A three-pole filter understanding of the average value of a Fourier series

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A three-pole filter understanding of the average value of a Fourier series and comparison with Hodrick-Prescott filter

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Abstract

This paper develops a method of analyzing average value of a complex-valued function that can be represented as a Fourier series satisfying a few realistic restrictions. This method may be useful when Discrete Fourier transform is highly inefficient, and comparison with Hodrick-Prescott filter is made.

1 Introduction: Hodrick-Prescott filter

Hodrick-Prescott filter has the following frequency response:

$$H(\omega) = \frac{4\lambda(1 - \cos(\omega))^2}{4\lambda(1 - \cos(\omega))^2 + 1} \tag{1}$$

as shown in [3]. The filter is high-pass filter - but one can convert it to become a low-pass filter, since the low-pass part that was filtered out can be obtained simply by subtraction.

The HP high-pass filter dependence on smoothing parameter λ implies a need for decision: one can allow more low-frequency parts to be on data by increasing λ and at first this decision seems good since the trend is obtained more clearly, but in many econometric usage, frequency is something related to error terms, and the linear trend at $\omega = 0$ may not be the trend to be detrended.

Even if one wishes to obtain very close approximation linear trend, though, complexities around Hodrick-Prescott filter often does not allow it.

In this paper, though, I consider the case where one wishes to obtain exact linear trend. It is not practically possible for all time processes, but possible when $f(t) : \mathbb{R} \to \mathbb{C}$ can be represented as fourier series involving finite harmonics, with other realistic assumptions on magnitude and negative/positive sign of individual amplitude of harmonics.

After performing analysis in z-transform, I consider a graph theory example.

2 z-transform analysis

2.1 Three-pole filter-based analysis

From now on, angular frequency will be represented by both ω and u. Let the digital filter based on z-transform be:

$$H(z) = \frac{1}{(z - p_1)(z - p_2)(z - p_3)} \tag{2}$$

where $p_1, p_2 \in \mathbb{R}$ will later be defined.

Definition 2.1 (f_1, f_2, μ_1, μ_2). $f_1(t) = f(t) - n_h$, $f_2(t) = n_h$, $\mu_1(t) = e^{iut}$, $\mu_2(t) = 1$.

Definition 2.2 (γ_1, γ_2). $\gamma_1 = 1 - p_1$, $\gamma_2 = 1 - p_2$.

Definition 2.3 (Constant terms). Constant terms are the ones not depend on angular frequency u.

Definition 2.4 (Atomic term). An atomic term refers to the term defined only by multiplication of positive powers of either γ_1, γ_2 or u, and constant $k \in \mathbb{C}$. For convenience, for B/C, with both B and C are sums of atomic terms, when referring to an atomic term, I will assume there should be no other atomic terms j in the equation with $k = k_j$ such that only k differ - in other words, $k = \sum_j k_j$.

Definition 2.5 (Expanding factor). Expanding factor refers to N/D, where D is the dominant (in magnitude) atomic term of C for B/C with B, C sums of atomic terms. By definition, any expanding factor is much less than 1 in magnitude.

   The idea of expanding factors is based on the following principle. Suppose that $d = c/(a + b + e + f + ...)$, and a is the dominant term. The first term will be c/a.

$$d - \frac{c}{a} = \frac{(-c/a)(a + b + e + ...)}{(a + b + e + f..)}$$

Notice that in the numerator, $b, e, f..$ can be noticed, and in the denominator, the dominant term a appears again. Thus $b/a, e/a, f/a$ are referred to as expanding factors.

Furthermore, additive inverse of these new terms will be multiplied by expanding factors to produce new terms, and this process continues indefinitely.

For input of $\mu_1(t) = e^{iut}$, $\mu_1(z)H(z)$ is:

$$H(z)\mu_1(z) = \frac{z}{(z - p_1)(z - p_2)(z - p_3)(z - e^{iu})} \tag{3}$$

Let $\gamma_2 = k_2 iu$, $\gamma_3 = iu/k_3$ Assume that

$$\gamma_1 \gg 1, \ \gamma_2 \ll 1, \ \frac{k_2}{k_3} \ll 1, \ |u| \ll 1 \tag{4}$$

The terms I will be looking for are of the form $k\gamma_3/(\gamma_1\gamma_2{}^2)$ with $k \in \mathbb{C}$. The idea is that it may be the case that the terms obtained for μ_1 and μ_2 may be different, because of Equation 4. Let us call this "discrepancy procedure."
Taking the inverse z-transform residue for pole p_1 and subtracting away $t = 0$ part at $t = 1$:

$$\frac{-\gamma_1}{(\gamma_2 - \gamma_1)(\gamma_1 - \gamma_3)(\gamma_1 + iu - \frac{u^2}{2} + ..)} \tag{5}$$

The initial term is:

$$\frac{1}{\gamma_1{}^2}$$

Notice that constant expanding factors and the initial term are shared in case of the input μ_2. For analyzing discrepancy procedure, the pole will be ignored. (I will later find the term being searched in case of pole p_1.)
For pole p_2, the inverse z-transform calculation:

$$\frac{-\gamma_2}{(\gamma_1 - \gamma_2)(\gamma_2 - \gamma_3)(\gamma_2 + iu - \frac{u^2}{2} - ...)} \tag{6}$$

The initial term is:

$$\frac{-1}{\gamma_1\gamma_2}$$

Again, constant expanding factors and the initial term are shared for both inputs μ_1 and μ_2. Thus, for discrepancy procedure analysis, the pole will be ignored. For pole p_3, the inverse z-transform calculation:

$$\frac{-\gamma_3}{(\gamma_1 - \gamma_3)(\gamma_3 - \gamma_2)(\gamma_3 + iu - \frac{u^2}{2} - ..)} \tag{7}$$

The initial term is:

$$\frac{\gamma_3}{\gamma_1\gamma_2 iu}$$

By Equation 4, multiplying the additive inverse of the initial term by the expanding factor with greatest magnitude is:

$$\frac{-\gamma_3}{\gamma_1\gamma_2 iu}\frac{\gamma_1\gamma_2\gamma_3}{-\gamma_1\gamma_2 iu} = \frac{\gamma_3{}^2}{\gamma_1\gamma_2 u^2}$$

By equation 4, this is less than the term being searched for in magnitude. For pole e^{iu}, inverse z-transform calculation:

$$\frac{iu - \frac{u^2}{2} - ..}{(\gamma_1 + iu - \frac{u^2}{2} - ..)(\gamma_2 + iu - \frac{u^2}{2} - ..)(\gamma_3 + iu - \frac{u^2}{2} - ..)} \tag{8}$$

The initial terms:

$$\frac{1}{\gamma_1\gamma_2}, \ \frac{-u}{2\gamma_1\gamma_2 i}, ..$$

The two expanding factors in order from greatest magnitude to least are:

$$\frac{\gamma_1(iu)(iu)}{\gamma_1\gamma_2 iu} = \frac{-u}{\gamma_2 i}, \ \frac{\gamma_1\gamma_2\gamma_3}{\gamma_1\gamma_2 iu} = \frac{\gamma_3}{iu}$$

These two expanding factors can be used to obtain:

$$2 \cdot \frac{1}{\gamma_1\gamma_2} \frac{-u}{\gamma_2 i} \frac{\gamma_3}{iu} = \frac{2\gamma_3}{\gamma_1{\gamma_2}^2} \tag{9}$$

(2 in Equation 9 comes from the fact that there are two ways to achieve this form, by re-arranging two expanding factors.) The first expanding factor may be used twice, but because the numerator then now contains u^2, this requires use of two expanding factors that are dependent on $1/u$.

Now one can restrict the search to a single use of a constant expanding factor.

$$\frac{-1}{\gamma_1\gamma_2} \frac{\gamma_1\gamma_3 iu}{\gamma_1\gamma_2 iu} = \frac{-\gamma_3}{\gamma_1{\gamma_2}^2} \tag{10}$$

Let us now consider the second initial term. Multiply by the expanding factor with greatest magnitude:

$$\frac{u}{2\gamma_1\gamma_2 i} \frac{-u}{\gamma_2 i} = \frac{u^2}{2\gamma_1{\gamma_2}^2}$$

This term is less than the term being searched for, since $u^2 \ll \gamma_3$.

Sum up Equation 9 and 10 and one obtains:

$$\frac{\gamma_3}{\gamma_1{\gamma_2}^2} \tag{11}$$

This completes the discrepancy analysis.

The additional requirements will be studied in the "separation principles" subsection.

2.2 Filter adjustments

Let us recall the formula of the filter:

$$H_a(z) = \frac{1}{(z - p_1)(z - p_2)(z - p_3)}$$

These filter forms in practice do need to change, since it is usually assumed that the filter input and output are both zero before $t = 0$. The following changes will be made:

$$H_c(z) = \frac{z^3}{(z - p_1)(z - p_2)(z - p_3)} \tag{12}$$

The analysis above works if we replace $t = 0$ by $t = -3$ for the two-pole filter case (with the filter replaced by $H_c(z)$). But we assumed that the filter outputs

and inputs are zero before $t = 0$, so the modified procedure is needed.

First, let us derive the difference equations. Let the filter output be $\nu(t)$.

$$\nu_c(t) = (p_1+p_2+p_3)\nu_c(t-1)-(p_1p_2+p_1p_3+p_2p_3)\nu_c(t-2)+p_1p_2p_3\nu_c(t-3)+f(t) \tag{13}$$

where $f(t)$ is initially treated as zero before $t = 0$.

For going backward in time, the filter needs to be adjusted as:

$$H_d(z) = \frac{z^3}{(z - 1/p_1)(z - 1/p_2)(z - 1/p_3)} \tag{14}$$

and with the input to the filter provided backward. This means that the difference equation is:

$$\nu_c(t) = \left(\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3}\right)\nu_c(t+1)-\left(\frac{1}{p_1p_2} + \frac{1}{p_1p_3} + \frac{1}{p_2p_3}\right)\nu_c(t+2)+\frac{1}{p_1p_2p_3}\nu_c(t+3)+f(t) \tag{15}$$

for $t < 0$ and $f(t)$ not assumed to be zero before $t = 0$.

These procedures allow one to retain z-transform analysis done in this section.

3 Example: graph theory grid

I will use the analysis developed above for finding the number of hamiltonian paths: here n_h. Hamiltonian path models have recently been used to study Nash equilibrium, as in [4] and [2].

Definition 3.1 (Z^+, Z_+, Z^-, Z_-)**.** Z^+ or Z_+ refers to the set of positive integers. Similarly, Z^- refers to the set of negative integers.

Definition 3.2 ("less than", "more than", "greater than", "smaller than")**.** Unless otherwise noted, these are all comparisons in magnitude/size/absolute value.

Definition 3.3 (Base-n expansion)**.** Base-n expansion of some number k is basically expressing k in base-n: $k = \pm\sum_{p=-\infty}^{\infty} a_p n^p$ with $0 \leq a_p < n$.

The power of base-n is that if important parameters are the finite sums (that is, $k = \pm\sum_{p=b_l}^{b_h} a_p n^p$, with b_l and b_h finite), instead of infinite sums, then analysis becomes much easier. For studying numerical approximation of k (if exact value cannot be known), one can just focus on finite number of numerical digits.

Definition 3.4 (graph, n)**.** A graph G is denoted with $G = (V, E)$ as done in the standard literature. $n = |V|$ is assumed whenever n appears.

Definition 3.5 (walk, n-walk, hamiltonian path)**.** A walk is defined as in the standard graph theory vocabulary. A walk that has n vertices is called n-walk. Let us represent a walk with a list (tuple) of vertices in a traversing order from the start vertex to the end vertex. By the definition of a walk, one vertex can appear more than once in a list. A hamiltonian path, as defined in the standard graph theory vocabulary, is a walk with n distinct vertices, where $|V| = n$.

Definition 3.6 (vertex)**.** A vertex is assigned a number. Each distinct vertex has a distinct number. Let $V = \{n, n^2, n^3, ..., n^n\}$. From now on, one can assume a vertex as a number whenever appropriate.

Definition 3.7 (n_h, n_p)**.** n_h is the number of hamiltonian paths of G. n_p is the total number of n-walks of G.

Definition 3.8 (Vertex-number)**.** The vertex-number of a walk is defined as the sum of all elements (vertices) in the list of a walk.

Note that the vertex-number of a walk represents the angular frequency of a walk in $x(t)$, as will be seen. It is certainly possible that two walks may occupy the same frequency. If there are k walks that occupy the same frequency ω_a, then the amplitude at the frequency would be k in Fourier series language, or $k\delta(\omega - \omega_a)$ in Fourier transform language where $\delta(\omega)$ is a dirac delta function. **The maximum number of vertices inside a walk is restricted to n, for sake of convenience.**

Definition 3.9 (Permutation of a list)**.** A permutation of a list is a re-ordering of list elements in ξ. For example, for $\xi = (\xi_1, \xi_2, .., \xi_n)$, $\xi_\alpha = (\xi_n, \xi_{n-3}, \xi_{n-4}, ..., \xi_1)$ is a permutation of ξ.

Lemma 3.1. *Given V as defined above, a vertex-number can only be formed out of a permutation of a single vertex-number list.*

Proof. The proof is simply the basis representation theorem, where basis are elements in V. One exception to this proof, though, arises when a list ξ representing a walk may be of $(k, k, ...k)$ with $|\xi| = n$ and $k = n^i$, or in words, there are n k's in ξ. In this case, $nk = n^{i+1}$, meaning the vertex-number ξ equals one of vertices in V. But this should not matter whenever walks one deals with have same number of vertices. $\qquad\square$

Following from above:

Definition 3.10 (Contribution of each n-walk to $x(t)$)**.** From above, each walk has a vertex number k. Each n-walk is said to contribute e^{ikt} to $x(t)$.

Definition 3.11 (Amplitude)**.** For any arbitrary function $\alpha(t)$ expressible as $\alpha(t) = \sum_{\omega=-\infty}^{\infty} A_\omega e^{i\omega t/d}$ where d is constant and does not vary with ω, A_ω is said to be amplitude of $\alpha(t)$ at angular frequency ω.

3.1 Grid: $x(t)$

Definition 3.12 (Grid, wires)**.** A grid consists of n depths, with each depth being equivalent to a column. Each depth contains n vertices as in V. Each wire connects a vertex v_α from ith depth to a vertex point of v_β in $i+1$th depth. A wire is connected between v_α to v_β if and only if $(v_\alpha, v_\beta) \in E$.

Definition 3.13 (Function transmission: first depth case)**.** In the first depth (first column), each vertex v_α transmits $e^{iv_\alpha t}$.

Definition 3.14 (Function transmission except for first and nth depth). Defining for each v_α in arbitrary ith depth. All incoming wire transmissions $w_\zeta(t)$ from each wire ζ from $i-1$th depth to v_α in ith depth are summed, or equivalently $w_\lambda = \sum_\zeta w_\zeta$. And then multiply by $e^{iv_\alpha t}$ and transmit $u_{v_\alpha} = e^{iv_\alpha t} w_\lambda$ to each wire starting from v_α.

Definition 3.15 (Vertex point function transmission: nth depth case). All incoming wire transmissions $w_\zeta(t)$ from each wire ζ from $n-1$th depth to v_α in nth depth are summed, or equivalently $w_\lambda = \sum_\zeta w_\zeta$. And then multiply by $e^{iv_\alpha t}$, resulting in $s_{v_\alpha} = e^{iv_\alpha t} w_\lambda$. $x_{ideal}(t) = \sum_{v \in V} s_v$ is the output of the grid, not considering quantization errors involved.

For each depth i, $\sum_{v \in V} u_v$ shows the sum of all vertex-numbers representing i-walk.

3.2 Post-grid: $y(t)$

Simply, this post-grid procedure is all about calculating $y(t) = x(t)e^{-iht}$ where $h = \sum_{i=1}^{n} n^i$, the hamiltonian frequency of $x(t)$. Thus, $y(t)$ has 0 has hamiltonian frequency.

3.3 Post-grid: $f(t)$

$f(t)$ is defined as $f(t) = y(ct)$. c will be defined later.
Let the angular frequencies of $f(t)$ be labelled with u. $u = 0$ refers to hamiltonian frequency.
From now on, when it is said "every u," this refers to every u with non-zero amplitude in $f(t)$.

3.4 Sinusoidal quantization errors

For every vertex v of each depth of the grid, the numbers from maximum of n vertices are added and then multiplied by e^{ivt}, for each t.
For each vertex v of each depth, the error occurred would be of the following form:

- (Sum of errors from previous depths) $\times$ (e^{ivt} + calculation error for e^{ivt}) + (The correct sum of previous depths) $\times$ (calculation error for e^{ivt}).

Here the purpose of error analysis is not to find out exact error but to derive the formula for the magnitude that is equal or bigger than actual possible maximum error.
Assume that the correct value of the previous depth is always $2n^n > |n^n + n^n i|$, and the correct value of e^{ivt} is $2 > |1 + 1i|$. This is bigger than it actually is, thus Equation 15 is an overestimate of the sum of errors.

Definition 3.16 (e_v, e_i). e_i representing total maximum sinusoidal quantization error in magnitude occurring from depth 1 to depth i of all vertices, and e_v represents the maximum error in magnitude that occurs from calculating e^{ivt}.

Note that e_v and e_i represents entirely different things, and v inside e_v is not an index, unlike i, which is an index, in e_i.
Thus, this will yield the following recurrence equation:

$$e_{i+1} = n^2 \left[(2 + e_v)e_i + 2n^n e_v\right] \tag{16}$$

Now let us simplify Equation 15 by the following substitutions:

$$\Lambda = 2n^2 + n^2 e_v, \ \Upsilon = 2n^n e_v \tag{17}$$

$$e_{i+1} = \Lambda e_i + \Upsilon \tag{18}$$

Assuming that we start from $e_0 = 0$ (for sure, depth 0 does not exist, but this can safely be used), by geometric series formula,

$$e_i = \Upsilon \frac{\Lambda^i - 1}{\Lambda - 1} \tag{19}$$

$$e_{i=n} \equiv e_n = \Upsilon \frac{\Lambda^n - 1}{\Lambda - 1} < \Upsilon \Lambda^n \tag{20}$$

Assuming that $e_v < 1/n^2$, we can assume that $\Lambda < 3n^2$.
To incorporate the errors occurring from further calculating $y(t)$,

$$e_{i=n+1} \equiv e_{n+1} = \Upsilon \frac{\Lambda^{n+1} - 1}{\Lambda - 1} < \Upsilon \Lambda^{n+1} \tag{21}$$

with $\Lambda^{n+1} \approx 3^{n+1} n^{2n+2} \approx n^{3n+3} \approx n^{4n}$, assuming $e_v < 1/n^2$. The required range of e_v will be determined in the "Separation Principles" subsection.

4 Conclusion

This paper presented an alternative way of deriving the average value of a complex-valued Fourier series with realistic restrictions in case Discrete Fourier transform becomes costly. The paper then presents a possible application of the analysis.

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