On the optimal investment

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Abstract

In 1988 Dybvig introduced the payoff distribution pricing model (PDPM) as an alternative to the capital asset pricing model (CAPM). Under this new paradigm agents preferences depend on the probability distribution of the payoff and for the same distribution agents prefer the payoff that requires less investment. In this context he gave the notion of efficient payoff. Both approaches run parallel to the theory of choice of von Neumann-Morgenstern (1947), known as the Expected Utility Theory and posterior axiomatic alternatives. In this paper we consider the notion of optimal payoff as that maximizing the terminal position for a chosen preference functional and we investigate the relationship between both concepts, optimal and efficient payoffs, as well as the behavior of the efficient payoffs under different market dynamics. We also show that path-dependent options can be efficient in some simple models.

Key words: Expected Utility, Prospect Theory, Risk Aversion, Law invariant preferences, Growth Optimal Portfolio, Portfolio Numeraire.

JEL-Classification G11, D03, D11, G02

1 Introduction

The capital asset pricing model (CAPM) can be seeing as an approach to investment analysis based on the following simple assumptions:

Agents preferences depend only on the mean and variance of the payoff.

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Between two payoffs with equal variance an agent will choose the one with higher return.

In 1988 Dybvig introduced the payoff distribution pricing model (PDPM) as an alternative to CAPM. His goal was to find another alternative to evaluate investment performance. He assumed that agents preferences depend on the probability distribution of the payoff and for the same distribution agents prefer the payoff that requires less investment.

Both approaches run parallel to the axiomatic theory of choice of von Neumann-Morgenstern (1947) and the posterior axiomatic alternatives; see for example Föllmer and Schied (2011).

The von Neumann-Morgenstern (1947) axiomatic theory together with the inclusion of risk aversion lead us to the expected utility theory (EUT).

The optimal payoff consists in choosing a payoff in such a way that we obtain the largest expected utility of the payoff for a fixed investment.

Alternatives to EUT are based on modifications or elimination of the *independence axiom.* The independence axiom of the EUT says the following:

A preference relation $\succ$ on a set of probability distributions $\mathcal{X}$ satisfies the independence axiom if for all $\mu, \nu \in \mathcal{X}$, $\mu \succ \nu$ implies

$$\alpha\mu + (1-\alpha)\tau \succ \alpha\nu + (1-\alpha)\tau$$

for all $\tau \in \mathcal{X}$ and $\alpha \in (0,1]$.

Many examples or paradoxes show that this axiom or principle is not followed by real agents. The following example is a well known paradox where the independence axiom is violated.

Example 1 *(Allais' paradox) You have to choose between:*

$$\mu_1 = 0.33\delta_{2500} + 0.66\delta_{2400} + 0.01\delta_0,$$

$$\mu_2 = \delta_{2400}$$

and later between

$$\nu_1 = 0.33\delta_{2500} + 0.67\delta_0,$$

$$\nu_2 = 0.34\delta_{2400} + 0.66\delta_0.$$

Allais showed that for 66% of people $\mu_2 \succ \mu_1$ and $\nu_1 \succ \nu_2$. However $\frac{1}{2}(\mu_2 + \nu_1) = \frac{1}{2}(\mu_1 + \nu_2)$ and this *violates the independence axiom. In fact if the independence is true and* $\mu_2 \succ \mu_1$ *and* $\nu_1 \succ \nu_2$ *we have*

$$\alpha\mu_2 + (1-\alpha)\nu_1 \succ \alpha\mu_1 + (1-\alpha)\nu_1 \succ \alpha\mu_1 + (1-\alpha)\nu_2,$$

and taking $\alpha = 1/2$ *we obtain*

$$\frac{\mu_2 + \nu_1}{2} \succ \frac{\mu_1 + \nu_2}{2}.$$

The Dual Theory of Choice (DTC) (Yaari (1987)) or the Cumulative Prospect Theory (CPT) (see Kahneman-Tverski (1979) and Tverski-Kahneman (1992)) are some of the alternatives to EUT. Both propose that the optimality of a payoff is a functional of its law. For instance Yaari proposed a preference functional of the form

$$V(X) = \int_0^1 h(1-t) F_X^{-1}(t) \mathrm{d}t,$$

where $h : [0,1] \longmapsto \mathbb{R}_+$ (distortion function). In the CPT

$$\begin{aligned} V(X) = &\int_0^1 h_1(1-t) u_1 \left(\left(F_X^{-1}(t) - x_0 \right)_+ \right) \mathrm{d}t \\ &- \int_0^1 h_2(t) u_2 \left(\left(F_X^{-1}(t) - x_0 \right)_- \right) \mathrm{d}t, \end{aligned}$$

with h_1, h_2 distortion functions and u_1 concave and u_2 convex, $x_0 \in \mathbb{R}$ is a reference level where consumers pass from being risk adverse to being risk takers. These functionals are particular cases of

$$V(X) = \int_0^1 L(t, F_X^{-1}(t)) \mathrm{d}t.$$

The EUT is included in the previous framework with

$$V(X) = \mathbb{E}\left(u(X)\right) = \int_0^1 u(F_X^{-1}(t)) \mathrm{d}t.$$

In this work we investigate the relationship between the concepts of efficient and optimal payoffs. In addition we study the behavior of the efficient portfolio for various derivatives and different assets' price dynamics.

The paper is organised as follows: Section 2 contains preliminary results on expected utility theory and payoff distribution pricing model. Section 3 studies efficient payoffs and law invariant preferences. Section 4 is devoted to efficient payoffs in a dynamic setting while Section 5 investigates conditional efficient payoffs.

2 EUT and PDPM

We start this section by recalling the definition of a utility function.

Definition 1 *A utility function is map $u : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$, which is strictly increasing and continuous on $\{u > -\infty\}$, of class $\mathbb{C}^2$ and strictly concave in the interior of $\{u > -\infty\}$, and such that marginal utility tends to zero when wealth tends to infinity, i.e.,*

$$u'(\infty) := \lim_{x \to \infty} u'(x) = 0.$$

Let us denote the interior of $\{u > -\infty\}$ by $\mathrm{dom}(u)$. We will only consider the two following cases:

Case 1 *$dom(u) = (0, \infty)$ and u satisfies*

$$u'(0) := \lim_{x \to 0^+} u'(x) = \infty.$$

Case 2 *$dom(u) = \mathbb{R}$ and u satisfies*

$$u'(-\infty) := \lim_{x \to -\infty} u'(x) = \infty.$$

The HARA utility functions $u(x) = \frac{x^{1-p}}{1-p}$ for $p \in \mathbb{R}_+ \backslash \{0, 1\}$ and the logarithmic utility $u(x) = \log(x)$ are important examples of **Case** 1 and the exponential utility function $u(x) = -\frac{1}{\alpha}e^{-\alpha x}$ is a typical example of **Case** 2.

Let us fix a pricing measure $\mathbb{Q}$. Given $w_0 > 0$ and a utility function u, we want to find a payoff X, with initial value w_0, that maximizes $\mathbb{E}(u(X))$ that is we consider the following optimization problem

$$\max \{\mathbb{E}(u(X)) : \mathbb{E}_{\mathbb{Q}}(X) = w_0\}. \tag{1}$$

Such X if it exists is said to be an *optimal payoff.* For the sake of simplicity we consider that interest rates are zero.

Proposition 1 *The optimal payoff is a decreasing function of $\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}$.*

Proof. The corresponding Lagrangian for (1) is

$$\mathbb{E}(u(X)) - \lambda \mathbb{E}_{\mathbb{Q}}(X - w_0) = \mathbb{E}\left(u(X) - \lambda\left(X\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} - w_0\right)\right).$$

Then, the obvious candidate to be the optimal terminal wealth is

$$X^* := (u')^{-1}\left(\lambda\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right), \tag{2}$$

where λ is the solution of the equation $\mathbb{E}_{\mathbb{Q}}\left[(u')^{-1}\left(\lambda\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right)\right] = w_0$. The existence of X^* follows from the fact that u is strictly concave, so $(u')^{-1}(\cdot)$ is a strictly decreasing, and λ is positive and u' takes values on $\mathbb{R}_+$ (in both cases 1 and 2). To see the optimality of X^* we can consider another payoff X and we obtain that

$$\begin{aligned}
&\mathbb{E}(u(X)) - \lambda\mathbb{E}_{\mathbb{Q}}(X - w_0) - (\mathbb{E}(u(X^*)) - \lambda\mathbb{E}_{\mathbb{Q}}(X^* - w_0)) \\
&= \mathbb{E}\left(u(X) - u(X^*)) - \lambda(X - X^*)\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right) \\
&= \frac{1}{2}\mathbb{E}\left(u''(\tilde{X})(X - X^*)^2\right) \le 0,
\end{aligned}$$

where $\tilde{X}$ is in between X and X^*. Since u is strictly concave, (a.s.) uniqueness follows. ■

Suppose that $Y = (u')^{-1}\left(\lambda\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right)$ is the payoff of certain contract, then this payoff is better than any other payoff X *with the same law* as Y if the risk neutral measure used to price derivatives is $\mathbb{Q}$ and the utility function that we choose is u. Then *a fortiori*

$$\mathbb{E}_{\mathbb{Q}}(X) \ge \mathbb{E}_{\mathbb{Q}}(Y).$$

In fact we have that

$$\mathbb{E}(u(Y)) = \mathbb{E}(u(X)),$$

so if $\mathbb{E}_{\mathbb{Q}}(Y) - \mathbb{E}_{\mathbb{Q}}(X) = h > 0$, we will have that $\mathbb{E}_{\mathbb{Q}}(X + h) = w_0$ and $\mathbb{E}(u(X + h)) > \mathbb{E}(u(Y))$ contradicting the optimality of Y. So among the payoffs with the same law as Y, Y is the payoff with the lowest price. This is the idea of efficient payoff introduced by Dybvig (1988a) and further developed in Dybvig (1988b). Recently a systematic study of efficient payoffs in different contexts has been done by Bernard et al. (2014) and Von Hammerstein et al. (2014) under the name of cost-efficient payoffs. Here we shall use the term *efficient payoff* for brevity.

Definition 2 *A payoff Y is said to be an efficient payoff if any other payoff X with the same law is more expensive.*

Therefore, we have proved, in the previous paragraph, the following proposition.

Proposition 2 *The optimal payoff w.r.t. the utility function u is an efficient payoff.*

Suppose $Y = (u')^{-1}\left(\lambda\frac{d\mathbb{Q}}{d\mathbb{P}}\right)$, and that u is as in **Case** 1 (a similar discussion can be done for **Case** 2), let $h : \mathbb{R}_+ \to \mathbb{R}$ be a non decreasing $\mathbb{C}^1$ function with $h(0) = 0$ and define $Z := h\left((u')^{-1}\left(\lambda\frac{d\mathbb{Q}}{d\mathbb{P}}\right)\right)$. Then we wonder if Z is an optimal payoff w.r.t. another utility function. Let V be such utility function, that is, it must satisfy

$$(V')^{-1}\left(\lambda\frac{d\mathbb{Q}}{d\mathbb{P}}\right) = h\left((u')^{-1}\left(\lambda\frac{d\mathbb{Q}}{d\mathbb{P}}\right)\right).$$

Therefore it is sufficient to have that $V(\cdot)$ is a primitive function of $u'(h^{-1}(\cdot))$. Hence $h(Y)$ is an efficient payoff by the argument in the paragraph before Definition 2. As a consequence, if we want to create efficient payoffs with a fixed distribution function $F : \mathbb{R}_+ \to [0,1)$ and we assume that $\frac{d\mathbb{Q}}{d\mathbb{P}}$ is a continuous random variable, then this efficient payoff is given by

$$F^{-1}\left(1 - F_{\frac{d\mathbb{Q}}{d\mathbb{P}}}\left(\frac{1}{\lambda}u'(Y)\right)\right) = F^{-1}\left(1 - F_{\frac{d\mathbb{Q}}{d\mathbb{P}}}\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)\right),$$

where it is assumed that $F^{-1} \in \mathbb{C}^1$, and $F_{\frac{d\mathbb{Q}}{d\mathbb{P}}}(\cdot)$ denotes the distribution function of $\frac{d\mathbb{Q}}{d\mathbb{P}}$. This efficient payoff is also an optimal payoff w.r.t. a utility function $V(\cdot)$ (belonging to **Case** 1) which is a primitive function of $\lambda F^{-1}_{\frac{d\mathbb{Q}}{d\mathbb{P}}}(1 - F(\cdot))$. The factor λ can obviously be omitted. We have derived the following result:

Proposition 3 *Assume that $\frac{d\mathbb{Q}}{d\mathbb{P}}$ has a continuous distribution and that F is a smooth distribution function, such that $F^{-1} \in \mathbb{C}^1$. Then*

$$X := F^{-1}\left(1 - F_{\frac{d\mathbb{Q}}{d\mathbb{P}}}\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)\right)$$

*is an efficient payoff. X is also an optimal payoff w.r.t. a utility function (belonging to **Case** 1 or **Case** 2) $V(\cdot)$ which is a primitive function of $F^{-1}_{\frac{d\mathbb{Q}}{d\mathbb{P}}}(1 - F(\cdot))$.*

Example 2 *It is easy to see that when F and $F_{\log\frac{d\mathbb{Q}}{d\mathbb{P}}}$ are Gaussian the corresponding utility funcion is the exponential utility. In fact, if $F_{\log\frac{d\mathbb{Q}}{d\mathbb{P}}}(z) = \Phi\left(\frac{z-\mu}{\sigma}\right)$ and $F(u) = \Phi\left(\frac{u-\alpha}{\gamma}\right)$, where $\Phi(\cdot)$ cumulative distribution function of the standard normal distribution, then*

$$F^{-1}_{\frac{d\mathbb{Q}}{d\mathbb{P}}}(1 - F(u)) = \exp\left\{\mu - \frac{\sigma}{\gamma}(u - \alpha)\right\},$$

and a primitive function, up to multiplicative constants, is given by

$$V(u) := -\frac{\gamma}{\sigma}\exp\left\{-\frac{\sigma}{\gamma}u\right\}, u \in \mathbb{R}.$$

As we shall see later this smoothness condition on F can be relaxed. The relationship between efficient and optimal payoffs has also been studied in a recent paper by Bernard *et al.* (2015b)

2.1 Inefficiency of path dependent options

In 1988 Dybvig wrote a paper entitled: Inefficient Dynamic Portfolio or How to Throw Away a Million Dollars in the Stock Market (Dybvig (1988b)). The title suggests a general or universal result about investment in stock markets. His claim is that path dependent options are inefficient in the sense that we can have a payoff depending only of the final price of the stock, say S_T, with higher terminal utility and the same initial price. Vanduffel *et al.* (2009) obtained the same inneficiency result in a Lévy market model and Kassberger-Liebmann (2012) explained when this phenomenon happens. The following simple lemma and theorem clarify the situation.

Lemma 1 *Let $X \geq 0$ be a payoff. Consider a model in which the risk neutral probability $\mathbb{Q}$ satisfies*

$$\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} \in \sigma(S_T).$$

Then

$$\mathbb{E}_{\mathbb{Q}}(X|S_T) = \mathbb{E}(X|S_T).$$

Proof. First, set $Z := \mathbb{E}_{\mathbb{Q}}(X|S_T)$, by definition of the conditional expectation:

$$\mathbb{E}_{\mathbb{Q}}(YZ) = \mathbb{E}_{\mathbb{Q}}(YX) \text{ for all } Y \geq 0,\, Y \in \sigma(S_T),$$

then

$$\mathbb{E}_{\mathbb{Q}}(YZ) = \int_\Omega YZ\mathrm{d}\mathbb{Q} = \int_\Omega Y\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}Z\mathrm{d}\mathbb{P} = \int_\Omega \bar{Y}Z\mathrm{d}\mathbb{P} = \int_\Omega \bar{Y}X\mathrm{d}\mathbb{P},$$

with $\bar{Y} \geq 0$ and $\bar{Y} \in \sigma(S_T)$ arbitrary, so $Z = \mathbb{E}(X|S_T)$. ■

Theorem 1 *If the risk neutral probability satisfies $\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} \in \sigma(S_T)$, and the savings account is deterministic, path-dependent payoffs are* dominated, *in the sense that there is another payoff with the same initial price and more terminal utility.*

Proof. Given a payoff X, define $\bar{X}$ by $\bar{X} := \mathbb{E}_{\mathbb{Q}}(X|S_T)$. Then, the price is the same, since the savings account $(B_t)_{t\geq 0}$ is deterministic,

$$\mathbb{E}_{\mathbb{Q}}\left(\frac{X}{B_T}\right) = \mathbb{E}_{\mathbb{Q}}\left(\frac{1}{B_T}\mathbb{E}_{\mathbb{Q}}(X|S_T)\right).$$

Now, by Lemma 1

$$\bar{X} = \mathbb{E}_{\mathbb{Q}}(X|S_T) = \mathbb{E}(X|S_T),$$

and given a utility function u

$$\mathbb{E}(u(\bar{X})) = \mathbb{E}(u(\mathbb{E}(X|S_T))) \geq \mathbb{E}(\mathbb{E}(u\left(X\right)|S_T)) = \mathbb{E}(u\left(X\right)),$$

where the inequality follows from Jensen's inequality since u is concave. ■

However, as shown in Example 4, the condition $\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} \in \sigma(S_T)$ is not satisfied in some simple models and the claim of Dybvig is not true in such cases. In the next section we consider a more general frame that includes EUT.

3 Efficient payoffs and law invariant preferences

Definition 3 *A preference functional $V(X) : L^\infty \to \mathbb{R}$ is called*

1. *monotone if $X \geq Y$ a.s. implies $V(X) \geq V(Y)$,*
2. *law invariant if $V(X) = V(Y)$ whenever $X \overset{d}{\sim} Y$.*

EUT, DTC and CPT use monotone and law invariant functionals and this law invariance is in agreement with the Dybvig approach.

Here we follow Carlier-Dana (2011). Choose an agent with preference functional V (strictly monotone and law invariant) and initial wealth w_0. Consider the optimization problem

$$\sup\left\{V(X), \mathbb{E}_{\mathbb{Q}}(X) = w_0, X \in L_+^\infty\right\}, \tag{3}$$

where $\mathbb{Q}$ is the pricing measure and let the interest rate be zero. Further, assume that $\psi := \frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}$ has continuous distribution function F_ψ.

Set

$$\mathcal{A} := \{x : (0,1) \to \mathbb{R}_+, x \text{ is increasing and right continuous}\},$$

and define $v(x) := V(x(U))$ where U is a uniform distribution on $(0,1)$. Note that $V(X) = v(F_X^{-1})$. Consider now X of the form

$$X = F_X^{-1}(1 - F_\psi(\psi)) = x(1 - F_\psi(\psi)), x \in \mathcal{A}. \tag{4}$$

Then the optimisation problem (3) is equivalent to

$$\sup\left\{v(x), x \in \mathcal{A}, x \text{ bounded}, \int_0^1 F_\psi^{-1}(1-t)x(t)\mathrm{d}t = w_0\right\}. \tag{5}$$

The condition (4) is not a restriction. In fact the solution to the optimal investment has to be in the set of efficient payoffs.

Theorem 2 *Given two random variables X, Y we have*

$$\mathbb{E}(F_X^{-1}(1-U)F_Y^{-1}(U)) \le \mathbb{E}(XY) \le \mathbb{E}(F_X^{-1}(U)F_Y^{-1}(U)),$$

where U is a uniform distribution on $(0,1)$.

So

$$\mathbb{E}_\mathbb{Q}(X) = \mathbb{E}\left(\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}X\right) = \mathbb{E}(\psi X) = \mathbb{E}(F_\psi^{-1}(1-U)F_X^{-1}(U)).$$

Proof. By the formula of Hoeffding (see Lemma 2 in Lehman (1966))

$$\begin{aligned}\mathrm{Cov}\,(X,Y) &= \mathbb{E}\,(XY) - \mathbb{E}\,(X)\mathbb{E}(Y)\\ &= \int_\mathbb{R}\int_\mathbb{R}(F_{X,Y}(x,y) - F_X(x)F_Y(y))\,\mathrm{d}x\mathrm{d}y.\end{aligned}$$

So, the minimum of $\mathbb{E}\,(XY)$, for fixed F_X and F_Y, is obtained when $F_{X,Y}$ is minimum and this minimum is given by the Fréchet (1935) lower bound for $F_{X,Y}$ fixed F_X and F_Y :

$$\min_{F_X(\cdot)=g(\cdot),F_Y(\cdot)=h(\cdot)} F_{X,Y}(x,y) = \max(g(x) + h(y) - 1, 0),$$

and this bound is reached if we take

$$(X,Y) = (F_X^{-1}(1-U), F_Y^{-1}(U)).$$

This is the approach in Bernard et al. (2014a) to prove the result. Another way of proving it is by using the Hardy-Littlewood inequalities directly (see for instance Theorem A.24 in Föllmer and Schied (2011)). ■

Note that if Y is continuous, we can choose $U = F_Y(Y)$ and we can write the random variable

$$\bar{X} := F_X^{-1}(1-U) = F_X^{-1}(1-F_Y(Y)) = \bar{x}(1-F_Y(Y)), \bar{x} \in \mathcal{A}.$$

Note that we have solved the problem

$$\min \left\{ \mathbb{E}_{\mathbb{Q}}(X) : X \sim F \right\},$$

and its solution is given by $X = F^{-1}(1-F_\psi(\psi)) = F^{-1}\left(1 - F_{\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}}\left(\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right)\right)$. Hence we have the following proposition.

Proposition 4 *The optimal payoff w.r.t. a law invariant and monotone functional $V(X)$ and initial wealth w_0, is the efficient payoff with distribution function F that satisfies*

$$F^{-1} = \arg\max_{x \in \mathcal{I}} V(x(U)),$$

where $\mathcal{I} = \left\{ x : (0,1) \to \mathbb{R}_+, x \text{ increasing, right continuous and bounded, } \int_0^1 F_\psi^{-1}(1-t)x(t)\mathrm{d}t = w_0 \right\}$ and U is a uniform distribution on $(0,1)$.

It is interesting to notice that we have not assumed any additional condition on the preference functional except the monotonicity and the law invariance. Then we cannot in general guarantee the existence of the solution to the problem (3). In the case that

$$v(x) = \int_0^1 h(1-t)u(x(t))\mathrm{d}t,$$

where u is a utility function. We also have the following theorem:

Theorem 3 *(Carlier-Dana 2011) The optimal payoff is an efficient payoff with an inverse distribution function F^{-1} that is strictly decreasing iff F_ψ^{-1}/h is strictly increasing. If F_ψ^{-1}/h is not increasing there are ranges of values of the pricing density for which F^{-1} is constant. If F_ψ^{-1}/h is decreasing then F^{-1} is constant.*

Let us stress that the problem

$$\min \left\{ \mathbb{E}_{\mathbb{Q}}(X) : X \sim F \right\},$$

is exactly what Dybvig considered. That is, for a given distribution of the payoffs; what is the cheapest one? This payoff is the efficient payoff that we defined in the previous section. We have seen that they have the form

$$X = F^{-1}\left(1 - F_{\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}}\left(\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right)\right).$$

Theorem 4 *A payoff X is efficient iff it is a decreasing function of $\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}$.*

Proof. If X is efficient then $X = h\left(\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right)$ with $h = F^{-1}\left(1 - F_{\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}}(\cdot)\right)$ that is decreasing, on the other hand if $X = h\left(\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right)$ with h decreasing then

$$\begin{aligned} F_X(x) &= 1 - \mathbb{P}(X > x) = 1 - \mathbb{P}\left(h\left(\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right) > x\right) \\ &= 1 - \mathbb{P}\left(\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} < h^{-1}(x)\right) = 1 - F_{\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}}\left(h^{-1}(x)\right), \end{aligned}$$

so

$$F_X(h(y)) = 1 - F_{\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}}\left(h^{-1}(h(y))\right) = 1 - F_{\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}}(y)$$

and

$$X = F_X^{-1}\left(1 - F_{\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}}\left(\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right)\right).$$

■

In the following examples, that can be found in Bernard et al. (2014a), we illustrate the efficiency or not of the payoff of certain derivatives and the case they are not, we find their corresponding efficient payoff.

Example 3 *Consider the Black-Scholes market model,* $\mathrm{d}S_t = S_t(\mu \mathrm{d}t + \sigma \mathrm{d}W_t)$ *and*

$$\mathrm{d}B_t = rB_t\mathrm{d}t.$$

Then

$$\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} = \exp\left\{\frac{r-\mu}{\sigma}W_T - \frac{1}{2}\left(\frac{r-\mu}{\sigma}\right)^2 T\right\},$$

and

$$S_T = S_0 \exp\left\{\left(\mu - \frac{1}{2}\sigma^2\right)T + \sigma W_T\right\}.$$

Hence

$$\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} = CS_T^{\frac{r-\mu}{\sigma^2}},$$

where C is a constant that depends on T. Then if we assume a bullish market: $\mu > r$, $\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}$ is a decreasing function of S_T. So, any efficient payoff has to be an increasing function of S_T. In this context, the payoffs

$$X_1 = (K - S_T)_+,\ X_2 = K - S_T$$

are not efficient since they are decreasing functions of S_T. Now

$$\begin{aligned}\log \frac{1}{S_T} &\overset{d}{=} -\log S_0 - \left(\mu - \frac{1}{2}\sigma^2\right) T + \sigma W_T \\ &= \log S_T - 2\log S_0 - 2\left(\mu - \frac{1}{2}\sigma^2\right) T\ .\end{aligned}$$

That is

$$S_T \overset{d}{=} \frac{c}{S_T}$$

with $c = S_0^2 e^{\left(2\mu - \sigma^2\right)T}$. As a consequence the corresponding efficient payoffs of a put option and a short forward are respectively,

$$\bar{X}_1 = \left(K - \frac{c}{S_T}\right)_+ = \frac{K}{S_T}\left(S_T - \frac{c}{K}\right)_+,\ \bar{X}_2 = K - \frac{c}{S_T}.$$

and the corresponding prices of the original and efficient payoffs are:

Short forward contract: $Ke^{-rT} - S_0$; *efficient:* $Ke^{-rT} - S_0 e^{(\mu - r)T}$

Put option : $Ke^{-rT}\Phi(d_-) - S_0\Phi(d_+)$;

Efficient:

$$Ke^{-rT}\Phi\left(d_- - \frac{2(\mu - r)\sqrt{T}}{\sigma}\right) - S_0 e^{(\mu - r)T}\Phi\left(d_+ - \frac{2(\mu - r)\sqrt{T}}{\sigma}\right),$$

with $d_\pm := \dfrac{\log \frac{K}{S_0} - (r \pm \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$.

Note that efficient prices depend on μ, so their estimation can be difficult.

Example 4 *Consider the path-dependent payoff*

$$X_3 = \left(e^{\frac{1}{T}\int_0^T \log(S_t)\mathrm{d}t} - K\right)_+.$$

It can be shown that, under a Black-Scholes model, the efficient payoff is

$$\bar{X}_3 = c\left(S_T^{1/\sqrt{3}} - \frac{K}{c}\right)_+, \ c = S_0^{1-\frac{1}{\sqrt{3}}} e^{\left(\frac{1}{2}-\frac{1}{\sqrt{3}}\right)(\mu-\frac{1}{2}\sigma^2)T}.$$

This is in agreement with Theorem 1: path dependent options have inefficient payoffs if $\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} = CS_T^{\frac{r-\mu}{\sigma^2}}$. *However if we a assume that the stock S evolves as*

$$\mathrm{d}S_t = S_t\left(\mu_t \mathrm{d}t + \sigma_t \mathrm{d}W_t\right),$$

and the savings bank account as

$$\mathrm{d}B_t = r_t B_t \mathrm{d}t,$$

with μ_t, σ_t, r_t deterministic and càdlàg, then

$$\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} = \exp\left\{\int_0^T \frac{r_t-\mu_t}{\sigma_t}\mathrm{d}W_t - \frac{1}{2}\int_0^T \left(\frac{r_t-\mu_t}{\sigma_t}\right)^2 \mathrm{d}t\right\},$$

so

$$\begin{aligned}\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} &= \exp\left\{\int_0^T \frac{r_t-\mu_t}{\sigma_t}\mathrm{d}W_t - \frac{1}{2}\int_0^T \left(\frac{r_t-\mu_t}{\sigma_t}\right)^2 \mathrm{d}t\right\} \\ &= \exp\left\{\int_0^T \frac{r_t-\mu_t}{\sigma_t^2}\frac{\mathrm{d}S_t}{S_t} - \frac{1}{2}\int_0^T \frac{r_t^2-\mu_t^2}{\sigma_t^2}\mathrm{d}t\right\} \\ &= C_T \exp\left\{\int_0^T \frac{r_t-\mu_t}{\sigma_t^2}\frac{\mathrm{d}S_t}{S_t}\right\}.\end{aligned}$$

Then, any payoff that is a decreasing function of

$$V_T = \exp\left\{\int_0^T \frac{r_t-\mu_t}{\sigma_t^2}\frac{\mathrm{d}S_t}{S_t}\right\}$$

will be efficient. Consider for instance a put option

$$(K-S_T)_+$$

$$\log S_T \sim N\left(\int_0^T \left(\mu_t - \frac{1}{2}\sigma_t^2\right)\mathrm{d}t, \int_0^T \sigma_t^2 \mathrm{d}t\right) := N\left(m_T, v_T^2\right)$$

and
$$\log V_T \sim N\left(\int_0^T \frac{\mu_t\left(r_t-\mu_t\right)}{\sigma_t^2}\mathrm{d}t, \int_0^T\left(\frac{r_t-\mu_t}{\sigma_t}\right)^2\mathrm{d}t\right) := N\left(a_T, b_T^2\right),$$
in such a way that an optimal payoff is
$$\left(K - V_T^{\frac{v_T}{b_T}} e^{\frac{v_T}{b_T}(m_T-a_T)}\right)_+,$$
since
$$V_T^{\frac{v_T}{b_T}} e^{\frac{v_T}{b_T}(m_T-a_T)} \overset{d}{=} S_T$$
and $K - V_T^{\frac{v_T}{b_T}} e^{\frac{v_T}{b_T}(m_T-a_T)}$ *is a decreasing function of* V_T. *In this situation a path dependent option is better than a vanilla option! contrarily to what the title of Dybvig (1988b) suggests, as explained in Section 2.1.*

4 Efficient payoffs in a dynamic setting

Here we follow Becherer (2001). Consider the set of strictly positive self-financing portfolios with initial value one:
$$\mathcal{N} := \left\{N > 0 : N_t = 1 + \int_0^t \varphi_u \mathrm{d}S_u\right\}.$$
$N \in \mathcal{N}$ is said to be the *numeraire portfolio* (NP) if, for all $V \in \mathcal{N}$, V/N is a supermartingale (w.r.t. the probability measure $\mathbb{P}$). We say that an element of $\mathcal{N}$ is the *growth-optimal portfolio* (GOP) if it solves the maximization problem
$$u := \sup_{V\in\mathcal{N}} \mathbb{E}\left(\log V_T\right).$$
We have the following important results.

Theorem 5 *Assume* $u < \infty$. *Then the numeraire portfolio and the growth-optimal portfolio are the same.*

Proof. See Proposition 4.3 in Becherer (2001) . ■

Theorem 6 *If the market is complete the numeraire portfolio is given by*
$$N_t = \mathbb{E}\left(\left.\frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\mathbb{Q}}\right| \mathcal{F}_t\right), 0 \le t \le T,$$
with $\mathcal{F}_t := \sigma(S_u, 0 \le u \le t)$.

Proof. See Example 1 in Becherer (2001). ■

In the Black-Scholes model

$$\begin{aligned} N_t &= \exp\left\{-\frac{r-\mu}{\sigma}W_t + \frac{1}{2}\left(\frac{r-\mu}{\sigma}\right)^2 t\right\} \\ &= \exp\left\{-\frac{r-\mu}{\sigma}\tilde{W}_t - \frac{1}{2}\left(\frac{r-\mu}{\sigma}\right)^2 t\right\}, \end{aligned}$$

where $\tilde{W}$ is $\mathbb{Q}$-Brownian motion. We have seen that any efficient payoff can be written as a decreasing function of $\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}$ and consequently as an increasing function of the final value of the numeraire portfolio N_T, say $\tilde{X} = h(N_T)$.

Then the (discounted) value of the replicating portfolio is given by

$$\tilde{V}_t = \mathbb{E}\left(\tilde{X}|\mathcal{F}_t\right) = \mathbb{E}\left(h(N_T)|\mathcal{F}_t\right) = \mathbb{E}\left(h\left(\frac{N_T}{N_t}x\right)\right)\Big|_{x=N_t} =: g(t, N_t),$$

from which (under smoothness assumptions on g), we get

$$\mathrm{d}\tilde{V}_t = \partial_2 g(t, N_t)\mathrm{d}N_t.$$

Hence V is a *locally optimal portfolio* in the sense that it has the largest discounted drift given a diffusion coefficient (Platten (2002)) and

$$\frac{\partial_2 g(t, N_t)N_t}{\tilde{V}_t}$$

can be interpreted as a *risk aversion coefficient* (Platten (2002)).

If the market is incomplete, one uses the numeraire portfolio to get arbitrage free prices of a payoff X by

$$\mathbb{E}\left(\frac{X}{N_T}\right).$$

The latter is referred as the *benchmark approach* where the numeraire is chosen in such a way that the corresponding risk-neutral measure coincides with the historical one (see Platen and Heath (2006)). In this case a payoff X is efficient iff X is an increasing function of N_T as above, but if we use a pricing measure $\mathbb{Q}$ a payoff X will be efficient iff it is a decreasing function of $\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}$. In the continuous case both approaches coincide if we use the minimal martingale measure (see Schweizer (1999)).

If we consider an exponential Lévy model for S :

$$\mathrm{d}S_t = S_{t-}\mathrm{d}Z_t, S_0 > 0,$$

where Z is a Lévy process with characteristics (d, c^2, ν) (with jumps strictly greater than -1) and the pricing measure $\mathbb{Q}$ is such that Z is a $\mathbb{Q}$-Lévy process it can be seen (see Corcuera et al. (2006)) that

$$\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} = aS_T^b e^{V_T}, \ a > 0, b \in \mathbb{R} \text{ (that depends on } \mathbb{Q}),$$

and

$$V_T = \int_{-\infty}^{\infty} (\log H(x) - b\log(1+x))\, \tilde{M}((0,t], \mathrm{d}x).$$

with $H(x) = \frac{\mathrm{d}\tilde{\nu}}{\mathrm{d}\nu}(x)$ and where $\tilde{M}((0,t], \mathrm{d}x)$ is the compensated Poisson random measure associated with Z. Tilde indicates the characteristics w.r.t. $\mathbb{Q}$ (see Corcuera et al. (2006) for more details).

In such cases an efficient payoff is an increasing function of $S_T^b e^{V_T}$ and only in the case that $V_T \equiv 0$ efficient payoffs are a monotone function of S_T. It corresponds to the case that $\mathbb{Q}$ is the Esscher measure, see Von Hammerstein et al. (2014).

The benchmark approach coincides with the pricing measure approach when

$$H(x) = \frac{1}{1-bx}, \quad \text{and}$$
$$c^2 b + d - r + b\int_{-\infty}^{\infty} \frac{x^2}{1-bx}\mathrm{d}\nu(x) = 0,$$

since in this case the optimal terminal wealth corresponding to the log-utility can be replicated by using stocks and bonds (see Corcuera et al. (2006), Example 4.1).

It will be also interesting to include optimal consumption problem in this context, as for example it is done in Fajardo (2003).

5 Conditional efficient payoffs

Reducing the importance of a payoff to its law is quite controversial. For instance when one buys a Call option he/she is buying a right to buy a stock at a certain price and this is lost if he/she takes another payoff with the same law but with different values. There are many other examples that suggest that, if there is no perfect correlation, the investor would like a fixed dependency w.r.t. some special payoff.

This approach was introduced by Takahashi-Yamamoto (2013). See also Bernard et al. (2014b) and Bernard et al. (2015a).

Suppose a benchmark payoff Y is given, and that the investor wishes to invest in another payoff X with a joint distribution (X,Y) fixed. In other words two payoffs X and Γ are equivalent if $(X,Y)\sim(\Gamma,Y)$, or, equivalently, if $X|Y=y\sim\Gamma|Y=y$ for all y. So, one wants to solve the problem

$$\min_{(X,Y)\sim F_{X,Y}} \mathbb{E}_{\mathbb{Q}}(X). \tag{6}$$

Firstly, given Z, we can find a function $g(Z,Y)$ such that $(X,Y)\sim(g(Z,Y),Y)$. In fact, if we assume that $F_{Z|Y}(z|y)$ is continuous, then, conditionally on $Y=y$, $F_{Z|Y}(Z|y)\sim U(0,1)$ (note that the random variable $F_{Z|Y}(Z|Y)$ is, therefore, independent of Y) and $F^{-1}_{X|Y}(F_{Z|Y}(Z|y)|y)$ (where $F^{-1}_{X|Y}(\cdot|y)$ is the pseudo-inverse of $F_{X|Y}(\cdot|y)$) will be a random variable such that conditionally on $Y=y$ has the same law as X, then

$$(F^{-1}_{X|Y}(F_{Z|Y}(Z|Y)|Y),Y)\sim(X,Y)$$

and the function we are looking for is $g(z,y)=F^{-1}_{X|Y}(F_{Z|Y}(z|y)|y)$.

Now we can solve the optimization problem (6). We know that

$$\mathbb{E}_{\mathbb{Q}}(X)=\mathbb{E}\left(\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}X\right),$$

so, since the law of X and $\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}$ are fixed, if $X\sim h\left(\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right)$ for some decreasing function h, we reach the lower bound for $\mathbb{E}\left(\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}X\right)$. But we have to fix the conditional law, that is, we need that $(X,Y)\sim\left(h\left(\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right),Y\right)$. Then, according to the previous step, we can take $h\left(\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right)=g\left(\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}},Y\right)$.

In fact we are solving the conditional problem: in the set of random variables X such that $X|Y=y$ is fixed, we solve the problem

$$\min_{X|Y=y\sim F_{X|Y}} \mathbb{E}\left(\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}X\,\middle|\,Y=y\right)$$

and the solution is $F^{-1}_{X|Y}\left(F_{\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}|Y}\left(\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\middle|\,y\right)\middle|\,y\right)=g\left(\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}},y\right)$. Consequently

$$\min_{X|Y=y\sim F_{X|Y}} \mathbb{E}\left(\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}X\right)=\mathbb{E}\left(\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\Gamma\right),$$

with $\Gamma=g\left(\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}},Y\right)$. Three elements interact in the expression: the conditional law of X given Y, the price state density $\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}$ and Y.

An additional reason to consider conditional efficient payoffs could be the existence of privileged information about a certain payoff Y. This might be object for future research.

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