mL_t(1-\alpha)}{mL_t(1-\alpha)+\varpi\alpha}\left[\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha} - n_t\right] - \theta\right\} ,$$

where ε is DRA, and α is a constant. Suppose for simplicity that the population is sufficiently large (i.e., $\frac{mL_t(1-\alpha)}{mL_t(1-\alpha)+\varpi\alpha} = 1$) and constant (i.e., $n_t = 0$), then the rate of growth on the balanced growth path is

$$\frac{\dot{c}_t}{c_t} = \varepsilon^{-1}\left\{\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha} - \theta\right\} . \qquad (10)$$

Equation (10) is identical to equation (A14) in Appendix A.

3.2 Unilateral path and production of technology

3.2.1 Productivity in generating technology on the unilateral path

If economy 1 chooses the unilateral path, economy 2 loses all capital. Moreover, it not only loses all ownership of all capital but also will owe a huge debt to economy 1. In this situation, economy 2 may violently resist politically. If this political resistance is easily overcome by economy 1, however, economy 2 will face miserable conditions. At the same time, this situation will reduce the capacity of economy 2 (i.e., the less advantaged economies) to generate new technologies, which will hurt economy 1.

Technology is generated by special labor inputs, that is, by well-educated and talented researchers. Hitherto, it has been assumed that the supply of these types of well-educated and talented researchers is always sufficient. However, if people in an economy are very poor, it will be very difficult for their children to receive enough education even if they are sufficiently talented. Education represents a type of investment and requires a considerable amount of money. If a household has no savings and a high debt level, it is difficult to invest in education. Even if the household's children are very talented, it will be difficult to raise funds to invest in their education. As a result, these talented children will have to choose jobs other than those that create innovations or new technologies. Of course, there are various outside sources to obtain funds for education (e.g., scholarships), but it is likely that poor people have fewer chances to obtain a higher education as compared to wealthier people.

On the unilateral path, the most talented children from less advantaged economies will not be able to attend higher education institutions because most of these economies will be in a distressed state. Educational institutions will instead enroll students who are less talented but belong to more advantaged (i.e., wealthier) economies. As a result, on the unilateral path, the supply of talented researchers will decrease. Because of the decreased supply of innovative thinkers, productivity in generating new technologies will also decrease on the unilateral path. The accumulation of technology will therefore be slower than that of the multilateral path and the growth rate of economy 1 will decrease as a side effect of its unilateral behavior.

3.2.2 Productivity in generating technology

In the model shown in Section 3.1, productivity in generating new technologies is represented by the parameter v. As assumed in Section 3.1.1, a unit of capital K_t and v^{-1} units of technology A_t are equivalent in the sense that they are produced using the same quantities of common resources (capital, labor, and technology). Let $\bar{a}(>0)$ and $\bar{k}(>0)$ be units of the common resources that are required to produce a unit of technology A_t and a unit of capital K_t, respectively. In other words, a unit of technology is equivalent to $\bar{a}$ units of the common resources, and a unit of capital is equivalent to $\bar{k}$ units of the common resources. Thereby, $v\bar{k} = \bar{a}$ because a unit of capital and v^{-1} units of technology require the same units of the common resources.

On the unilateral path, the supply of talented researchers is short. Consequently, production of a unit of new technology requires more units of common resources than on the multilateral path. Suppose that on the unilateral path, a unit of technology requires $\bar{a}_{Uni}$ units of common resources and $\bar{a} < \bar{a}_{Uni}$. In other words, with the same units of common resources, fewer units of technology are produced on the unilateral path than on the multilateral path. Because $v\bar{k} = \bar{a}$, then $v\bar{k} = \bar{a}_{Uni}\dfrac{\bar{a}}{\bar{a}_{Uni}}$ and thus

$$\bar{a}_{Uni} = v\bar{k}\frac{\bar{a}_{Uni}}{\bar{a}} = \bar{k}\left(v\frac{\bar{a}_{Uni}}{\bar{a}}\right) . \tag{11}$$

Equation (11) indicates that, on the unilateral path, a unit of technology is equivalent to $v\dfrac{\bar{a}_{Uni}}{\bar{a}}$ units of capital. In other words, a unit of capital and $\left(v\dfrac{\bar{a}_{Uni}}{\bar{a}}\right)^{-1}$ units of technology are equivalent. Therefore, if economy 1 chooses the unilateral path, v should be replaced with $v\dfrac{\bar{a}_{Uni}}{\bar{a}}$ in the model in Section 3.1.

3.2.3 The rate of growth on the unilateral path

On the unilateral path, therefore, the rate of growth is expressed by replacing v with $v\dfrac{\bar{a}_{Uni}}{\bar{a}}$ in equation (10) such that

$$\frac{\dot{c}_t}{c_t} = \varepsilon^{-1}\left\{\left(\frac{\bar{a}}{\bar{a}_{Uni}}\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha} - \theta\right\} . \tag{12}$$

In comparing equations (10) and (12), the growth rate on the unilateral path is clearly lower as a result of the negative side effect because $\bar{a} < \bar{a}_{Uni}$.

3.3 *A path that yields a higher rate of growth*

3.3.1 Comparison of growth rates

The path that is better for the most advantaged economy is likely to be the one that yields the higher rate of growth. Let the average RTP of the two economies be $\hat{\theta}$ and the RTP of the most advantaged economy be $\tilde{\theta}$; hence, $\tilde{\theta} < \hat{\theta}$. By equations (1), (5), (6), and (12), the growth rate of the most advantaged economy on the multilateral path is

$$\lim_{t\to\infty}\frac{\dot{c}_t}{c_t} = \varepsilon^{-1}\left\{\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha} - \hat{\theta}\right\} ,$$

and that on the unilateral path is

$$\lim_{t\to\infty}\frac{\dot{c}_t}{c_t} = \varepsilon^{-1}\left\{\left(\frac{\bar{a}}{\bar{a}_{Uni}}\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha} - \tilde{\theta}\right\} ,$$

On the multilateral path, the rate of growth depends on $\hat{\theta}$, whereas on the unilateral path, it is dominated by $\tilde{\theta}$. A unit of technology is equivalent to $\bar{a}_{Uni}$ on the unilateral path, but it is $\bar{a}$ on the multilateral path. Hence, the choice between the multilateral and unilateral paths depends not only on the difference between $\hat{\theta}$ and $\tilde{\theta}$ but also on that between $\bar{a}$ and $\bar{a}_{Uni}$. If the effect of the difference between $\bar{a}$ and $\bar{a}_{Uni}$, that is,

$$\left[\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha}\right] - \left[\left(\frac{\bar{a}}{\bar{a}_{Uni}}\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha}\right]$$
$$= \left(1-\frac{\bar{a}}{\bar{a}_{Uni}}\right)\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha} > 0$$

prevails over the effect of the difference between $\hat{\theta}$ and $\tilde{\theta}$, that is,

$$-\hat{\theta} + \tilde{\theta} < 0 ,$$

then the growth rate on the multilateral path is higher than that on the unilateral path. That is,

$$\left[\varepsilon^{-1}\left\{\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha} - \hat{\theta}\right\}\right] - \left[\varepsilon^{-1}\left\{\left(\frac{\bar{a}}{\bar{a}_{Uni}}\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha} - \tilde{\theta}\right\}\right] > 0 .$$

Because all economies except the most advantaged economy become very poor eventually on the unilateral path, it is likely that the effect of the difference between $\bar{a}$ and $\bar{a}_{Uni}$ prevails over the effect of the difference between $\hat{\theta}$ and $\tilde{\theta}$. Therefore, the unilateral path does not necessarily guarantee a higher rate of growth for the most advantaged economy. Even if political resistance of less advantaged economies can be easily overcome, choosing the unilateral path is not always better for the most advantaged economy from the point of view of the growth rate.

3.3.2 The better path at the present time

As technology progresses, that is, as accumulated technology A_t increases, education will become more important for researchers who engage in the generation of new technologies because researchers need to possess and understand larger amounts of knowledge. In addition, it becomes more difficult for untalented people to sufficiently learn everything they need to know about technology as knowledge accumulates. The previously discussed problem in which the most talented people are unable to obtain higher education will therefore become more serious as knowledge accumulates. An economy in which greater amounts of knowledge about technology accumulates will be more severely damaged by a shortage of educated talented researchers. In such an economy, the value of $\bar{a}_{Uni}$ will be very large, and as the economy develops, $\bar{a}_{Uni}$ will become far larger. Therefore, it will be likely that as an economy becomes more highly developed, the effect of the difference between $\bar{a}$ and $\bar{a}_{Uni}$ will generally prevail over the effect of the difference between $\hat{\theta}$ and $\tilde{\theta}$.

In the modern world, technology has vastly progressed and education is critically important to generate new technologies. Hence, it is likely that the effect of the difference between $\bar{a}$ and $\bar{a}_{Uni}$ generally currently prevails over the effect of the difference between $\hat{\theta}$ and $\tilde{\theta}$. That is, choosing sustainable heterogeneity (i.e., the multilateral path) will be generally better for the most advantaged economy as well as all the other economies.

4 CONCLUDING REMARKS

Becker (1980) showed that the most patient household eventually will own all capital in an economy. That is, heterogeneity in RTP results in extreme circumstances, but if relatively more advantaged households behave multilaterally, sustainable heterogeneity can be achieved. This paper showed that the unilateral path is not always a better choice for more advantaged households than the multilateral path because innovations (generation of new technologies) diminish because of a negative side effect of the unilateral path. Thereby, the rate of growth on the unilateral path is not always higher than that on the multilateral path. At the present time, because technology has vastly progressed as compared to the pre-modern period and education is very important for generating new technologies, the negative side effect of reducing educational opportunities for talented individuals on the unilateral path will be significant. Therefore, the rate of growth on the multilateral path will be generally higher than that on the unilateral path.

APPENDIX A

A1 The model

A1.1 Production of technologies

Let Y_t (≥ 0) be outputs, K_t (≥ 0) be capital input, L_t (≥ 0) be labor input, A_t be technology, and C_t (≥ 0) be consumption in period t. In addition, $y_t = \frac{Y_t}{L_t}$, $k_t = \frac{K_t}{L_t}$, and $c_t = \frac{C_t}{L_t}$. Outputs Y_t are the sum of consumption C_t, the increase in capital K_t, and the increase in technology A_t such that

$$Y_t = C_t + \dot{K}_t + v\dot{A}_t \quad . \tag{A1}$$

Thus,

$$\dot{k}_t = y_t - c_t - \frac{v\dot{A}_t}{L_t} - n_t k_t \quad , \tag{A2}$$

where $v(> 0)$ is a constant, and a unit of K_t and v^{-1} of a unit of A_t are equivalent; that is, they are produced using the same quantities of inputs (capital, labor, and technology). This means that technologies are produced with capital, labor, and technology in the same way as consumer goods and services and capital.

Because balanced growth paths are the focal point of this paper, Harrod-neutral technical progress is assumed.[1] Hence, the production function is $Y_t = K_t^{1-\alpha}(A_t L_t)^{\alpha}$; thus,

$$y_t = A_t^{\alpha} k_t^{1-\alpha} \quad .$$

It is assumed for simplicity that the population growth rate (n_t) is constant and not negative such that $n_t = n \geq 0$.

A1.2 Substitution between investments in K_t and A_t

For any period,

$$m = \frac{M_t}{L_t} \quad , \tag{A3}$$

where M_t is the number of firms (which are assumed to be identical) and m (> 0) is a constant. Equation (A3) presents a natural assumption that the population and number of firms are proportional to each other. Equation (A3) therefore indicates that any firm consists of the same number of employee regardless of L_t. Note that, unlike the arguments in Young (1998), Peretto (1998), Aghion and Howitt (1998), and Dinopoulos and Thompson (1998), M_t is not implicitly assumed to be proportional to the number of sectors or researchers in the economy (see also Jones, 1999). Equation (A3) merely indicates that the average number of employees per firm in an economy is independent of the population. Hence, M_t is not essential for the amount of production of A_t. As will be shown by equations (7) and (8), production of A_t does not depend

[1] As is well known, only Harrod-neutral technological progress matches the stylized facts presented by Kaldor (1961). As Barro and Sala-i-Martin (1995) argue, technological progress must take the labor-augmenting form in the production function if the models are to display a steady state.

on the number of researchers but on investments in technology. In contrast, M_t plays an important role in the amount of uncompensated knowledge spillovers.

The constant m implicitly indicates that the size of a firm is, on average, unchanged even if the population increases. This assumption can be justified by Coase (1937) who argued that the size of a firm is limited by the overload of administrative information. In addition, Williamson (1967) argued that there can be efficiency losses in larger firms (see also Grossman and Hart, 1986 and Moore, 1992). Their arguments equally imply that there is an optimal firm size that is determined by factors that are basically independent of population.

Next, for any period,

$$\frac{\partial Y_t}{\partial K_t} = \frac{\varpi}{M_t^{1-\rho}} \frac{\partial Y_t}{\partial (vA_t)} \quad ; \tag{A4}$$

thus,

$$\frac{\partial y_t}{\partial k_t} = \frac{\varpi L_t^{\rho}}{m^{1-\rho} v} \frac{\partial y_t}{\partial A_t} \tag{A5}$$

is always kept, where $\varpi(>1)$ and $\rho(0 \le \rho < 1)$ are constants. The parameter ρ describes the effect of uncompensated knowledge spillovers, and the parameter ϖ indicates the effect of patent protection. With patents, incomes are distributed not only to capital and labor but also to technology. For simplicity, the patent period is assumed to be indefinite, and no capital depreciation is assumed.

Equations (A4) and (A5) indicate that returns on investing in capital and technology for the investing firm are kept equal. The driving force behind the equations is that firms exploit all opportunities and select the most profitable investments at all times. Through arbitrage, this behavior leads to equal returns on investments in capital and technology. With substitution between investments in capital and technology, the model exhibits endogenous balanced growth. Because $\frac{\varpi}{mv} \frac{\partial y_t}{\partial A_t} = \frac{\partial y_t}{\partial k_t} \Leftrightarrow \frac{\varpi L_t^{\rho} \alpha}{m^{1-\rho} v} A_t^{\alpha-1} k_t^{\,1-\alpha} = (1-\alpha) A_t^{\alpha} k_t^{\,-\alpha}$, $A_t = \frac{\varpi L_t^{\rho} \alpha}{m^{1-\rho} v (1-\alpha)} k_t$ by equations (A3) and (A4), which lucidly indicates that $\frac{A_t}{k_t}$= constant, and the model can therefore show balanced endogenous growth.
endogenous growth.

A1.3 Uncompensated knowledge spillovers

Equations (A4) and (A5) also indicate that the investing firm cannot obtain all of the returns on its investment in technology. That is, although investment in technology increases Y_t, the investing firm's returns are only a fraction of the increase in Y_t, such that $\frac{\varpi}{M_t^{1-\rho}} \frac{\partial Y_t}{\partial (vA_t)}$, because knowledge spills over to other firms without compensation and other firms possess complementary technologies.

Broadly speaking, there are two types of uncompensated knowledge spillovers: intra-sectoral knowledge spillovers (MAR externalities: Marshall, 1890; Arrow, 1962; Romer, 1986) and inter-sectoral knowledge spillovers (Jacobs externalities: Jacobs, 1969). MAR theory assumes that knowledge spillovers between homogenous firms are the most effective and that spillovers will primarily emerge within sectors. As a result, uncompensated knowledge spillovers will be more active if the number of firms within a sector is larger. On the other hand, Jacobs (1969) argues that knowledge spillovers are most effective among firms that practice different activities and that diversification (i.e., a variety of sectors) is more important in influencing spillovers. As a result, uncompensated knowledge spillovers will be more active if the number of sectors in the economy is larger. If all sectors have the same number of firms, an increase in the number of firms in the economy results in more knowledge spillovers in any case, as a result of either MAR or Jacobs externalities.

As uncompensated knowledge spillovers increase, the investing firm's returns on investment in technology decrease. $\frac{\partial Y_t}{\partial A_t}$ indicates the total increase in Y_t in the economy by an increase in A_t, which consists of increases in both outputs of the firm that invested in the new technologies and outputs of other firms that utilize the newly invented technologies, regardless of whether the firms obtained the technologies by compensating the originating firm or through uncompensated knowledge spillovers. If the number of firms increases and uncompensated knowledge spillovers increase, the compensated fraction in $\frac{\partial Y_t}{\partial A_t}$ that the investing firm can obtain becomes smaller, as do its returns on the investment in technology. The parameter ρ describes the magnitude of this effect. If $\rho = 0$, the investing firm's returns are reduced at the same rate as the increase of the number of firms. $0 < \rho < 1$ indicates that the investing firm's returns diminish as the number of firms increase but not to the same extent as when $\rho = 0$.

Both types of externalities predict that uncompensated knowledge spillovers will increase as the number of firms increases, and scale effects have not actually been observed (Jones, 1995), which implies that scale effects are almost canceled out by the effects of MAR and Jacobs externalities. Thus, the value of ρ is quite likely to be very small. From the point of view of a firm's behavior, a very small ρ appears to be quite natural. Because firms intrinsically seek profit opportunities, newly established firms work as hard as existing firms to profit from knowledge spillovers. An increase in the number of firms therefore indicates that more firms are trying to obtain the investing firm's technologies.

Because of the non-rivalness of technology, all firms can equally benefit from uncompensated knowledge spillovers, regardless of the number of firms. Because the size of firms is independent of population and thus constant as argued in Section A1.2, each firm's ability to utilize the knowledge that has spilled over from each of the other firms will not be reduced by an increase in population. In addition, competition over technologies will increase as the number of firms increases, and any firm will completely exploit all opportunities to utilize uncompensated knowledge spillovers as competition increases.[2] Hence, it is quite likely that the probability that a firm can utilize a unit of new technologies developed by each of the other firms without compensation will be kept constant even if the population and the number of firms increase. As a result, uncompensated knowledge spillovers will increase eventually to the point that they increase at the same rate as the increase in the number of firms.

The investing firm's fraction of $\frac{\partial Y_t}{\partial A_t}$ that it can obtain will thereby be reduced at the same rate as the increase in the number of firms, which means that ρ will naturally decrease to zero as a result of firms' profit-seeking behavior. Based on $\rho = 0$,

$$\frac{\partial Y_t}{\partial K_t} = \frac{\varpi}{M_t} \frac{\partial Y_t}{\partial (\upsilon A_t)} \tag{A6}$$

[2] Moreover, a larger number of firms indicates that firms are more specialized. More specialized and formerly neglected technologies may become valuable to the larger number of specialized firms. Hence, knowledge spillovers will increase.

by equations (A4) and (A5); thus,

$$\frac{\partial y_t}{\partial k_t} = \frac{\varpi}{mv} \frac{\partial y_t}{\partial A_t} \tag{A7}$$

is always maintained.

Complementary technologies also reduce the fraction of $\frac{\partial Y_t}{\partial A_t}$ that the investing firm can obtain. If a new technology is effective only if it is combined with other technologies, the returns on investment in the new technology will belong not only to the investing firm but also to the firms that possess the other technologies. For example, an innovation in computer software technology generated by a software company increases the sales and profits of computer hardware companies. The economy's productivity increases because of the innovation but the increased incomes are attributed not only to the firm that generated the innovation but also to the firms that possess complementary technologies. A part of $\frac{\partial Y_t}{\partial A_t}$ leaks to these firms, and the leaked income is a kind of rent revenue that unexpectedly became obtainable because of the original firm's innovation. Most new technologies will have complementary technologies. Because of both complementary technologies and uncompensated knowledge spillovers, the fraction of $\frac{\partial Y_t}{\partial A_t}$ that an investing firm can obtain on average will be very small; that is, ϖ will be far smaller than M_t except when M_t is very small.[3]

A1.4 The optimization problem

Because $A_t = \frac{\varpi \alpha}{mv(1-\alpha)} k_t$, then $y_t = \left(\frac{\varpi \alpha}{mv}\right)^{\alpha} (1-\alpha)^{-\alpha} k_t$ and $\dot{A}_t = \frac{\varpi}{mv} \dot{k}_t \left(\frac{\alpha}{1-\alpha}\right)$. Hence, $\dot{k}_t = y_t - c_t - \frac{v\dot{A}_t}{L_t} - n_t k_t = \left(\frac{\varpi \alpha}{mv}\right)^{\alpha} (1-\alpha)^{-\alpha} k_t - c_t - \frac{\varpi}{mL_t} \dot{k}_t \left(\frac{\alpha}{1-\alpha}\right) - n_t k_t$ and

$$\dot{k}_t = \frac{mL_t(1-\alpha)}{mL_t(1-\alpha) + \varpi \alpha} \left[\left(\frac{\varpi \alpha}{mv}\right)^{\alpha} (1-\alpha)^{-\alpha} k_t - c_t - n_t k_t \right] . \tag{A8}$$

As a whole, the optimization problem of the representative household is to maximize the expected utility

$$E \int_0^{\infty} u(c_t) \exp(-\theta t) dt$$

[3] If M_t is very small, the value of ϖ will be far smaller than that for sufficiently large M_t because the number of firms that can benefit from an innovation is constrained owing to the very small M_t. The very small number of firms indicates that the economy is not sufficiently sophisticated, and thereby the benefit of an innovation cannot be fully realized. This constraint can be modeled as $\varpi = \tilde{\varpi}\left[1-\left(1-\tilde{\varpi}^{-1}\right)^{M_t}\right]$, where $\tilde{\varpi}(\geq 1)$ is a constant. Nevertheless, for sufficiently large M_t (i.e., in sufficiently sophisticated economies), the constraint is removed such that $\lim_{M_t \to \infty} \tilde{\varpi}\left[1-\left(1-\tilde{\varpi}^{-1}\right)^{M_t}\right] = \tilde{\varpi} = \varpi$.

subject to equation (A8) where $u(\bullet)$ is a constant relative risk aversion (CRRA) utility function and E is the expectation operator.

A2 An asymptotically non-scale balanced growth path

A2.1 Growth rate and transversality condition

Let Hamiltonian H be

$$H = u(c_t)\exp(-\theta t) + \lambda_t \frac{mL_t(1-\alpha)}{mL_t(1-\alpha)+\varpi\alpha}\left[\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha}k_t - c_t - n_t k_t\right] ,$$

where λ_t is a costate variable. The optimality conditions for the optimization problem shown in the previous section are

$$\frac{\partial u(c_t)}{\partial c_t}\exp(-\theta t) = \lambda_t \frac{mL_t(1-\alpha)}{mL_t(1-\alpha)+\varpi\alpha} \tag{A9}$$

$$\dot{\lambda}_t = -\frac{\partial H}{\partial k_t} \tag{A10}$$

$$\dot{k}_t = \frac{mL_t(1-\alpha)}{mL_t(1-\alpha)+\varpi\alpha}\left[\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha}k_t - c_t - n_t k_t\right] \tag{A11}$$

$$\lim_{t\to\infty}\lambda_t k_t = 0 \ . \tag{A12}$$

By equation (A10),

$$\dot{\lambda}_t = -\lambda_t \frac{mL_t(1-\alpha)}{mL_t(1-\alpha)+\varpi\alpha}\left[\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha} - n_t\right] . \tag{A13}$$

Hence, by equations (A9) and (A13), the growth rate of consumption is

$$\frac{\dot{c}_t}{c_t} = \varepsilon^{-1}\left\{\frac{mL_t(1-\alpha)}{mL_t(1-\alpha)+\varpi\alpha}\left[\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha} - n_t\right] - \theta\right\}, \tag{A14}$$

where $\varepsilon = -\frac{c_t u''}{u'}$. Note that usually $\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha} - n_t > 0$, so this is the case examined in this paper.

By equation (A11), $\frac{\dot{k}_t}{k_t} = \frac{mL_t(1-\alpha)}{mL_t(1-\alpha)+\varpi\alpha}\left[\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha} - n_t - \frac{c_t}{k_t}\right]$, and by equation (A13), $\frac{\dot{\lambda}_t}{\lambda_t} = -\frac{mL_t(1-\alpha)}{mL_t(1-\alpha)+\varpi\alpha}\left[\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha} - n_t\right]$. Hence,

$$\frac{\dot{\lambda}_t}{\lambda_t} + \frac{\dot{k}_t}{k_t} = -\frac{mL_t(1-\alpha)}{mL_t(1-\alpha)+\varpi\alpha}\left(\frac{c_t}{k_t}\right) .$$

Therefore, if $\frac{c_t}{k_t} > 0$ for any period, then $\frac{\dot{\lambda}_t}{\lambda_t} + \frac{\dot{k}_t}{k_t} < 0$, and transversality condition (A12) is satisfied. Conversely, if $\frac{c_t}{k_t} = 0$ for any period after a certain period, the transversality condition is not satisfied.

A2.2 Balanced growth path

There is a balanced growth path on which all the optimality conditions are satisfied.

Lemma: If and only if $\lim_{t\to\infty} \frac{\dot{c}_t}{c_t} = \lim_{t\to\infty} \frac{\dot{k}_t}{k_t}$, all the conditions (equations [A8]–[A11]) are satisfied.

Proof: (Step 1) $\lim_{t\to\infty} \frac{\dot{c}_t}{c_t} = \varepsilon^{-1} \left\{ \frac{m \lim_{t\to\infty} L_t (1-\alpha) \left[\left(\frac{\varpi\alpha}{mv} \right)^{\alpha} (1-\alpha)^{-\alpha} - n_t \right] - \alpha n_t}{m \lim_{t\to\infty} L_t (1-\alpha) + \varpi\alpha} - \theta \right\} =$ $\varepsilon^{-1} \left[\left(\frac{\varpi\alpha}{mv} \right)^{\alpha} (1-\alpha)^{-\alpha} - n - \theta \right]$. Therefore, $\lim_{t\to\infty} \frac{\dot{c}_t}{c_t} =$ constant. On the other hand, $\lim_{t\to\infty} \frac{\dot{k}_t}{k_t} =$ $\frac{m \lim_{t\to\infty} L_t (1-\alpha)}{m \lim_{t\to\infty} L_t (1-\alpha) + \varpi\alpha} \left[\left(\frac{\varpi\alpha}{mv} \right)^{\alpha} (1-\alpha)^{-\alpha} - n - \lim_{t\to\infty} \frac{c_t}{k_t} \right]$.

(Step 2) If $\lim_{t\to\infty} \frac{\dot{k}_t}{k_t} > \lim_{t\to\infty} \frac{\dot{c}_t}{c_t}$, then $\frac{c_t}{k_t}$ diminishes as time passes because $\lim_{t\to\infty} \frac{\dot{c}_t}{c_t} =$ constant by (Step 1) while $\lim_{t\to\infty} \frac{\dot{k}_t}{k_t}$ increases by (Step 1). Thus, eventually $\frac{c_t}{k_t}$ diminishes to zero, and as shown in Section A2.1, transversality condition (A12) is not satisfied.

If $\lim_{t\to\infty} \frac{\dot{k}_t}{k_t} < \lim_{t\to\infty} \frac{\dot{c}_t}{c_t}$, then $\frac{c_t}{k_t}$ increases indefinitely as time passes because $\lim_{t\to\infty} \frac{\dot{c}_t}{c_t} =$ constant by (Step 1) while $\lim_{t\to\infty} \frac{\dot{k}_t}{k_t}$ diminishes and eventually becomes negative by (Step 1). Hence, k_t decreases and eventually equation (A11) is violated because $k_t \geq 0$.

On the other hand, if $\lim_{t\to\infty} \frac{\dot{c}_t}{c_t} = \lim_{t\to\infty} \frac{\dot{k}_t}{k_t}$, then $\lim_{t\to\infty} \frac{c_t}{k_t}$ is constant; thus, $\lim_{t\to\infty} \frac{\dot{k}_t}{k_t}$ and $\lim_{t\to\infty} \frac{\dot{c}_t}{c_t}$ are identical and constant because $\lim_{t\to\infty} \frac{\dot{c}_t}{c_t} =$ constant by (Step 1). ■

Rational households will set an initial consumption that leads to the growth path that satisfies all the conditions. The Lemma therefore indicates that, given an initial A_0 and k_0, rational households will set the initial consumption c_0 so as to achieve the growth path that satisfies $\lim_{t\to\infty}\frac{\dot{c}_t}{c_t}=\lim_{t\to\infty}\frac{\dot{k}_t}{k_t}$, while firms will adjust k_t so as to achieve $\frac{\partial Y_t}{\partial K_t}=\frac{\varpi}{M_t}\frac{\partial Y_t}{\partial(vA_t)}$.[4] With this household behavior, the growth rates of technology, per capita output, consumption, and capital converge at the same rate.

Proposition: If all of the optimality conditions (equations [A8]–[A11]) are satisfied,

$$\lim_{t\to\infty}\frac{\dot{y}_t}{y_t}=\lim_{t\to\infty}\frac{\dot{A}_t}{A_t}=\lim_{t\to\infty}\frac{\dot{c}_t}{c_t}=\lim_{t\to\infty}\frac{\dot{k}_t}{k_t} .$$

Proof: Because $y_t=A_t^{\alpha}k_t^{1-\alpha}$, $\dot{y}_t=\left(\frac{A_t}{k_t}\right)^{\alpha}\left[(1-\alpha)\dot{k}_t+\alpha\frac{k_t}{A_t}\dot{A}_t\right]$. Since $\dot{A}_t=\frac{\varpi}{mv}\dot{k}_t\left(\frac{\alpha}{1-\alpha}\right)$, then $\dot{y}_t=\dot{k}_t\left(\frac{A_t}{k_t}\right)^{\alpha}\left[(1-\alpha)+\frac{\varpi\alpha^2}{mv(1-\alpha)}\frac{k_t}{A_t}\right]$; thus, $\frac{\dot{y}_t}{y_t}=\ \frac{\dot{k}_t}{k_t}\left[(1-\alpha)+\frac{\varpi\alpha^2}{mv(1-\alpha)}\frac{k_t}{A_t}\right]$. Since $A_t=\frac{\varpi\alpha}{mv(1-\alpha)}k_t$, $\frac{\dot{y}_t}{y_t}=\frac{\dot{k}_t}{k_t}[(1-\alpha)+\alpha]=\frac{\dot{k}_t}{k_t}$. Therefore, $\lim_{t\to\infty}\frac{\dot{y}_t}{y_t}=\lim_{t\to\infty}\frac{\dot{c}_t}{c_t}=\lim_{t\to\infty}\frac{\dot{k}_t}{k_t}$. In addition, since $\dot{y}_t=\dot{A}_t\left(\frac{A_t}{k_t}\right)^{\alpha}\left[\frac{mv(1-\alpha)^2}{\varpi\alpha}+\alpha\frac{k_t}{A_t}\right]$ by $y_t=A_t^{\alpha}k_t^{1-\alpha}$ and $\dot{A}_t=\frac{\varpi\alpha}{mv(1-\alpha)}\dot{k}_t$, then $\frac{\dot{y}_t}{y_t}=\frac{\dot{A}_t}{k_t}\frac{mv(1-\alpha)^2}{\varpi\alpha}+\alpha\frac{\dot{A}_t}{A_t}$. Because $A_t=\frac{\varpi\alpha}{mv(1-\alpha)}k_t$, then $\frac{\dot{y}_t}{y_t}=(1-\alpha)\frac{\dot{k}_t}{k_t}+\alpha\frac{\dot{A}_t}{A_t}$. Thereby, $\frac{\dot{y}_t}{y_t}=\frac{\dot{k}_t}{k_t}=(1-\alpha)\frac{\dot{k}_t}{k_t}+\alpha\frac{\dot{A}_t}{A_t}$ and $\lim_{t\to\infty}\frac{\dot{k}_t}{k_t}=\lim_{t\to\infty}\frac{\dot{A}_t}{A_t}=\lim_{t\to\infty}\frac{\dot{A}_t}{A_t}=$ constant. Hence, by the Lemma, $\lim_{t\to\infty}\frac{\dot{y}_t}{y_t}=\lim_{t\to\infty}\frac{\dot{A}_t}{A_t}=\lim_{t\to\infty}\frac{\dot{c}_t}{c_t}=\lim_{t\to\infty}\frac{\dot{k}_t}{k_t}$ if all the optimality conditions are satisfied. ■

By Proposition and Lemma, the balanced growth path is

$$\lim_{t\to\infty}\frac{\dot{y}_t}{y_t}=\lim_{t\to\infty}\frac{\dot{A}_t}{A_t}=\lim_{t\to\infty}\frac{\dot{c}_t}{c_t}=\lim_{t\to\infty}\frac{\dot{k}_t}{k_t}=\varepsilon^{-1}\left[\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha}-n-\theta\right] .$$

This balanced growth path can be seen as a natural extension of the steady state in the conventional Ramsey growth model with exogenous technology.

[4] Arbitrage conditions (A4) and (A5) indicate that until $A_t=\frac{\varpi\alpha}{mv(1-\alpha)}k_t$ is achieved, no investment is made in technology if $A_0>\frac{\varpi\alpha}{mv(1-\alpha)}k_0$ and in capital if $A_0<\frac{\varpi\alpha}{mv(1-\alpha)}k_0$.

APPENDIX B

B1 The model

B1.1 The base model

The model shown in Appendix A is used as the base model.

B1.2 Models with heterogeneous households

Three heterogeneities—heterogeneous time preference, risk aversion, and productivity—are examined in endogenous growth models, which are modified versions of the model shown in Appendix A. First, suppose that there are two economies— economy 1 and economy 2—that are identical except for time preference, risk aversion, or productivity. The population growth rate is zero (i.e., $n_t = 0$). The economies are fully open to each other, and goods, services, and capital are freely transacted between them, but labor is immobilized in each economy.

Each economy can be interpreted as representing either a country (the international interpretation) or a group of identical households in a country (the national interpretation). Because the economies are fully open, they are integrated through trade and form a combined economy. The combined economy is the world economy in the international interpretation and the national economy in the national interpretation. In the following discussion, a model based on the international interpretation is called an international model and that based on the national interpretation is called a national model. Usually, the concept of the balance of payments is used only for the international transactions. However, because both national and international interpretations are possible, this concept and terminology are also used for the national models in this paper.

B1.2.1 Heterogeneous time preference model

First, a model in which the two economies are identical except for time preference is constructed. The rate of time preference of the representative household in economy 1 is θ_1 and that in economy 2 is θ_2, and $\theta_1 < \theta_2$. The production function in economy 1 is $y_{1,t} = A_t^{\alpha} f(k_{1,t})$ and that in economy 2 is $y_{2,t} = A_t^{\alpha} f(k_{2,t})$, where $y_{i,t}$ and $k_{i,t}$ are, respectively, output and capital per capita in economy *i* in period *t* for *i* = 1, 2. The population of each economy is $\frac{L_t}{2}$; thus, the total for both is L_t, which is sufficiently large. Firms operate in both economies, and the number of firms is M_t. The current account balance in economy 1 is τ_t and that in economy 2 is $-\tau_t$. Because a balanced growth path requires Harrod neutral technological progress, the production functions are further specified as

$$y_{i,t} = A_t^{\alpha} k_{i,t}^{1-\alpha} \quad ;$$

thus, $Y_{i,t} = K_{i,t}^{1-\alpha} (A_t L_t)^{\alpha} \ (i = 1,2)$.

Because both economies are fully open, returns on investments in each economy are kept equal through arbitration such that

$$\frac{\partial y_{1,t}}{\partial k_{1,t}} = \frac{\varpi}{2mv} \frac{\partial (y_{1,t} + y_{2,t})}{\partial A_t} = \frac{\partial y_{2,t}}{\partial k_{2,t}} \quad . \tag{B1}$$

Equation (B1) indicates that an increase in A_t enhances outputs in both economies such that

$\frac{\partial Y_{i,t}}{\partial K_{i,t}} = \frac{\varpi}{M_t} \frac{\partial (Y_{1,t} + Y_{2,t})}{\partial (vA_t)}$, and because the population is equal ($\frac{L_t}{2}$), $\frac{\partial Y_{i,t}}{\partial K_{i,t}} = \frac{\partial y_{i,t}}{\partial k_{i,t}} = \frac{\varpi}{M_t} \frac{\partial (Y_{1,t} + Y_{2,t})}{\partial (vA_t)} = \frac{\varpi}{mL_t} \frac{\partial (y_{1,t} + y_{2,t})}{\partial (vA_t)} \frac{L_t}{2} = \frac{\varpi}{2mv} \frac{\partial (y_{1,t} + y_{2,t})}{\partial A_t}$. Therefore,

$$A_t = \frac{\varpi \alpha \lfloor f(k_{1,t}) + f(k_{2,t}) \rfloor}{2mvf'(k_{1,t})} = \frac{\varpi \alpha \lfloor f(k_{1,t}) + f(k_{2,t}) \rfloor}{2mvf'(k_{2,t})} .$$

Because equation (B1) is always held through arbitration, equations $k_{1,t} = k_{2,t}$, $\dot{k}_{1,t} = \dot{k}_{2,t}$, $y_{1,t} = y_{2,t}$ and $\dot{y}_{1,t} = \dot{y}_{2,t}$ are also held. Hence,

$$A_t = \frac{\varpi \alpha f(k_{1,t})}{mvf'(k_{1,t})} = \frac{\varpi \alpha f(k_{2,t})}{mvf'(k_{2,t})} .$$

In addition, because $\frac{\partial (y_{1,t} + y_{2,t})}{\partial A_{1,t}} = \frac{\partial (y_{1,t} + y_{2,t})}{\partial A_{2,t}}$ through arbitration, then $\dot{A}_{1,t} = \dot{A}_{2,t}$ is held.

The accumulated current account balance $\int_0^t \tau_s ds$ mirrors capital flows between the two economies. The economy with current account surpluses invests them in the other economy. Since $\frac{\partial y_{1,t}}{\partial k_{1,t}} \left(= \frac{\partial y_{2,t}}{\partial k_{2,t}} \right)$ are returns on investments, $\frac{\partial y_{1,t}}{\partial k_{1,t}} \int_0^t \tau_s ds$ and $\frac{\partial y_{2,t}}{\partial k_{2,t}} \int_0^t \tau_s ds$ represent income receipts or payments on the assets that an economy owns in the other economy. Hence,

$$\tau_t - \frac{\partial y_{2,t}}{\partial k_{2,t}} \int_0^t \tau_s ds$$

is the balance on goods and services of economy 1, and

$$\frac{\partial y_{1,t}}{\partial k_{1,t}} \int_0^t \tau_s ds - \tau_t$$

is that of economy 2. Because the current account balance mirrors capital flows between the economies, the balance is a function of capital in both economies such that

$$\tau_t = g(k_{1,t}, k_{2,t}) .$$

The representative household in economy 1 maximizes its expected utility

$$E \int_0^\infty u_1(c_{1,t}) \exp(-\theta_1 t) dt ,$$

subject to

$$\dot{k}_{1,t} = y_{1,t} + \frac{\partial y_{2,t}}{\partial k_{2,t}} \int_0^t \tau_s \, ds - \tau_t - c_{1,t} - v\dot{A}_{1,t}\left(\frac{L_t}{2}\right)^{-1} \quad , \tag{B2}$$

and the representative household in economy 2 maximizes its expected utility

$$E\int_0^\infty u_2(c_{2,t})\exp(-\theta_2 t)dt \quad ,$$

subject to

$$\dot{k}_{2,t} = y_{2,t} - \frac{\partial y_{1,t}}{\partial k_{1,t}} \int_0^t \tau_s ds + \tau_t - c_{2,t} - v\dot{A}_{2,t}\left(\frac{L_t}{2}\right)^{-1} \quad , \tag{B3}$$

where $u_{i,t}$, $c_{i,t}$, and $\dot{A}_{i,t}$, respectively, are the utility function, per capita consumption, and the increase in A_t by R&D activities in economy i in period t for i = 1, 2; E is the expectation operator; and $\dot{A}_t = \dot{A}_{1,t} + \dot{A}_{2,t}$. Equations (B2) and (B3) implicitly assume that each economy does not have foreign assets or debt in period t = 0.

Because the production function is Harrod neutral and because $A_t = \frac{\varpi \alpha f(k_{1,t})}{mvf'(k_{1,t})} = \frac{\varpi \alpha f(k_{2,t})}{mvf'(k_{2,t})}$ and $f = k_{i,t}^{1-\alpha}$, then

$$A_t = \frac{\varpi \alpha}{mv(1-\alpha)} k_{i,t}$$

and

$$\frac{\partial y_{i,t}}{\partial k_{i,t}} = \left(\frac{\varpi \alpha}{mv}\right)^\alpha (1-\alpha)^{1-\alpha} \quad .$$

Since $\dot{A}_{1,t} = \dot{A}_{2,t}$ and $\frac{\partial y_{1,t}}{\partial k_{1,t}} = \frac{\partial y_{2,t}}{\partial k_{2,t}}$, then

$$\begin{aligned}\dot{k}_{1,t} &= y_{1,t} + \frac{\partial y_{1,t}}{\partial k_{1,t}} \int_0^t \tau_s ds - \tau_t - c_{1,t} - \frac{v\dot{A}_t}{2}\left(\frac{L_t}{2}\right)^{-1} \\ &= \left(\frac{\varpi \alpha}{mv}\right)^\alpha (1-\alpha)^{-\alpha} k_{1,t} + \left(\frac{\varpi \alpha}{mv}\right)^\alpha (1-\alpha)^{1-\alpha} \int_0^t \tau_s ds - \tau_t - c_{1,t} - \frac{\varpi \alpha}{mL_t(1-\alpha)} \dot{k}_{1,t}\end{aligned}$$

and

$$\dot{k}_{1,t} = \frac{mL_t(1-\alpha)}{mL_t(1-\alpha) + \varpi \alpha}\left[\left(\frac{\varpi \alpha}{mv}\right)^\alpha (1-\alpha)^{-\alpha} k_{1,t} + \left(\frac{\varpi \alpha}{mv}\right)^\alpha (1-\alpha)^{1-\alpha} \int_0^t \tau_s ds - \tau_t - c_{1,t}\right] \quad .$$

Because L_t is sufficiently large and ϖ is far smaller than M_t, the problem of scale effects vanishes and thereby $\frac{mL_t(1-\alpha)}{mL_t(1-\alpha)+\varpi\alpha}=1$.

Putting the above elements together, the optimization problem of economy 1 can be rewritten as

$$Max\ E\int_0^\infty u_1(c_{1,t})\exp(-\theta_1 t)dt \quad ,$$

subject to

$$\dot{k}_{1,t}=\left(\frac{\varpi\alpha}{mv}\right)^\alpha(1-\alpha)^{-\alpha}k_{1,t}+\left(\frac{\varpi\alpha}{mv}\right)^\alpha(1-\alpha)^{1-\alpha}\int_0^t\tau_s ds-\tau_t-c_{1,t} \quad .$$

Similarly, that of economy 2 can be rewritten as

$$Max\ E\int_0^\infty u_2(c_{2,t})\exp(-\theta_2 t)dt \quad ,$$

subject to

$$\dot{k}_{2,t}=\left(\frac{\varpi\alpha}{mv}\right)^\alpha(1-\alpha)^{-\alpha}k_{2,t}-\left(\frac{\varpi\alpha}{mv}\right)^\alpha(1-\alpha)^{1-\alpha}\int_0^t\tau_s ds+\tau_t-c_{2,t} \quad .$$

B1.2.2 Heterogeneous risk aversion model

The basic structure of the model with heterogeneous risk aversion is the same as that of heterogeneous time preference. The two economies are identical except in regard to risk aversion. The degree of relative risk aversion of economy 1 is $\varepsilon_1=-\frac{c_{1,t}\,u_1''}{u_1'}$ and that of economy 2 is $\varepsilon_2=-\frac{c_{2,t}\,u_2''}{u_2'}$, which are constant, and $\varepsilon_1<\varepsilon_2$. The optimization problem of economy 1 is

$$Max\ E\int_0^\infty u_1(c_{1,t})\exp(-\theta t)dt \quad ,$$

subject to

$$\dot{k}_{1,t}=\left(\frac{\varpi\alpha}{mv}\right)^\alpha(1-\alpha)^{-\alpha}k_{1,t}+\left(\frac{\varpi\alpha}{mv}\right)^\alpha(1-\alpha)^{1-\alpha}\int_0^t\tau_s ds-\tau_t-c_{1,t} \quad ,$$

and that of economy 2 is

$$Max\ E\int_0^\infty u_2(c_{2,t})\exp(-\theta t)dt \quad ,$$

subject to

$$\dot{k}_{2,t} = \left(\frac{\varpi\alpha}{mv}\right)^{\alpha} (1-\alpha)^{-\alpha} k_{2,t} - \left(\frac{\varpi\alpha}{mv}\right)^{\alpha} (1-\alpha)^{1-\alpha} \int_0^t \tau_s ds + \tau_t - c_{2,t} \ .$$

B1.2.3 Heterogeneous productivity model

With heterogeneous productivity, the production function is heterogeneous, not the utility function. Because technology A_t is common to both economies, a heterogeneous production function requires heterogeneity in elements other than technology. Prescott (1998) argues that unknown factors other than technology have made total factor productivity (TFP) heterogeneous across countries. Harashima (2009) argues that average workers' innovative activities are an essential element of productivity and make TFP heterogeneous across workers, firms, and economies. Since average workers are human and capable of creative intellectual activities, they can create innovations even if their innovations are minor. It is rational for firms to exploit all the opportunities that these ordinary workers' innovative activities offer. Furthermore, innovations created by ordinary workers are indispensable for efficient production. A production function incorporating average workers' innovations has been shown to have a Cobb-Douglas functional form with a labor share of about 70% (Harashima 2009), such that

$$Y_t = \bar{\sigma}\omega_A\omega_L A_t^{\alpha} K_t^{1-\alpha} L_t^{\alpha} \ , \tag{B4}$$

where ω_A and ω_L are positive constant parameters with regard to average workers' creative activities, and $\bar{\sigma}$ is a parameter that represents a worker's accessibility limit to capital with regard to location. The parameters ω_A and ω_L are independent of A_t but are dependent on the creative activities of average workers. Thereby, unlike with technology A_t, these parameters can be heterogeneous across workers, firms, and economies.

In this model of heterogeneous productivity, it is assumed that workers whose households belong to different economies have different values of ω_A and ω_L. In addition, only productivity that is represented by $\bar{\sigma}\omega_A\omega_L A_t^{\alpha}$ in equation (B4) is heterogeneous between the two economies. The production function of economy 1 is $y_{1,t} = \omega_1^{\alpha} A_t^{\alpha} f(k_{1,t})$ and that of economy 2 is $y_{2,t} = \omega_2^{\alpha} A_t^{\alpha} f(k_{2,t})$, where $\omega_1\,(0 < \omega_1 \le 1)$ and $\omega_2\,(0 < \omega_2 \le 1)$ are constants and $\omega_2 < \omega_1$. Since $\frac{\partial Y_{i,t}}{\partial K_{i,t}} = \frac{\partial y_{i,t}}{\partial k_{i,t}} = M_t^{-1} \frac{\partial (Y_{1,t} + Y_{2,t})}{\partial (vA_t)} = \frac{\varpi}{mL_t} \frac{\partial (y_{1,t} + y_{2,t})}{\partial (vA_t)} \frac{L_t}{2} = \frac{\varpi}{2mL_t} \frac{\partial (y_{1,t} + y_{2,t})}{\partial A_t}$ by equation (B1), then

$$A_t = \frac{\varpi\alpha\left[\omega_1^{\alpha} f(k_{1,t}) + \omega_2^{\alpha} f(k_{2,t})\right]}{2mv\,\omega_1^{\alpha} f'(k_{1,t})} = \frac{\varpi\alpha\left[\omega_1^{\alpha} f(k_{1,t}) + \omega_2^{\alpha} f(k_{2,t})\right]}{2mv\,\omega_2^{\alpha} f'(k_{2,t})} \ . \tag{B5}$$

Because equation (B1) is always held through arbitration, equations $k_{1,t} = \frac{\omega_1}{\omega_2} k_{2,t}$, $\dot{k}_{1,t} = \frac{\omega_1}{\omega_2} \dot{k}_{2,t}$, $y_{1,t} = \frac{\omega_1}{\omega_2} y_{2,t}$, and $\dot{y}_{1,t} = \frac{\omega_1}{\omega_2} \dot{y}_{2,t}$ are also held. In addition, since $\frac{\partial (y_{1,t} + y_{2,t})}{\partial A_{1,t}} = \frac{\partial (y_{1,t} + y_{2,t})}{\partial A_{2,t}}$ by arbitration, $\dot{A}_{1,t} = \frac{\omega_1}{\omega_2} \dot{A}_{2,t}$ is held. Because of equation (B5) and $f = \omega_i^{\alpha} k_{i,t}^{1-\alpha}$, then $A_t =$ $\frac{\varpi\alpha}{2mv(1-\alpha)\omega_1^{\alpha}} \left(\omega_1^{\alpha} k_1 + \omega_2^{\alpha} k_1^{\alpha} k_2^{1-\alpha}\right) = \frac{\varpi\alpha}{2mv(1-\alpha)\omega_2^{\alpha}} \left(\omega_1^{\alpha} k_1^{1-\alpha} k_2^{\alpha} + \omega_2^{\alpha} k_2\right)$, $\frac{\omega_1^{\alpha} k_1 + \omega_2^{\alpha} k_1^{\alpha} k_2^{1-\alpha}}{\omega_1^{\alpha}} = \frac{\omega_1^{\alpha} k_1^{1-\alpha} k_2^{\alpha} + \omega_2^{\alpha} k_2}{\omega_2^{\alpha}}$, and $\frac{\partial y_{i,t}}{\partial k_{i,t}} = \left(\frac{\varpi\alpha}{2mv}\right)^{\alpha}(1-\alpha)^{1-\alpha}\left(\omega_1^{\alpha}k_1 + \omega_2^{\alpha}k_1^{\alpha}k_2^{1-\alpha}\right)^{\alpha}k_1^{-\alpha} = \left(\frac{\varpi\alpha}{2mv}\right)^{\alpha}(1-\alpha)^{1-\alpha}\left(\omega_1^{\alpha}k_1^{1-\alpha}k_2^{\alpha} + \omega_2^{\alpha}k_2\right)^{\alpha}k_2^{-\alpha}$. Since $\frac{\omega_2}{\omega_1}k_{1,t} = k_{2,t}$, then $\frac{\omega_1^{\alpha}k_1 + \omega_2^{\alpha}k_1^{\alpha}k_2^{1-\alpha}}{\omega_1^{\alpha}} = \frac{\omega_1^{\alpha}k_1 + \omega_2^{\alpha}k_1^{\alpha}\left(\frac{\omega_2}{\omega_1}\right)^{1-\alpha}k_1^{1-\alpha}}{\omega_1^{\alpha}} = k_1\left(1+\omega_1^{-1}\omega_2\right)$ and $\frac{\omega_1^{\alpha}k_1^{1-\alpha}k_2^{\alpha} + \omega_2^{\alpha}k_2}{\omega_2^{\alpha}} = \frac{\omega_1^{\alpha}k_1^{1-\alpha}\left(\frac{\omega_2}{\omega_1}\right)^{\alpha}k_1^{\alpha} + \omega_2^{\alpha}\frac{\omega_2}{\omega_1}k_1}{\omega_2^{\alpha}} = k_1 + \frac{\omega_2}{\omega_1}k_1 = k_1\left(1+\omega_1^{-1}\omega_2\right) = k_2\left(1+\omega_1\omega_2^{-1}\right)$. Hence,

$$A_t = k_1\frac{\varpi\alpha\left(1+\omega_1^{-1}\omega_2\right)}{2mv(1-\alpha)} = k_2\frac{\varpi\alpha\left(1+\omega_1\omega_2^{-1}\right)}{2mv(1-\alpha)} \quad ,$$

and

$$\frac{\partial y_{i,t}}{\partial k_{i,t}} = \left(\frac{\omega_1+\omega_2}{2}\right)^{\alpha}\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{1-\alpha}$$

for i = 1, 2. Because $\dot{A}_{1,t} = \left(\frac{\omega_2}{\omega_1}\right)^{-\frac{1}{\alpha}}\dot{A}_{2,t}$ (i.e., $\dot{A}_t = \dot{A}_{1,t} + \dot{A}_{2,t} = \left(1+\omega_1^{-1}\omega_2\right)\dot{A}_{1,t}$) and $\frac{\partial y_{1,t}}{\partial k_{1,t}} = \frac{\partial y_{2,t}}{\partial k_{2,t}}$, then

$$\begin{aligned}
\dot{k}_{1,t} &= y_{1,t} + \frac{\partial y_{1,t}}{\partial k_{1,t}}\int_0^t \tau_s ds - \tau_t - c_{1,t} - v\dot{A}_{1,t}\left(\frac{L_t}{2}\right)^{-1} \\
&= y_{1,t} + \frac{\partial y_{1,t}}{\partial k_{1,t}}\int_0^t \tau_s ds - \tau_t - c_{1,t} - v\dot{A}_t\left(1+\omega_1^{-1}\omega_2\right)^{-1}\left(\frac{L_t}{2}\right)^{-1} \\
&= \omega_1^{\alpha}\left[\frac{\left(1+\omega_1^{-1}\omega_2\right)\varpi\alpha}{2mv(1-\alpha)}\right]^{\alpha}k_{1,t} + \left[\frac{(\omega_1+\omega_2)\varpi\alpha}{2mv}\right]^{\alpha}(1-\alpha)^{1-\alpha}\int_0^t \tau_s ds - \tau_t - c_{1,t} - \frac{\varpi\alpha}{mL_t(1-\alpha)}\dot{k}_{1,t} \quad ,
\end{aligned}$$

and

$$\dot{k}_{1,t} = \frac{mL_t(1-\alpha)}{mL_t(1-\alpha)+\varpi\alpha}\left\{\left[\frac{(\omega_1+\omega_2)\varpi\alpha}{2mv(1-\alpha)}\right]^{\alpha}k_{1,t} + \left[\frac{(\omega_1+\omega_2)\varpi\alpha}{2mv}\right]^{\alpha}(1-\alpha)^{1-\alpha}\int_0^t \tau_s ds - \tau_t - c_{1,t}\right\} \quad .$$

Because L_t is sufficiently large and ϖ is far smaller than M_t and thus $\frac{mL_t(1-\alpha)}{mL_t(1-\alpha)+\varpi\alpha} = 1$, the optimization problem of economy 1 is

$$Max\ E\int_0^{\infty} u_1\left(c_{1,t}\right)\exp(-\theta t)dt \quad ,$$

subject to

$$\dot{k}_{1,t} = \left[\frac{(\omega_1+\omega_2)\varpi\alpha}{2mv(1-\alpha)}\right]^{\alpha} k_{1,t} + \left[\frac{(\omega_1+\omega_2)\varpi\alpha}{2mv}\right]^{\alpha} (1-\alpha)^{1-\alpha}\int_0^t \tau_s ds - \tau_t - c_{1,t} \quad ,$$

and similarly, that of economy 2 is

$$Max\ E\int_0^{\infty} u_2(c_{2,t})\exp(-\theta t)dt \quad ,$$

subject to

$$\dot{k}_{2,t} = \left[\frac{(\omega_1+\omega_2)\varpi\alpha}{2mv(1-\alpha)}\right]^{\alpha} k_{2,t} - \left[\frac{(\omega_1+\omega_2)\varpi\alpha}{2mv}\right]^{\alpha} (1-\alpha)^{1-\alpha}\int_0^t \tau_s ds + \tau_t - c_{2,t} \quad .$$

B2 Sustainability of heterogeneity

Heterogeneity is defined as being sustainable if all the optimality conditions of all heterogeneous households are satisfied indefinitely. Although the previously discussed state of Becker (1980) is Pareto efficient, by this definition, the heterogeneity is not sustainable because only the most patient household can achieve optimality. Sustainability is therefore the stricter criterion for welfare than Pareto efficiency.

In this section, the growth path that makes heterogeneity sustainable is examined. First, the basic natures of the models presented in Section B1 are examined and then sustainability is examined.

B2.1 The consumption growth rate

B2.1.1 Heterogeneous time preference model

Let Hamiltonian H_1 be

$$H_1 = u_1(c_{1,t})\exp(-\theta_1 t) + \lambda_{1,t}\left\{\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha} k_{1,t} + \left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{1-\alpha}\int_0^t \tau_s ds - \tau_t - c_{1,t}\right\} \quad ,$$

where λ_{1t} is a costate variable. The optimality conditions for economy 1 are

$$\frac{\partial u_1(c_{1,t})}{\partial c_{1,t}}\exp(-\theta_1 t) = \lambda_{1,t} \quad , \tag{B6}$$

$$\dot{\lambda}_{1,t} = -\frac{\partial H_1}{\partial k_{1,t}} \quad , \tag{B7}$$

$$\dot{k}_{1,t} = \left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha}k_{1,t} + \left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{1-\alpha}\int_0^t \tau_s ds - \tau_t - c_{1,t} \quad \text{, and} \tag{B8}$$

$$\lim_{t\to\infty}\lambda_{1,t}\, k_{1,t} = 0 \quad . \tag{B9}$$

Similarly, let Hamiltonian H_2 be

$$H_2 = u_2(c_{2,t})\exp(-\theta_1 t) + \lambda_{2,t}\left\{\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha}k_{2,t} - \left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{1-\alpha}\int_0^t \tau_s ds + \tau_t - c_{2,t}\right\} \ ,$$

where λ_{2t} is a costate variable. The optimality conditions for economy 2 are

$$\frac{\partial u_2(c_{2,t})}{\partial c_{2,t}}\exp(-\theta_1 t) = \lambda_{2,t} \ , \tag{B10}$$

$$\dot{\lambda}_{2,t} = -\frac{\partial H_2}{\partial k_{2,t}} \ , \tag{B11}$$

$$\dot{k}_{2,t} = \left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha}k_{2,t} - \left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{1-\alpha}\int_0^t \tau_s\, ds + \tau_t - c_{2,t}\text{, and} \tag{B12}$$

$$\lim_{t\to\infty}\lambda_{2,t}k_{2,t} = 0 \ . \tag{B13}$$

By equations (B6), (B7), and (B8), the consumption growth rate in economy 1 is

$$\frac{\dot{c}_{1,t}}{c_{1,t}} = \varepsilon^{-1}\left[\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha} + \left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{1-\alpha}\frac{\partial\left(\int_0^t \tau_s ds\right)}{\partial k_{1,t}} - \frac{\partial\tau_t}{\partial k_{1,t}} - \theta_1\right] \ , \tag{B14}$$

and by equations (B10), (B11), and (B12), that in economy 2 is

$$\frac{\dot{c}_{2,t}}{c_{2,t}} = \varepsilon^{-1}\left[\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha} - \left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{1-\alpha}\frac{\partial\left(\int_0^t \tau_s ds\right)}{\partial k_{2,t}} + \frac{\partial\tau_t}{\partial k_{2,t}} - \theta_2\right] \ , \tag{B15}$$

where $\varepsilon = -\frac{c_{1,t}\,u_1''}{u_1'} = -\frac{c_{2,t}\,u_2''}{u_2'}$ is the degree of relative risk aversion, which is constant. A constant growth rate such that $\frac{\dot{c}_{1,t}}{c_{1,t}} = \frac{\dot{c}_{2,t}}{c_{2,t}}$ is possible if

$$\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{1-\alpha}\left[\frac{\partial\left(\int_0^t \tau_s ds\right)}{\partial k_{1,t}} + \frac{\partial\left(\int_0^t \tau_s ds\right)}{\partial k_{2,t}}\right] - \left(\frac{\partial\tau_t}{\partial k_{1,t}} + \frac{\partial\tau_t}{\partial k_{2,t}}\right) = \theta_1 - \theta_2 \tag{B16}$$

is satisfied.

B2.1.2 Heterogeneous risk aversion model

By using similar procedures as were used with the heterogeneous time preference model, the consumption growth rate in economy 1 in this model is

$$\frac{\dot{c}_{1,t}}{c_{1,t}} = \varepsilon_1^{-1}\left[\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha} + \left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{1-\alpha}\frac{\partial\left(\int_0^t \tau_s ds\right)}{\partial k_{1,t}} - \frac{\partial \tau_t}{\partial k_{1,t}} - \theta\right] , \tag{B17}$$

and that in economy 2 is

$$\frac{\dot{c}_{2,t}}{c_{2,t}} = \varepsilon_2^{-1}\left[\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha} - \left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{1-\alpha}\frac{\partial\left(\int_0^t \tau_s ds\right)}{\partial k_{2,t}} + \frac{\partial \tau_t}{\partial k_{2,t}} - \theta\right] . \tag{B18}$$

A constant growth rate such that $\frac{\dot{c}_{1,t}}{c_{1,t}} = \frac{\dot{c}_{2,t}}{c_{2,t}}$ is possible if

$$\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{1-\alpha}\left[\varepsilon_2\frac{\partial\left(\int_0^t \tau_s ds\right)}{\partial k_{1,t}} + \varepsilon_1\frac{\partial\left(\int_0^t \tau_s ds\right)}{\partial k_{2,t}}\right] + (\varepsilon_2 - \varepsilon_1)\left[\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha} - \theta\right]$$
$$= \varepsilon_2\frac{\partial \tau_t}{\partial k_{1,t}} + \varepsilon_1\frac{\partial \tau_t}{\partial k_{2,t}} \tag{B19}$$

is satisfied.

B2.1.3 Heterogeneous productivity model

By similar procedures, the consumption growth rate in economy 1 in this model is

$$\frac{\dot{c}_{1,t}}{c_{1,t}} = \varepsilon^{-1}\left\{\left[\frac{(\omega_1+\omega_2)\varpi\alpha}{2mv(1-\alpha)}\right]^{\alpha} + \left[\frac{(\omega_1+\omega_2)\varpi\alpha}{2mv}\right]^{\alpha}(1-\alpha)^{1-\alpha}\frac{\partial\int_0^t \tau_s ds}{\partial k_{1,t}} - \frac{\partial \tau_t}{\partial k_{1,t}} - \theta\right\} , \tag{B20}$$

and that in economy 2 is

$$\frac{\dot{c}_{2,t}}{c_{2,t}} = \varepsilon^{-1}\left\{\left[\frac{(\omega_1+\omega_2)\varpi\alpha}{2mv(1-\alpha)}\right]^{\alpha} - \left[\frac{(\omega_1+\omega_2)\varpi\alpha}{2mv}\right]^{\alpha}(1-\alpha)^{1-\alpha}\frac{\partial\int_0^t \tau_s ds}{\partial k_{2,t}} + \frac{\partial \tau_t}{\partial k_{2,t}} - \theta\right\} . \tag{B21}$$

A constant growth rate such that $\frac{\dot{c}_{1,t}}{c_{1,t}} = \frac{\dot{c}_{2,t}}{c_{2,t}}$ is possible if

$$\left[\frac{(\omega_1+\omega_2)\varpi\alpha}{2mv(1-\alpha)}\right]^{\alpha}(1-\alpha)\left(\frac{\partial\int_0^t \tau_s ds}{\partial k_{1,t}} + \frac{\partial\int_0^t \tau_s ds}{\partial k_{2,t}}\right) = \frac{\partial \tau_t}{\partial k_{1,t}} + \frac{\partial \tau_t}{\partial k_{2,t}} \tag{B22}$$

is satisfied.

B2.2 Transversality conditions

B2.2.1 Heterogeneous time preference model

Transversality conditions are satisfied if the following conditions are satisfied.

Lemma 1-1: In the model of heterogeneous time preference, unless $\lim_{t\to\infty}\frac{\dot{\lambda}_{1,t}}{\lambda_{1,t}} < -1$, $\lim_{t\to\infty}\frac{\dot{\lambda}_{2,t}}{\lambda_{2,t}} < -1$, $\lim_{t\to\infty}\frac{\dot{k}_{1,t}}{k_{1,t}} < -1$, or $\lim_{t\to\infty}\frac{\dot{k}_{2,t}}{k_{2,t}} < -1$, the transversality conditions (equations [B9] and [B13]) are satisfied if

$$\lim_{t\to\infty}\left\{\left(\frac{\partial\tau_t}{\partial k_{1,t}}-\frac{\tau_t}{k_{1,t}}\right)-\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{1-\alpha}\left[\frac{\partial\left(\int_0^t\tau_s ds\right)}{\partial k_{1,t}}-\frac{\int_0^t\tau_s ds}{k_{1,t}}\right]-\frac{c_{1,t}}{k_{1,t}}\right\}<0 \qquad \text{(B23)}$$

and

$$\lim_{t\to\infty}\left\{\left(\frac{\tau_t}{k_{2,t}}-\frac{\partial\tau_t}{\partial k_{2,t}}\right)-\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{1-\alpha}\left[\frac{\int_0^t\tau_s ds}{k_{2,t}}-\frac{\partial\left(\int_0^t\tau_s ds\right)}{\partial k_{2,t}}\right]-\frac{c_{2,t}}{k_{2,t}}\right\}<0 \; . \qquad \text{(B24)}$$

Proof: See Harashima (2010).

B2.2.2 Heterogeneous risk aversion model

Lemma 1-2: In the model of heterogeneous risk aversion, unless $\lim_{t\to\infty}\frac{\dot{\lambda}_{1,t}}{\lambda_{1,t}} < -1$, $\lim_{t\to\infty}\frac{\dot{\lambda}_{2,t}}{\lambda_{2,t}} < -1$, $\lim_{t\to\infty}\frac{\dot{k}_{1,t}}{k_{1,t}} < -1$, or $\lim_{t\to\infty}\frac{\dot{k}_{2,t}}{k_{2,t}} < -1$, the transversality conditions are satisfied if

$$\lim_{t\to\infty}\left\{\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha}\left[\frac{\int_0^t\tau_s ds}{k_{1,t}}-\frac{\partial\left(\int_0^t\tau_s ds\right)}{\partial k_{1,t}}\right]-\left(\frac{\tau_t}{k_{1,t}}-\frac{\partial\tau_t}{\partial k_{1,t}}\right)-\frac{c_{1,t}}{k_{1,t}}\right\}<0$$

and

$$-\lim_{t\to\infty}\left\{\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha}\left[\frac{\int_0^t\tau_s ds}{k_{2,t}}-\frac{\partial\left(\int_0^t\tau_s ds\right)}{\partial k_{2,t}}\right]-\left(\frac{\tau_t}{k_{2,t}}-\frac{\partial\tau_t}{\partial k_{2,t}}\right)+\frac{c_{2,t}}{k_{2,t}}\right\}<0 \cdot$$

B2.2.3 Heterogeneous productivity model

Lemma 1-3: In the model of heterogeneous productivity, unless $\lim_{t\to\infty}\frac{\dot{\lambda}_{1,t}}{\lambda_{1,t}} < -1$, $\lim_{t\to\infty}\frac{\dot{\lambda}_{2,t}}{\lambda_{2,t}} < -1$, $\lim_{t\to\infty}\frac{\dot{k}_{1,t}}{k_{1,t}} < -1$, or $\lim_{t\to\infty}\frac{\dot{k}_{2,t}}{k_{2,t}} < -1$, the transversality conditions are satisfied if

$$\lim_{t\to\infty}\left\{\left[\frac{(\omega_1+\omega_2)\varpi\alpha}{2mv}\right]^{\alpha}(1-\alpha)^{1-\alpha}\left(\frac{\int_0^t \tau_s ds}{k_{1,t}} - \frac{\partial\int_0^t \tau_s ds}{\partial k_{1,t}}\right) - \left(\frac{\tau_t}{k_{1,t}} - \frac{\partial \tau_t}{\partial k_{1,t}}\right) - \frac{c_{1,t}}{k_{1,t}}\right\} < 0$$

and

$$-\lim_{t\to\infty}\left\{\left[\frac{(\omega_1+\omega_2)\varpi\alpha}{2mv}\right]^{\alpha}(1-\alpha)^{1-\alpha}\left(\frac{\int_0^t \tau_s ds}{k_{2,t}} - \frac{\partial\int_0^t \tau_s ds}{\partial k_{2,t}}\right) - \left(\frac{\tau_t}{k_{2,t}} - \frac{\partial \tau_t}{\partial k_{2,t}}\right) + \frac{c_{2,t}}{k_{2,t}}\right\} < 0 \quad .$$

In all three models, the occurrence of $\lim_{t\to\infty}\frac{\dot{\lambda}_{1,t}}{\lambda_{1,t}} < -1$, $\lim_{t\to\infty}\frac{\dot{\lambda}_{2,t}}{\lambda_{2,t}} < -1$, $\lim_{t\to\infty}\frac{\dot{k}_{1,t}}{k_{1,t}} < -1$, or $\lim_{t\to\infty}\frac{\dot{k}_{2,t}}{k_{2,t}} < -1$ is extremely unusual, and these cases are excluded in the following discussion.

B2.3 Sustainability

Because balanced growth is the focal point for the growth path analysis, the following analyses focus on the steady state such that $\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}}$, $\lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}}$, $\lim_{t\to\infty}\frac{\dot{k}_{1,t}}{k_{1,t}}$, $\lim_{t\to\infty}\frac{\dot{k}_{2,t}}{k_{2,t}}$, and $\lim_{t\to\infty}\frac{\dot{\tau}_t}{\tau_t}$ are constants.

B2.3.1 Heterogeneous time preference model

The balanced growth path in the heterogeneous time preference model has the following properties.

Lemma 2-1: In the model of heterogeneous time preference, if $\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}} = \lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}} =$ constant, then

$$\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}} = \lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}} = \lim_{t\to\infty}\frac{\dot{k}_{1,t}}{k_{1,t}} = \lim_{t\to\infty}\frac{\dot{k}_{2,t}}{k_{2,t}} = \lim_{t\to\infty}\frac{\dot{\tau}_t}{\tau_t} = \lim_{t\to\infty}\frac{\frac{d\left(\int_0^t \tau_s ds\right)}{dt}}{\int_0^t \tau_s ds} \quad .$$

Proof: See Harashima (2010).

Proposition 1-1: In the model of heterogeneous time preference, if and only if $\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}}$ $=\lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}}=$constant, all the optimality conditions of both economies are satisfied at steady state.

Proof: By Lemma 2-1, if $\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}}=\lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}}=$ constant,

$$\lim_{t\to\infty}\frac{\tau_t}{k_{1,t}}=\lim_{t\to\infty}\frac{\tau_t}{k_{2,t}}=\Xi \quad ,$$

where Ξ is a constant. In addition, because $\lim_{t\to\infty}\frac{\frac{d\left(\int_0^t \tau_s ds\right)}{dt}}{\int_0^t \tau_s ds}=\lim_{t\to\infty}\frac{\tau_t}{\int_0^t \tau_s ds}=\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}}$,

$$\lim_{t\to\infty}\frac{\int_0^t \tau_s ds}{k_{1,t}}=\lim_{t\to\infty}\frac{\int_0^t \tau_s ds}{k_{2,t}}=\Xi\left(\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}}\right)^{-1} \quad .$$

Thus, $\lim_{t\to\infty}\frac{\tau_t}{k_{1,t}}=\lim_{t\to\infty}\frac{\partial\tau_t}{\partial k_{1,t}}=\lim_{t\to\infty}\frac{\tau_t}{k_{2,t}}=\lim_{t\to\infty}\frac{\partial\tau_t}{\partial k_{2,t}}$ and $\lim_{t\to\infty}\frac{\int_0^t \tau_s ds}{k_{1,t}}=\lim_{t\to\infty}\frac{\partial\int_0^t \tau_s ds}{\partial k_{1,t}}=$ $\lim_{t\to\infty}\frac{\int_0^t \tau_s ds}{k_{2,t}}=\lim_{t\to\infty}\frac{\partial\int_0^t \tau_s ds}{\partial k_{2,t}}$, and

$$\lim_{t\to\infty}\left\{\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha}\left[\frac{\int_0^t \tau_s ds}{k_{1,t}}-\frac{\partial\left(\int_0^t \tau_s ds\right)}{\partial k_{1,t}}\right]-\left(\frac{\tau_t}{k_{1,t}}-\frac{\partial\tau_t}{\partial k_{1,t}}\right)-\frac{c_{1,t}}{k_{1,t}}\right\}=-\lim_{t\to\infty}\frac{c_{1,t}}{k_{1,t}}<0 \quad ,$$

and

$$-\lim_{t\to\infty}\left\{\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha}\left[\frac{\int_0^t \tau_s ds}{k_{2,t}}-\frac{\partial\left(\int_0^t \tau_s ds\right)}{\partial k_{2,t}}\right]-\left(\frac{\tau_t}{k_{2,t}}-\frac{\partial\tau_t}{\partial k_{2,t}}\right)+\frac{c_{2,t}}{k_{2,t}}\right\}=-\lim_{t\to\infty}\frac{c_{2,t}}{k_{2,t}}<0 \quad .$$

Hence, by Lemma 1-1, the transversality conditions are satisfied while all the other optimality conditions are also satisfied.

On the other hand, if $\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}}\neq\lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}}$, then $\lim_{t\to\infty}\frac{\dot{\tau}_t}{\tau_t}\neq\lim_{t\to\infty}\frac{\frac{d\left(\int_0^t \tau_s ds\right)}{dt}}{\int_0^t \tau_s ds}$. Thus, by Lemma 1-1, for both economies to satisfy the transverality conditions, it is necessary that

$\lim_{t\to\infty}\frac{c_{1,t}}{k_{1,t}}=\infty$ or $\lim_{t\to\infty}\frac{\dot{c}_{2,t}}{k_{2,t}}=\infty$, which violates equation (B8) or (B12). ■

The path on which $\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}}=\lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}}=$ constant has the following properties.

Corollary 1-1: In the model of heterogeneous time preference, if and only if $\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}}=$ $\lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}}=$ constant, then

$$\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}}=\lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}}=\lim_{t\to\infty}\frac{\dot{k}_{1,t}}{k_{1,t}}=\lim_{t\to\infty}\frac{\dot{k}_{2,t}}{k_{2,t}}=\lim_{t\to\infty}\frac{\dot{y}_{1,t}}{y_{1,t}}=\lim_{t\to\infty}\frac{\dot{y}_{2,t}}{y_{2,t}}=\lim_{t\to\infty}\frac{\dot{A}_{t}}{A_{t}}=\text{constant.}$$

Proof: See Harashima (2010).

Note that the limit of the growth rate on this path is

$$\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}}=\lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}}=\varepsilon^{-1}\left[\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha}-\frac{\theta_1+\theta_2}{2}\right] \quad \text{(B25)}$$

by equations (B14) and (B15).

Corollary 2-1: In the model of heterogeneous time preference, if and only if $\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}}=$ $\lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}}=$ constant,

$$\lim_{t\to\infty}\frac{\dot{\tau}_{t}}{\tau_{t}}=\lim_{t\to\infty}\frac{\frac{d\int_0^t\tau_s ds}{dt}}{\int_0^t\tau_s ds}=\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}}=\lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}}=\lim_{t\to\infty}\frac{\dot{k}_{1,t}}{k_{1,t}}=\lim_{t\to\infty}\frac{\dot{k}_{2,t}}{k_{2,t}}$$
$$=\lim_{t\to\infty}\frac{\dot{y}_{1,t}}{y_{1,t}}=\lim_{t\to\infty}\frac{\dot{y}_{2,t}}{y_{2,t}}=\lim_{t\to\infty}\frac{\dot{A}_{t}}{A_{t}}=\text{constant.} \quad \text{(B26)}$$

Proof: By Lemma 2-1, $\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}}=\lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}}=\lim_{t\to\infty}\frac{\dot{\tau}_{t}}{\tau_{t}}=\lim_{t\to\infty}\frac{\frac{d\int_0^t\tau_s ds}{dt}}{\int_0^t\tau_s ds}$. Therefore, by Corollary 1-1, equation (B26) holds. ■

Because current account imbalances eventually grow at the same rate as output, consumption, and capital on the multilateral path, the ratios of the current account balance to output, consumption, and capital do not explode, but they stabilize as shown in the proof of Proposition

1-1; that is, $\lim_{t\to\infty}\frac{\tau_t}{k_{1,t}}=\lim_{t\to\infty}\frac{\tau_t}{k_{2,t}}=\varXi$.

On the balanced growth path satisfying Proposition 1-1 and Corollaries 1-1 and 2-1, heterogeneity in time preference is sustainable by definition because all the optimality conditions of the two economies are indefinitely satisfied. The balanced growth path satisfying Proposition 1-1 and Corollaries 1-1 and 2-1 is called the "multilateral balanced growth path" or (more briefly) the "multilateral path" in the following discussion. The term "multilateral" is used even though there are only two economies, because the two-economy models shown can easily be extended to the multi-economy models shown in Section B2.6.

Because technology will not decrease persistently (i.e., $\lim_{t\to\infty}\frac{\dot{A}_t}{A_t}>0$), only the case such that $\lim_{t\to\infty}\frac{\dot{A}_t}{A_t}>0$ (i.e., $\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}}=\lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}}>0$ on the multilateral path by Corollary 1-1) is examined in the following discussion.

B2.3.2 Heterogeneous risk aversion model

On the multilateral path in the heterogeneous risk aversion model, the same Proposition, Lemmas, and Corollaries are proved by arguments similar to those shown in Section B2.3.1.

Lemma 2-2: In the model of heterogeneous risk aversion, if $\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}}=\lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}}=$ constant,

$$\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}}=\lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}}=\lim_{t\to\infty}\frac{\dot{k}_{1,t}}{k_{1,t}}=\lim_{t\to\infty}\frac{\dot{k}_{2,t}}{k_{2,t}}=\lim_{t\to\infty}\frac{\dot{\tau}_t}{\tau_t}=\lim_{t\to\infty}\frac{\frac{d\left(\int_0^t\tau_s ds\right)}{dt}}{\int_0^t\tau_s ds}\ .$$

Proposition 1-2: In the model of heterogeneous risk aversion, if and only if $\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}}=\lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}}=$ constant, all the optimality conditions of both economies are satisfied at steady state.

Corollary 1-2: In the model of heterogeneous risk aversion, if and only if $\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}}=\lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}}=$ constant,

$$\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}}=\lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}}=\lim_{t\to\infty}\frac{\dot{k}_{1,t}}{k_{1,t}}=\lim_{t\to\infty}\frac{\dot{k}_{2,t}}{k_{2,t}}=\lim_{t\to\infty}\frac{\dot{y}_{1,t}}{y_{1,t}}\lim_{t\to\infty}\frac{\dot{y}_{2,t}}{y_{2,t}}=\lim_{t\to\infty}\frac{\dot{A}_t}{A_t}=\text{constant.}$$

Corollary 2-2: In the model of heterogeneous risk aversion, if and only if $\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}}=$

$\lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}} = \text{constant}$,

$$\lim_{t\to\infty}\frac{\dot{\tau}_t}{\tau_t} = \lim_{t\to\infty}\frac{\frac{d\int_0^t \tau_s ds}{dt}}{\int_0^t \tau_s ds} = \lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}} = \lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}} = \lim_{t\to\infty}\frac{\dot{k}_{1,t}}{k_{1,t}} = \lim_{t\to\infty}\frac{\dot{k}_{2,t}}{k_{2,t}}$$

$$= \lim_{t\to\infty}\frac{\dot{y}_{1,t}}{y_{1,t}} = \lim_{t\to\infty}\frac{\dot{y}_{2,t}}{y_{2,t}} = \lim_{t\to\infty}\frac{\dot{A}_t}{A_t} = \text{constant}.$$

On the balanced growth path satisfying Proposition 1-2 and Corollaries 1-2 and 2-2, heterogeneity in risk aversion is also sustainable by definition because all the optimality conditions of the two economies are indefinitely satisfied, and this path is the multilateral path.

B2.3.3 Heterogeneous productivity model

Similar Proposition, Lemmas, and Corollaries also hold in the heterogeneous productivity model. However, unlike heterogeneous preferences, $\lim_{t\to\infty}\tau_t = 0$ and $\lim_{t\to\infty}\int_0^t \tau_s ds = 0$ are possible even if $\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}} = \lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}}$ as equations (B20) and (B21) indicate. Therefore, the case of $\lim_{t\to\infty}\tau_t = 0$ and $\lim_{t\to\infty}\int_0^t \tau_s ds = 0$ will be dealt with separately from the case of $\lim_{t\to\infty}\tau_t \neq 0$ and $\lim_{t\to\infty}\int_0^t \tau_s ds \neq 0$ if necessary.

Lemma 2-3: In the model of heterogeneous productivity, if $\lim_{t\to\infty}\tau_t = 0$ and $\lim_{t\to\infty}\int_0^t \tau_s ds = 0$, then if $\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}} = \lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}} = \text{constant}$,

$$\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}} = \lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}} = \lim_{t\to\infty}\frac{\dot{k}_{1,t}}{k_{1,t}} = \lim_{t\to\infty}\frac{\dot{k}_{2,t}}{k_{2,t}} ,$$

and if $\lim_{t\to\infty}\tau_t \neq 0$ and $\lim_{t\to\infty}\int_0^t \tau_s ds \neq 0$, then if $\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}} = \lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}} = \text{constant}$,

$$\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}} = \lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}} = \lim_{t\to\infty}\frac{\dot{k}_{1,t}}{k_{1,t}} = \lim_{t\to\infty}\frac{\dot{k}_{2,t}}{k_{2,t}}$$

and

$$\lim_{t\to\infty}\frac{\dot{\tau}_t}{\tau_t} = \lim_{t\to\infty}\frac{\frac{d\left(\int_0^t \tau_s ds\right)}{dt}}{\int_0^t \tau_s ds} = \left[\frac{(\omega_1+\omega_2)\varpi\alpha}{2mv}\right]^{\alpha}(1-\alpha)^{1-\alpha} .$$

By Lemma 2-3, if all the optimality conditions of both economies are satisfied, either

$$\lim_{t\to\infty}\frac{\tau_t}{k_{1,t}}=\lim_{t\to\infty}\frac{\int_0^t\tau_s ds}{k_{1,t}}=0 \tag{B27}$$

or

$$\lim_{t\to\infty}\frac{\dot{\tau}_t}{\tau_t}=\lim_{t\to\infty}\frac{\frac{d\left(\int_0^t\tau_s ds\right)}{dt}}{\int_0^t\tau_s ds}=\left[\frac{(\omega_1+\omega_2)\varpi\alpha}{2mv}\right]^{\alpha}(1-\alpha)^{1-\alpha}\quad . \tag{B28}$$

Proposition 1-3: If and only if $\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}}=\lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}}=\text{constant}$, all the optimality conditions of both economies are satisfied at steady state.

Corollary 1-3: In the model of heterogeneous productivity, if and only if $\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}}=$ $\lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}}=\text{constant}$,

$$\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}}=\lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}}=\lim_{t\to\infty}\frac{\dot{k}_{1,t}}{k_{1,t}}=\lim_{t\to\infty}\frac{\dot{k}_{2,t}}{k_{2,t}}=\lim_{t\to\infty}\frac{\dot{y}_{1,t}}{y_{1,t}}=\lim_{t\to\infty}\frac{\dot{y}_{2,t}}{y_{2,t}}=\lim_{t\to\infty}\frac{\dot{A}_t}{A_t}=\text{constant.}$$

Corollary 2-3: In the model of heterogeneous productivity, if $\lim_{t\to\infty}\tau_t\neq 0$ and $\lim_{t\to\infty}\int_0^t\tau_s ds\neq 0$, then if and only if $\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}}=\lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}}=\text{constant}$,

$$\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}}=\lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}}=\lim_{t\to\infty}\frac{\dot{k}_{1,t}}{k_{1,t}}=\lim_{t\to\infty}\frac{\dot{k}_{2,t}}{k_{2,t}}=\lim_{t\to\infty}\frac{\dot{y}_{1,t}}{y_{1,t}}=\lim_{t\to\infty}\frac{\dot{y}_{2,t}}{y_{2,t}}=\lim_{t\to\infty}\frac{\dot{A}_t}{A_t}=\text{constant}$$

and

$$\lim_{t\to\infty}\frac{\dot{\tau}_t}{\tau_t}=\lim_{t\to\infty}\frac{\frac{d\left(\int_0^t\tau_s ds\right)}{dt}}{\int_0^t\tau_s ds}=\left[\frac{(\omega_1+\omega_2)\varpi\alpha}{2mv}\right]^{\alpha}(1-\alpha)^{1-\alpha}\quad .$$

On the two balanced growth paths satisfying Proposition 1-3 and Corollaries 1-3 and 2-3, heterogeneity in productivity is sustainable by definition because all the optimality conditions of the two economies are indefinitely satisfied.

By equations (B20) and (B21), the limit of the growth rate on these sustainable paths is

$$\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}}=\lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}}=\varepsilon^{-1}\left\{\left[\frac{(\omega_1+\omega_2)\varpi\alpha}{2mv(1-\alpha)}\right]^{\alpha}-\theta\right\} .$$

B2.5 A model with heterogeneities in multiple elements

The three heterogeneities are not exclusive. It is particularly likely that heterogeneities in time preference and productivity coexist. Many empirical studies conclude that the rate of time preference is negatively correlated with income (e.g., Lawrance, 1991; Samwick, 1998; Ventura, 2003); this indicates that the economy with the higher productivity has a lower rate of time preference and vice versa. In this section, the models are extended to include heterogeneity in multiple elements.

Suppose that economies 1 and 2 are identical except for time preference, risk aversion, and productivity. The Hamiltonian for economy 1 is

$$H_1=u_1(c_{1,t})\exp(-\theta_1 t)+\lambda_{1t}\left\{\left[\frac{(\omega_1+\omega_2)\varpi\alpha}{2mv(1-\alpha)}\right]^{\alpha}k_{1,t}+\left[\frac{(\omega_1+\omega_2)\varpi\alpha}{2mv}\right]^{\alpha}(1-\alpha)^{1-\alpha}\int_0^t\tau_s ds-\tau_t-c_{1,t}\right\} ,$$

and that for economy 2 is

$$H_2=u_2(c_{2,t})\exp(-\theta_2 t)+\lambda_{2,t}\left\{\left[\frac{(\omega_1+\omega_2)\varpi\alpha}{2mv(1-\alpha)}\right]^{\alpha}k_{2,t}-\left[\frac{(\omega_1+\omega_2)\varpi\alpha}{2mv}\right]^{\alpha}(1-\alpha)^{1-\alpha}\int_0^t\tau_s ds+\tau_t-c_{2,t}\right\} .$$

The growth rates are

$$\frac{\dot{c}_{1,t}}{c_{1,t}}=\varepsilon_1^{-1}\left\{\left[\frac{(\omega_1+\omega_2)\varpi\alpha}{2mv(1-\alpha)}\right]^{\alpha}+\left[\frac{(\omega_1+\omega_2)\varpi\alpha}{2mv}\right]^{\alpha}(1-\alpha)^{1-\alpha}\frac{\partial\int_0^t\tau_s d s}{\partial k_{1,t}}-\frac{\partial\tau_t}{\partial k_{1,t}}-\theta_1\right\} ,$$

and

$$\frac{\dot{c}_{2,t}}{c_{2,t}}=\varepsilon_2^{-1}\left\{\left[\frac{(\omega_1+\omega_2)\varpi\alpha}{2mv(1-\alpha)}\right]^{\alpha}-\left[\frac{(\omega_1+\omega_2)\varpi\alpha}{2mv}\right]^{\alpha}(1-\alpha)^{1-\alpha}\frac{\partial\int_0^t\tau_s d s}{\partial k_{2,t}}+\frac{\partial\tau_t}{\partial k_{2,t}}-\theta_2\right\} .$$

Here, $\lim_{t\to\infty}\frac{\tau_t}{k_{1,t}}=\Xi$, $\lim_{t\to\infty}\frac{\tau_t}{k_{2,t}}=\frac{\omega_1}{\omega_2}\Xi$, $\lim_{t\to\infty}\frac{\int_0^t\tau_s ds}{k_{1,t}}=\Xi\left(\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}}\right)^{-1}$, and $\lim_{t\to\infty}\frac{\int_0^t\tau_s ds}{k_{2,t}}=$

$\dfrac{\omega_1}{\omega_2}\Xi\left(\lim_{t\to\infty}\dfrac{\dot{c}_{1,t}}{c_{1,t}}\right)^{-1}$ at steady state. Thus, $\Xi = \dfrac{\dfrac{\varepsilon_2}{\varepsilon_1}\theta_1 - \theta_2 - \left(\dfrac{\varepsilon_2}{\varepsilon_1} - 1\right)\left[\dfrac{(\omega_1+\omega_2)\varpi\alpha}{2mv(1-\alpha)}\right]^{\alpha}}{\left(\dfrac{\varepsilon_2}{\varepsilon_1} + \dfrac{\omega_1}{\omega_2}\right)\left\{\left[\dfrac{(\omega_1+\omega_2)\varpi\alpha}{2mv}\right]^{\alpha}(1-\alpha)^{1-\alpha}\left(\lim_{t\to\infty}\dfrac{\dot{c}_{1,t}}{c_{1,t}}\right)^{-1} - 1\right\}}$,

and the limit of the growth rate on the multilateral path is

$$\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}} = \lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}} = \left(\frac{\varepsilon_1\omega_1 + \varepsilon_2\omega_2}{\omega_1+\omega_2}\right)^{-1}\left\{\left[\frac{(\omega_1+\omega_2)\varpi\alpha}{2mv(1-\alpha)}\right]^{\alpha} - \frac{\theta_1\omega_1+\theta_2\omega_2}{\omega_1+\omega_2}\right\} .$$

Clearly, if $\varepsilon_1 = \varepsilon_2$ and $\omega_1 = \omega_2 = 1$, then $\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}} = \lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}} = \varepsilon_1^{-1}\left[\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha} - \frac{\theta_1+\theta_2}{2}\right]$; if $\theta_1 = \theta_2$ and $\omega_1 = \omega_2 = 1$, then $\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}} = \lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}} = \left(\frac{\varepsilon_1+\varepsilon_2}{2}\right)^{-1}\left[\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha} - \theta_1\right]$; and if $\theta_1 = \theta_2$ and $\varepsilon_1 = \varepsilon_2$, then $\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}} = \lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}} = \varepsilon_1^{-1}\left\{\left[\frac{(\omega_1+\omega_2)\varpi\alpha}{2mv(1-\alpha)}\right]^{\alpha} - \theta_1\right\}$ as shown in Sections B2.3 and B2.4.

The sign of Ξ on the multilateral path depends on the relative values between θ_1 and θ_2, ε_1 and ε_2, and ω_1 and ω_2. Nevertheless, if the rate of time preference and productivity are negatively correlated, as argued above (i.e., if $\theta_1 < \theta_2$ and $\omega_1 > \omega_2$ while $\varepsilon_1 = \varepsilon_2$), then by similar proofs as those presented for Proposition 2-1 and Corollary 3-1, if $\left[\frac{(\omega_1+\omega_2)\varpi\alpha}{2mv}\right]^{\alpha}\left[1-(1-\alpha)^{1-\alpha}\varepsilon_1\right] < \frac{\omega_1\theta_1+\omega_2\theta_2}{\omega_1+\omega_2}$, then $\Xi < 0$ and $\lim_{t\to\infty}\left(\tau_t - \frac{\partial y_{2,t}}{\partial k_{2,t}}\int_0^t \tau_s ds\right) > 0$ on the multilateral path; that is, the current account deficits and trade surpluses of economy 1 continue indefinitely. The condition $\left[\frac{(\omega_1+\omega_2)\varpi\alpha}{2mv}\right]^{\alpha}\left[1-(1-\alpha)^{1-\alpha}\varepsilon_1\right] < \frac{\omega_1\theta_1+\omega_2\theta_2}{\omega_1+\omega_2}$ is generally satisfied for reasonable parameter values.

B2.6 Multi-economy models

The two-economy models can be extended to include numerous economies that have differing degrees of heterogeneity.

B2.6.1 Heterogeneous time preference model

Suppose that there are H economies that are identical except for time preference. Let θ_i be the rate of time preference of economy i and $\tau_{i,j,t}$ be the current account balance of economy i with economy j, where i = 1, 2, … , H, j = 1, 2, … , H, and $i \neq j$. Because the total population is L_t, the population in each economy is $\frac{L_t}{H}$. The representative household of economy i maximizes its expected utility

$$E\int_0^{\infty} u_i(c_{i,t})\exp(-\theta_i t)dt \quad ,$$

subject to

$$\dot{k}_{i,t} = y_{i,t} + \sum_{j=1}^{H} \frac{\partial y_{j,t}}{\partial k_{j,t}} \int_0^t \tau_{i,j,s} ds - \sum_{j=1}^{H} \tau_{i,j,t} - c_{i,t} - v\dot{A}_{i,t} \left(\frac{L_t}{H} \right)^{-1}$$

for $i \neq j$.

Proposition 3-1: In the multi-economy model of heterogeneous time preference, if and only if

$$\lim_{t \to \infty} \frac{\dot{c}_{i,t}}{c_{i,t}} = \varepsilon^{-1} \left[\left(\frac{\varpi \alpha}{mv} \right)^{\alpha} (1-\alpha)^{-\alpha} - \frac{\sum_{q=1}^{H} \theta_q}{H} \right]$$

for any i, all the optimality conditions of all heterogeneous economies are satisfied at steady state, and

$$\lim_{t \to \infty} \frac{\dot{c}_{i,t}}{c_{i,t}} = \lim_{t \to \infty} \frac{\dot{k}_{i,t}}{k_{i,t}} = \lim_{t \to \infty} \frac{\dot{y}_{i,t}}{y_{i,t}} = \lim_{t \to \infty} \frac{\dot{A}_t}{A_t} = \lim_{t \to \infty} \frac{\dot{\tau}_{i,j,t}}{\tau_{i,j,t}} = \lim_{t \to \infty} \frac{\frac{d \int_0^t \tau_{i,j,s} ds}{dt}}{\int_0^t \tau_{i,j,s} ds}$$

for any i and j ($i \neq j$).
Proof: See Harashima (2010).

B2.6.2 Heterogeneous risk aversion model

The heterogeneous risk aversion model can be extended to the multi-economy model by a proof similar to that for Proposition 3-1. Suppose that H economies are identical except for risk aversion, and their degrees of risk aversion are ε_i ($i = 1, 2, \dots, H$).

Proposition 3-2: In the multi-economy model of heterogeneous risk aversion, if and only if

$$\lim_{t \to \infty} \frac{\dot{c}_{i,t}}{c_{i,t}} = \left(\frac{\sum_{q=1}^{H} \varepsilon_q}{H} \right)^{-1} \left[\left(\frac{\varpi \alpha}{mv} \right)^{\alpha} (1-\alpha)^{-\alpha} - \theta \right]$$

for any i, all the optimality conditions of all heterogeneous economies are satisfied at steady state, and

$$\lim_{t \to \infty} \frac{\dot{c}_{i,t}}{c_{i,t}} = \lim_{t \to \infty} \frac{\dot{k}_{i,t}}{k_{i,t}} = \lim_{t \to \infty} \frac{\dot{y}_{i,t}}{y_{i,t}} = \lim_{t \to \infty} \frac{\dot{A}_t}{A_t} = \lim_{t \to \infty} \frac{\dot{\tau}_{i,j,t}}{\tau_{i,j,t}} = \lim_{t \to \infty} \frac{\frac{d \int_0^t \tau_{i,j,s} ds}{dt}}{\int_0^t \tau_{i,j,s} ds}$$

for any i and j ($i \neq j$).

B2.6.3 Heterogeneous productivity model

The heterogeneous productivity model can also be extended by a proof similar to that for Proposition 3-1. Suppose that H economies are identical except for productivity, and their productivities are ω_i (i = 1, 2, … , H). Note that, because $k_{1+2,t} = k_{1,t} + k_{2,t} = k_{2,t}\left[\frac{\omega_1}{\omega_2} + 1\right]$, the productivity of economy 1+2 is $y_{1+2,t} = A_t^{\alpha}\left(\omega_1^{\alpha} k_{1,t}^{1-\alpha} + \omega_2^{\alpha} k_{2,t}^{1-\alpha}\right) = (\omega_1 + \omega_2)^{\alpha} A_t^{\alpha} k_{1+2,t}^{1-\alpha}$.

Proposition 3-3: In the multi-economy model of heterogeneous productivity, if and only if

$$\lim_{t\to\infty} \frac{\dot{c}_{i,t}}{c_{i,t}} = \varepsilon^{-1}\left\{\left[\frac{\left(\sum_{q=1}^{H}\omega_q\right)\varpi\alpha}{Hmv(1-\alpha)}\right]^{\alpha} - \theta\right\}$$

for any i, all the optimality conditions of all heterogeneous economies are satisfied at steady state, and

$$\lim_{t\to\infty} \frac{\dot{c}_{i,t}}{c_{i,t}} = \lim_{t\to\infty} \frac{\dot{k}_{i,t}}{k_{i,t}} = \lim_{t\to\infty} \frac{\dot{y}_{i,t}}{y_{i,t}} = \lim_{t\to\infty} \frac{\dot{A}_t}{A_t}$$

for any i and j ($i \neq j$).

B2.6.4 Heterogeneity in multiple elements

Similarly, the multi-economy model can be extended to heterogeneity in multiple elements, as follows.

Proposition 3-4: In the multi-economy model of heterogeneity in multiple elements, if and only if

$$\lim_{t\to\infty} \frac{\dot{c}_{i,t}}{c_{i,t}} = \left(\frac{\sum_{q=1}^{H}\varepsilon_q\omega_q}{\sum_{q=1}^{H}\omega_q}\right)^{-1}\left\{\left[\frac{\varpi\alpha\sum_{q=1}^{H}\omega_q}{Hmv(1-\alpha)}\right]^{\alpha} - \frac{\sum_{q=1}^{H}\theta_q\omega_q}{\sum_{q=1}^{H}\omega_q}\right\} \qquad \text{(B29)}$$

for any i (= 1, 2, … , H), all the optimality conditions of all heterogeneous economies are satisfied at steady state, and

$$\lim_{t\to\infty} \frac{\dot{c}_{i,t}}{c_{i,t}} = \lim_{t\to\infty} \frac{\dot{k}_{i,t}}{k_{i,t}} = \lim_{t\to\infty} \frac{\dot{y}_{i,t}}{y_{i,t}} = \lim_{t\to\infty} \frac{\dot{A}_t}{A_t} = \lim_{t\to\infty} \frac{\dot{\tau}_{i,j,t}}{\tau_{i,j,t}} = \lim_{t\to\infty} \frac{\frac{d\int_0^t \tau_{i,j,s}ds}{dt}}{\int_0^t \tau_{i,j,s}ds}$$

for any i and j ($i \neq j$).

Proposition 3-4 implies that the concept of the representative household in a heterogeneous population implicitly assumes that all households are on the multilateral path.

B2.7 Degeneration to an exogenous technology model

The multilateral paths in the endogenous growth models imply that similar sustainable states exist in exogenous technology models. However, this is true only for the heterogeneous time preference model, because, in exogenous technology models, the steady state means that $\frac{\partial y_t}{\partial k_t} = \theta$; that is, the heterogeneity in risk aversion is irrelevant to the steady state, and the heterogeneous productivities do not result in permanent trade imbalances due to $\frac{\partial y_{1,t}}{\partial k_{1,t}} = \frac{\partial y_{2,t}}{\partial k_{2,t}}$. Thereby, only heterogeneous time preference is relevant to sustainable heterogeneity in exogenous growth models.

If technology is exogenously given and constant ($A_t = A$), Hamiltonians for the heterogeneous time preference model shown in Section B1.2.1 degenerate to

$$H_1 = u_1\left(c_{1,t}\right)\exp\left(-\theta_1 t\right) + \lambda_{1t}\left[A^\alpha k_{1,t}^{1-\alpha} + (1-\alpha)A^\alpha k_{1,t}^{-\alpha}\int_0^t \tau_s ds - \tau_t - c_{1,t}\right]$$

and

$$H_2 = u_2\left(c_{2,t}\right)\exp\left(-\theta_2 t\right) + \lambda_{2,t}\left[A^\alpha k_{2,t}^{1-\alpha} - (1-\alpha)A^\alpha k_{2,t}^{-\alpha}\int_0^t \tau_s ds + \tau_t - c_{2,t}\right] .$$

By equations (B6), (B7), and (B8), the growth rate of consumption in economy 1 is $\frac{\dot{c}_{1,t}}{c_{1,t}} = \varepsilon^{-1}\left\{(1-\alpha)A^\alpha k_{1,t}^{-\alpha} + (1-\alpha)A^\alpha k_{1,t}^{-\alpha}\frac{\partial\int_0^t \tau_s ds}{\partial k_{1,t}} - \alpha(1-\alpha)A^\alpha k_{1,t}^{-\alpha-1}\int_0^t \tau_s ds - \frac{\partial \tau_t}{\partial k_{1,t}} - \theta_1\right\}$. Hence,

$$\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}} = \varepsilon^{-1}\lim_{t\to\infty}\left\{(1-\alpha)A^\alpha k_{1,t}^{-\alpha} + (1-\alpha)A^\alpha k_{1,t}^{-\alpha}\frac{\partial\int_0^t \tau_s ds}{\partial k_{1,t}} - \alpha(1-\alpha)A^\alpha k_{1,t}^{-\alpha-1}\int_0^t \tau_s ds - \frac{\partial \tau_t}{\partial k_{1,t}} - \theta_1\right\}$$

= 0 and thereby $\lim_{t\to\infty}(1-\alpha)A^\alpha k_{1,t}^{-\alpha}\left[1+(1-\alpha)\Psi\right] - \Xi - \theta_1 = 0$, where $\Psi = \lim_{t\to\infty}\frac{\int_0^t \tau_s ds}{k_{1,t}}$ and Ψ is constant at steady state and $\lim_{t\to\infty}\frac{\dot{y}_{1,t}}{y_{1,t}} = \lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}} = \lim_{t\to\infty}\frac{\dot{k}_{1,t}}{k_{1,t}} = 0$. For Ψ to be constant at steady state, it is necessary that $\lim_{t\to\infty}\tau_t = 0$ and thus $\Xi = 0$. Therefore, $\lim_{t\to\infty}(1-\alpha)A^\alpha k_{1,t}^{-\alpha}\left[1+(1-\alpha)\Psi\right] - \theta_1 = 0$, and $\lim_{t\to\infty}(1-\alpha)A^\alpha k_{2,t}^{-\alpha}\left[1-(1-\alpha)\Psi\right] - \theta_2 = 0$ because

$$\lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}} = \varepsilon^{-1}\lim_{t\to\infty}\left\{(1-\alpha)A^\alpha k_{2,t}^{-\alpha} - (1-\alpha)A^\alpha k_{2,t}^{-\alpha}\frac{\partial\int_0^t \tau_s ds}{\partial k_{2,t}} + \alpha(1-\alpha)A^\alpha k_{2,t}^{-\alpha-1}\int_0^t \tau_s ds + \frac{\partial \tau_t}{\partial k_{2,t}} - \theta_2\right\}$$

= 0.

Because $\lim_{t\to\infty}(1-\alpha)A^{\alpha}k_{1,t}^{-\alpha}[1+(1-\alpha)\Psi]=\theta_1$, $\lim_{t\to\infty}(1-\alpha)A^{\alpha}k_{2,t}^{-\alpha}[1-(1-\alpha)\Psi]=\theta_2$, and $\frac{\partial y_{1,t}}{\partial k_{1.t}}=\frac{\partial y_{2,t}}{\partial k_{2.t}}=A^{\alpha}k_{1,t}^{-\alpha}=A^{\alpha}k_{2,t}^{-\alpha}$, then

$$\Psi=\frac{\theta_1-\theta_2}{2(1-\alpha)\lim_{t\to\infty}\frac{\partial y_{1,t}}{\partial k_{1.t}}} . \tag{B30}$$

By $\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}}=\varepsilon^{-1}\lim_{t\to\infty}\left\{\frac{\partial y_{1,t}}{\partial k_{1.t}}+\frac{\partial y_{1,t}}{\partial k_{1.t}}\frac{\partial\int_0^t\tau_s ds}{\partial k_{1,t}}-\alpha\frac{\partial y_{1,t}}{\partial k_{1.t}}\frac{\int_0^t\tau_s ds}{k_{1,t}}-\frac{\partial\tau_t}{\partial k_{1,t}}-\theta_1\right\}=0$ and equation (B30), then $\lim_{t\to\infty}\frac{\partial y_{1,t}}{\partial k_{1.t}}+\lim_{t\to\infty}\frac{\partial y_{1,t}}{\partial k_{1.t}}(1-\alpha)\Psi=\theta_1$; thus,

$$\lim_{t\to\infty}\frac{\partial y_{1,t}}{\partial k_{1.t}}=\frac{\theta_1+\theta_2}{2}=\lim_{t\to\infty}\frac{\partial y_{2,t}}{\partial k_{2.t}} . \tag{B31}$$

If equation (B31) holds, all the optimality conditions of both economies are indefinitely satisfied. This result is analogous to equation (B25) and corresponds to the multilateral path in the endogenous growth models. The state indicated by equation (B31) is called the "multilateral steady state" in the following discussion.

If both economies are not open and are isolated, $\lim_{t\to\infty}\frac{\partial y_{1,t}}{\partial k_{1.t}}=\theta_1$ and $\lim_{t\to\infty}\frac{\partial y_{2,t}}{\partial k_{2.t}}=\theta_2$ at steady state instead of the conditions shown in equation (B31). Hence, at the multilateral steady state with $\theta_1<\theta_2$, the amount of capital in economy 1 is smaller than when the economy is isolated and vice versa. As a result, output and consumption in economy 1 are also smaller in the multilateral steady state with $\theta_1<\theta_2$ than when the economy is isolated. Furthermore, $\Psi=\frac{\theta_1-\theta_2}{2(1-\alpha)\lim_{t\to\infty}\frac{\partial y_{1,t}}{\partial k_{1.t}}}=\frac{\theta_1-\theta_2}{(1-\alpha)(\theta_1+\theta_2)}<0$ by equation (B31). Thus, by

$\lim_{t\to\infty}\frac{\int_0^t\tau_s ds}{k_{1,t}}=\Psi<0$,

$$\lim_{t\to\infty}\int_0^t\tau_s ds<0 \quad ;$$

that is, economy 1 possesses accumulated debts owed to economy 2 at steady state, and economy 1 has to export goods and services to economy 2 by

$$\left|(1-\alpha)A^{\alpha}k_{1,t}^{-\alpha}\int_0^t\tau_s ds\right|$$

in every period to pay the debts. Nevertheless, because $\lim_{t\to\infty}\tau_t = 0$ and $\Xi = 0$, the debts do not explode but stabilize at steady state.

In the multilateral steady state, all the optimality conditions of both economies are satisfied, and heterogeneity is therefore sustainable. However, this state will be economically less preferable for economy 1 as compared with the state of Becker (1980), because consumption is smaller and debts are owed. Which state should economy 1 select? A similar dilemma—whether to give priority to simultaneous optimality with economy 2 or to unilaterally optimal higher utility—will also arise in the endogenous growth models; this is examined in the following sections.

B3 Unilateral balanced growth

The multilateral path satisfies all the optimality conditions, but that does not mean that the two economies naturally select the multilateral path. Ghiglino (2002) predicts that it is likely that, under appropriate assumptions, the results of Becker (1980) still hold in endogenous growth models. Farmer and Lahiri (2005) show that balanced growth equilibria do not exist in a multi-agent economy in general, except in the special case that all agents have the same constant rate of time preference. How the economies behave in the environments described in Sections B1 and B2 is examined in this section.

B3.1 Heterogeneous time preference model

The multilateral path is not the only path on which all the optimality conditions of economy 1 are satisfied. Even if economy 1 behaves unilaterally, it can achieve optimality, but economy 2 cannot.

Lemma 4-1: In the heterogeneous time preference model, if each economy sets τ_t without regarding the other economy's optimality conditions, then it is not possible to satisfy all the optimality conditions of both economies.
Proof: See Harashima (2010).

Since $\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}} = \varepsilon^{-1}\left\{\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha} + \left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{1-\alpha}\lim_{t\to\infty}\frac{\tau_t}{k_{1,t}}\left(\lim_{t\to\infty}\frac{\dot{\tau}_t}{\tau_t}\right)^{-1} - \lim_{t\to\infty}\frac{\tau_t}{k_{1,t}} - \theta_1\right\}$

at steady state, all the optimality conditions of economy 1 can be satisfied only if either

$$\lim_{t\to\infty}\frac{\dot{\tau}_t}{\tau_t} = \lim_{t\to\infty}\frac{\frac{d\left(\int_0^t \tau_s ds\right)}{dt}}{\int_0^t \tau_s ds} = \lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}} \tag{B32}$$

or

$$\lim_{t\to\infty}\frac{\dot{\tau}_t}{\tau_t} = \lim_{t\to\infty}\frac{\frac{d\left(\int_0^t \tau_s ds\right)}{dt}}{\int_0^t \tau_s ds} = \left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{1-\alpha} \quad . \tag{B33}$$

That is, $\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}}$ can be constant only when either equation (B32) or (B33) is satisfied. Conversely, economy 1 has two paths on which all its optimality conditions are satisfied. Equation (B32) indicates that $\lim_{t\to\infty}\frac{\tau_t}{k_{1,t}}=$ constant, and equation (B33) indicates that $\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{1-\alpha}\left(\lim_{t\to\infty}\frac{\dot{\tau}_t}{\tau_t}\right)^{-1}-1=0$ for any $\lim_{t\to\infty}\frac{\tau_t}{k_{1,t}}$. Equation (B32) corresponds to the multilateral path. On the path satisfying equation (B33), $\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}}\neq\lim_{t\to\infty}\frac{\dot{\tau}_t}{\tau_t}=\lim_{t\to\infty}\frac{\frac{d\left(\int_0^t\tau_s ds\right)}{dt}}{\int_0^t\tau_s ds}$, and $\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}}>\lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}}$. Here, by equations (B2) and (B3),

$$c_{1,t}-c_{2,t}=2\left(\frac{\partial y_{1,t}}{\partial k_{1,t}}\int_0^t\tau_s ds-\tau_t\right)=2\left[\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{1-\alpha}\int_0^t\tau_s ds-\tau_t\right],$$

and

$$\lim_{t\to\infty}(c_{1,t}-c_{2,t})=0$$

is required because $\lim_{t\to\infty}\frac{\tau_t}{\int_0^t\tau_s ds}=\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{1-\alpha}$. However, because $\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}}>\lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}}$, economy 2 must initially set consumption such that $c_{2,0}=\infty$, which violates the optimality condition of economy 2. Therefore, unlike with the multilateral path, all the optimality conditions of economy 2 cannot be satisfied on the path satisfying equation (B33) even though those of economy 1 can. Hence, economy 2 has only one path on which all its optimality conditions can be satisfied—the multilateral path. The path satisfying equation (B33) is called the "unilateral balanced growth path" or the "unilateral path" in the following discussion. Clearly, heterogeneity in time preference is not sustainable on the unilateral path.

How should economy 2 respond to the unilateral behavior of economy 1? Possibly, both economies negotiate for the trade between them, and some agreements may be reached. If no agreement is reached, however, and economy 1 never regards economy 2's optimality conditions, economy 2 generally will fall into the following unfavorable situation.

Remark 1-1: In the model of heterogeneous time preference, if economy 1 does not regard the optimality conditions of economy 2, the ratio of economy 2's debts (owed to economy 1) to its consumption explodes to infinity while all the optimality conditions of economy 1 are satisfied.

The reasoning behind Remark 1-1 is as follows. When economy 1 selects the unilateral path and sets $c_{1,0}$ so as to achieve this path, there are two options for economy 2. The first option is for economy 2 to also pursue its own optimality without regarding economy 1: that is, to select its own unilateral path. The second option is to adapt to the behavior of economy 1 as a follower. If economy 2 takes the first option, it sets $c_{2,0}$ without regarding $c_{1,0}$. As the proof of Lemma 4-1 indicates, unilaterally optimal growth rates are different between the two economies and

$\frac{\dot{c}_{1,t}}{c_{1,t}} > \frac{\dot{c}_{2,t}}{c_{2,t}}$; thus, the initial consumption should be set as $c_{1,0} < c_{2,0}$. Because $\frac{\partial y_{1,t}}{\partial k_{1,t}} = \frac{\varpi}{2mv}\frac{\partial(y_{1,t}+y_{2,t})}{\partial A_t} = \frac{\partial y_{2,t}}{\partial k_{2,t}}$ and $k_{1,t} = k_{2,t}$ must be kept, capital and technology are equal and grow at the same rate in both economies. Hence, because $c_{1,0} < c_{2,0}$, more capital is initially produced in economy 1 than in economy 2 and some of it will need to be exported to economy 2. As a result, $\frac{\dot{c}_{1,t}}{c_{1,t}} > \frac{\dot{k}_{1,t}}{k_{1,t}} = \frac{\dot{k}_{2,t}}{k_{2,t}} > \frac{\dot{c}_{2,t}}{c_{2,t}}$, which means that all the optimality conditions of both economies cannot be satisfied. Since $\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}} > \lim_{t\to\infty}\frac{\dot{k}_{1,t}}{k_{1,t}} = \lim_{t\to\infty}\frac{\dot{k}_{2,t}}{k_{2,t}} > \lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}}$, capital soon becomes abundant in economy 2, and excess goods and services are produced in that economy. These excess products are exported to and utilized in economy 1. This process escalates as time passes because $\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}} > \lim_{t\to\infty}\frac{\dot{k}_{1,t}}{k_{1,t}} = \lim_{t\to\infty}\frac{\dot{k}_{2,t}}{k_{2,t}} > \lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}}$, and eventually almost all consumer goods and services produced in economy 2 are consumed by households in economy 1. These consequences will be unfavorable for economy 2.

If economy 2 takes the second option, it should set $c_{2,0} = \infty$ to satisfy all its optimality conditions, as the proof of Lemma 4-1 indicates. Setting $c_{2,0} = \infty$ is impossible, but economy 2 as the follower will initially set $c_{2,t}$ as large as possible. This action gives economy 2 a higher expected utility than that of the first option, because consumption in economy 2 in the second case is always higher. As a result, economy 2 imports as many goods and services as possible from economy 1, and the trade deficit of economy 2 continues until $\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{1-\alpha}\int_0^t \tau_s ds = \tau_t$ is achieved; this is, $\frac{\dot{\tau}_t}{\tau_t} = \frac{\frac{d\left(\int_0^t \tau_s ds\right)}{dt}}{\int_0^t \tau_s ds}$ is achieved. The current account deficits and the accumulated debts of economy 2 will continue to increase indefinitely. Furthermore, they will increase more rapidly than the growth rate of outputs ($\lim_{t\to\infty}\frac{\dot{y}_{2,t}}{y_{2,t}}$) because, in general, $\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}} < \lim_{t\to\infty}\frac{\dot{\tau}_t}{\tau_t}$; that is, $(1-\varepsilon)\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{1-\alpha} < \theta_1(<\theta_2)$. If no disturbance occurs, the expansion of debts may be sustained forever, but economy 2 becomes extremely vulnerable to even a very tiny negative disturbance. If such a disturbance occurs, economy 2 will lose all its capital and will no longer be able to repay its debts. This result corresponds to the state shown by Becker (1980), and it will also be unfavorable for economy 2. Because $\lim_{t\to\infty}\left[\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{1-\alpha}\int_0^t \tau_s ds - \tau_t\right] = 0$, inequality (B23) holds, and the transversality condition for economy 1 is satisfied by Lemma 1-1. Thus, all the optimality conditions of economy 1 are satisfied if economy 2 takes the second option.

As a result, all the optimality conditions of economy 2 cannot be satisfied in any case if economy 1 takes the unilateral path. Both options to counter the unilateral behavior of economy 1 are unfavorable for economy 2. However, the expected utility of economy 2 is higher if it takes the second option rather than the first, and economy 2 will choose the second option. Hence, if economy 1 does not regard economy 2's optimality conditions, the debts owed by economy 2 to economy 1 increase indefinitely at a higher rate than consumption.

B3.2 Heterogeneous risk aversion model

The same consequences are observed in this model.

Lemma 4-2: In the model of heterogeneous risk aversion, if each economy sets τ_t without regard for the other economy's optimality conditions, then all the optimality conditions of both economies cannot be satisfied.

Therefore, heterogeneity in risk aversion is not sustainable on the unilateral path.

Remark 1-2: In the model of heterogeneous risk aversion, if economy 1 does not regard economy 2's optimality conditions, the ratio of economy 2's debts (owed to economy 1) to its consumption explodes to infinity while all the optimality conditions of economy 1 are satisfied.

B3.3 Heterogeneous productivity model

Unlike the heterogeneous preferences shown in Sections B3.1 and B3.2, heterogeneity in productivity can be sustainable even on the unilateral path.

Lemma 4-3: In the heterogeneous productivity model, even if each economy sets τ_t without regard for the other economy's optimality conditions, it is possible that all the optimality conditions of both economies are satisfied if

$$\lim_{t\to\infty}\frac{\dot{\tau}_t}{\tau_t}=\lim_{t\to\infty}\frac{\dfrac{d\left(\int_0^t \tau_s ds\right)}{dt}}{\int_0^t \tau_s ds}=\left[\frac{(\omega_1+\omega_2)\varpi\alpha}{2mv}\right]^{\alpha}(1-\alpha)^{1-\alpha} .$$

Proof: See Harashima (2010).

All the optimality conditions of economy 1 can be satisfied only if either equation (B27) or (B28) holds, because $\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}}$ can be constant only when equation (B27) or (B28) holds. Equation (B27) corresponds to the multilateral path, and equation (B28) corresponds to the unilateral path. Unlike the heterogeneity in preferences, Lemma 4-3 shows that, even on the unilateral path, all the optimality conditions of both economies are satisfied because the limit of both economies' growth rates is identical on the path of either equation (B27) or (B28), such that $\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}}=\lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}}=\varepsilon^{-1}\left\{\left[\frac{(\omega_1+\omega_2)\varpi\alpha}{2mv(1-\alpha)}\right]^{\alpha}-\theta\right\}$. Therefore, heterogeneity in productivity is sustainable even on the unilateral path.

Nevertheless, on the unilateral path, current account imbalances generally grow steadily at a higher rate than consumption; this is not the case on the multilateral path. How does economy 1 set τ? If economy 1 imports as many goods and services as possible before reaching the steady state at which $\lim_{t\to\infty}\frac{\dot{\tau}_t}{\tau_t}=\lim_{t\to\infty}\frac{\frac{d\left(\int_0^t \tau_s ds\right)}{dt}}{\int_0^t \tau_s ds}=\left[\frac{(\omega_1+\omega_2)\varpi\alpha}{2mv}\right]^{\alpha}(1-\alpha)^{1-\alpha}$ (i.e., if it initially sets τ_t as $\tau_t<0$ and $\tau_t-\frac{\partial y_{2,t}}{\partial k_{2,t}}\int_0^t \tau_s ds<0$), the expected utility of economy 1 will be higher than it is in either case where $\tau_t>0$ or in the multilateral path. However, the debts economy 1 owes to economy 2 will grow indefinitely at a higher rate than consumption, and the ratio of debt to consumption explodes to infinity. If there is no disturbance, this situation will be sustained forever, but economy 1 will become extremely vulnerable to even a very tiny negative disturbance. Hence, the unilateral path will not necessarily be favorable for economy 1 although all its optimality conditions are satisfied on this path, and economy 1 will prefer the multilateral path.

Remark 1-3: In the heterogeneous productivity model, even though economy 1 does not regard economy 2's optimality conditions, the multilateral balanced growth path will be selected.

Hence, the state shown by Becker (1980) will not be observed in the case of heterogeneous productivity.

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