Financial Methods: A Quantitative Approach

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Abstract: The study analyses finite difference methods and stochastic volatility for option pricing model till Asian and Barrier options. Simulated result is presented with VBA code. The interest rate models is analyzed in arbitrage setting e simulated environment by using an affine term structure and the drift condition in combination with inflation model by measuring the liquidity and risk premium by presenting an efficient Monte Carlo simulator. Structural Model is presented in single time maturity and default barrier in first passage model. The intensity model is also faced by analyzing the liquidity e risk premium with copula approaches as well

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Stochastic Differential Equations

In finance is made a large use of Wiener process and geometric Brown process, the name came from George Brown in the 1827 that noted that the volatility of a small particle suspended in a liquid increases with the time, Wiener gave a mathematical formal assumption on the phenomena from this the term Wiener process. The geometric Brown process is used in finance to indicate a formal assumption for the dynamic of the prices that does not permit to assume negative value, formally we have:

$$\frac{dS(t)}{S(t)} = \mu(t)dt + \sigma dWs$$

Where $\mu(t)$ denotes the drift of the distribution and it is the average in the dt, σ denotes the volatility of the distribution and dWs denotes a Wiener process such that it may be decomposed by the following:

$$dWs = N[0,1]\sqrt{dt}$$

We may assume the following for the Wiener process:

$$E\left[\oint_t^T dWs^2\right] = T - t \implies dWs^2 \sim dt$$

This means that a Wiener process is a forward process, the uncertainty is to the end of the process in $T + dt$. From this we may obtain explanation of Ito's lemma by using Taylor series, if we take a function of S as F(S) we may write Ito's lemma in the following way:

$$dF(S) = \frac{\partial F}{\partial t}dt + \frac{\partial F}{\partial S}dS + \frac{1}{2}\left(\frac{\partial^2 F}{\partial^2 S}dS^2 + \frac{\partial^2 F}{\partial^2 t}dt^2 + 2\frac{\partial^2 F}{\partial S \partial t}dt\, dS\right) + Q$$

We may note that:

$$dS^2 = \mu^2 S^2 dt^2 + 2\mu\sigma S^2 dt^{3/2} + \sigma^2 S^2 dt$$

$$dS\, dt = \mu S\, dt^2 + \sigma S\, dt^{3/2}$$

From this we obtain as *dt* tends to zero:

$$dF(S) = \frac{\partial F}{\partial t}dt + \frac{\partial F}{\partial S}dS + \frac{1}{2}\left(\frac{\partial^2 F}{\partial^2 S}S^2\sigma^2 dt\right)$$

By substituting *d*S we obtain Ito's lemma:

$$dF(S) = \left(\frac{\partial F}{\partial t} + \mu\frac{\partial F}{\partial S}S + \frac{1}{2}\frac{\partial^2 F}{\partial^2 S}S^2\sigma^2\right)dt + \sigma\frac{\partial F}{\partial S}SdZ$$

We may see now as to obtain the expected value of a normal distribution as such we have the following:

$$\int f(z) = \int z f(z)$$

As such we have the following:

$$\int \frac{1}{\sigma\sqrt{2\pi}} z e^{-\frac{1}{2}z^2}$$

This may be rewritten by:

$$\int \frac{1}{\sigma} z \frac{1}{2} z^2 \sim \frac{1}{2} \sigma^2$$

From this we may obtain explanation for Ito's lemma, if we take a function of S as F(S) we may write Ito's lemma in the following way:

$$dF(S) = \left(\frac{\partial F}{\partial t} + \mu \frac{\partial F}{\partial S} S + \frac{1}{2} \frac{\partial^2 F}{\partial^2 S} S^2 \sigma^2 \right) dt + \sigma \frac{\partial F}{\partial S} S dZ$$

Where:

$$\sigma dZ = N[\frac{z - \mu}{\sigma\sqrt{dt}} ; \sigma^2 dt]$$

As result:

$$E[\sigma dZ] = 0$$

Because:

$$\int f(z) = \int \frac{z - \mu}{\sigma\sqrt{dt}} f(z) = 0$$

Now we may analyze the following parabolic problem:

$$\frac{\partial F}{\partial t} + \mu \frac{\partial F}{\partial S} S + \frac{1}{2} \frac{\partial^2 F}{\partial^2 S} S^2 \sigma^2 = 0$$

Subject to the following constraint:

$$F(S(T)) = F(S)$$

The solution it is easy to solve, because if we take Ito's lemma and we take the expectation we obtain that the solution to the parabolic problem is given by:

$$F(S(T)) = E[F(S)]$$

As result we may rewrite Ito's lemma in the following form:

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$$dF(S) = \left(\frac{\partial F}{\partial t} + \mu \frac{\partial F}{\partial S} S - \frac{1}{2} \frac{\partial^2 F}{\partial^2 S} S^2 \sigma^2 \right) dt - \sigma \frac{\partial F}{\partial S} SdZ$$

We may see now the solution of geometric Brown process:

$$F(S) = \ln S(t) \quad dF(S) = \ln(dS(t))$$

$$dF(S) = \frac{1}{S} dS + \frac{1}{2}\left(-\frac{1}{S^2} \right) dS^2$$

Because:

$$\frac{\partial F}{\partial t} = 0$$

As such we have:

$$dF(S) = \left(\mu(t) - \frac{1}{2} \sigma^2 \right) dt + \sigma dWs$$

$$F(S(T)) = F(S(t)) + dF(S)$$

$$\ln S(T) = \ln S(t) + \left(\mu(t) - \frac{1}{2} \sigma^2 \right) dt + \sigma dWs$$

$$S(T) = S(t)e^{\left(\mu(t) - \frac{1}{2}\sigma^2 \right) dt + \sigma dWs}$$

Interest Rate Models

The price of a zero coupon bond P(T) is given by the following:

$$P(T) = E\left[e^{-\int_t^T r(t)dt} \right]$$

Where *r(t)* denotes the short rate that is given by the following stochastic differential equation:

$$dr(t) = \mu(t)dt + \sigma_r \, dWr$$

$$r(t) = R(t)$$

As result by applying Ito's lemma we have the following for the price of a zero coupon bond:

$$P(T) = e^{-(r(t)T + \mu(t)T^2)}$$

From this we may see that the internal compounded *R(T)* interest rate is given by the following:

$$R(T) = r(t) + \mu(t)T$$

We may investigate the drift by using the forward process as result we have the following:

$$Forward = -\frac{\partial LogP(T)}{\partial T}$$

By applying Ito's lemma we have the following process:

$$dLogP(T) = \left(r(t) - \frac{1}{2} \sigma_r^2 T \right)dt - \sigma_r T dWr$$

As such we have:

$$-\frac{\partial dLogP(T)}{\partial T} = r(t) + \sigma_r^2 dt + \sigma_r dWr$$

As result we have:

$$R(T) = r(t) + \sigma_r^2 T$$

That is the drift condition. Now if we build a portfolio of default free bonds by shorting the bonds overvalued and acquiring the bonds undervalued we obtain a relation rule that the yields curve must respect given by the following:

$$\gamma = \frac{R(T) - r(t)}{\sigma_r T}$$

From this we may derive that in absence of arbitrage opportunities we have by assumption the following:

$$\gamma = \sigma_r$$

This is the risk premium requested by the markets, and it is a function of the risk associated with the volatility of the short rate. In absence of arbitrage opportunities the stochastic differential equation that a default free bond must satisfy is given by the following for P(T) = F(r,t):

$$\frac{\partial F}{\partial t} + (\mu - \gamma\sigma_r)\frac{\partial F}{\partial r} + \frac{1}{2}\frac{\partial^2 F}{\partial^2 r}\sigma^2 - rF(r,t) = 0$$

The solution to this parabolic problem is given by the integrant factor, as such we have the following:

$$F(r,t) = E(e^{-\int_t^T r(t)\,dt} \; x \; 1)$$

Where:

$$dR(T) = (\mu - \gamma\sigma_r)dt + \sigma_r\,dWr$$

$$r(t) = R(t)$$

As result F(r,t) is the risk free as such we have the following solution:

$$R(T) = r(t) + \sigma_r^2 T$$

This is the future value of the short rate in fact if take the average of R(T) for each maturities we have that the risk free rate is given by:

$$r(T) = r(t) + \frac{1}{2}\sigma_r^2 T$$

As such we have that $\frac{1}{2}\sigma_r^2 T$ represents the liquidity risk. Because:

$$dR(t) = (\mu + \gamma\sigma_r)dt + \sigma_r\,dWr$$

As result we have without arbitrage conditions the following:

$$R(T) = r(t) + \sigma_r^2 T + \sigma_r^2 T$$

Where $\sigma_r^2 T$ denotes the risk premium, so in the yield curve it is possible to have liquidity risk and risk premium. If we take in consideration the inflation we have the following;

$$\sigma_I^2 T = \sigma_r^2 T + \sigma_r^2 T$$

Where $\sigma_I^2 T$ is the expected inflation for each maturities, where σ_I^2 is the variance of the inflation, that is the drift condition such that we have:

$$r(T) = r(t) + \sigma_I^2 T$$

As result we have that:

$$Average\,(\sigma_I^2 T) = \sigma_r^2 T$$

The compounded inflation is equal to $\sigma_r^2 T$, so there is an equilibrium relation between the volatility of inflation and the volatility of the short rate. Now it is interesting to analyse the inflation, as result we have the following for the inflation swap as expectation;

$$Inflation \ Swap = \sigma_I^2 T + \frac{1}{2}\sigma_I^2 T$$

and the following for the inflation simulated:

$$Inflation \ Simulated = \sigma_I^2 T + \sigma_I^2 T$$

We may have an inflation, decreasing, or stable, i.e. there are different kind of equilibrium with the short rate, and it is possible to have just liquidity risk or risk premium as well. Now we may assume the following affine form that F(r,t) must satisfy:

$$F(r,t) = A(t,T)e^{-B(t,T)r(t)}$$

Where:

$$dr(t) = (b - ar(t))dt + \sigma_r \, dWr$$

$$r(t) = R(t)$$

We may note that:

$$\mu(t) = b - ar(t)$$

At this point we may solve the stochastic differential equation:

$$A_t - rAB_t - ABb + ABar + \frac{1}{2} AB^2\sigma^2 - rA = 0$$

Where:

$$A(T,T) = 1$$

$$B(T,T) = 0$$

$$A_t - ABb + \frac{1}{2} AB^2\sigma^2 = 0$$

$$B_t + Ba - 1 = 0$$

From this we obtain:

$$B(t,T) = \frac{1 - e^{-a \, (T-t)}}{a}$$

$$A(t,T) = exp \int_t^T Bb - \frac{1}{2} B^2\sigma^2 = exp \frac{(B(T,t) - T + t)(ab - \frac{1}{2}\sigma^2)}{a^2} - \sigma^2 \frac{B(t,T)^2}{4a}$$

We may now assume the following distribution for the short rate:

$$dr(t) = (b - ar(t))dt + \sqrt{r(t)}\sigma_r \, dWr$$

$$r(t) = R(t)$$

At this point we may solve the stochastic differential equation:

$$A_t - rAB_t - ABb + ABar + \frac{1}{2}AB^2r\sigma^2 - rA = 0$$

As such we have the following system:

$$A_t - ABb = 0$$

$$B_t - Ba - \frac{1}{2}B^2\sigma^2 + 1 = 0$$

From this we may obtain a simplified solution:

$$B(t,T) = \frac{2\left(e^{\gamma(T-t)} - 1\right)}{(\gamma + a)\left(e^{\gamma(T-t)} - 1\right) + 2\gamma}$$

$$A(t,T) = \left(\frac{2\gamma \, e^{\frac{(\gamma+a)(T-t)}{2}}}{(\gamma + a)\left(e^{\gamma(T-t)} - 1\right) + 2\gamma}\right)^{\frac{2ab}{\sigma^2}}$$

$$\gamma = a + 2\sigma$$

We may note that:

$$R(T) = -\frac{lnA(t,T) - B(t,T)r(t)}{T-t}$$

We may note that if we take the expected value of the distribution we obtain directly the risk free rate, as such we have the following pricing formula:

$$B(t,T) = \frac{2\left(e^{\gamma(T-t)} - 1\right)}{(\gamma + a)\left(e^{\gamma(T-t)} - 1\right) + 2\gamma}$$

$$A(t,T) = \left(\frac{2\gamma \, e^{\frac{(\gamma+a)(T-t)}{2}}}{(\gamma + a)\left(e^{\gamma(T-t)} - 1\right) + 2\gamma}\right)^{\frac{2b}{\sigma^2}}$$

$$\gamma = \sqrt{a^2 + 2\sigma^2}$$

On this we have to add liquidity and risk premium to obtain the yield curve without arbitrage condition, the simplified solution is the forward of this solution. Indeed, we do not know how much risk premium there is embedded in the yield curve, for instance if there is just liquidity premium or risk premium as well, or if the yield curve is the risk free such that the forward rate is the forward. Indeed, the expected value of the solution does not calibrate the yields curve if there is risk premium or a different forward structure with respect the assumptions of the model, instead the simplified solution permits to calibrate the yields curve where the forward rate is the forward such that does not consider the liquidity and risk premium, it is just the case of risk free rate. The interest rate models developed in the literature are based on arbitrage setting such that the yield curve is based on the expectation of the market on the future short rates. To assume that the expectation is the future does not consider the uncertainty associated with the expectation that may be reassumed in liquidity and risk premium on the average of the short rate. As result we may assume the following:

$$\frac{dP(T)}{P(T)} = (r(t) - \sigma dWr)dt$$

The expectation of σdWr is equal to $\frac{1}{2}\sigma^2 dt$, but if we simulate $\frac{dP(T)}{P(T)}$ cannot assume negative value because is a geometric Brown process, as result we have that $-\sigma dWr$ simulated is equal to $-\frac{-2}{2}\sigma^2 dt$ As result in absence of arbitrage opportunities we have that the yield curve is given by the following;

$$r(T) = r(t) + \sigma_r^2 T$$

Where r(T) is the value of the future short rate and represents the yield curve with the liquidity risk, in fact if we take the average of r(T) for each maturities we have that the risk free rate is equal to:

$$r(T) = r(t) + \frac{1}{2}\sigma_r^2 T$$

For a yield curve decreasing we have:

$$r(T) = Ass(\,r(t) - \sigma_r^2 T)$$

We are assuming that the interest rate cannot assume negative value, we may remove this hypothesis in the case we have deflation. Now by taking the average of r(T) we may obtain the risk free rate, on this we may add liquidity risk to obtain the yield curve. Again if we take the yield curve as average we have:

$$\frac{dP(T)}{P(T)} = (r(T) - \sigma dWr)dt$$

As result we have the following:

$$r(T) = r(t) + \sigma_r^2 T + \sigma_r^2 T$$

Where $\sigma_r^2 T$ denotes the risk premium, so in the yield curve it is possible to have liquidity risk and risk premium. Now it is interesting to note that the variation of short rate may be simulated by the following process that is in the midline between a Martingale and a geometric Brown process that does not permit to the short rate to assume negative value, as such we have the following:

$$dr(t) = \frac{k}{2}\big(R - r(t)\big)dt + \sqrt{r(t)}\sigma_r dWr$$

Where R denotes the average of the short rate and k is given by $\frac{\sigma_I^2}{\text{Ass}(R-r(t))}$, from this it is possible to derive that the risk premium is given approximate by $\frac{\sigma_r^2 T}{3}$, as such the liquidity risk is given by $\frac{\sigma_r^2 T}{6}$. Furthermore, the short rate may be simulated by using the following modification of the diffusion process $\sqrt{r(t)}\sigma_r dWr$, as such we have the following;

$$dr(T) = \frac{k}{2}\big(R - r(t)\big)T + 100\, r(t)\ \frac{\sigma_r^2 T}{2}$$

We may note that R(T) = r(t) + dr(T) converges to the same result of the $r(T) = r(t) + \sigma_r^2 T$. By taking the average we obtain approximate the same risk free rate for rational value of parameters. Again in the case we have a decreasing curve such that r(t) > R we may obtain the yield curve by the following:

$$dr(T) = \frac{k}{2}\big(R - r(t)\big)T - 100\, r(t)\ \frac{\sigma_r^2 T}{2}$$

We may note that the short rate cannot assume negative value. We may see an example of equilibrium between interest rate, inflation and simulation of the short rate:

Vol Inflation	5,94%
Vol Rate	4,20%
Average	2,00%
Rate	1,25%
Parameter	0,47

Maturity (Years)	Rate MKT	Short Rate Down	Risk Free Down	Swap UP & Short Rate UP	Risk Free UP	STD Inflation	Inflation	Short Rate CIR
0	1,25%	1,25%	1,25%	1,25%	1,25%	0,00%	3,53%	1,25%
1	1,60%	1,07%	1,16%	1,43%	1,34%	0,35%	3,18%	1,54%
2	1,96%	0,90%	1,07%	1,60%	1,43%	0,71%	2,82%	1,74%
3	2,31%	0,72%	0,99%	1,78%	1,51%	1,06%	2,47%	1,89%
4	2,66%	0,54%	0,90%	1,96%	1,60%	1,41%	2,12%	2,02%
5	3,01%	0,37%	0,81%	2,13%	1,69%	1,76%	1,76%	2,12%
6	3,37%	0,19%	0,72%	2,31%	1,78%	2,12%	1,41%	2,20%
7	3,72%	0,02%	0,63%	2,48%	1,87%	2,47%	1,06%	2,28%
8	4,07%	0,16%	0,58%	2,66%	1,96%	2,82%	0,71%	2,34%
9	4,43%	0,34%	0,56%	2,84%	2,04%	3,18%	0,35%	2,39%
10	4,78%	0,51%	0,55%	3,01%	2,13%	3,53%	0,00%	2,44%
	Average	0,55%	Average	2,13%	Comp. Inflation	1,76%	1,76%	2,02%

Now we may see an example of simulation of the short rate decreasing:

Vol Inflation	5,94%			
Vol Rate	4,20%			
Average	0,25%			
Rate	1,25%			
Parameter	0,35			

Maturity (Years)	Rate MKT	Short Rate Down	Risk Free Down	Swap UP & Short Rate UP	Risk Free UP	STD Inflation	Inflation	Short Rate CIR
0	1,25%	1,25%	1,25%	1,25%	1,25%	0,00%	3,53%	1,25%
1	1,60%	1,07%	1,16%	1,43%	1,34%	0,35%	3,18%	0,96%
2	1,96%	0,90%	1,07%	1,60%	1,43%	0,71%	2,82%	0,83%
3	2,31%	0,72%	0,99%	1,78%	1,51%	1,06%	2,47%	0,72%
4	2,66%	0,54%	0,90%	1,96%	1,60%	1,41%	2,12%	0,66%
5	3,01%	0,37%	0,81%	2,13%	1,69%	1,76%	1,76%	0,60%
6	3,37%	0,19%	0,72%	2,31%	1,78%	2,12%	1,41%	0,56%
7	3,72%	0,02%	0,63%	2,48%	1,87%	2,47%	1,06%	0,51%
8	4,07%	0,16%	0,58%	2,66%	1,96%	2,82%	0,71%	0,52%
9	4,43%	0,34%	0,56%	2,84%	2,04%	3,18%	0,35%	0,41%
10	4,78%	0,51%	0,55%	3,01%	2,13%	3,53%	0,00%	0,60%
	Average	0,55%	Average	2,13%	Comp. Inflation	1,76%	1,76%	0,69%

As we may see from the examples the two approaches converge to the same result. We may note now that there are two factors in the determination of the yields curve, the short rate and the inflation in combination with the uncertainty, as such we may assume the following distributions:

$$dr(t) = -a\, r(t) \; + \; \sigma_r dWr$$

$$d\,\sigma_r^2 T \; = \; -a\,\sigma_r^2 T \; + \; \sigma_r dW\sigma$$

As such we have the following simplified pricing formula for a default free zero coupon bond:

$$P(t,T) = Exp\left[-\frac{1-e^{-a\,(T-t)}}{a}\, r(t) - \frac{1-e^{-a\,(T-t)}}{a}\,\sigma_r^2 T + G(t,T)\right]$$

Where:

$$G(t,T) = \frac{\sigma_r^2}{a^2}\left(T-t\;+\;\frac{1-e^{-a\,(T-t)}}{a} - \frac{e^{-2a\,(T-t)}}{2a} - \frac{3}{2a}\right)$$
$$+\frac{\sigma_r^2}{2a^2}\left(T-t\;+\;\frac{1-e^{-a\,(T-t)}}{a} - \frac{e^{-2a\,(T-t)}}{2a} - \frac{3}{2a}\right)$$

The evolution of the stock prices may be denoted by the following stochastic continuous process:

$$\frac{dS(t)}{S(t)} = (r(t) + \delta - q)dt + \sigma_S\, dWs$$

δ may be the risk premium or the liquidity premium, depends from the equilibrium in the treasury market, instead q denotes the dividend yield, so by putting $\delta = q$ it is possible to get the prices of the stocks for the different kind of risk premiums that are in the interest rate models, by putting:

$$Price = \frac{Dividend}{\delta}\frac{1}{100}$$

Instead, the risk premium is given by:

$$\frac{1}{2}\sigma_S^2 T$$

Usually is given by the average of the rate of return computed with respect the initial value of the prices. Because if we simulate:

$$S(T) = S(t)e^{(r(t)-\frac{1}{2}\sigma^2)dt+ \sigma dWs}$$

We obtain:

$$S(T) = S(t)e^{(r(t)-\frac{1}{2}\sigma^2)T+ \sigma^2 T}$$

Because:

$$\frac{dS(t)}{S(t)} = (r(t) - \frac{1}{2}\sigma^2)dt + \frac{2}{2}\sigma^2 dt$$

As such by applying Ito's lemma and simulating we obtain the same result of expected value. Now it is possible to note that if GDP increases the dividend yield increases, but there is an equilibrium relation, as such we have:

$$\partial GDP = \partial q = \partial r(T) = \partial \sigma_r^2 T$$

$$\delta = q \implies \partial S$$

As GDP increases, dividend increases, but stocks prices will increase as well, so the dividend yield after increase will decrease, this movement is embedded in the term structure of interest rates through the equilibrium with the inflation, because the increase of the demand for product will increase the consumer prices and the inflation through the increase of GDP. In the term structure of interest rates there is embedded, through the dynamic of GDP and the equilibrium with the term structure of the inflation, the policy on the movement of the short rate as expectation of the market.

Monte Carlo Simulations

We may obtain a normal distribution by doing the standardized normal inverse of a random (RND) that produce in a stochastic way a series of number from zero to one, in notation: $N^{-1}[RND]$. With this approach we obtain a normal distribution with zero average and one volatility with zero kurtosis. The lognormal distribution is obtained by doing the exponential of the normal distribution, in notation:$e^{\mp N^{-1}[RND]}$, this produce a lognormal distribution with one volatility by applying Ito's lemma and a very high kurtosis. This is a problem because it will infer problems in every kind of simulations by giving wrong result for the tails. Indeed, it is possible to eliminate the excess kurtosis, we may construct a lognormal distribution by doing the following: round(1/N[RND]);round(1/N[RND+1]);round(1/N[RND]+2);− round(1/N[RND]); − round(1/N[RND +1]); − round(1/N[RND]+2), from this approach we obtain a kind of normal distribution with zero average and 1.2 volatility and low kurtosis. Now if we take the exponential of this distribution we obtain a lognormal distribution that keeps the same average with one volatility and zero kurtosis. We may see an example:

Rate	Vol	TIME				
1,250%	4,20%	5				
				Logormal	Simulator	
SWAP(log Mean)	Mean		Mean	Kurtosi	Kurtosi	
1,90%	1,019143		1,846807	-0,50954	-1,812744457	RND
				Vol	Vol	Vol
				0,965638	1,213716374	0,873525
	0,899313			3,536367	1,2631	0,812319
	0,86112			5,613644	1,7252	0,200988
	0,89048			3,928273	1,3682	0,615449
	0,852702			6,23264	1,8298	0,116857
	0,881825			4,358814	1,4722	0,46568
	0,904089			3,342436	1,2067	0,949057
	0,875545			4,703467	1,5483	0,374245
	0,899271			3,538136	1,2636	0,811345
	0,902613			3,401104	1,2241	0,903773
	0,884197			4,235918	1,4436	0,503571
	0,873599			4,816271	1,572	0,348147
	0,872181			4,900318	1,5893	0,329728
	0,893882			3,77198	1,3276	0,684654
	0,898992			3,549831	1,2669	0,804017
	0,900057			3,505384	1,2543	0,831768
	0,88571			4,159521	1,4254	0,528901
	0,879567			4,479449	1,4995	0,431296
	0,898005			3,591608	1,2786	0,779372
	0,865221			5,33666	1,6746	0,24599
	0,879922			4,460229	1,4952	0,436644
	0,847799			6,627317	1,8912	0,072157
	0,869392			5,0703	1,6234	0,294978

0,902817	3,392951	1,2217	0,909741
0,905551	3,285438	1,1895	0,997196
0,887608	4,065757	1,4026	0,562002
0,873976	4,794167	1,5674	0,353162
0,900133	3,50223	1,2534	0,833808
0,879262	4,496053	1,5032	0,426793
0,888843	4,006027	1,3878	0,584551
0,899972	3,508891	1,2553	0,829587
0,847083	6,687232	1,9002	0,065843
0,859875	5,700763	1,7406	0,187898
0,887491	4,071453	1,404	0,55999
0,861031	5,619822	1,7263	0,200062
0,878832	4,519494	1,5084	0,420565
0,873385	4,82881	1,5746	0,345365
0,867044	5,218448	1,6522	0,266937
0,902486	3,406209	1,2256	0,900049
0,895807	3,686583	1,3047	0,727223
0,890305	3,936531	1,3703	0,612136
0,886409	4,124728	1,417	0,54095
0,89379	3,776131	1,3287	0,682695
0,901868	3,431166	1,2329	0,881912
0,90375	3,355833	1,2107	0,938225
0,846892	6,7033	1,9026	0,064197
0,898191	3,583715	1,2764	0,783965
0,897853	3,598079	1,2804	0,775507
0,856651	5,933415	1,7806	0,155023
0,904098	3,342102	1,2066	0,949359
0,850303	6,422452	1,8598	0,094617
0,852878	6,218943	1,8276	0,11847
0,912552	3,026782	1,1075	1,298504
0,919554	2,790163	1,0261	1,952887
0,915676	2,918588	1,0711	1,503395
0,916485	2,891282	1,0617	1,570441
0,919598	2,788768	1,0256	1,959954
0,91845	2,826107	1,0389	1,781355
0,912423	3,031326	1,109	1,291506
0,91131	3,07099	1,122	1,233363
0,91578	2,915088	1,0699	1,511642
0,919641	2,787374	1,0251	1,968755
0,919062	2,806112	1,0318	1,868331
0,916485	2,891282	1,0617	1,571116
0,909173	3,148733	1,147	1,135221
0,915092	2,938502	1,0779	1,458922
0,913083	3,008074	1,1013	1,328549
0,918622	2,82046	1,0369	1,804886
0,918105	2,837434	1,0429	1,737326
0,915857	2,912466	1,069	1,517795

0,906146	3,26252	1,1825	1,0179
0,917743	2,849376	1,0471	1,695204
0,912243	3,037698	1,1111	1,281511
0,910617	3,095966	1,1301	1,199874
0,911763	3,054757	1,1167	1,256308
0,918407	2,82752	1,0394	1,776013
0,917665	2,851942	1,048	1,686654
0,917932	2,843114	1,0449	1,71744
0,917148	2,869105	1,054	1,633294
0,905917	3,271341	1,1852	1,009988
0,915418	2,927357	1,0741	1,483515
0,912183	3,039825	1,1118	1,278327
0,915676	2,918588	1,0711	1,503578
0,910249	3,109307	1,1344	1,182514
0,919675	2,786259	1,0247	1,974786
0,919157	2,803027	1,0307	1,884016
0,909224	3,146844	1,1464	1,137384
0,9182	2,834314	1,0418	1,749399
0,917157	2,868818	1,0539	1,633969
0,908294	3,181332	1,1573	1,098704
0,908166	3,186108	1,1588	1,093771
0,90779	3,200157	1,1632	1,078871
0,908157	3,186426	1,1589	1,093388
0,908532	3,172437	1,1545	1,108336
0,914379	2,962993	1,0862	1,409427
0,915848	2,912757	1,0691	1,516677
0,907995	3,192486	1,1608	1,08695
0,915711	2,917421	1,0707	1,506076
0,908763	3,163883	1,1518	1,117858
0,9106	3,096585	1,1303	1,198975
0,918717	2,817359	1,0358	1,817209
0,917096	2,870827	1,0546	1,628091
0,920176	2,770146	1,0189	2,083865
0,921586	2,725359	1,0026	2,799973
0,921508	2,727812	1,0035	2,702235
0,921655	2,723179	1,0018	2,91476
0,920868	2,748073	1,0109	2,296628
0,921664	2,722907	1,0017	2,925621
0,921647	2,723451	1,0019	2,893709
0,921595	2,725086	1,0025	2,804761
0,921629	2,723996	1,0021	2,860567
0,921526	2,727267	1,0033	2,720888
0,921076	2,741486	1,0085	2,390512
0,920833	2,749173	1,0113	2,284528
0,921153	2,739019	1,0076	2,431789
0,921413	2,730815	1,0046	2,604019
0,92092	2,746425	1,0103	2,319842

0,921577	2,725631	1,0027	2,78701
0,921041	2,742583	1,0089	2,373023
0,921629	2,723996	1,0021	2,85825
0,920073	2,773472	1,0201	2,059245
0,920972	2,744777	1,0097	2,340652
0,920738	2,752198	1,0124	2,247811
0,919952	2,777358	1,0215	2,032607
0,920609	2,75633	1,0139	2,204973
0,921526	2,727267	1,0033	2,721495
0,921621	2,724269	1,0022	2,850231
0,921586	2,725359	1,0026	2,794015
0,921595	2,725086	1,0025	2,81218
0,9199	2,779025	1,0221	2,020895
0,92169	2,72209	1,0014	2,991468
0,921119	2,740115	1,008	2,413126
0,921629	2,723996	1,0021	2,87077
0,920012	2,775414	1,0208	2,046474
0,921396	2,731361	1,0048	2,590842
0,920228	2,768484	1,0183	2,097932
0,921612	2,724541	1,0023	2,832039
0,920955	2,745326	1,0099	2,33283
0,919969	2,776802	1,0213	2,036367
0,921119	2,740115	1,008	2,412251
0,920176	2,770146	1,0189	2,084091
0,921171	2,738472	1,0074	2,439102
0,921595	2,725086	1,0025	2,81303
0,921448	2,729723	1,0042	2,637781
0,921352	2,732727	1,0053	2,556975
0,921664	2,722907	1,0017	2,93729
0,921491	2,728358	1,0037	2,675379
0,92041	2,762677	1,0162	2,146461
0,921621	2,724269	1,0022	2,844377
0,920782	2,750823	1,0119	2,264757
0,920142	2,771254	1,0193	2,075747
0,921638	2,723724	1,002	2,871759
0,92169	2,72209	1,0014	2,998588
1,140109	0,282776	-1,2631	
1,190677	0,178137	-1,7252	
1,151418	0,254565	-1,3682	
1,202431	0,160446	-1,8298	
1,162719	0,22942	-1,4722	
1,134086	0,299183	-1,2067	
1,171059	0,212609	-1,5483	
1,140163	0,282635	-1,2636	
1,135941	0,294022	-1,2241	
1,1596	0,236076	-1,4436	
1,173668	0,20763	-1,572	

1,175577	0,204068	-1,5893
1,147036	0,265113	-1,3276
1,140516	0,281704	-1,2669
1,139167	0,285275	-1,2543
1,15762	0,240412	-1,4254
1,165704	0,223242	-1,4995
1,14177	0,278427	-1,2786
1,185032	0,187383	-1,6746
1,165233	0,224204	-1,4952
1,209385	0,150891	-1,8912
1,179348	0,197227	-1,6234
1,135685	0,294729	-1,2217
1,132256	0,304373	-1,1895
1,155144	0,245957	-1,4026
1,173161	0,208587	-1,5674
1,139071	0,285532	-1,2534
1,166109	0,222417	-1,5032
1,153539	0,249624	-1,3878
1,139274	0,28499	-1,2553
1,210407	0,149539	-1,9002
1,1924	0,175415	-1,7406
1,155296	0,245613	-1,404
1,1908	0,177942	-1,7263
1,166679	0,221264	-1,5084
1,173955	0,20709	-1,5746
1,182542	0,191628	-1,6522
1,136101	0,293581	-1,2256
1,144572	0,271254	-1,3047
1,151645	0,254031	-1,3703
1,156707	0,24244	-1,417
1,147155	0,264821	-1,3287
1,13688	0,291446	-1,2329
1,134512	0,297989	-1,2107
1,21068	0,14918	-1,9026
1,141534	0,27904	-1,2764
1,141963	0,277926	-1,2804
1,196888	0,168537	-1,7806
1,134075	0,299213	-1,2066
1,205824	0,155704	-1,8598
1,202183	0,160799	-1,8276
1,12357	0,330384	-1,1075
1,115013	0,358402	-1,0261
1,119735	0,342631	-1,0711
1,118747	0,345867	-1,0617
1,114961	0,358581	-1,0256
1,116354	0,353844	-1,0389
1,123728	0,329889	-1,109

1,125101	0,325628	-1,122
1,119609	0,343043	-1,0699
1,114908	0,358761	-1,0251
1,11561	0,356365	-1,0318
1,118747	0,345867	-1,0617
1,127745	0,317588	-1,147
1,12045	0,340309	-1,0779
1,122916	0,332439	-1,1013
1,116144	0,354552	-1,0369
1,116774	0,352431	-1,0429
1,119514	0,343352	-1,069
1,131511	0,306512	-1,1825
1,117214	0,350954	-1,0471
1,123949	0,329197	-1,1111
1,125957	0,323001	-1,1301
1,124541	0,327358	-1,1167
1,116407	0,353667	-1,0394
1,117309	0,350638	-1,048
1,116983	0,351727	-1,0449
1,117938	0,348541	-1,054
1,131798	0,305685	-1,1852
1,120051	0,341605	-1,0741
1,124023	0,328966	-1,1118
1,119735	0,342631	-1,0711
1,126412	0,321615	-1,1344
1,114866	0,358904	-1,0247
1,115495	0,356757	-1,0307
1,127682	0,317779	-1,1464
1,116658	0,352819	-1,0418
1,117928	0,348576	-1,0539
1,128837	0,314334	-1,1573
1,128996	0,313863	-1,1588
1,129462	0,312485	-1,1632
1,129006	0,313831	-1,1589
1,12854	0,315215	-1,1545
1,121324	0,337497	-1,0862
1,119525	0,343317	-1,0691
1,129208	0,313235	-1,1608
1,119693	0,342768	-1,0707
1,128254	0,316067	-1,1518
1,125978	0,322936	-1,1303
1,116029	0,354942	-1,0358
1,118001	0,348332	-1,0546
1,114259	0,360992	-1,0189
1,112555	0,366924	-1,0026
1,112649	0,366594	-1,0035
1,112471	0,367218	-1,0018

1,113422	0,363891	-1,0109
1,112461	0,367255	-1,0017
1,112482	0,367181	-1,0019
1,112544	0,366961	-1,0025
1,112503	0,367108	-1,0021
1,112628	0,366667	-1,0033
1,113171	0,364766	-1,0085
1,113464	0,363746	-1,0113
1,113077	0,365094	-1,0076
1,112764	0,366191	-1,0046
1,11336	0,36411	-1,0103
1,112565	0,366888	-1,0027
1,113213	0,36462	-1,0089
1,112503	0,367108	-1,0021
1,114385	0,360559	-1,0201
1,113297	0,364328	-1,0097
1,113579	0,363346	-1,0124
1,114531	0,360054	-1,0215
1,113736	0,362801	-1,0139
1,112628	0,366667	-1,0033
1,112513	0,367071	-1,0022
1,112555	0,366924	-1,0026
1,112544	0,366961	-1,0025
1,114594	0,359838	-1,0221
1,112429	0,367365	-1,0014
1,113119	0,364948	-1,008
1,112503	0,367108	-1,0021
1,114458	0,360307	-1,0208
1,112785	0,366118	-1,0048
1,114196	0,361208	-1,0183
1,112524	0,367034	-1,0023
1,113318	0,364255	-1,0099
1,11451	0,360126	-1,0213
1,113119	0,364948	-1,008
1,114259	0,360992	-1,0189
1,113056	0,365167	-1,0074
1,112544	0,366961	-1,0025
1,112722	0,366338	-1,0042
1,112837	0,365935	-1,0053
1,112461	0,367255	-1,0017
1,11267	0,366521	-1,0037
1,113977	0,361968	-1,0162
1,112513	0,367071	-1,0022
1,113527	0,363528	-1,0119
1,114301	0,360847	-1,0193
1,112492	0,367144	-1,002
1,112429	0,367365	-1,0014

Appendix

To run the simulation we may use the following VBA code that works with 100 ITER:

```
Function sim(r, sigma, T, ITER)

Dim Path() As Double, e() As Double

simPath = 0

ReDim Path(ITER * 6) As Double, e(ITER * 6) As Double

For itcount = 1 To ITER

e(itcount) = Application.Round(1 / Application.NormSDist(Rnd), 4)

e(itcount) = Application.Round(1 / Application.NormSDist(Rnd + 1), 4)

e(itcount) = Application.Round(1 / Application.NormSDist(Rnd + 2), 4)

e(itcount) = Application.Round(-1 / Application.NormSDist(Rnd), 4)

e(itcount) = Application.Round(-1 / Application.NormSDist(Rnd + 1), 4)

e(itcount) = Application.Round(-1 / Application.NormSDist(Rnd + 2), 4)

Path(itcount) = Exp( - sigma * Sqr(T) * e(itcount))

Next itcount

simPath = simPath + r + Path(ITER) / ITER

sim = simPath

End Function


Function Swap(r, sigma, T, ITER)

Dim Swapt() As Double

simPath = 0

ReDim Swapt(ITER) As Double

For itcount = 1 To ITER

e = Application.NormSInv(Rnd)
```

Swapt(itcount) = Exp(-(sigma * Sqr(T) * e))

Next itcount

simPath = simPath + Application.Average(Swapt(ITER))

Swap = r + simPath / ITER

End Function

Explicit Finite Difference Methods

The option pricing model is based on the arbitrage setting, the main idea is that the pay off and the price of the option may be replicated so its value is directly determinate to avoid arbitrage opportunity called hedging relation. Further application was about the dividend because when the stock pays the dividend its prices will decrease for the same amount but we have to observe that speculators in the hedged portfolio will income the dividend as risk premium. Now we assume the following distribution for the stock prices:

$$\frac{dS(t)}{S(t)} = (r(t) + \delta - q)dt + \sigma_S\, dWs$$

δ may be the risk premium or the liquidity premium, depends from the equilibrium in the treasury market, instead q denotes the dividend yield, This let us to introduce arbitrage theory, in practice if we built the following portfolio we have:

$$V_t = \pm\, S\, \frac{\partial F(S)}{\partial S} - F(S)$$

The portfolio is risk free, as such by using Ito's lemma we obtain the following stochastic differential equation:

$$\frac{\partial F}{\partial t} + r\frac{\partial F}{\partial S}S + \frac{1}{2}\frac{\partial^2 F}{\partial^2 S}\,S^2\sigma^2 - rF(S) = 0$$

We may solve the stochastic differential equation by using the integrant factor:

$$F(S) = Z(S)e^{-r\,T}$$

By solving the stochastic differential equation for Z(S) we obtain that the solution is given by:

$$Z(S) = E(S(T))$$

By solving we obtain:

$$F(S) = E(S(T)e^{-r\,T})$$

This means that if we replicate the prices of options by using delta hedging the value of options are given by the expected value of the pay off discounted where the drift of the process is given by the short rate, this is what it is called risk neutral world. The final pay off of a Call and Put option is given by the following:

$$Call = Max[S(T) - K\,; 0]$$

$$Put = Max[K - S(T)\,; 0]$$

The prices of the options are given by the expectation of the final pay off discounted:

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$$C(S,T,K) = P(T)E[Max(S(T) - K\,;0)]$$

$$P(S,T,K) = P(T)E[Max(K - S(T)\,;0)]$$

Instead, we assume the following process for the default free zero coupon bond:

$$\frac{dP(T)}{P(T)} = r(t)dt + \sigma_P dWp$$

$$\sigma_P = \sigma_r T$$

Now to compute the value of the option is a problem because we have stochastic interest rate so the solution is to take the default free zero coupon bond as forward measure, so by using it as numeraire we have the following process in equilibrium between financial markets:

$$\frac{dN(t)}{N(t)} = \sigma_N dWn$$

Where:

$$N(t) = \frac{S(t)}{P(T)}$$

$$\sigma_N^2 = \frac{\int_t^T \sigma_S^2 + \sigma_P^2 - 2\rho\sigma_S\sigma_P \; dt}{T - t}$$

Now we derive the price of a Call option, as such we have the following:

$$\frac{C(S,T,K)}{P(T)} = \int_{-\infty}^{+\infty} Max\left[N(t)e^{-\frac{1}{2}\sigma_N^2 T + \sigma_N z\sqrt{T}} - K;0\right] f(z)dz$$

The integral vanishes when N(T) < K, thus by solving for z we have:

$$z^{\circ} = \frac{\ln\left(\frac{KP(T)}{S}\right) + \frac{1}{2}\sigma_N^2 T}{\sigma_N\sqrt{T}}$$

As result we may rewrite the integral in the following form:

$$\int_{z^{\circ}}^{+\infty} [N(t)e^{-\frac{1}{2}\sigma_N^2 T + \sigma_N z\sqrt{T}}] f(z)dz \int_{z^{\circ}}^{+\infty} K f(z)dz$$

By using the symmetry property of a normal distribution we obtain the following pricing formula:

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$$\frac{C(S,T,K)}{P(T)} = N(t)N\left[-z^\circ + \sigma_N\sqrt{T}\right] - KN\left[-z^\circ\right]$$

As result we obtain the following pricing formula for the options by using the respective numeraire:

$$C(S,T,K) = S(t)N[d1] - P(T)KN[d2]$$

Where:

$$d1 = \frac{\ln\left(\frac{S}{KP(T)}\right) + \frac{1}{2}\sigma_N^2 T}{\sigma_N\sqrt{T}}$$

$$d2 = \frac{\ln\left(\frac{S}{KP(T)}\right) - \frac{1}{2}\sigma_N^2 T}{\sigma_N\sqrt{T}}$$

$$P(S,T,K) = P(T)KN[-d2] - S(t)N[-d1]$$

Where:

$$-d2 = \frac{\ln\left(\frac{KP(T)}{S}\right) + \frac{1}{2}\sigma_N^2 T}{\sigma_N\sqrt{T}}$$

$$-d1 = \frac{\ln\left(\frac{KP(T)}{S}\right) - \frac{1}{2}\sigma_N^2 T}{\sigma_N\sqrt{T}}$$

We may note that between the two formulations there is a parity relation such that we have:

$$P(S,T,K) - C(S,T,K) + S(t) = P(T)K$$

Now we may note that the Put option formula may be less than its pay off, this not a good news because we may have European and American options, the European options may be exercised just at maturity, instead the American options may be exercised before of maturity as such if the value of the options is greater or equal to the pays off they will not be exercised before of maturity, furthermore, the early exercised opportunity may bring in the Put Call parity to have a greater earning with respect the risk free rate so we may value the American Put option such that there is parity relation in the world of numeraire i.e. with interest rate nil. As such we have the following pricing formula for the American Put option that is greater or equal to the pay off:

$$P(S,T,K) = KN[h1] - S(t)N[h2]$$

Where:

$$h1 = \frac{\ln\left(\frac{K}{S}\right) + \frac{1}{2}\sigma_N^2 T}{\sigma_N\sqrt{T}}$$

$$h2 = \frac{\ln\left(\frac{K}{S}\right) - \frac{1}{2}\sigma_N^2 T}{\sigma_N\sqrt{T}}$$

Indeed, we have got the same formulation of Black, Scholes (1973) with the changes of measure and by considering that the dividend is income so to have the same final pay off in the hedge portfolio. Indeed, we may estimate the prices of the options by using a numerical procedure, as such we may rewrite the stochastic differential equation that an option must satisfy by using the following notation:

$$S_i = ih \quad T_j = jdt$$

$$\frac{\partial F}{\partial t} = \frac{C(i,j) - C(i,j-1)}{dt}$$

$$\frac{\partial F}{\partial S} = \frac{C(i+1,j) - C(i,j)}{h}$$

$$\frac{\partial^2 F}{\partial S^2} = \frac{C(i+1,j) - 2C(i,j) - C(i-1,j)}{h^2}$$

By substituting these values in the stochastic differential equation we obtain the following:

$$aC(i-1,j) + bC(i,j) + cC(i+1,j) = (1 + r\,dt)\,C(i,j-1)$$

Where:

$$a = -\frac{1}{2}\sigma^2 i^2 dt$$

$$b = 1 - \sigma^2 i^2 dt - dt\,r\,i$$

$$c = dt\,r\,i + \frac{1}{2}\sigma^2 i^2 dt$$

As such for the Put options we have the following:

$$S_i = ih \quad T_j = jdt$$

$$\frac{\partial F}{\partial t} = \frac{P(i,j) - P(i,j-1)}{dt}$$

$$\frac{\partial F}{\partial S} = \frac{P(i-1,j) - P(i,j)}{h}$$

$$\frac{\partial^2 F}{\partial S^2} = \frac{P(i-1,j) - 2P(i,j) - P(i+1,j)}{h^2}$$

By substituting these values in the stochastic differential equation we obtain the following:

$$cP(i-1,j) + bP(i,j) + aP(i+1,j) = (1 + r\, dt)\, P(i,j-1)$$

Where:

$$c = \frac{1}{2}\sigma^2 i^2 dt + dt\, r\, i$$

$$b = 1 - \sigma^2 i^2 dt - dt\, r\, i$$

$$a = -\frac{1}{2}\sigma^2 i^2 dt$$

We assume the following:

$$u = e^{\sigma\sqrt{\Delta t}} \qquad d = e^{-\sigma\sqrt{\Delta t}}$$

As result the pays off are given by:

$$C_{N,i} = Max\left[Su^i d^{N-i} - K, 0\right]$$

$$P_{N,i} = Max\left[K - Su^i d^{N-i}, 0\right]$$

As result the prices for European options are given by:

$$C_{j,i} = \left(\frac{1}{(1 + r\, dt)}\right) a\, C_{j+1,i-1} + b\, C_{j+1,i} + c\, C_{j+1,i+1}$$

$$P_{j,i} = \left(\frac{1}{(1 + r\, dt)}\right) a\, P_{j+1,i+1} + b\, P_{j+1,i} + c\, P_{j+1,i-1}$$

Instead, for American options are given by:

$$C_{j,i} = Max\left[Su^i d^{j-i} - K, \left(\frac{1}{(1 + r\, dt)}\right) a\, C_{j+1,i-1} + b\, C_{j+1,i} + c\, C_{j+1,i+1}\right]$$

$$P_{j,i} = Max\left[K - Su^i d^{j-i}, \left(\frac{1}{(1 + r\, dt)}\right) a\, P_{j+1,i+1} + b\, P_{j+1,i} + c\, P_{j+1,i-1}\right]$$

We may compare now the model with the European Call options:

σ_S	K	r	S_t	T	Numerical 3 Grids	Expected
0,07	1	0,03	1,5	0,25	0,52365	0,50747
0,07	1	0,03	1,25	0,25	0,27204	0,25747
0,07	1	0,03	1	0,25	0,02040	0,01796
0,07	1	0,03	0,75	0,25	0,00000	0,00000
0,07	1	0,03	0,5	0,25	0,00000	0,00000
0,07	1	0,03	1,5	0,5	0,52965	0,51489
0,07	1	0,03	1,25	0,5	0,27935	0,26489
0,07	1	0,03	1	0,5	0,02896	0,02793
0,07	1	0,03	0,75	0,5	0,00000	0,00000
0,07	1	0,03	0,5	0,5	0,00000	0,00000
0,07	1	0,03	1,5	0,75	0,53282	0,52225
0,07	1	0,03	1,25	0,75	0,28428	0,27225
0,07	1	0,03	1	0,75	0,03557	0,03666
0,07	1	0,03	0,75	0,75	0,00000	0,00000
0,07	1	0,03	0,5	0,75	0,00000	0,00000

We may compare now the model with the European Put option:

σ_S	K	r	S_t	T	Numerical 3 Grids	Expected
0,07	1	0,03	1,5	0,25	0,00000	0,00000
0,07	1	0,03	1,25	0,25	0,00000	0,00000
0,07	1	0,03	1	0,25	0,00023	0,01049
0,07	1	0,03	0,75	0,25	0,23332	0,24253
0,07	1	0,03	0,5	0,25	0,48641	0,49253
0,07	1	0,03	1,5	0,5	0,00000	0,00000
0,07	1	0,03	1,25	0,5	0,00000	0,00000
0,07	1	0,03	1	0,5	0,00066	0,01305
0,07	1	0,03	0,75	0,5	0,22594	0,23511
0,07	1	0,03	0,5	0,5	0,47902	0,48511
0,07	1	0,03	1,5	0,75	0,00000	0,00000
0,07	1	0,03	1,25	0,75	0,00000	0,00000
0,07	1	0,03	1	0,75	0,00120	0,01442
0,07	1	0,03	0,75	0,75	0,22027	0,22775
0,07	1	0,03	0,5	0,75	0,47279	0,47775

We may compare now the model with the American Put options:

σ_S	K	r	S_t	T	Numerical 3 Grids	Expected
0,07	1	0,03	1,5	0,25	0,00000	0,00000
0,07	1	0,03	1,25	0,25	0,00000	0,00000
0,07	1	0,03	1	0,25	0,01995	0,01396
0,07	1	0,03	0,75	0,25	0,26434	0,25000
0,07	1	0,03	0,5	0,25	0,50873	0,50000
0,07	1	0,03	1,5	0,5	0,00000	0,00000
0,07	1	0,03	1,25	0,5	0,00000	0,00000
0,07	1	0,03	1	0,5	0,02803	0,01974

0,07	1	0,03	0,75	0,5	0,26978	0,25000
0,07	1	0,03	0,5	0,5	0,51153	0,50000
0,07	1	0,03	1,5	0,75	0,00000	0,00000
0,07	1	0,03	1,25	0,75	0,00000	0,00000
0,07	1	0,03	1	0,75	0,03414	0,02418
0,07	1	0,03	0,75	0,75	0,27374	0,25000
0,07	1	0,03	0,5	0,75	0,51335	0,50000

The numerical results gives the proof that the solution to the partial differential equation is given by the solution of Black, Scholes (1973). We may further develop the numerical methods by assuming the following:

$$S_i = ih \quad T_j = jdt$$

$$\frac{\partial F}{\partial t} = \frac{C(i,j) - C(i,j-1)}{dt}$$

$$\frac{\partial F}{\partial S} = \frac{\big(C(i+1,j) - C(i,j)\big) + \big(C(i,j) - C(i-1,j)\big)}{h}$$

$$\frac{\partial^2 F}{\partial S^2} = \frac{\big(C(i+1,j) - C(i,j)\big) - \big(C(i,j) - C(i-1,j)\big)}{h^2}$$

By substituting these values in the stochastic differential equation we obtain the following:

$$aC(i-1,j) + bC(i,j) + cC(i+1,j) = (1 + r\,dt)\,C(i,j-1)$$

$$cP(i-1,j) + bP(i,j) + aP(i+1,j) = (1 + r\,dt)\,P(i,j-1)$$

Where:

$$a = \frac{1}{2}\sigma^2 i^2 dt - dt\,r\,i$$

$$b = 1 - \sigma^2 i^2 dt$$

$$c = dt\,r\,i + \frac{1}{2}\sigma^2 i^2 dt$$

We may compare the result with European Call options:

σ_S	K	r	S_t	T	Numerical 3 Grids	Expected
0,07	1	0,03	1,5	0,25	0,52577	0,50747
0,07	1	0,03	1,25	0,25	0,27327	0,25747
0,07	1	0,03	1	0,25	0,02061	0,01796
0,07	1	0,03	0,75	0,25	0,00000	0,00000
0,07	1	0,03	0,5	0,25	0,00000	0,00000
0,07	1	0,03	1,5	0,5	0,53427	0,51489
0,07	1	0,03	1,25	0,5	0,28213	0,26489
0,07	1	0,03	1	0,5	0,02954	0,02793

σ_S	K	r	S_t	T	Numerical 3 Grids	Expected
0,07	1	0,03	0,75	0,5	0,00000	0,00000
0,07	1	0,03	0,5	0,5	0,00000	0,00000
0,07	1	0,03	1,5	0,75	0,54017	0,52225
0,07	1	0,03	1,25	0,75	0,28881	0,27225
0,07	1	0,03	1	0,75	0,03663	0,03666
0,07	1	0,03	0,75	0,75	0,00000	0,00000
0,07	1	0,03	0,5	0,75	0,00000	0,00000

We may compare the result with European Put options:

σ_S	K	r	S_t	T	Numerical 3 Grids	Expected
0,07	1	0,03	1,5	0,25	0,00000	0,00000
0,07	1	0,03	1,25	0,25	0,00000	0,00000
0,07	1	0,03	1	0,25	0,00023	0,01049
0,07	1	0,03	0,75	0,25	0,23498	0,24253
0,07	1	0,03	0,5	0,25	0,48972	0,49253
0,07	1	0,03	1,5	0,5	0,00000	0,00000
0,07	1	0,03	1,25	0,5	0,00000	0,00000
0,07	1	0,03	1	0,5	0,00066	0,01305
0,07	1	0,03	0,75	0,5	0,22926	0,23511
0,07	1	0,03	0,5	0,5	0,48562	0,48511
0,07	1	0,03	1,5	0,75	0,00000	0,00000
0,07	1	0,03	1,25	0,75	0,00000	0,00000
0,07	1	0,03	1	0,75	0,00122	0,01442
0,07	1	0,03	0,75	0,75	0,22527	0,22775
0,07	1	0,03	0,5	0,75	0,48269	0,47775

We may compare the result with American Put options:

σ_S	K	r	S_t	T	Numerical 3 Grids	Expected
0,07	1	0,03	1,5	0,25	0,00000	0,00000
0,07	1	0,03	1,25	0,25	0,00000	0,00000
0,07	1	0,03	1	0,25	0,01995	0,01396
0,07	1	0,03	0,75	0,25	0,26434	0,25000
0,07	1	0,03	0,5	0,25	0,50873	0,50000
0,07	1	0,03	1,5	0,5	0,00000	0,00000
0,07	1	0,03	1,25	0,5	0,00000	0,00000
0,07	1	0,03	1	0,5	0,02803	0,01974
0,07	1	0,03	0,75	0,5	0,26978	0,25000
0,07	1	0,03	0,5	0,5	0,51153	0,50000
0,07	1	0,03	1,5	0,75	0,00000	0,00000
0,07	1	0,03	1,25	0,75	0,00000	0,00000
0,07	1	0,03	1	0,75	0,03414	0,02418
0,07	1	0,03	0,75	0,75	0,27374	0,25000
0,07	1	0,03	0,5	0,75	0,51335	0,50000

We may further develop the numerical methods by assuming the following:

$$S_i = ih \ T_j = jdt$$

$$\frac{\partial F}{\partial t} = \frac{C(i,j) - C(i,j-1)}{dt}$$

$$\frac{\partial F}{\partial S} = \frac{\big(C(i+1,j) - C(i,j)\big) + \big(C(i,j) - C(i-1,j)\big)}{h}$$

$$\frac{\partial^2 F}{\partial S^2} = \frac{\big(C(i,j) - C(i-1,j)\big) - \big(C(i+1,j) - C(i,j)\big)}{h^2}$$

By substituting these values in the stochastic differential equation we obtain the following:

$$aC(i-1,j) + bC(i,j) + cC(i+1,j) = (1 + r \ dt) \ C(i,j-1)$$

$$cP(i-1,j) + bP(i,j) + aP(i+1,j) = (1 + r \ dt) \ P(i,j-1)$$

Where:

$$a = -\frac{1}{2}\sigma^2 i^2 dt - dt \ r \ i$$

$$b = 1 + \sigma^2 i^2 dt$$

$$c = dt \ r \ i - \frac{1}{2}\sigma^2 i^2 dt$$

We may compare the result with European Call options:

σ_S	K	r	S_t	T	Numerical 3 Grids	Expected
0,07	1	0,03	1,5	0,25	0,52673	0,50747
0,07	1	0,03	1,25	0,25	0,27380	0,25747
0,07	1	0,03	1	0,25	0,02064	0,01796
0,07	1	0,03	0,75	0,25	0,00000	0,00000
0,07	1	0,03	0,5	0,25	0,00000	0,00000
0,07	1	0,03	1,5	0,5	0,53629	0,51489
0,07	1	0,03	1,25	0,5	0,28328	0,26489
0,07	1	0,03	1	0,5	0,02963	0,02793
0,07	1	0,03	0,75	0,5	0,00000	0,00000
0,07	1	0,03	0,5	0,5	0,00000	0,00000
0,07	1	0,03	1,5	0,75	0,54332	0,52225
0,07	1	0,03	1,25	0,75	0,29063	0,27225
0,07	1	0,03	1	0,75	0,03680	0,03666
0,07	1	0,03	0,75	0,75	0,00000	0,00000
0,07	1	0,03	0,5	0,75	0,00000	0,00000

We may compare the result with European Put options:

σ_S	K	r	S_t	T	Numerical 3 Grids	Expected
0,07	1	0,03	1,5	0,25	0,00000	0,00000
0,07	1	0,03	1,25	0,25	0,00000	0,00000
0,07	1	0,03	1	0,25	0,00017	0,01049
0,07	1	0,03	0,75	0,25	0,23532	0,24253
0,07	1	0,03	0,5	0,25	0,49048	0,49253
0,07	1	0,03	1,5	0,5	0,00000	0,00000
0,07	1	0,03	1,25	0,5	0,00000	0,00000
0,07	1	0,03	1	0,5	0,00048	0,01305
0,07	1	0,03	0,75	0,5	0,22987	0,23511
0,07	1	0,03	0,5	0,5	0,48711	0,48511
0,07	1	0,03	1,5	0,75	0,00000	0,00000
0,07	1	0,03	1,25	0,75	0,00000	0,00000
0,07	1	0,03	1	0,75	0,00088	0,01442
0,07	1	0,03	0,75	0,75	0,22611	0,22775
0,07	1	0,03	0,5	0,75	0,48486	0,47775

We may compare the result with American Put options:

σ_S	K	r	S_t	T	Numerical 3 Grids	Expected
0,07	1	0,03	1,5	0,25	0,00000	0,00000
0,07	1	0,03	1,25	0,25	0,00000	0,00000
0,07	1	0,03	1	0,25	0,01995	0,01396
0,07	1	0,03	0,75	0,25	0,26434	0,25000
0,07	1	0,03	0,5	0,25	0,50873	0,50000
0,07	1	0,03	1,5	0,5	0,00000	0,00000
0,07	1	0,03	1,25	0,5	0,00000	0,00000
0,07	1	0,03	1	0,5	0,02803	0,01974
0,07	1	0,03	0,75	0,5	0,26978	0,25000
0,07	1	0,03	0,5	0,5	0,51153	0,50000
0,07	1	0,03	1,5	0,75	0,00000	0,00000
0,07	1	0,03	1,25	0,75	0,00000	0,00000
0,07	1	0,03	1	0,75	0,03414	0,02418
0,07	1	0,03	0,75	0,75	0,27374	0,25000
0,07	1	0,03	0,5	0,75	0,51335	0,50000

We may further develop the numerical methods by assuming the following:

$$S_i = ih \quad T_j = jdt$$

$$\frac{\partial F}{\partial t} = \frac{C(i,j) - C(i,j-1)}{dt}$$

$$\frac{\partial F}{\partial S} = \frac{\big(C(i+1,j) - C(i,j)\big) + \big(C(i,j) - C(i-1,j)\big)}{h}$$

$$\frac{\partial^2 F}{\partial S^2} = \frac{\left(C(i,j) - C(i-1,j)\right) - \left(C(i+1,j) - C(i,j)\right)}{h^2}$$

By substituting these values in the stochastic differential equation we obtain the following:

$$cP(i-1,j) + bP(i,j) + aP(i+1,j) = P(i,j-1)$$

$$aC(i-1,j) + bC(i,j) + cC(i+1,j) = C(i,j-1)$$

Where:

$$a = -\frac{1}{2}\sigma^2 i^2 dt - dt\, r\, i$$

$$b = 1 + \sigma^2 i^2 dt + r\, dt$$

$$c = dt\, r\, i - \frac{1}{2}\sigma^2 i^2 dt$$

We may compare the result with European Call options:

σ_S	K	r	S_t	T	Numerical 3 Grids	Expected
0,07	1	0,03	1,5	0,25	0,53202	0,50747
0,07	1	0,03	1,25	0,25	0,27655	0,25747
0,07	1	0,03	1	0,25	0,02084	0,01796
0,07	1	0,03	0,75	0,25	0,00000	0,00000
0,07	1	0,03	0,5	0,25	0,00000	0,00000
0,07	1	0,03	1,5	0,5	0,54711	0,51489
0,07	1	0,03	1,25	0,5	0,28899	0,26489
0,07	1	0,03	1	0,5	0,03022	0,02793
0,07	1	0,03	0,75	0,5	0,00000	0,00000
0,07	1	0,03	0,5	0,5	0,00000	0,00000
0,07	1	0,03	1,5	0,75	0,55982	0,52225
0,07	1	0,03	1,25	0,75	0,29945	0,27225
0,07	1	0,03	1	0,75	0,03790	0,03666
0,07	1	0,03	0,75	0,75	0,00000	0,00000
0,07	1	0,03	0,5	0,75	0,00000	0,00000

We may compare the result with European Put options:

σ_S	K	r	S_t	T	Numerical 3 Grids	Expected
0,07	1	0,03	1,5	0,25	0,00000	0,00000
0,07	1	0,03	1,25	0,25	0,00000	0,00000
0,07	1	0,03	1	0,25	0,00017	0,01049
0,07	1	0,03	0,75	0,25	0,23767	0,24253
0,07	1	0,03	0,5	0,25	0,49540	0,49253
0,07	1	0,03	1,5	0,5	0,00000	0,00000
0,07	1	0,03	1,25	0,5	0,00000	0,00000
0,07	1	0,03	1	0,5	0,00048	0,01305
0,07	1	0,03	0,75	0,5	0,23448	0,23511
0,07	1	0,03	0,5	0,5	0,49689	0,48511

σ_S	K	r	S_t	T		
0,07	1	0,03	1,5	0,75	0,00000	0,00000
0,07	1	0,03	1,25	0,75	0,00000	0,00000
0,07	1	0,03	1	0,75	0,00090	0,01442
0,07	1	0,03	0,75	0,75	0,23293	0,22775
0,07	1	0,03	0,5	0,75	0,49950	0,47775

We may compare the result with American Put options:

σ_S	K	r	S_t	T	Numerical 3 Grids	Expected
0,07	1	0,03	1,5	0,25	0,00000	0,00000
0,07	1	0,03	1,25	0,25	0,00000	0,00000
0,07	1	0,03	1	0,25	0,02000	0,01396
0,07	1	0,03	0,75	0,25	0,26500	0,25000
0,07	1	0,03	0,5	0,25	0,51000	0,50000
0,07	1	0,03	1,5	0,5	0,00000	0,00000
0,07	1	0,03	1,25	0,5	0,00000	0,00000
0,07	1	0,03	1	0,5	0,02817	0,01974
0,07	1	0,03	0,75	0,5	0,27113	0,25000
0,07	1	0,03	0,5	0,5	0,51409	0,50000
0,07	1	0,03	1,5	0,75	0,00000	0,00000
0,07	1	0,03	1,25	0,75	0,00000	0,00000
0,07	1	0,03	1	0,75	0,03439	0,02418
0,07	1	0,03	0,75	0,75	0,27580	0,25000
0,07	1	0,03	0,5	0,75	0,51720	0,50000

Appendix

To run the simulation we used the following VBA code:

```
Function NumeriCallAmerican(Spot, k, T, r, sigma, n)

Dim dt As Double, u As Double, d As Double, p As Double

   dt = T / n

   u = Exp(sigma * (dt ^ 0.5))

   d = 1 / u

Dim S() As Double

ReDim S(n + 1, n + 1) As Double

 For i = 1 To n + 1

   For j = i To n + 1

      S(i, j) = Spot * u ^ (j - i) * d ^ (i - 1)

   Next j

Next i

Dim Opt() As Double

ReDim Opt(n + 1, n + 1) As Double

For i = 1 To n + 1

    Opt(i, n + 1) = Application.Max(S(i, n + 1) - k, 0)

Next i

Dim a() As Double, b() As Double, c() As Double

ReDim a(n + 1) As Double, b(n + 1) As Double, c(n + 1) As Double

For i = 1 To n + 1

   a(i) = (- 0.5 * sigma ^ 2 * i ^ 2 * dt)

   b(i) = (1 - (sigma ^ 2 * i ^ 2 * dt) - (dt * r * i))

   c(i) = (dt * r * i + 0.5 * sigma ^ 2 * i ^ 2 * dt)

Next i
```

```vba
For j = n To 1 Step -1

    For i = 2 To j

    Opt(i, j) = (1 / (1 + r * dt)) * Application.Max(S(i, j) - k, (a(i) * Opt(i + 1, j + 1) + b(i) * Opt(i, j + 1) +
c(i) * Opt(i - 1, j + 1)))

    NumeriCallAmerican = Opt(i, j)

  Next i

 Next j

End Function

Function NumeriCallEuropean(Spot, k, T, r, sigma, n)

Dim dt As Double, u As Double, d As Double, p As Double

   dt = T / n

   u = Exp(sigma * (dt ^ 0.5))

   d = 1 / u

Dim S() As Double

ReDim S(n + 1, n + 1) As Double

 For i = 1 To n + 1

   For j = i To n + 1

     S(i, j) = Spot * u ^ (j - i) * d ^ (i - 1)

   Next j

Next i

Dim Opt() As Double

ReDim Opt(n + 1, n + 1) As Double

For i = 1 To n + 1

    Opt(i, n + 1) = Application.Max(S(i, n + 1) - k, 0)

Next i

Dim a() As Double, b() As Double, c() As Double

ReDim a(n + 1) As Double, b(n + 1) As Double, c(n + 1) As Double
```
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```vba
For i = 1 To n + 1

    a(i) = (-0.5 * sigma ^ 2 * i ^ 2 * dt)

b(i) = (1 - (sigma ^ 2 * i ^ 2 * dt) - (dt * r * i))

    c(i) = (dt * r * i + 0.5 * sigma ^ 2 * i ^ 2 * dt)

Next i

For j = n To 1 Step -1

    For i = 2 To j

    Opt(i, j) = (1 / (1 + r * dt)) * (a(i) * Opt(i + 1, j + 1) + b(i) * Opt(i, j + 1) + c(i) * Opt(i - 1, j + 1))

    NumeriCallEuropean = Opt(i, j)

  Next i

 Next j

End Function

Function NumeriPutAmerican(Spot, k, T, r, sigma, n)

Dim dt As Double, u As Double, d As Double, p As Double

    dt = T / n

    u = Exp(sigma * (dt ^ 0.5))

    d = 1 / u

Dim S() As Double

ReDim S(n + 1, n + 1) As Double

 For i = 1 To n + 1

    For j = i To n + 1

        S(i, j) = Spot * u ^ (j - i) * d ^ (i - 1)

    Next j

Next i

Dim Opt() As Double

ReDim Opt(n + 1, n + 1) As Double
```

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```vba
For i = 1 To n + 1

    Opt(i, n + 1) = Application.Max(k - S(i, n + 1) , 0)

Next i

Dim a() As Double, b() As Double, c() As Double

ReDim a(n + 1) As Double, b(n + 1) As Double, c(n + 1) As Double

For i = 1 To n + 1

    a(i) = (-0.5 * sigma ^ 2 * i ^ 2 * dt)

    b(i) = (1 - (sigma ^ 2 * i ^ 2 * dt) - (dt * r * i))

    c(i) = (dt * r * i + 0.5 * sigma ^ 2 * i ^ 2 * dt)

Next i

For j = n To 1 Step -1

    For i = 2 To j

    Opt(i, j) = (1 / (1 + r * dt)) * Application.Max(k - S(i, j), (a(i) * Opt(i - 1, j + 1) + b(i) * Opt(i, j + 1) +
c(i) * Opt(i + 1, j + 1)))

    NumeriPutAmerican = Opt(i, j)

  Next i

 Next j

End Function


Function NumeriPutEuropean(Spot, k, T, r, sigma, n)

Dim dt As Double, u As Double, d As Double, p As Double

    dt = T / n

    u = Exp(sigma * (dt ^ 0.5))

    d = 1 / u

Dim S() As Double

ReDim S(n + 1, n + 1) As Double

 For i = 1 To n + 1
```

```
    For j = i To n + 1

        S(i, j) = Spot * u ^ (j - i) * d ^ (i - 1)

    Next j

Next i

Dim Opt() As Double

ReDim Opt(n + 1, n + 1) As Double

For i = 1 To n + 1

    Opt(i, n + 1) = Application.Max(k - S(i, n + 1) , 0)

Next i

Dim a() As Double, b() As Double, c() As Double

ReDim a(n + 1) As Double, b(n + 1) As Double, c(n + 1) As Double

For i = 1 To n + 1

    a(i) = (-0.5 * sigma ^ 2 * i ^ 2 * dt)

    b(i) = (1 - (sigma ^ 2 * i ^ 2 * dt) - (dt * r * i))

    c(i) = (dt * r * i + 0.5 * sigma ^ 2 * i ^ 2 * dt)

Next i

For j = n To 1 Step -1

    For i = 2 To j

    Opt(i, j) = (1 / (1 + r * dt)) * (a(i) * Opt(i - 1, j + 1) + b(i) * Opt(i, j + 1) + c(i) * Opt(i + 1, j + 1))

    NumeriPutEuropean = Opt(i, j)

    Next i

 Next j

End Function
```

A Note:

```
Function NumericallS(Spot, k, T, r, sigma, n)

Dim dt As Double

dt = T / n

Dim Spt As Double

    Spt = Spot * 2 / n

Dim S() As Double

ReDim S(n + 1) As Double

For i = 1 To n + 1

   S(i) = (i * Spt)

Next i

Dim Opt() As Double

ReDim Opt(n + 1) As Double

For i = 1 To n + 1

    Opt(i) = Application.Max(S(i) - k, 0)

Next i

Dim a() As Double, b() As Double, c() As Double

ReDim a(n + 1) As Double, b(n + 1) As Double, c(n + 1) As Double

For i = 1 To n + 1

   a(i) = (-0.5 * sigma ^ 2 * i ^ 2 * dt)

   b(i) = (1 - (sigma ^ 2 * i ^ 2 * dt) - (dt * r * i))

   c(i) = (dt * r * i + 0.5 * sigma ^ 2 * i ^ 2 * dt)

Next i

For i = n To 2 Step -1

   Opt(i) = (1 / (1 + r * dt)) * Application.Max(S(i) - k, (a(i) * Opt(i + 1) + b(i) * Opt(i) + c(i) * Opt(i - 1)))

   NumericallS = Opt(i)

 Next i
```

```vba
End Function

Function NumeriputS(Spot, k, T, r, sigma, n)

Dim dt As Double

dt = T / n

Dim Spt As Double

    Spt = Spot * 2 / n

Dim S() As Double

ReDim S(n + 1) As Double

For i = 1 To n + 1

   S(i) = (i * Spt)

Next i

Dim Opt() As Double

ReDim Opt(n + 1) As Double

For i = 1 To n + 1

   Opt(i) = Application.Max(k - S(i), 0)

Next i

Dim a() As Double, b() As Double, c() As Double

ReDim a(n + 1) As Double, b(n + 1) As Double, c(n + 1) As Double

For i = 1 To n + 1

   a(i) = (-0.5 * sigma ^ 2 * i ^ 2 * dt)

   b(i) = (1 - (sigma ^ 2 * i ^ 2 * dt) - (dt * r * i))

   c(i) = (dt * r * i + 0.5 * sigma ^ 2 * i ^ 2 * dt)

Next i

For i = n To 2 Step -1

   Opt(i) = (1 / (1 + r * dt)) * Application.Max(k - S(i), (c(i) * Opt(i + 1) + b(i) * Opt(i) + a(i) * Opt(i - 1)))

   NumeriputS = Opt(i)

 Next i

End Function
```
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Stochastic Volatility

It is interesting to introduce the concept of stochastic volatility, as such we may write Ito's lemma in the following form:

$$dF(S) = \left(\frac{\partial F}{\partial t} + \mu \frac{\partial F}{\partial S} S + \alpha \frac{\partial F}{\partial \sigma} + \frac{1}{2} \frac{\partial^2 F}{\partial^2 S} S^2 \sigma^2 + \rho \sigma \delta S \frac{\partial F}{\partial S \partial \sigma} + \frac{1}{2} \frac{\partial^2 F}{\partial^2 \sigma} \delta^2 \right) dt + \sigma \frac{\partial F}{\partial S} S dZ$$

Where dZ denotes a standardized bivariate normal distribution with the following form:

$$\frac{1}{\sigma_{x,y} \sqrt{2\pi}} e^{-\frac{1}{2}x^2 - \frac{1}{2}y^2} \sim \frac{1}{\sigma_{x,y} \sqrt{2\pi}} (e^{-\frac{1}{2}x^2} + e^{-\frac{1}{2}y^2})$$

The PDE that an option must satisfy by assuming stochastic volatility is given by the following:

$$\frac{\partial F}{\partial t} + r \frac{\partial F}{\partial S} S + (\gamma \delta - \alpha) \frac{\partial F}{\partial \sigma} + \frac{1}{2} \frac{\partial^2 F}{\partial^2 S} S^2 \sigma^2 + \rho \sigma \delta S \frac{\partial F}{\partial S \partial \sigma} + \frac{1}{2} \frac{\partial^2 F}{\partial^2 \sigma} \delta^2 - rF(S) = 0$$

Where:

$$\frac{dS(t)}{S(t)} = r \, dt + \sigma(t) dW_S$$

$$d\sigma(t) = (\gamma \delta - \alpha) dt + \delta \, dW_\sigma$$

$$dW_S \, dW_\sigma = \rho \, dt$$

The solution it is easy to solve, because if we take Ito's lemma and we take the expectation we obtain that the solution to the parabolic problem is given by the following by using the integrant factor e^{-rT}:

$$F(S(T)) = e^{-rT} E[F(S, \sigma)]$$

The final pay off of a Call and Put option is given by the following:

$$Call = Max[S(T) - K ; 0]$$

$$Put = Max[K - S(T) ; 0]$$

The prices of the options are given by the expectation of the final pay off discounted:

$$C(S, T, K) = P(T)E[Max(S(T) - K ; 0)]$$

$$P(S, T, K) = P(T)E[Max(K - S(T) ; 0)]$$

As result we obtain the following pricing formula for the options by using the respective numeraire:

$$C(S, T, K) = S(t)N[d1] - P(T)KN[d2] + (S(t) - K)N[d3]$$

Where:

$$d1 = \frac{\ln\left(\frac{S}{KP(T)}\right) + \frac{1}{2}\sigma_N^2 T}{\sigma_N \sqrt{T}}$$

$$d2 = \frac{\ln\left(\frac{S}{KP(T)}\right) - \frac{1}{2}\sigma_N^2 T}{\sigma_N \sqrt{T}}$$

$$d3 = \frac{(\gamma\delta - \alpha)T - \sigma(t)}{\sigma_N \sqrt{T}}$$

$$P(S,T,K) = P(T)KN[-d2] - S(t)N[-d1] + \big(K - S(t)\big)N[d3]$$

Where:

$$-d2 = \frac{\ln\left(\frac{KP(T)}{S}\right) + \frac{1}{2}\sigma_N^2 T}{\sigma_N \sqrt{T}}$$

$$-d1 = \frac{\ln\left(\frac{KP(T)}{S}\right) - \frac{1}{2}\sigma_N^2 T}{\sigma_N \sqrt{T}}$$

$$\sigma_N^2 = \frac{\int_t^T \sigma_{x,y}^2 + \sigma_P^2 - 2\rho\sigma_{x,y}\sigma_P \, dt}{T - t}$$

$$\sigma_{x,y}^2 = \sigma(t)^2 + \delta^2 + 2\,\rho\sigma\delta$$

It is interesting to analyses the numerical result respect the Black, Scholes (1973) to analyse the skew implied by the formulation developed, as such we have the following result for European Call options:

σ_S	K	r	S_t	T	Bivariate	Expected
0,07	1	0,03	1,5	0,5	0,55421	0,51489
0,07	1	0,03	1,25	0,5	0,28455	0,26489
0,07	1	0,03	1	0,5	0,02793	0,02793
0,07	1	0,03	0,75	0,5	0,00000	0,00000
0,07	1	0,03	0,5	0,5	0,00000	0,00000
0,07	1	0,03	1,5	1	0,60888	0,52955
0,07	1	0,03	1,25	1	0,31922	0,27956
0,07	1	0,03	1	1	0,04477	0,04477
0,07	1	0,03	0,75	1	0,00000	0,00000
0,07	1	0,03	0,5	1	0,00000	0,00000
0,07	1	0,03	1,5	2	0,67811	0,55824

σ_S	K	r	S_t	T		
0,07	1	0,03	1,25		2	0,36824	0,30830
0,07	1	0,03	1		2	0,07427	0,07427
0,07	1	0,03	0,75		2	0,00030	0,00030
0,07	1	0,03	0,5		2	0,00000	0,00000

And the following result for European Put options:

σ_S	K	r	S_t	T	Bivariate	Expected
0,07	1	0,03	1,5	0,5	0,00000	0,00000
0,07	1	0,03	1,25	0,5	0,00000	0,00000
0,07	1	0,03	1	0,5	0,01305	0,01305
0,07	1	0,03	0,75	0,5	0,25477	0,23511
0,07	1	0,03	0,5	0,5	0,52444	0,48511
0,07	1	0,03	1,5	1	0,00000	0,00000
0,07	1	0,03	1,25	1	0,00000	0,00000
0,07	1	0,03	1	1	0,01522	0,01522
0,07	1	0,03	0,75	1	0,26011	0,22045
0,07	1	0,03	0,5	1	0,54977	0,47045
0,07	1	0,03	1,5	2	0,00000	0,00000
0,07	1	0,03	1,25	2	0,00007	0,00007
0,07	1	0,03	1	2	0,01603	0,01603
0,07	1	0,03	0,75	2	0,25201	0,19207
0,07	1	0,03	0,5	2	0,56164	0,44176

The model seems to be able to capture the skew in the implied volatility of option pricing model, so it is a good candidate in period of crisis, we have assumed the volatility premium to be zero and zero correlation, indeed, the model may be much more sensitive to the grade it is in the money. Indeed, it is possible to obtain the skew with numerical methods as well such that we have the following formulation in the valuation of the Black, Scholes (!973) partial differential equation, as such we may rewrite the stochastic differential equation that an option must satisfy by using the following notation:

$$cP(i-1,j) + bP(i,j) + aP(i+1,j) = P(i,j-1)$$

$$aC(i-1,j) + bC(i,j) + cC(i+1,j) = C(i,j-1)$$

Where:

$$a = \frac{1}{2}\sigma^2 i^2 dt - dt\, r\, i$$

$$b = 1 - \sigma^2 i^2 dt + r\, dt$$

$$c = dt\, r\, i + \frac{1}{2}\sigma^2 i^2 dt$$

We assume the following:

$$u = e^{\sigma\sqrt{\Delta t}} \quad d = e^{-\sigma\sqrt{\Delta t}}$$

As result the pays off are given by:

$$C_{N,i} = Max\left[Su^i d^{N-i} - K, 0\right]$$

$$P_{N,i} = Max\left[K - Su^i d^{N-i}, 0\right]$$

As result the prices for European options are given by:

$$C_{j,i} = a\,C_{j+1,i-1} + b\,C_{j+1,i} + c\,C_{j+1,i+1}$$

$$P_{j,i} = a\,P_{j+1,i+1} + b\,P_{j+1,i} + c\,P_{j+1,i-1}$$

Seems that the stochastic volatility model overvalues the risk, but we may set a positive parameter $a = 0.03$ with zero volatility premium for a better calibration. We may compare the result with European Call options:

σ_S	K	r	S_t	T	Numerical 3 Grids	Bivariate
0,07	1	0,03	1,5	0,5	0,54506	0,53637
0,07	1	0,03	1,25	0,5	0,28782	0,27563
0,07	1	0,03	1	0,5	0,03013	0,02793
0,07	1	0,03	0,75	0,5	0,00000	0,00000
0,07	1	0,03	0,5	0,5	0,00000	0,00000
0,07	1	0,03	1,5	1	0,56699	0,56784
0,07	1	0,03	1,25	1	0,30639	0,29870
0,07	1	0,03	1	1	0,04451	0,04477
0,07	1	0,03	0,75	1	0,00000	0,00000
0,07	1	0,03	0,5	1	0,00000	0,00000
0,07	1	0,03	1,5	2	0,60406	0,60551
0,07	1	0,03	1,25	2	0,33790	0,33194
0,07	1	0,03	1	2	0,06826	0,07427
0,07	1	0,03	0,75	2	0,00000	0,00030
0,07	1	0,03	0,5	2	0,00000	0,00000

We may compare the result with European Put options:

σ_S	K	r	S_t	T	Numerical 3 Grids	Bivariate
0,07	1	0,03	1,5	0,5	0,00000	0,00000
0,07	1	0,03	1,25	0,5	0,00000	0,00000
0,07	1	0,03	1	0,5	0,00067	0,01305
0,07	1	0,03	0,75	0,5	0,23386	0,24585
0,07	1	0,03	0,5	0,5	0,49539	0,50659
0,07	1	0,03	1,5	1	0,00000	0,00000
0,07	1	0,03	1,25	1	0,00000	0,00000
0,07	1	0,03	1	1	0,00194	0,01522
0,07	1	0,03	0,75	1	0,23120	0,23959

0,07	1	0,03	0,5	1	0,49980	0,50873
0,07	1	0,03	1,5	2	0,00000	0,00000
0,07	1	0,03	1,25	2	0,00000	0,00007
0,07	1	0,03	1	2	0,00571	0,01603
0,07	1	0,03	0,75	2	0,23301	0,21571
0,07	1	0,03	0,5	2	0,51352	0,48904

As we may note the two approaches converge by setting the volatility premium to zero and the parameters drift equal 0.03. But if we increases the volatility at 0.1 the two approaches converge with zero parameters drift. We may compare the result with European Call options:

σ_S	K	r	S_t	T	Numerical 3 Grids	Bivariate
0,1	1	0,03	1,5	0,5	0,56399	0,55421
0,1	1	0,03	1,25	0,5	0,30389	0,28456
0,1	1	0,03	1	0,5	0,04326	0,03606
0,1	1	0,03	0,75	0,5	0,00000	0,00000
0,1	1	0,03	0,5	0,5	0,00000	0,00000
0,1	1	0,03	1,5	1	0,59480	0,60888
0,1	1	0,03	1,25	1	0,33014	0,31942
0,1	1	0,03	1	1	0,06403	0,05582
0,1	1	0,03	0,75	1	0,00000	0,00013
0,1	1	0,03	0,5	1	0,00000	0,00000
0,1	1	0,03	1,5	2	0,64653	0,67813
0,1	1	0,03	1,25	2	0,37446	0,36947
0,1	1	0,03	1	2	0,09845	0,08869
0,1	1	0,03	0,75	2	0,00000	0,00269
0,1	1	0,03	0,5	2	0,00000	0,00000

We may compare the result with European Put options:

σ_S	K	r	S_t	T	Numerical 3 Grids	Bivariate
0,1	1	0,03	1,5	0,5	0,00000	0,00000
0,1	1	0,03	1,25	0,5	0,00000	0,00001
0,1	1	0,03	1	0,5	0,00110	0,02118
0,1	1	0,03	0,75	0,5	0,22451	0,25478
0,1	1	0,03	0,5	0,5	0,48858	0,52444
0,1	1	0,03	1,5	1	0,00000	0,00000
0,1	1	0,03	1,25	1	0,00000	0,00020
0,1	1	0,03	1	1	0,00315	0,02626
0,1	1	0,03	0,75	1	0,21838	0,26024
0,1	1	0,03	0,5	1	0,49011	0,54977
0,1	1	0,03	1,5	2	0,00000	0,00002
0,1	1	0,03	1,25	2	0,00010	0,00129
0,1	1	0,03	1	2	0,00921	0,03046
0,1	1	0,03	0,75	2	0,21647	0,25440
0,1	1	0,03	0,5	2	0,50018	0,56164

We may note that the numerical methods for the Put option do not capture the skew for higher volatility. We may see now an example with higher volatility at 0.15 and in this case a volatility premium of 0.03. We may compare the result with European Call options:

σ_S	K	r	S_t	T	Numerical 3 Grids	Bivariate
0,15	1	0,03	1,5	0,5	0,59473	0,56566
0,15	1	0,03	1,25	0,5	0,33020	0,29078
0,15	1	0,03	1	0,5	0,06538	0,04984
0,15	1	0,03	0,75	0,5	0,00000	0,00015
0,15	1	0,03	0,5	0,5	0,00000	0,00000
0,15	1	0,03	1,5	1	0,63912	0,63558
0,15	1	0,03	1,25	1	0,36847	0,33563
0,15	1	0,03	1	1	0,09703	0,07485
0,15	1	0,03	0,75	1	0,00000	0,00224
0,15	1	0,03	0,5	1	0,00000	0,00000
0,15	1	0,03	1,5	2	0,71234	0,72733
0,15	1	0,03	1,25	2	0,43211	0,40185
0,15	1	0,03	1	2	0,14975	0,11437
0,15	1	0,03	0,75	2	0,00558	0,01285
0,15	1	0,03	0,5	2	0,00000	0,00006

We may note that the two approaches converge so they are very good candidate to capture the skew embedded in the smile of implied volatility, unfortunately the stochastic volatility approach depends from the volatility premium so introduce a qualitative valuation in the pricing of options. It is now interesting to introduce the Heston model, the distributions of the stock prices are given by the following:

$$\frac{dS(t)}{S(t)} = r\,dt + \sqrt{\sigma(t)^2}dW_S$$

$$d\sigma(t)^2 = (k\alpha - (k+\gamma)\sigma(t)^2)dt + \delta\sqrt{\sigma(t)^2}dW_\sigma$$

$$dW_S\,dW_\sigma = \rho\,dt$$

Heston derived a semi closed form solution via inverse Fourier transformation but the computational effort to solve the integral via numerical integration arise many problems. Indeed, it is possible to obtain the skew in the Heston model by simulating, in combination, the dynamic of the stock prices and the dynamic of the variance. We may compare the result for European Call options:

1.1

Spot Price (S)	1
Strike Price (K)	1
Risk Free Rate (r)	0,03
Time to Maturity (Days)	730
Rho (ρ)	-0,5
Kappa (κ)	0,2

Theta (θ)	0,03
Lambda (λ)	2
Volatility of Variance (σ)	0,1
Current variance (*v*)	0,01
Number of Simulations	1.000
Heston Call Price	0,0833
Black Scholes Call Price	0,0887

1.2

Spot Price (*S*)	1,5
Strike Price (*K*)	1
Risk Free Rate (*r*)	0,03
Time to Maturity (Days)	730
Rho (ρ)	-0,5
Kappa (κ)	0,2
Theta (θ)	0,03
Lambda (λ)	2
Volatility of Variance (σ)	0,1
Current variance (*v*)	0,01
Number of Simulations	1.000
Heston Call Price	0,5647
Black Scholes Call Price	0,5583

1.3

Spot Price (*S*)	2
Strike Price (*K*)	1
Risk Free Rate (*r*)	0,03
Time to Maturity (Days)	730
Rho (ρ)	-0,5
Kappa (κ)	0,2
Theta (θ)	0,03
Lambda (λ)	2
Volatility of Variance (σ)	0,1
Current variance (*v*)	0,04
Number of Simulations	1.000
Heston Call Price	1,0717
Black Scholes Call Price	1,0582

We may compare the result for European Put options by setting a positive correlation coefficient:

2.1

Spot Price (S)	1
Strike Price (K)	1
Risk Free Rate (r)	0,03
Time to Maturity (Days)	730
Rho (ρ)	0,5
Kappa (κ)	0,2
Theta (θ)	0,03
Lambda (λ)	2
Volatility of Variance (σ)	0,1
Current variance (v)	0,01
Number of Simulations	1.000
Heston Put Price	0,0318
Black Scholes Put Price	0,0305

2.2

Spot Price (S)	0,75
Strike Price (K)	1
Risk Free Rate (r)	0,03
Time to Maturity (Days)	730
Rho (ρ)	0,5
Kappa (κ)	0,2
Theta (θ)	0,03
Lambda (λ)	2
Volatility of Variance (σ)	0,1
Current variance (v)	0,01
Number of Simulations	1.000
Heston Put Price	0,2039
Black Scholes Put Price	0,1945

2.3

Spot Price (S)	0,5
Strike Price (K)	1
Risk Free Rate (r)	0,03
Time to Maturity (Days)	730
Rho (ρ)	0,5
Kappa (κ)	0,2
Theta (θ)	0,03

Lambda (λ)	2
Volatility of Variance (σ)	0,1
Current variance (ν)	0,01
Number of Simulations	1.000
Heston Put Price	0,4421
Black Scholes Put Price	0,4418

We may note that the simulated approach is able to capture the skew of option pricing model for long maturities and for options deep in the money but for short maturities the simulation converges to the normal distribution, this is the same problem of numerical integration by assuming normal distribution so a good candidate is the quadrature approach as trapezoidal rule, but it is a field of recent research that has still to be refined to be more consistent with itself.

Appendix

To run the simulation we used the following VBA code:

```
Function HestonMCCall(kappa, theta, lambda, rho, sigmav, daynum, startS, r, startv, K, ITER)

Dim allS() As Double

simPath = 0

ReDim allS(daynum) As Double

deltat = (1 / 365)

For itcount = 1 To ITER

   lnSt = Log(startS)

   lnvt = Log(startv)

   curv = startv

   curS = startS

     For daycnt = 1 To daynum

        e = Application.NormSInv(Rnd)

        eS = Application.NormSInv(Rnd)

        ev = rho * eS + Sqr(1 - rho ^ 2) * e

        lnSt = lnSt + (r - 0.5 * curv) * deltat + Sqr(curv) * Sqr(deltat) * eS

        curS = Exp(lnSt)

     lnvt = lnvt + (kappa * theta – (kappa + lambda) * startv) * deltat + Sqr(startv) * sigmav * Sqr(deltat) * ev

        curv = Exp(lnvt)

        allS(daycnt) = curS

     Next daycnt

   simPath = simPath + Exp((-daynum / 365) * r) * Application.Max(allS(daynum) - K, 0)

Next itcount

 HestonMCCall = simPath / ITER

End Function
```

```
Function HestonMCPut(kappa, theta, lambda, rho, sigmav, daynum, startS, r, startv, K, ITER)

Dim allS() As Double

simPath = 0

ReDim allS(daynum) As Double

deltat = (1 / 365)

For itcount = 1 To ITER

  lnSt = Log(startS)

  lnvt = Log(startv)

  curv = startv

  curS = startS

    For daycnt = 1 To daynum

       e = Application.NormSInv(Rnd)

       eS = Application.NormSInv(Rnd)

       ev = rho * eS + Sqr(1 - rho ^ 2) * e

       lnSt = lnSt + (r - 0.5 * curv) * deltat + Sqr(curv) * Sqr(deltat) * eS

       curS = Exp(lnSt)

     lnvt = lnvt + (kappa * theta – (kappa + lambda) * startv) * deltat + Sqr(startv) * sigmav * Sqr(deltat) * ev

       curv = Exp(lnvt)

       allS(daycnt) = curS

     Next daycnt

   simPath = simPath + Exp((-daynum / 365) * r) * Application.Max(K - allS(daynum) , 0)

Next itcount

 HestonMCPut = simPath / ITER

End Function
```

Caplet and Floorlet

The price of Caplet and Floorlet may be derived directly from the arbitrage condition between Cap, Floor and Swap that is given by the following relation:

$$Nominal\ x\ \big[(1 + r(t)) + Floor - Cap\big]xT = Nominal\ x\ [1 + Swap\ Rate]xT$$

From the relation we may note that we have a fix rate $(1 + r(t))$ because if the interest rate decreases the Floor goes in the money so to have a fix rate, instead if interest rate increases the Cap goes in the money to subtract the greater earning to obtain a fix rate. Indeed the fix rate is not $(1 + r(t))$ because we income the price of the Cap less the price of the Floor, As such we have a fix rate that to avoid arbitrage opportunities must give the following prices for Caplet and Floorlet:

$$Caplet = P(T)\big[\ F(T)N[h1] - KN[h2]\big]$$

$$Floorlet = P(T)\big[\ KN[-h2] - F(T)N[-h1]\big]$$

Where:

$$h1 = \frac{\ln\left(\frac{F(T)}{K}\right) + \frac{1}{2}\sigma_F^2(T - t)}{\sigma_F\sqrt{T - t}}$$

$$h2 = \frac{\ln\left(\frac{F(T)}{K}\right) - \frac{1}{2}\sigma_F^2(T - t)}{\sigma_F\sqrt{T - t}}$$

$$\sigma_F = \sigma_r/F(T)$$

F(T) denotes the Forward of the Swap rate given by:

$$-\frac{\ln\left(\frac{exp - (Swap(T)xT)}{exp - (Swap(t)xt)}\right)}{T - t}$$

This may be considered the fair value as well but if we go in the OTC market we will not get the market prices because there isn't arbitrage and the price of Cap and Floor are equals, this suggests that they are priced with a geometric martingale such that we have the following pricing formula:

$$Caplet = P(T)\big[\ r(t)N[h1] - KN[h2]\big]$$

$$Floorlet = P(T)\big[\ KN[-h2] - r(t)N[-h1]\big]$$

Where:

$$h1 = \frac{\ln\left(\frac{r(t)}{K}\right) + \frac{1}{2}\sigma_r^2 T}{\sigma_r\sqrt{T}}$$

$$h2 = \frac{\ln\left(\frac{r(t)}{K}\right) - \frac{1}{2}\sigma_r^2 T}{\sigma_r\sqrt{T}}$$

Asian Options

We start with the presentation of a jump diffusion process:

$$\frac{dS(t)}{S(t)} = \mu(t)dt + \sigma(t)dWs + P(dt)N[a, \sigma]$$

P(dt) denotes a Poisson distribution and counts the number of jumps that are measured by the Normal distribution that is perfectly correlated with the Wiener process, the jump has the same direction. The problem it is easy to solve because in real markets the jump happens in every instant because the markets prices are not continuous as result we may solve the equation in the following way:

$$\frac{dS(t)}{S(t)} = (\mu - a)dt + \sigma(J)dWj$$

Where:

$$\sigma(J) = \sigma(t) + \sigma$$

So the effective volatility may be decomposed in two parts, a continuous part and a jump part. The pay off of an Asian option is not replicable because we do not have the underlying value, so the continuous time model drops as result the value of an Asian option is determined with approximation of the Average underlying on the effective volatility, as such we have the following pricing formula.

$$Call\ asian(S, T, K) = S(t)N[d1] - P(T)KN[d2]$$

$$Put\ asian(S, T, K) = P(T)KN[-d2] - S(t)N[-d1]$$

Where:

$$d1 = \frac{\ln\left(\frac{S}{KP(T)}\right) + \frac{2}{3}\sigma_N^2 T}{\sigma_N\sqrt{\frac{4}{3}T}}$$

$$d2 = \frac{\ln\left(\frac{S}{KP(T)}\right) - \frac{2}{3}\sigma_N^2 T}{\sigma_N\sqrt{\frac{4}{3}T}}$$

$$-d1 = \frac{\ln\left(\frac{KP(T)}{S}\right) - \frac{2}{3}\sigma_N^2 T}{\sigma_N\sqrt{\frac{4}{3}T}}$$

$$-d2 = \frac{\ln\left(\frac{KP(T)}{S}\right) + \frac{2}{3}\sigma_N^2 T}{\sigma_N\sqrt{\frac{4}{3}T}}$$

We may see deeply how to achieve a pricing formula for Asian option, we will make use of geometric average to approximate the arithmetic average, as we know the pay off of an Asian option is given by the average prices less the strike for Asian Call option and the strike less the average prices for Asian Put option, so the problem to obtain a closed form solution for the pricing is to approximate the average, we will show that the problem it is easy to solve if we use the forward for each calendar days, as result we have the following:

$$(S_1 F_2 \dots\dots F_n)^{\frac{1}{n}}$$

$$F^{\frac{n}{n}}(e^{-\frac{1}{2}\sigma^2 T_i\, n + \sigma\sqrt{T_i n}\, dW})^{\frac{1}{n}}$$

Now if we put the following:

$$T_i n = T$$

From which:

$$T_i = T/n$$

We obtain the following for $n \to \infty$:

$$Fe^{-\frac{1}{2}\sigma^2 T/2 + \sigma\sqrt{\frac{T}{3}}\, dW}$$

From this we may obtain a formulation by taking the expectation, and it is a formulation that make use of Ito's lemma, indeed the result is not market consistent although we may approximate F by using the actual price in the world of numeraire. Indeed, if we want to use the result of keeping the drift as average, and we assume that the continuous time model drops we may assume that we have the following distribution:

$$\frac{S}{P(T)}e^{-\frac{2}{3}\sigma_N^2 T + \sigma_N\sqrt{\frac{4T}{3}}\, dW}$$

We have to note that the lattice methods may be generalized to price Asian options as such we have the following:

$$C_{N,i} = Max\left[Average\left(Su^i d^{N-i}\right) - K, 0\right]$$

$$P_{N,i} = Max\left[K - Average\left(Su^i d^{N-i}\right), 0\right]$$

As such we assume the following:

$$u = e^{\sigma\sqrt{\Delta t}} \quad d = e^{-\sigma\sqrt{\Delta t}} \quad a = e^{r\Delta t}$$

The risk neutral probability is given by the following for up and down respectively:

$$p = \frac{a - d}{u - d} \quad 1 - p$$

The prices are given by:

$$C_{j,i} = e^{-r\Delta t}\left[p\, C_{j+1,i+1} + (1 - p)C_{J+1,i}\right]$$

$$P_{j,i} = e^{-r\Delta t}\left[p\, P_{j+1,i+1} + (1 - p)P_{J+1,i}\right]$$

The pricing formula depends from the number of steps that is a function of time of maturity so we do run the simulated result by assuming nodes equal to time for two that is equal to assume that the average will be computed two times for years, as such we have the following results for Asian Call options with respect the European options:

σ_S	K	r	S_t	T	Binomial Asian	Expected
0,07	1	0,03	1,5	1	0,42186	0,52955
0,07	1	0,03	1,25	1	0,21075	0,27956
0,07	1	0,03	1	1	0,02070	0,04477
0,07	1	0,03	0,75	1	0,00000	0,00000
0,07	1	0,03	0,5	1	0,00000	0,00000
0,07	1	0,03	1,5	2	0,48482	0,55824
0,07	1	0,03	1,25	2	0,24969	0,30830
0,07	1	0,03	1	2	0,04406	0,07427
0,07	1	0,03	0,75	2	0,00000	0,00030
0,07	1	0,03	0,5	2	0,00000	0,00000
0,07	1	0,03	1,5	5	0,56863	0,63930
0,07	1	0,03	1,25	5	0,33150	0,38975
0,07	1	0,03	1	5	0,11394	0,15236
0,07	1	0,03	0,75	5	0,00518	0,01308
0,07	1	0,03	0,5	5	0,00000	0,00001

And the following results for Asian Put options:

σ_S	K	r	S_t	T	Binomial Asian	Expected
0,07	1	0,03	1,5	1	0,00000	0,00000
0,07	1	0,03	1,25	1	0,00000	0,00000
0,07	1	0,03	1	1	0,02107	0,01522
0,07	1	0,03	0,75	1	0,21149	0,22045
0,07	1	0,03	0,5	1	0,42260	0,47045
0,07	1	0,03	1,5	2	0,00000	0,00000
0,07	1	0,03	1,25	2	0,00000	0,00007
0,07	1	0,03	1	2	0,02951	0,01603
0,07	1	0,03	0,75	2	0,22058	0,19207
0,07	1	0,03	0,5	2	0,45572	0,44176
0,07	1	0,03	1,5	5	0,00002	0,00001
0,07	1	0,03	1,25	5	0,00111	0,00046
0,07	1	0,03	1	5	0,02176	0,01306
0,07	1	0,03	0,75	5	0,15121	0,12379
0,07	1	0,03	0,5	5	0,38425	0,36071

Appendix

To run the simulation we used the following VBA code:

```
Function BinomialAsianCall(Spot, k, T, r, sigma, n)

Dim dt As Double, u As Double, d As Double, p As Double

    dt = T / n

    u = Exp(sigma * (dt ^ 0.5))

    d = 1 / u

    p = (Exp(r * dt) - d) / (u - d)

Dim S() As Double

ReDim S(n + 1, n + 1) As Double

    For i = 1 To n + 1

    For j = i To n + 1

        S(i, j) = Spot * u ^ (j - i) * d ^ (i - 1)

    Next j

Next i

Dim Opt() As Double

ReDim Opt(n + 1, n + 1) As Double

For i = 1 To n

    Opt(i, n + 1) = Application.Max(Application.Average(S(i, n + 1), S(i + 1, n + 1)) - k, 0)

Next i

For j = n To 1 Step -1

    For i = 1 To j

    Opt(i, j) = Exp(-r * dt) * (p * Opt(i, j + 1) + (1 - p) * Opt(i + 1, j + 1))

    BinomialAsianCall = Opt(i, j)

    Next i

Next j
```

```
End Function

Function BinomialAsianPut(Spot, k, T, r, sigma, n)

Dim dt As Double, u As Double, d As Double, p As Double

   dt = T / n

   u = Exp(sigma * (dt ^ 0.5))

   d = 1 / u

   p = (Exp(r * dt) - d) / (u - d)

Dim S() As Double

ReDim S(n + 1, n + 1) As Double

     For i = 1 To n + 1

   For j = i To n + 1

     S(i, j) = Spot * u ^ (j - i) * d ^ (i - 1)

   Next j

Next i

Dim Opt() As Double

ReDim Opt(n + 1, n + 1) As Double

For i = 1 To n

   Opt(i, n + 1) = Application.Max(k - Application.Average(S(i, n + 1), S(i + 1, n + 1)) , 0)

Next i

For j = n To 1 Step -1

   For i = 1 To j

   Opt(i, j) = Exp(-r * dt) * (p * Opt(i, j + 1) + (1 - p) * Opt(i + 1, j + 1))

     BinomialAsianPut = Opt(i, j)

   Next i

Next j

End Function
```

Valuing Barrier Options

Now it is interesting to analyses first passage model:

$$f(z) - f(z - H)$$

Where:

$$f(z - H) \; \exists \; \forall \; z > H$$

As result we have:

$$\int_{-\infty}^{+\infty} f(z) - f(z - H) = \int_{-\infty}^{+\infty} f(z) - \int_{H}^{+\infty} z \, f(z) = 1 - f(H)^{-1} \, N[-H]$$

We may assume the following affine process for the barrier:

$$\frac{dH}{H} = \pm \sigma_N \sqrt{T}$$

As such we may write the value of a Barrier knock out option as follows by using the numeraire $P(T)$:

$$Call \; out(S < H, K < H, T) = S \left(1 - \left(\frac{S}{HP(T)}\right)^{-1} N[h1] \right)$$

$$- P(T)K \left(1 - \frac{S}{HP(T)} \; N[h2] \right)$$

$$Put \; out(S > H, K > H, T) = - S \left(1 - \frac{HP(T)}{S} \; N[-h1] \right)$$

$$+ P(T)K \left(1 - \left(\frac{HP(T)}{S}\right)^{-1} \; N[-h2] \right)$$

$$h1 = \frac{\ln\left(\frac{S}{HP(T)}\right) - \frac{1}{2}\sigma_N^2 T}{\sigma_N \sqrt{T}}$$

$$h2 = \frac{\ln\left(\frac{S}{HP(T)}\right) + \frac{1}{2}\sigma_N^2 T}{\sigma_N \sqrt{T}}$$

$$Call \; out(S > H, K < H, T) = S \left(1 - \left(\frac{HP(T)}{S}\right)^{-1} N[h3] \right)$$

$$- P(T)K \left(1 - \frac{HP(T)}{S} \ N[h4]\right)$$

$$Put \ out(S < H, K > H, T) \ = - S \left(1 - \frac{S}{HP(T)} \ N[-h3]\right)$$

$$+ P(T)K \left(1 - (\frac{S}{HP(T)})^{-1} \ N[-h4]\right)$$

$$h3 = \frac{\ln\left(\frac{HP(T)}{S}\right) - \frac{1}{2}\sigma_N^2 T}{\sigma_N \sqrt{T}}$$

$$h4 = \frac{\ln\left(\frac{HP(T)}{S}\right) + \frac{1}{2}\sigma_N^2 T}{\sigma_N \sqrt{T}}$$

As such we may obtain the value of in barrier alive from the following equalities:

$$C(S, K, T) = Call \ out(S, K < H, T) + \ Call \ in \ alive(S, K < H, T)$$

$$P(S, K, T) = Put \ out(S, K > H, T) + \ Put \ in \ alive(S, K > H, T)$$

As result the price of a Call option is given by:

$$C(S, T, K) = S(t)N[d1] - \ P(T)KN[d2]$$

Where:

$$d1 = \frac{\ln\left(\frac{S}{KP(T)}\right) + \frac{1}{2}\sigma_N^2 T}{\sigma_N \sqrt{T}}$$

$$d2 = \frac{\ln\left(\frac{S}{KP(T)}\right) - \frac{1}{2}\sigma_N^2 T}{\sigma_N \sqrt{T}}$$

As result the value of a Put option is given by:

$$P(S, T, K) = P(T)KN[-d2] - \ S(t)N[-d1]$$

Where:

$$-d2 = \frac{\ln\left(\frac{KP(T)}{S}\right) + \frac{1}{2}\sigma_N^2 T}{\sigma_N \sqrt{T}}$$

$$-d1 = \frac{\ln\left(\frac{KP(T)}{S}\right) - \frac{1}{2}\sigma_N^2 T}{\sigma_N \sqrt{T}}$$

As result we have the following:

$$Call\ in\ alive(S < H, K < H, T) = S\left(-N[-d1] + \left(\frac{S}{HP(T)}\right)^{-1} N[h1]\right)$$

$$- P(T)K\left(-N[-d2] + \frac{S}{HP(T)}\ N[h2]\right)$$

$$Call\ in\ alive(S > H, K < H, T) = S\left(-N[-d1] + \left(\frac{HP(T)}{S}\right)^{-1} N[h3]\right)$$

$$- P(T)K\left(-N[-d2] + \frac{HP(T)}{S}\ N[h4]\right)$$

$$Put\ in\ alive(S > H, K > H, T)$$
$$= -S\left(-N[d1] + \frac{HP(T)}{S}\ N[-h1]\right)$$
$$+ P(T)K\left(-N[d2] + \left(\frac{HP(T)}{S}\right)^{-1} N[-h2]\right)$$

$$Put\ in\ alive(S < H, K > H, T)$$
$$= -S\left(-N[d1] + \frac{S}{HP(T)}\ N[-h3]\right) + P(T)K\left(-N[d2] + \left(\frac{S}{HP(T)}\right)^{-1} N[-h4]\right)$$

As such we may obtain the following survival probabilities:

$$\left(N[-d2] - \frac{S}{HP(T)}\ N[h2]\right)$$

$$\left(N[-d2] - \frac{HP(T)}{S}\ N[h4]\right)$$

$$\left(N[d1] - \frac{HP(T)}{S}\ N[-h1]\right)$$

$$\left(N[d1] - \frac{S}{HP(T)}\ N[-h3]\right)$$

Structural Model

It is interesting now to analyse structural model for credit risk, the basic idea is that equity value of a levered firm may be seen as Call option on the assets of the firm with strike price equal to the face value of the debts, in particular the balance sheet of the company may be decomposed by the following:

$$E(t) = C(A, L, T)$$

$$L(t) = L(T)P(T) - P(A, L, T)$$

The Equity value E(t) is a Call option C(A,L,T) on the firm's under laying value A(t), that represents the asset portfolio, with strike prices equal to the final value of the debts L(T). Instead, the initial value of the debts L(t) is given by the debts value discounted by using the risk free discount factor P(T) with a short position on the Put option P(A,L,T) on firm's under laying value A(t), with strike prices equal to the final value of the debts. It reflects the option of stake holder to walk away if things go wrong by leaving the liabilities holders with the residual value of the company. We may note that the PD probability is given by:

$$PD = N[d1]$$

Where:

$$d1 = \frac{\ln\left(\frac{LP(T)}{A(t)}\right) - \frac{1}{2}\sigma_N^2 T}{\sigma_N\sqrt{T}}$$

We may note that we are assuming that there is just a single debt with time of maturity but we may assume that the company may default before to maturity by assuming a default barrier, as such we have the following:

$$E(t) = A(t) - L(t)$$

$$L(t) = L(T)P(T) - Put\ out(A, L, T) - (L - H)P(T)\left(N[d1] - \frac{HP(T)}{A(t)} N[h1]\right)$$

$$Put\ out(A > H, L > H, T) = -A(t)\left(1 - \frac{HP(T)}{A(t)} N[h1]\right)$$

$$+ P(T)L(T)\left(1 - (\frac{HP(T)}{A(t)})^{-1} N[h2]\right)$$

$$h1 = \frac{\ln\left(\frac{HP(T)}{A(t)}\right) + \frac{1}{2}\sigma_N^2 T}{\sigma_N\sqrt{T}}$$

$$h2 = \frac{\ln\left(\frac{HP(T)}{A(t)}\right) - \frac{1}{2}\sigma_N^2 T}{\sigma_N\sqrt{T}}$$

$$d1 = \frac{\ln\left(\frac{A(t)}{HP(T)}\right) + \frac{1}{2}\sigma_N^2 T}{\sigma_N\sqrt{T}}$$

We may note that the PD probability is given by:

$$PD = \left(1 - \frac{HP(T)}{A(t)} N[h1]\right) - \left(N[d1] - \frac{HP(T)}{A(t)} N[h1]\right)$$

Intensity Models

Now we will present an intensity model approach that is based on the instantaneous probability of default. The price of a credit with face value 1 may be expressed by the following formulation:

$$1 - h(t)(1 - R)dt$$

Where $h(t)$ denotes the probability of default and R the recovery rate The formulation may be rewritten by the following:

$$e^{-h(t)(1-R)T}$$

This is the survival probability, so by weighting the face value of the credit with the survival probability we may obtain a credit value adjustment (CVA). On the other side we have the following for the probability of default:

$$1 - e^{-h(t)(1-R)T}$$

We have along the interest rate models the following:

$$Credit\ Risk\ Yield = r(T) + h(t)(1 - R) + \frac{1}{2}\sigma_h^2 T + \sigma_h^2 T \quad \sigma_h = \sqrt{PD * (1 - PD)}\,(1 - R)$$

On the credit risk yield we may have liquidity and risk premium as in the interest rate models. We have assumed no drift for the probability of default that is equal to assume that it is stable. From this we may have the CVA by considering the Liquidity and risk premium that may be considered the Credit Value at Risk, so the main point is the variance such that we may consider the case of bilateral credit risk and the wrong way risk by taking the variance between the two references and computing the correlation coefficient and the PD spread between the two references. Indeed, we may have a greater credit risk yield due to the systemic risk, but we may obtain the information with the copula approach. The systemic risk may be estimated on the base of deco relation risk, the main idea is that an entity is very correlated with other entities the systemic risk is low because the system will cover each entities with each others, it is the case for example when assets and liabilities between different entities is mixed but for instance if an entity is deco related from the others that have the liabilities of the deco related entity in their assets, i.e. negative correlation, we have the systemic risk, i.e. the deco related entity may bring a systemic risk in the mixed entities or the group, as such we have the following measure of systemic risk:

$$N\left[\frac{\left(N^{-1}\left(1 - e^{-h(t)(1-R)T}\right) - \rho\right)}{\sqrt{1 - \rho^2}}\right]$$

If the correlation with the group increases the systemic risk decreases, instead, if the correlation with the group decreases the systemic risk increases such that we may have negative correlation that is really systemic risk as the deco relation increases the systemic risk increases and after begins to decrease as the entities become totally different from the group, i.e. without relationing.

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