Downturn LGD: A Spot Recovery Approach

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13 January 2010

From Spot recovery to Downturn LGD

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First version: January 2010
Current version: April 2013

Basel II suggests that banks estimate downturn loss given default (DLGD) to capture the systemic correlation between default rate and loss given default through economic cycles. However, previous approaches in the literature may not be internally consistent and may have bias in capital calculation. In this paper, we propose a new consistent model framework based on our recent work on stochastic spot recovery. We also compare numerically the downturn LGD in our model with some of the previous approaches.

1. Introduction

Evidence from historic data suggests that recovery rates on corporate defaults tend to decrease when default rates increase in an economic downturn [1]. This phenomenon leads the BIS to suggest banks estimate downturn loss given default (DLGD) for capital requirement calculation [4, 5]. The main reason for this requirement is that the Vasicek model [22] used in the Basel Accord does not have systemic correlation between probability of default (PD) and loss given default (LGD), which would underestimate downturn risk.

There have been several attempts to model the dependence between PD and LGD, see for example [2, 3, 7, 8, 9, 10, 12, 17, 18, 19, 20]. Most of the approaches model the term LGD (LGD in a period of time) driven by a latent variable that is correlated with the latent variable driving default. This kind of approach has some drawbacks, as will be discussed in section 3 of the current paper. The key point is that the specification of the correlation between PD and LGD may not be internally consistent in a multi-period or continuous time setting or may lead to bias in capital calculation. Similar problems in CDO pricing with stochastic recovery have led to the direct modeling of spot recovery (or recovery at time of default) [6, 15]. The purpose of this paper is to use our recently proposed stochastic spot recovery model to build a consistent downturn LGD model for Basel II capital calculation. If we view the previous approaches as based on a structured model where default can only occur at the end of a period, then spot recovery model is related to the fact that default can actually be triggered any time during the period. Essentially, term LGD is a weighted average of spot LGD such that direct specification of term LGD may lead to inconsistent spot LGD. Our approach shows that a term recovery model may not lead to a consistent spot recovery model, but the reverse is true:

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a consistent spot recovery model leads to a consistent term recovery model. Although capital allocation is typically done for one year horizon, it should be part of a consistent multi-period or continuous time credit model and thus restriction on modeling assumptions may apply. The granularity of default at any time and recovery conditional on default time requires more consistent modeling of term LGD.

The paper is organized as follows. In section 2, we discuss the general concept of spot recovery and term recovery, and present the consistent condition for a term recovery model. In section 3, we discuss the Tasche model and the Chabaane-Laurent-Salomon model, and show the inconsistency or bias they may have if not used properly. In section 4, we present our stochastic spot recovery model using a two-factor setup. We also derive the copula that correlates both default time and recovery rate. In section 5, we derive the large homogeneous pool limit for the Tasche model, the Chabaane-Laurent-Salomon model and our spot recovery model in a single systemic factor case. Then we show how VaR can be calculated and define downturn LGD for all these models. In section 6, we give numerical examples to compare downturn LGD and show the bias in the Chabaane-Laurent-Salomon model. Section 7 concludes the paper.

2. The concept of spot recovery and term recovery

The spot recovery rate is the recovery rate paid on a debt at the time when the issuer defaults. It happens at the time of default and is conditional on default. To simplify the model, we ignore the time delay between issuer default and the actual recovery payout, which is common in the literature and may be relaxed by adding a time-delay. Assume τ is the default time random variable in a probability space $(\Omega, \Sigma, \mathrm{P})$ endowed with a filtration $(\Sigma_t)_{0\le t<\infty}$ modeling the flow of market information through time. We do not limit τ by the maximum maturity of an issuer's debts. τ is a stopping time under the filtration such that $\{\omega : \tau(\omega) \le t\} \in \Sigma_t$. Assume there is a stochastic process r that specifies the spot recovery rate at time of default as r_τ such that the mapping $\omega \to r_{\tau(\omega)}(\omega)$ is measurable and r_τ is indeed a random variable. Similarly, we can define the LGD random variable as $l_\tau = 1 - r_\tau$. The spot recovery at time t is the conditional random variable $r_\tau | \tau = t$. The term recovery rate in the time period $(0, t]$ is the conditional random variable $r_\tau | \tau \le t$. Note that t can be anytime in the future in a continuous-time model. In the traditional credit capital modeling, people tend to look at a fixed time horizon where term recovery is normally considered. But, for proper modeling of credit risk, a continuous time model is preferred where default can happen at any time and recovery is conditional on default time.

In a factor model, we may have a large number of issuers and the probability space is extended to include some random factors representing the economic state. Following the discussion in [6], the conditional expected spot recovery rate for an issuer at time t is defined as

$$\bar{r}(t,Z) = E\left[r_\tau \middle| \tau = t, Z\right] \tag{1}$$

and the conditional expected term recovery rate up to time t is defined as

$$\bar{R}(t,Z) = E\left[r_\tau \middle| \tau \le t, Z\right] \tag{2}$$

where Z is a random variable in an extended probability space, representing some systemic factor affecting default rate and recovery of all the issuers. As we will see later, Z is introduced to correlate the credit quality of the issuers, but it will not affect marginal distributions of default or recovery rate as in the copula models. The loss up to time t as a stochastic process is defined as

$$L(t) = (1 - r_\tau) \cdot 1_{\tau \le t} \tag{3}$$

Assume the conditional default probability is given by $p(t,Z) = P(\tau \le t | Z)$, then the conditional expected loss is

$$\begin{aligned}\bar{L}(t,Z) &= E\left[(1 - r_\tau) \cdot 1_{\tau \le t} \middle| Z\right] \\ &= \int_0^t (1 - \bar{r}(s,Z)) \cdot dp(s,Z) \\ &= (1 - \bar{R}(t,Z)) \cdot p(t,Z)\end{aligned} \tag{4}$$

where we have assumed that, conditional on Z, default and recovery are independent. From this we have the following relationship between conditional expected spot recovery and conditional expected term recovery

$$\bar{R}(t,Z) = \int_0^t \bar{r}(s,Z) \frac{dp(s,Z)}{p(t,Z)} \tag{5}$$

If we define the conditional expected spot recovery first and derive the conditional expected term recovery from the above relationship, term recovery will naturally fall in the range of $[0,1]$ as long as spot recovery is in the same range. However, if we define the conditional expected term recovery first, the derived conditional expected spot recovery is not guaranteed to be in the range $[0,1]$. As

$$\bar{r}(t,Z) = \frac{\partial_t (\bar{R}(t,Z) \cdot p(t,Z))}{\partial_t p(t,Z)} = \bar{R}(t,Z) + p(t,Z) \cdot \frac{\partial_t \bar{R}(t,Z)}{\partial_t p(t,Z)} \tag{6}$$

the consistent condition would be

$$0 \le \bar{r}(t,Z) = \bar{R}(t,Z) + p(t,Z) \cdot \frac{\partial_t \bar{R}(t,Z)}{\partial_t p(t,Z)} \le 1 \quad (7)$$

Note that the condition should also hold without conditioning on Z, or by integrating out Z. However, the consistent condition may be broken more for the reason that the dependency on Z is not defined correctly, in which case, the spot recovery may no longer be measurable in the extended probability space including Z.

If we look at the conditional expected spot LGD defined as

$$\bar{l}_d(t,Z) = E\left[(1-r_\tau) \middle| \tau = t, Z\right] \quad (8)$$

and the conditional expected term LGD defined as

$$\bar{L}_d(t,Z) = E\left[(1-r_\tau) \middle| \tau \le t, Z\right] \quad (9)$$

then the conditional expected loss is

$$\bar{L}(t,Z) = \bar{L}_d(t,Z) \cdot p(t,Z) = \int_0^t \bar{l}_d(s,Z) dp(s,Z) \quad (10)$$

and

$$\bar{l}_d(t,Z) = \frac{\partial_t \bar{L}(t,Z)}{\partial_t p(t,Z)} = \bar{L}_d(t,Z) + p(t,Z) \cdot \frac{\partial_t \bar{L}_d(t,Z)}{\partial_t p(t,Z)} \quad (11)$$

So the consistent condition for a term loss model can be stated as

$$0 \le \frac{\partial_t \bar{L}(t,Z)}{\partial_t p(t,Z)} = \bar{L}_d(t,Z) + p(t,Z) \cdot \frac{\partial_t \bar{L}_d(t,Z)}{\partial_t p(t,Z)} \le 1 \quad (12)$$

The consistent conditions can be extended beyond conditional expectation to conditional probability distribution. The conditional probability distribution of term recovery can be used to derive the conditional probability distribution of spot recovery, which can then be used to derive a relationship between spot recovery and the factors using the quantile function defined in the next section.

3. Issues with current LGD factor models

The models proposed for downturn LGD are mostly factor models. The Tasche model [20] assumes the same latent variable drives both default and loss give default so that the latent variable is actually driving the unconditional loss. All other models assume a correlated latent variable drives the LGD, where the difference is in the number of systemic or idiosyncratic factors. Frye [9] uses a single systemic factor with an independent idiosyncratic factor to drive the LGD. Pykhtin [18] also uses a single systemic factor but with an idiosyncratic factor that is correlated with the idiosyncratic factor driving default. Hillebrand [12] and Barco [3] assume two systemic factors but no idiosyncratic factor. Andersen and Sidenius [2] discuss two systemic factors with one independent idiosyncratic factor. Chabaane, Laurent and Salomon [7] discuss a more general factor correlation structure that is equivalent to two correlated systemic factors and two correlated idiosyncratic factors. The two types of models both may have some internal problems if not used carefully, which will be discussed below.

3.1. The Tasche Model

First we discuss the Tasche model following our previous work [14]. Let $L(t)$ be the loss up to time t defined in equation (3). Then $L(t)$ will be zero with probability $1-p(t)$ when the obligor is not in default before time t. $L(t)$ may take positive values with probability $p(t)$ if the obligor defaults before time t. Formally, the cumulative distribution function F_L of $L(t)$ has the following general form

$$F_L(x) = P(L(t) \le x) = 1 - p(t) + p(t) \cdot F_d(x) \qquad \text{for } x \in [0,1] \tag{13}$$

where $F_d(x) = P(L(t) \le x \mid \tau \le t) = P(l_\tau \le x \mid \tau \le t)$ is the cumulative distribution of term LGD. Note that $F_d(x)$ could be independent of $p(t)$, e.g., when it is a Beta distribution calibrated to historic data through economic cycles. Define the quantile function F_L^{-1} of F_L as

$$F_L^{-1}(y) = \min\{x \in [0,1] : F_L(x) \ge y\} \qquad \text{for } y \in [0,1] \tag{14}$$

Assume default of an obligor is linked to the latent variable $V = \sqrt{\rho}Z + \sqrt{1-\rho}\varepsilon$ through a default threshold $v = \Phi^{-1}(p(t))$, where Z and ε are independent standard normal random variables, Z is the systemic factor, ρ is the correlation coefficient and $\Phi(x)$ is the cumulative normal distribution function. Then we can model the dependence of loss and default by relating loss to the latent variable as

$$L(t) = F_L^{-1}(\Phi(-V)) \tag{15}$$

where the negative sign introduces a negative correlation between loss and asset value represented by the latent variable V. Note that this representation will not change the distribution of $L(t)$.

Conditional on $Z = z$, the probability of default is

$$p(t,z) = P(V \le \Phi^{-1}(p(t)) \,|\, Z = z) = \Phi\left(\frac{\Phi^{-1}(p(t)) - \sqrt{\rho}z}{\sqrt{1-\rho}}\right) \tag{16}$$

The loss distribution conditional on $Z = z$ is

$$\begin{aligned} P(L(t) = F_L^{-1}(\Phi(-V)) \le x | Z = z) &= \Phi\left(\frac{\Phi^{-1}(F_L(x)) + \sqrt{\rho}z}{\sqrt{1-\rho}}\right) \\ &= \Phi\left(\frac{-\Phi^{-1}(p(t)\cdot(1-F_d(x))) + \sqrt{\rho}z}{\sqrt{1-\rho}}\right) \\ &\equiv 1 - p(t,z) + p(t,z)\cdot P(L(t) \le x | \tau \le t, Z = z) \end{aligned} \tag{17}$$

So the distribution of term LGD conditional on $Z = z$ is

$$P(L(t) \le x | \tau \le t, Z = z) = 1 - p(t,z)^{-1} \cdot \Phi\left(\frac{\Phi^{-1}(p(t)\cdot(1-F_d(x))) - \sqrt{\rho}z}{\sqrt{1-\rho}}\right) \tag{18}$$

The expected loss conditional on $Z = z$ is

$$\begin{aligned} \overline{L}(t,z) = E(L(t)|Z = z) &= \int_0^1 x \cdot d_x P(L(t) \le x | Z = z) \\ &= p(t,z)\left(1 - \int_0^1 P(L(t) \le x | \tau \le t, Z = z)\cdot dx\right) \\ &= \int_0^1 \Phi\left(\frac{\Phi^{-1}(p(t)\cdot(1-F_d(x))) - \sqrt{\rho}z}{\sqrt{1-\rho}}\right)\cdot dx \end{aligned} \tag{19}$$

Next we prove that the Tasche model cannot easily be derived from a consistent multi-period model if the dependence on time is only through the default probability $p(t)$. For this purpose, we look at the conditional expected spot LGD defined in equation (8),

$$
\begin{aligned}
\bar{l}_d(t,z) &= \frac{\partial_t \bar{L}(t,z)}{\partial_t p(t,z)} = \frac{\partial_p E(L(t)|Z=z)}{\partial_p p(t,z)} \\
&= \int_0^1 \frac{d\Phi\left(\frac{\Phi^{-1}(p(t)\cdot(1-F_d(x)))-\sqrt{\rho}z}{\sqrt{1-\rho}}\right)/dp(t)}{d\Phi\left(\frac{\Phi^{-1}(p(t))-\sqrt{\rho}z}{\sqrt{1-\rho}}\right)/dp(t)}\cdot dx \\
&= \int_0^1 \exp\left(\frac{\Phi^{-1}(p(t))^2-\Phi^{-1}(p(t)\cdot(1-F_d(x)))^2-2\sqrt{\rho}z\cdot(\Phi^{-1}(p(t))-\Phi^{-1}(p(t)\cdot(1-F_d(x))))}{2(1-\rho)}\right. \\
&\qquad \left.-\frac{\Phi^{-1}(p(t))^2-\Phi^{-1}(p(t)\cdot(1-F_d(x)))^2}{2}\right)\cdot(1-F_d(x))\cdot dx
\end{aligned} \tag{20}
$$

As mentioned earlier, $F_d(x)$ can be independent of p as they are the marginal distributions out of the joint distribution of LGD and default. It is obvious that, when $z \to -\infty$, the conditional expected spot LGD will go to $+\infty$ and the conditional expected spot recovery will be negative. This would break the consistency condition defined in equation (12) such that spot recovery is not a well-defined random variable. This problem was initially found in the stochastic recovery models for CDOs [14] as the Tasche model was extended beyond its intended use for one-period capital calculation to a multi-period credit loss model. The other issue with the Tasche model is that it is very restrictive as the same latent variable drives both default and loss, which can be extended as discussed in the appendix.

3.2. The Chabaane-Laurent-Salomon Model

Chabaane, Laurent and Salomon [7] discussed the general factor structure for the underlying latent variables driving default and term LGD under the assumption of a homogeneous credit portfolio. Here we will discuss the problem with this model.

Again we assume $V = \sqrt{\rho}Z + \sqrt{1-\rho}\varepsilon$ drives the default of an obligor. The latent variable driving term LGD has the following form

$$
W = \sqrt{\beta}(\eta Z + \sqrt{1-\eta^2}Z_r) + \sqrt{1-\beta}(\gamma\varepsilon + \sqrt{1-\gamma^2}\xi) \tag{21}
$$

where Z, Z_r are independent systemic factors and ε, ξ are independent idiosyncratic factors. The parameters β and η are used to control the correlation with the two systemic factors, while γ is used to control the correlation between the idiosyncratic variables. The term LGD is linked to W through $L(t)|\tau \le t \sim F_d^{-1}(\Phi(-W))|V \le \Phi^{-1}(p(t))$.

Conditional on Z and Z_r, default and term LGD will be independent between obligors, although they are still correlated through the idiosyncratic factors for each obligor. The distribution of term LGD conditional on Z and Z_r will be

$$
\begin{aligned}
&P(L(t) \le x \mid \tau \le t, Z = z, Z_r = z_r) \\
&= P(F_d^{-1}(\Phi(-W)) \le x \mid V \le \Phi^{-1}(p(t)), Z = z, Z_r = z_r) \\
&= P\left(\gamma\varepsilon + \sqrt{1-\gamma^2}\,\xi \ge -\frac{\Phi^{-1}(F_d(x)) + \sqrt{\beta}(\eta z + \sqrt{1-\eta^2}\,z_r)}{\sqrt{1-\beta}} \middle| \varepsilon \le \frac{\Phi^{-1}(p(t)) - \sqrt{\rho} z}{\sqrt{1-\rho}}, Z = z, Z_r = z_r \right) \\
&= p(t,z)^{-1} \cdot \Phi_2\left(\frac{\Phi^{-1}(F_d(x)) + \sqrt{\beta}(\eta z + \sqrt{1-\eta^2}\,z_r)}{\sqrt{1-\beta}}, \frac{\Phi^{-1}(p(t)) - \sqrt{\rho} z}{\sqrt{1-\rho}}; -\gamma \right)
\end{aligned}
\tag{22}
$$

where $\Phi_2(x, y; \rho)$ is the cumulative bivariate normal distribution with correlation ρ.

So the loss distribution conditional on Z and Z_r is

$$
\begin{aligned}
&P(L(t) \le x \mid Z = z, Z_r = z_r) \\
&= 1 - p(t,z) + \Phi_2\left(\frac{\Phi^{-1}(F_d(x)) + \sqrt{\beta}(\eta z + \sqrt{1-\eta^2}\,z_r)}{\sqrt{1-\beta}}, \frac{\Phi^{-1}(p(t)) - \sqrt{\rho} z}{\sqrt{1-\rho}}; -\gamma \right)
\end{aligned}
\tag{23}
$$

such that, after integration over z and z_r,

$$
P(L(t) \le x) = 1 - p(t) + \Phi_2\left(\Phi^{-1}(F_d(x)), \Phi^{-1}(p(t)); -K \right) \tag{24}
$$

where $K = \eta\sqrt{\rho\beta} + \gamma\sqrt{(1-\rho)(1-\beta)}$ is the correlation between V and W. We have used the following formula to derive equation (24), see Appendix in [14],

$$
\int_{-\infty}^{+\infty} \Phi_2(az+b, cz+d; \rho) \cdot \phi(z)\,dz = \Phi_2\left(\frac{b}{\sqrt{1+a^2}}, \frac{d}{\sqrt{1+c^2}}; \frac{ac+\rho}{\sqrt{(1+a^2)(1+c^2)}} \right) \tag{25}
$$

where $\phi(z)$ is the density function of the standard normal distribution.

So, combining equations (13) and (24), the term LGD distribution is

$$
F_d^M(x) = P(L(t) \le x \mid \tau \le t) = p(t)^{-1} \cdot \Phi_2\left(\Phi^{-1}(F_d(x)), \Phi^{-1}(p(t)); -K \right) \tag{26}
$$

Note that the term LGD distribution F_d^M implied by the model is different from the calibrated marginal LGD distribution F_d unless the correlation K is zero. The two distributions should be the same if the model is to be fitted to observed data, but they are not the same due to incorrect construction of the model. The correlation term is always non-positive, which makes sense since an increase in F_d means a decrease in the expected LGD. When correlation is negative, F_d^M is smaller than F_d such that the expected loss from F_d^M is higher than the expected loss from F_d . So the model tends to overestimate VaR or downturn LGD. This kind of bias was also discussed in [11] and will be confirmed in section 6.

It should be noted that the bias problem is due to the incorrect way of linking term LGD to W as W no longer follows a normal distribution conditional on $V \le \Phi^{-1}(p)$. In the Appendix, we will clarify the source of the bias and construct a correlated term LGD model without the bias. It covers the models of Hillebrand [12] and Barco [3], and is also a generalization of the Tasche model to multiple latent variables. However, that model may still have the same inconsistency as the Tasche model.

4. Stochastic Recovery in the Default Time Copula Framework

The problems of the previous section can be resolved through a consistent stochastic spot recovery model in a default time copula framework. The way is to model the spot LGD or spot recovery directly to make sure it is in the range $[0,1]$. Here we generalize our one-factor Gaussian model of spot recovery [15, 16] to two systemic factors with correlation between idiosyncratic variables. We will follow the factor structure of Chabaane, Laurent and Salomon [7]. It is straightforward to extend the model to multi-factor or non-Gaussian copula cases.

In the default time copula framework of D. X. Li [13], the joint distribution of default times is determined by the individual default time distributions (given by default probability curve) and a copula function. In the Gaussian Copula setup, the latent variable $V = \sqrt{\rho}Z + \sqrt{1-\rho}\varepsilon$ drives the default of an obligor. The default event $1_{\tau \le t}$ can be characterized by $V \le v = \Phi^{-1}(p(t))$, where τ is the default time random variable, $p(t)$ is the default probability of the obligor. We relate the default time random variable τ to V as

$$\tau = p^{-1}(\Phi(V)) \tag{27}$$

Note that τ is not limited to a time range. It is the default time of an issuer, not that of a loan with finite maturity. We assume that the stochastic spot recovery is driven by the latent variable $W = \sqrt{\beta}(\eta Z + \sqrt{1-\eta^2} Z_r) + \sqrt{1-\beta}(\gamma\varepsilon + \sqrt{1-\gamma^2}\xi)$ through a time-independent cumulative distribution function $F_R(x) = P(r_\tau \le x | \tau = t)$ for spot recovery r_τ

at time $\tau = t$. It turns out that spot recovery and term recovery share the same distribution, as shown in equation (34).

Conditional on $\tau = t$ or $V = \Phi^{-1}(p(t))$, W follows a normal distribution with mean $K \cdot \Phi^{-1}(p(t))$ and standard deviation $\sqrt{1-K^2}$, where $K = \eta\sqrt{\rho\beta} + \gamma\sqrt{(1-\rho)(1-\beta)}$. To ensure that $F_R(x)$ is indeed the cumulative distribution for the spot recovery at time t, we specify the stochastic spot recovery at time t as

$$r_\tau \big| \tau = t \sim F_R^{-1}\left(\Phi\left(\frac{W - K \cdot \Phi^{-1}(p(t))}{\sqrt{1-K^2}}\right)\right) \Bigg| V = \Phi^{-1}(p(t)) \tag{28}$$

The special case $K = 1$ or $V = W$ is excluded as spot recovery is deterministic. Thus

$$P(r_\tau \le x \big| \tau = t) = P\left(F_R^{-1}\left(\Phi\left(\frac{W - K \cdot \Phi^{-1}(p(t))}{\sqrt{1-K^2}}\right)\right) \le x \,\Big|\, V = \Phi^{-1}(p(t))\right) = F_R(x) \tag{29}$$

If we fix $Z = z$ and $Z_r = z_r$, then $\varepsilon = \dfrac{\Phi^{-1}(p(t)) - \sqrt{\rho}z}{\sqrt{1-\rho}}$ for default time $\tau = t$ and the conditional spot recovery distribution will be

$$\begin{aligned}
&P(r_\tau \le x \,|\, \tau = t, Z = z, Z_r = z_r) \\
&= P(F_R^{-1}\left(\Phi\left(\frac{W - K \cdot \Phi^{-1}(p(t))}{\sqrt{1-K^2}}\right)\right) \le x \,|\, \varepsilon = \frac{\Phi^{-1}(p(t)) - \sqrt{\rho}z}{\sqrt{1-\rho}}, Z = z, Z_r = z_r) \\
&= \Phi\left(\frac{-Dz - \sqrt{\beta(1-\rho)(1-\eta^2)}z_r + \sqrt{(1-\rho)(1-K^2)} \cdot \Phi^{-1}(F_R(x)) + D\sqrt{\rho} \cdot \Phi^{-1}(p(t))}{\sqrt{(1-\rho)(1-\beta)(1-\gamma^2)}}\right)
\end{aligned} \tag{30}$$

where $D = \eta\sqrt{\beta(1-\rho)} - \gamma\sqrt{\rho(1-\beta)}$. If $D = 0$, then W is algebraically linear in V. We may require $D > 0$ such that, when z increases, the conditional cumulative distribution decreases and conditional expected recovery will increase. The conditional default probability for each obligor is given by

$$p(t, z) = P(\tau \le t \,|\, Z = z) = \Phi\left(\frac{\Phi^{-1}(p(t)) - \sqrt{\rho}z}{\sqrt{1-\rho}}\right) \tag{31}$$

Now we can derive the distribution for the term recovery conditional on Z and Z_r as

$$
\begin{aligned}
&P\left(r_\tau \le x \mid \tau \le t, Z = z, Z_r = z_r\right) \\
&= P\left(\tau \le t, r_\tau \le x \mid Z = z, Z_r = z_r\right) / P\left(\tau \le t \mid Z = z, Z_r = z_r\right) \\
&= \frac{1}{p(t,z)} \cdot \int_0^t P\left(s < \tau \le s + ds, r_\tau \le x \mid Z = z, Z_r = z_r\right) \\
&= \frac{1}{p(t,z)} \cdot \int_0^t \Phi\left(\frac{-Dz - \sqrt{\beta(1-\rho)(1-\eta^2)}z_r + \sqrt{(1-\rho)(1-K^2)} \cdot \Phi^{-1}(F_R(x)) + D\sqrt{\rho} \cdot \Phi^{-1}(p(s))}{\sqrt{(1-\rho)(1-\beta)(1-\gamma^2)}} \right) \cdot d\,p(s,z) \\
&= \frac{1}{p(t,z)} \cdot \Phi_2\left(\frac{-D\sqrt{1-\rho}z - \sqrt{\beta(1-\eta^2)}z_r + \sqrt{1-K^2}\Phi^{-1}(F_R(x))}{\sqrt{(1-\beta)(1-\gamma^2) + D^2\rho}}, c(p,z); -\tilde{\rho} \right)
\end{aligned}
\tag{32}
$$

where

$$
c(p,z) = \frac{\Phi^{-1}(p(t)) - \sqrt{\rho}z}{\sqrt{1-\rho}} = \Phi^{-1}(p(t,z)) \text{ and } \tilde{\rho} = \frac{D\sqrt{\rho}}{\sqrt{(1-\beta)(1-\gamma^2) + D^2\rho}}
$$

We also have

$$
\begin{aligned}
&P(\tau \le t, r_\tau \le x \mid Z = z, Z_r = z_r) \\
&= P(r_\tau \le x \mid \tau \le t, Z = z, Z_r = z_r) \cdot P(\tau \le t \mid Z = z) \\
&= \Phi_2\left(\frac{-D\sqrt{1-\rho}z - \sqrt{\beta(1-\eta^2)}z_r + \sqrt{1-K^2}\Phi^{-1}(F_R(x))}{\sqrt{(1-\beta)(1-\gamma^2) + D^2\rho}}, c(p,z); -\tilde{\rho} \right)
\end{aligned}
\tag{33}
$$

The term recovery distribution is as follows

$$
\begin{aligned}
&P(r_\tau \le x | \tau \le t) = \frac{P(\tau \le t, r_\tau \le x)}{P(\tau \le t)} \\
&= \frac{1}{p(t)} \iint P(\tau \le t, r_\tau \le x \mid Z = z, Z_r = z_r) \cdot \phi(z)\phi(z_r) dz dz_r \\
&= F_R(x)
\end{aligned}
\tag{34}
$$

where we have used the formula in equation (25).

So the distribution of term recovery rate is the same as the distribution of spot recovery rate and is time-independent. Note that, if the spot recovery distribution $F_R(x)$ is time dependent, then the integration in equation (32) would be more complicated.

Consider two obligors with correlated default and recovery rate, here we derive the copula of default time and term recovery rate. The one factor case has been discussed in

[16]. Conditional on Z and Z_r, the default and term recovery processes are independent for the two obligors, and we have

$$
\begin{aligned}
&P(\tau_1 \le t_1, (r_1)_{\tau_1} \le x_1; \tau_2 \le t_2, (r_2)_{\tau_2} \le x_2 \mid Z = z, Z_r = z_r) \\
&= \Phi_2\left(\frac{-D_1\sqrt{1-\rho_1}\, z - \sqrt{\beta_1(1-\eta_1^2)}\, z_r + \sqrt{1-K_1^2}\,\Phi^{-1}(F_{R_1}(x_1))}{\sqrt{(1-\beta_1)(1-\gamma_1^2) + D_1^2\rho_1}}, c_1(p_1(t_1), z); -\tilde{\rho}_1 \right) \\
&\quad \cdot \Phi_2\left(\frac{-D_2\sqrt{1-\rho_2}\, z - \sqrt{\beta_2(1-\eta_2^2)}\, z_r + \sqrt{1-K_2^2}\,\Phi^{-1}(F_{R_2}(x_2))}{\sqrt{(1-\beta_2)(1-\gamma_2^2) + D_2^2\rho_2}}, c_2(p_2(t_2), z); -\tilde{\rho}_2 \right)
\end{aligned}
\tag{35}
$$

Integrating over z and z_r, we will have the copula as

$$
\begin{aligned}
C(p_1(t_1), F_{R_1}(x_1); p_2(t_2), F_{R_2}(x_2)) &= P(\tau_1 \le t_1, (r_1)_{\tau_1} \le x_1; \tau_2 \le t_2, (r_2)_{\tau_2} \le x_2) \\
&= \iint P(\tau_1 \le t_1, (r_1)_{\tau_1} \le x_1; \tau_2 \le t_2, (r_2)_{\tau_2} \le x_2 \mid Z = z, Z_r = z_r) \cdot \phi(z)\phi(z_r)\, dz\, dz_r \\
&= \Phi_4(\Phi^{-1}(p_1(t_1)), \Phi^{-1}(F_{R_1}(x_1)), \Phi^{-1}(p_2(t_2)), \Phi^{-1}(F_{R_2}(x_2)); \Sigma_\rho)
\end{aligned}
\tag{36}
$$

where Φ_4 is the 4-variable cumulative normal distribution and the correlation matrix is defined as

$$
\Sigma_\rho = \begin{pmatrix}
1 & 0 & \sqrt{\rho_1\rho_2} & \dfrac{D_2\sqrt{\rho_1(1-\rho_2)}}{\sqrt{1-K_2^2}} \\
0 & 1 & \dfrac{D_1\sqrt{\rho_2(1-\rho_1)}}{\sqrt{1-K_1^2}} & \dfrac{D_1 D_2\sqrt{(1-\rho_1)(1-\rho_2)} + \sqrt{\beta_1\beta_2(1-\eta_1^2)(1-\eta_2^2)}}{\sqrt{(1-K_1^2)(1-K_2^2)}} \\
\sqrt{\rho_1\rho_2} & \dfrac{D_1\sqrt{\rho_2(1-\rho_1)}}{\sqrt{1-K_1^2}} & 1 & 0 \\
\dfrac{D_2\sqrt{\rho_1(1-\rho_2)}}{\sqrt{1-K_2^2}} & \dfrac{D_1 D_2\sqrt{(1-\rho_1)(1-\rho_2)} + \sqrt{\beta_1\beta_2(1-\eta_1^2)(1-\eta_2^2)}}{\sqrt{(1-K_1^2)(1-K_2^2)}} & 0 & 1
\end{pmatrix}
$$

Equation (36) can be compared with the standard Gaussian copula of default times with fixed recovery

$$
C(p_1(t_1), p_2(t_2)) = P(\tau_1 \le t_1, \tau_2 \le t_2) = \Phi_2(\Phi^{-1}(p_1(t_1)), \Phi^{-1}(p_2(t_2)); \sqrt{\rho_1\rho_2}) \tag{37}
$$

Note that, in equation (36), default time and recovery of the same obligor have correlation coefficient zero. This is because recovery is always conditional on default such that loss is the direct product of PD and LGD. This again shows a constraint on constructing a copula factor model of correlated default time and recovery, which may not be obvious as PD and LGD are correlated through the systemic factor Z. The copula for default time and recovery is still Gaussian. Equation (36) can be extended to more than two obligors, multi-factors and other types of copulas.

For numeric purpose, we consider the recovery distribution discussed in [15], which has the same form as the limiting portfolio loss distribution found by Vasicek [22]:

$$F_R(x) = P(r_\tau \le x | \tau = t) = \Phi(a \cdot \Phi^{-1}(x) - \sqrt{1+a^2}\Phi^{-1}(r_0)) \tag{38}$$

where $a \ge 0$ and $r_0 \in [0,1]$ is a constant. This distribution will simplify calculation for Gaussian Copula model. The expected recovery rate is r_0 and the variance of recovery rate is

$$V_R = \Phi_2\left(\Phi^{-1}(r_0), \Phi^{-1}(r_0); \frac{1}{1+a^2}\right) - r_0^{\ 2} \tag{39}$$

When a goes to zero, the variance goes to the maximum value $r_0(1-r_0)$, which corresponds to the case where recovery takes the extreme value 0 or 1. When a goes to infinity, the variance goes to zero and the distribution reduces to a constant recovery r_0.

The original spot recovery equation (28) can be written as

$$r_\tau | \tau = t \sim \Phi\left(\frac{W - K\Phi^{-1}(p(t))}{a\sqrt{1-K^2}} + \sqrt{1+\frac{1}{a^2}}\Phi^{-1}(r_0)\right) \Bigg| V = \Phi^{-1}(p(t)) \tag{40}$$

Then we have

$$P(r_\tau \le x \mid \tau = t, Z = z, Z_r = z_r)$$
$$= \Phi\left(\frac{-Dz - \sqrt{\beta(1-\rho)(1-\eta^2)}z_r + \sqrt{(1-\rho)(1-K^2)} \cdot (a\Phi^{-1}(x) - \sqrt{1+a^2}\Phi^{-1}(r_0)) + D\sqrt{\rho} \cdot \Phi^{-1}(p(t))}{\sqrt{(1-\rho)(1-\beta)(1-\gamma^2)}}\right) \tag{41}$$

The expected spot recovery conditional on Z and Z_r is

$$\bar{r}(t,z,z_r)=\int_0^1 x\cdot d_x P(r_\tau \le x \mid \tau=t, Z=z, Z_r=z_r)$$

$$=\Phi\left(\frac{Dz+\sqrt{\beta(1-\rho)(1-\eta^2)}z_r+\sqrt{(1-\rho)(1-K^2)}\cdot\sqrt{1+a^2}\Phi^{-1}(r_0)-D\sqrt{\rho}\cdot\Phi^{-1}(p(t))}{\sqrt{(1-\rho)(1-\beta)(1-\gamma^2)+a^2(1-\rho)(1-K^2)}}\right) \tag{42}$$

The expected loss up to time t conditional on Z and Z_r is

$$\bar{L}(t,z,z_r)=\int_0^t (1-\bar{r}(s,z,z_r))\cdot dp(s,z)=\Phi_2(c(p,z),b(z,z_r);-\hat{\rho}) \tag{43}$$

where $c(p,z)$ is defined in equation (32) and

$$b(z,z_r)=-\frac{D\sqrt{1-\rho}z+\sqrt{\beta(1-\eta^2)}z_r+\sqrt{1-K^2}\sqrt{1+a^2}\Phi^{-1}(r_0)}{\sqrt{(1-\beta)(1-\gamma^2)+D^2\rho+a^2(1-K^2)}}$$

$$\hat{\rho}=\frac{D\sqrt{\rho}}{\sqrt{1-\beta)(1-\gamma^2)+D^2\rho+a^2(1-K^2)}}$$

5. Large Homogeneous Pool Limit and Downturn LGD

In the Basel II capital requirement calculation, the portfolio is normally assumed to be fully granular which corresponds to the large homogeneous pool (LHP) limit. We look at the LHP limit for the Tasche model, the Chabaane-Laurent-Salomon model and our spot recovery model, and compare them to the standard Vasicek model.

In all these models, conditional on the systemic factors, loss of each obligor is independent. So in the LHP limit with total exposure equal to 1, the portfolio loss can be described by the expected loss of one obligor conditional on the systemic factors, $\bar{L}(Z)$ or $\bar{L}(Z,Z_r)$, see [7] for a proof.

In the Tasche model, the conditional expected loss is shown in equation (19). We will use the recovery distribution in equation (38) as an example for calculation purpose. Since $l_\tau=1-r_\tau$, we have

$$F_d(x)=P(l_\tau \le x | \tau \le t)=P(r_\tau \ge 1-x | \tau \le t)=1-F_R(1-x) \tag{44}$$

So the conditional expected loss is

$$\overline{L}(Z)=\int_0^1 \Phi\left(\frac{\Phi^{-1}(p(t)\cdot(1-F_d(x)))-\sqrt{\rho}Z}{\sqrt{1-\rho}}\right)\cdot dx=\int_0^1 \Phi\left(\frac{\Phi^{-1}(p(t)\cdot F_R(x))-\sqrt{\rho}Z}{\sqrt{1-\rho}}\right)\cdot dx \tag{45}$$

The portfolio loss in the LHP limit is $L_p=\overline{L}(Z)$. The portfolio loss distribution can be calculated as

$$F_{L_p}(x)=P(L_p\le x)=P(\overline{L}(Z)\le x)=\Phi(-\overline{L}^{-1}(x)) \tag{46}$$

where the negative sign is because $\overline{L}(z)$ is a monotonically decreasing function of z. Here we abuse the notion $\overline{L}$ temporarily by treating it also as a function of z to arrive at the correct formula for VaR. Equivalently, we have $x=\overline{L}(-\Phi^{-1}(F_{L_p}(x)))$. This gives an easy way to calculate VaR (see [7]) by replacing z in $\overline{L}(z)$ with $-\Phi^{-1}(\alpha)$:

$$VaR(\alpha)=F_{L_p}{}^{-1}(\alpha)=\overline{L}(-\Phi^{-1}(\alpha))=\overline{L}(\Phi^{-1}(1-\alpha)) \tag{47}$$

where α is the confidence level.

For the Tasche model with the recovery distribution defined in equation (38), we have

$$\begin{aligned} VaR(\alpha)&=\int_0^1 \Phi\left(\frac{\Phi^{-1}\big(p(t)\cdot F_R(r)\big)+\sqrt{\rho}\Phi^{-1}(\alpha)}{\sqrt{1-\rho}}\right)\cdot dr \\ &=\int_0^1 \Phi\left(\frac{\Phi^{-1}\Big(p(t)\cdot\Phi(a\Phi^{-1}(r)-\sqrt{1+a^2}\Phi^{-1}(r_0))\Big)+\sqrt{\rho}\Phi^{-1}(\alpha)}{\sqrt{1-\rho}}\right)\cdot dr \end{aligned} \tag{48}$$

The integration in equation (48) does not have analytical solution and numerical integration or Monte Carlo method has to be used for calculation.

Next, we look at the Chabaane-Laurent-Salomon model as discussed in section 3.2. For the two-factor model, loss is no longer a monotonic function and calculation is more complicated. Here we will confine to the special case of a single systemic factor with $\eta=1$ where loss is again monotonic. The loss distribution conditional on Z is from equation (22)

$$\begin{aligned} P(L(t)\le x|Z=z)&=P(\tau\le t|Z=z)\cdot P(L(t)\le x|\tau\le t,Z=z) \\ &=\Phi_2\left(\frac{\Phi^{-1}(F_d(x))+\sqrt{\beta}z}{\sqrt{1-\beta}},\frac{\Phi^{-1}(p(t))-\sqrt{\rho}z}{\sqrt{1-\rho}};-\gamma\right) \end{aligned} \tag{49}$$

The conditional expected loss is

$$\begin{aligned}\overline{L}(Z) &= \int_0^1 x \cdot d_x \Phi_2\left(\frac{\Phi^{-1}(F_d(x)) + \sqrt{\beta}Z}{\sqrt{1-\beta}}, \frac{\Phi^{-1}(p(t)) - \sqrt{\rho}Z}{\sqrt{1-\rho}}; -\gamma\right) \\ &= P(t,z) - \int_0^1 \Phi_2\left(\frac{\Phi^{-1}(F_d(x)) + \sqrt{\beta}Z}{\sqrt{1-\beta}}, \frac{\Phi^{-1}(p(t)) - \sqrt{\rho}Z}{\sqrt{1-\rho}}; -\gamma\right) \cdot dx \\ &= \int_0^1 \Phi_2\left(\frac{\Phi^{-1}(F_R(x)) - \sqrt{\beta}Z}{\sqrt{1-\beta}}, \frac{\Phi^{-1}(p(t)) - \sqrt{\rho}Z}{\sqrt{1-\rho}}; \gamma\right) \cdot dx\end{aligned} \tag{50}$$

So VaR will be

$$VaR(\alpha) = \int_0^1 \Phi_2\left(\frac{\Phi^{-1}(F_R(x)) + \sqrt{\beta}\Phi^{-1}(\alpha)}{\sqrt{1-\beta}}, \frac{\Phi^{-1}(p(t)) + \sqrt{\rho}\Phi^{-1}(\alpha)}{\sqrt{1-\rho}}; \gamma\right) \cdot dx \tag{51}$$

In the special case of the recovery distribution in equation (38), we have

$$VaR(\alpha) = \Phi_2\left(\frac{-\sqrt{1+a^2}\Phi^{-1}(r_0) + \sqrt{\beta}\Phi^{-1}(\alpha)}{\sqrt{1-\beta+a^2}}, \frac{\Phi^{-1}(p(t)) + \sqrt{\rho}\Phi^{-1}(\alpha)}{\sqrt{1-\rho}}; \frac{\gamma\sqrt{1-\beta}}{\sqrt{1-\beta+a^2}}\right) \tag{52}$$

This can be derived by plugging equation (38) into equation (51), changing variable $x = \Phi(y)$ and integrating out y.

For our new model, again we assume $\eta = 1$. Equation (33) simplifies to

$$P(\tau \le t, r_\tau \le x | Z = z) = \Phi_2\left(\frac{-D\sqrt{1-\rho}z + \sqrt{1-K^2}\Phi^{-1}(F_R(x))}{\sqrt{(1-\beta)(1-\gamma^2) + D^2\rho}}, \frac{\Phi^{-1}(p(t)) - \sqrt{\rho}z}{\sqrt{1-\rho}}; \frac{-D\sqrt{\rho}}{\sqrt{(1-\beta)(1-\gamma^2) + D^2\rho}}\right) \tag{53}$$

where $D = \sqrt{\beta(1-\rho)} - \gamma\sqrt{\rho(1-\beta)}$. So the conditional expected loss is

$$\begin{aligned}\overline{L}(Z) &= \int_0^1 (1-x) \cdot d_x P(\tau \le t, r_\tau \le x | Z) = \int_0^1 P(\tau \le t, r_\tau \le x | Z) dx \\ &= \int_0^1 \Phi_2\left(\frac{-D\sqrt{1-\rho}Z + \sqrt{1-K^2}\Phi^{-1}(F_R(x))}{\sqrt{(1-\beta)(1-\gamma^2) + D^2\rho}}, \frac{\Phi^{-1}(p(t)) - \sqrt{\rho}Z}{\sqrt{1-\rho}}; \frac{-D\sqrt{\rho}}{\sqrt{(1-\beta)(1-\gamma^2) + D^2\rho}}\right) \cdot dx\end{aligned} \tag{54}$$

So VaR will be

$$VaR(\alpha)=\int_0^1 \Phi_2\left(\frac{D\sqrt{1-\rho}\Phi^{-1}(\alpha)+\sqrt{1-K^2}\Phi^{-1}(F_R(x))}{\sqrt{(1-\beta)(1-\gamma^2)+D^2\rho}},\frac{\Phi^{-1}(p(t))+\sqrt{\rho}\Phi^{-1}(\alpha)}{\sqrt{1-\rho}};\frac{-D\sqrt{\rho}}{\sqrt{(1-\beta)(1-\gamma^2)+D^2\rho}}\right)\cdot dx \tag{55}$$

In the special case of the recovery distribution in equation (38), we have

$$VaR(\alpha)$$
$$=\Phi_2\left(\frac{-\sqrt{1-K^2}\sqrt{1+a^2}\Phi^{-1}(r_0)+D\sqrt{1-\rho}\Phi^{-1}(\alpha)}{\sqrt{(1-\beta)(1-\gamma^2)+D^2\rho+a^2(1-K^2)}},\frac{\Phi^{-1}(p(t))+\sqrt{\rho}\Phi^{-1}(\alpha)}{\sqrt{1-\rho}};\frac{-D\sqrt{\rho}}{\sqrt{(1-\beta)(1-\gamma^2)+D^2\rho+a^2(1-K^2)}}\right) \tag{56}$$

This can be derived by plugging equation (38) into equation (55), changing variable $x=\Phi(y)$ and integrating out y.

In the limit $a\to\infty$, the recovery distribution converges to the constant case, which is just the original Basel II formulation based on Vasicek [22] with no correlation between default and LGD:

$$VaR_{Vasicek}(\alpha)=ELGD\cdot\Phi\left(\frac{\Phi^{-1}(p(t))+\sqrt{\rho}\Phi^{-1}(\alpha)}{\sqrt{1-\rho}}\right) \tag{57}$$

where $ELGD=1-r_0$ is the expected LGD of each obligor. This limit can also be obtained if $K=0$, which is equivalent to $\beta=0$ and $\gamma=0$.

The downturn LGD ($DLGD$) for a general LGD model can be defined as (see [3])

$$DLGD(\alpha)=\frac{VaR(\alpha)}{\Phi\left(\frac{\Phi^{-1}(p)+\sqrt{\rho}\Phi^{-1}(\alpha)}{\sqrt{1-\rho}}\right)} \tag{58}$$

which is the same as $ELGD$ for the Vasicek model, and may be greater than $ELGD$ for correlated loss models with more tail risk. We will study this phenomenon in the next section. For expected shortfall calculation, see previous version of the current paper.

6. Numeric Examples

We present some numerical examples here to compare downturn LGD in our model with those of Tasche and Chabaane-Laurent-Salomon models. The confidence level is set to $\alpha=99.9\%$ and p is for 1-year probability of default. Below is a table showing the ratio between $DLGD$ and $ELGD=1-r_0$ under various parameter combinations (any parameter change from the first case is colored in yellow). The ratio is equivalent to the ratio between VaR of the correlated model and VaR of the Vasicek model.

$$ratio = \frac{DLGD(\alpha)}{ELGD} = \frac{VaR(\alpha)}{VaR_{Vasicek}(\alpha)}$$

r_0	a	p	ρ	β	γ	**Tasche**	**Chabaane**	**Ours**
40%	10	1%	15%	15%	25%	101.9%	110.0%	104.0%
	1					112.3%	154.4%	127.0%
	0.1					116.1%	163.3%	136.7%
40%	10	**5%**	15%	15%	25%	102.6%	109.2%	104.4%
	1					116.7%	151.9%	129.3%
	0.1					121.7%	161.8%	139.5%
40%	10	1%	**50%**	15%	25%	103.2%	108.9%	101.2%
	1					120.9%	150.9%	108.5%
	0.1					127.4%	161.1%	111.9%
40%	10	1%	15%	**50%**	25%	101.9%	115.4%	109.0%
	1					112.3%	165.1%	153.4%
	0.1					116.1%	166.7%	164.1%
40%	10	1%	15%	15%	**50%**	101.9%	112.4%	103.0%
	1					112.3%	160.8%	120.8%
	0.1					116.1%	166.1%	128.7%
70%	10	1%	15%	15%	25%	103.5%	119.0%	107.3%
	1					127.5%	245.4%	155.8%
	0.1					139.6%	293.5%	180.7%
40%	10	1%	**50%**	15%	**50%**	103.2%	110.2%	99.4%
	1					120.9%	156.0%	95.4%
	0.1					127.4%	164.6%	93.6%

From the table we can see that, in general, the Chabaane-Laurent-Salomon model overstates the ratio and the Tasche model understates the ratio comparing to our model. The ratio is less than 100% for our model in the case $D < 0$ which leads to negative correlation between default and LGD, and should be avoided. The bias in the Chabaane-Laurent-Salomon type model was also discussed in [11].

As for the estimation of the parameters, we note that it may follow similar schemes using maximum likelihood method as discussed in other term LGD models, as our model also includes formulas for term LGD. The difference is that our model allows for consistent spot recovery specification while previous approaches may lead to bias or inconsistency in multi-period setting. The difference is more in the correlation structure involving recovery such that the marginal distributions of default rate and recovery rate should be calibrated in the same way for all models. The default correlation is also the same. Thus the comparison with the same parameters r_0, a, p, ρ makes sense. The parameters β and γ will be hard to calibrate and could be different between our model and the Chabaane-Laurent-Salomon model.

7. Conclusion

In this paper, we present a new model framework for the quantification of downturn LGD due to systemic correlation between default and loss in the Basel II capital requirement. We show that previous approaches may not be internally consistent if not used properly and may lead to bias in downturn LGD calculation. The inconsistency and bias are avoided in our new model, which directly models stochastic spot recovery in a default time copula framework. We also discuss the large homogeneous pool limit and derive analytic formula for VaR for a single systemic factor given a specific form of recovery distribution. The downturn LGD in the new model is compared with two previous models with numerical examples to demonstrate the bias of one of them.

Appendix Extension of the Tasche Model to Multiple Latent Variables

In the Tasche model, default and LGD are driven by the same latent variable. Here we extend the Tasche model to allow LGD be driven by a different latent variable, which is also the correct way to construct the Chabaane-Laurent-Salomon model. This will remove the bias discussed in Sec. 3.2, but it may still have the inconsistency discussed in Sec. 3.1. Note that the models by Hillebrand [12] and Barco [3] belong to this class.

We assume $V = \sqrt{\rho}Z + \sqrt{1-\rho}\varepsilon$ drives the default of an obligor. The obligor default before time t ($\tau \le t$) is equivalent to $V \le \Phi^{-1}(p(t))$. The latent variable driving LGD has the same form as in equation (21)

$$W = \sqrt{\beta}(\eta Z + \sqrt{1-\eta^2}Z_r) + \sqrt{1-\beta}(\gamma\varepsilon + \sqrt{1-\gamma^2}\xi) \tag{A1}$$

Conditional on $V \le \Phi^{-1}(p(t))$, the distribution of $-W$ is

$$P(-W \le w \,|\, V \le \Phi^{-1}(p(t))) = \frac{P(-W \le w, V \le \Phi^{-1}(p(t)))}{P(V \le \Phi^{-1}(p(t)))} = \frac{1}{p(t)} \cdot \Phi_2(w, \Phi^{-1}(p(t)); -K) \equiv F_{p,-K}(w) \tag{A2}$$

Note that $F_{p,-K}(w)$ depends on $p(t)$, thus could be time-dependent. This distribution is not normal so $L(t)\big|\tau \le t \sim F_d^{-1}(\Phi(-W))\big|V \le \Phi^{-1}(p(t))$ will introduce bias to $F_d(x)$. The correct specification of term LGD is $L(t)\big|\tau \le t \sim F_d^{-1}(F_{p,-K}(-W))\big|V \le \Phi^{-1}(p(t))$. Conditional on Z and Z_r, default and loss will be independent between obligors, although they are still correlated through the idiosyncratic factors within each obligor. The LGD distribution conditional on Z and Z_r will be

$$P(L(t) \le x | \tau \le t, Z = z, Z_r = z_r) = P(F_d^{-1}(F_{p,-K}(-W)) \le x | V \le \Phi^{-1}(p(t)), Z = z, Z_r = z_r)$$
$$= P\left(\gamma\varepsilon + \sqrt{1-\gamma^2}\xi \ge -\frac{F_{p,-K}^{-1}(F_d(x)) + \sqrt{\beta}(\eta z + \sqrt{1-\eta^2} z_r)}{\sqrt{1-\beta}} \Big| V \le \Phi^{-1}(p(t)), Z = z, Z_r = z_r \right)$$
$$= p(t,z)^{-1} \cdot \Phi_2\left(\frac{F_{p,-K}^{-1}(F_d(x)) + \sqrt{\beta}(\eta z + \sqrt{1-\eta^2} z_r)}{\sqrt{1-\beta}}, \frac{\Phi^{-1}(p(t)) - \sqrt{\rho} z}{\sqrt{1-\rho}}; -\gamma \right) \tag{A3}$$

So the loss distribution conditional on Z and Z_r is

$$P(L(t) \le x | Z = z, Z_r = z_r)$$
$$= 1 - p(t,z) + \Phi_2\left(\frac{F_{p,-K}^{-1}(F_d(x)) + \sqrt{\beta}(\eta z + \sqrt{1-\eta^2} z_r)}{\sqrt{1-\beta}}, \frac{\Phi^{-1}(p(t)) - \sqrt{\rho} z}{\sqrt{1-\rho}}; -\gamma \right) \tag{A4}$$

such that, after integration over z and z_r,

$$\begin{aligned} P(L(t) \le x) &= 1 - p(t) + \Phi_2\left(F_{p,-K}^{-1}(F_d(x)), \Phi^{-1}(p(t)); -K\right) \\ &= 1 - p(t) + p(t) \cdot F_d(x) \end{aligned} \tag{A5}$$

So the model implied term LGD distribution is indeed $F_d(x)$, which does not have the bias in equation (26). In the limit $K = 1$, the model reduces to the Tasche model.

Similar to equation (19), we have the expected loss conditional on Z and Z_r as

$$E(L(t) | Z = z, Z_r = z_r) = p(t,z) \cdot \left(1 - \int_0^1 P(L(t) \le x | \tau \le t, Z = z, Z_r = z_r) \cdot dx \right)$$
$$= p(t,z) - \int_0^1 \Phi_2\left(\frac{F_{p,-K}^{-1}(F_d(x)) + \sqrt{\beta}(\eta z + \sqrt{1-\eta^2} z_r)}{\sqrt{1-\beta}}, \frac{\Phi^{-1}(p(t)) - \sqrt{\rho} z}{\sqrt{1-\rho}}; -\gamma \right) \cdot dx \tag{A6}$$
$$= p(t,z) - \int_0^1 \Phi_2\left(g(p,x,z,z_r), c(p,z); -\gamma\right) \cdot dx$$

where

$$g(p,x,z,z_r)=\frac{F^{-1}_{p,-K}(F_d(x))+\sqrt{\beta}(\eta z+\sqrt{1-\eta^2}z_r)}{\sqrt{1-\beta}}$$

and $c(p,z)$ is defined in equation (32).

If the only dependence on time in a multi-period model is through the default probability $p(t)$, the conditional expected spot LGD will be

$$\begin{aligned}\bar{l}_d(t,z)&=\frac{\partial_t E(L(t)|Z=z,Z_r=z_r)}{\partial_t p(t,z)}\\&=1-\int_0^1\frac{\partial_p g(p,x,z,z_r)}{\partial_p c(p,z)}\cdot e^{\frac{c(p,z)^2-g(p,x,z,z_r)^2}{2}}\cdot\Phi\left(\frac{\gamma\cdot g(p,x,z,z_r)+c(p,z)}{\sqrt{1-\gamma^2}}\right)\cdot dx\\&\quad-\int_0^1\Phi\left(\frac{g(p,x,z,z_r)+\gamma\cdot c(p,z)}{\sqrt{1-\gamma^2}}\right)\cdot dx\end{aligned}\tag{A7}$$

where

$$\frac{\partial g(p,x,z,z_r)}{\partial p}=\frac{F_d(x)-\Phi\left(\frac{F^{-1}_{p,-K}(F_d(x))+K\cdot\Phi^{-1}(p)}{\sqrt{1-K^2}}\right)}{\sqrt{1-\beta}\cdot\phi(F^{-1}_{p,-K}(F_d(x)))\cdot\Phi\left(\frac{\Phi^{-1}(p)+K\cdot F^{-1}_{p,-K}(F_d(x))}{\sqrt{1-K^2}}\right)}$$

and

$$\frac{\partial c(p,z)}{\partial p}=\frac{1}{\sqrt{1-\rho}\cdot\phi(\Phi^{-1}(p))}$$

The exponential term $e^{\frac{c(p,z)^2-g(p,x,z,z_r)^2}{2}}$ in equation (A7) determines if the spot LGD is inconsistently defined. When $\eta\neq 1$, z_r can cancel out z in $g(p,x,z,z_r)$ but $c(p,z)$ can be unbounded such that the model becomes inconsistent. When $\eta=1$, if $\rho>\beta$, then $c(p,z)^2$ will dominate $g(p,x,z,z_r)^2$ in the exponential term, which leads to inconsistency. If $\eta=1$ and $\rho=\beta$, there is a linear term in z in the exponential term similar to the Tasche model, so the model is inconsistent. If $\eta=1$ and $\rho<\beta$, then the exponential term is bounded and the model may be consistent. Both the Hillebrand model and the Barco model have $\eta\neq 1$, so they may not be consistent as multi-period models.

Another way to extend the Tasche model to multiple latest variables was discussed in [21].

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