The Oil Price Crash in 2014/15: Was There a (Negative) Financial Bubble?

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Abstract

This paper suggests that there was a negative bubble in oil prices in 2014/15, which decreased them beyond the level justified by economic fundamentals. This proposition is corroborated by two sets of bubble detection strategies: the first set consists of tests for financial bubbles, while the second set consists of the log-periodic power law (LPPL) model for negative financial bubbles. Despite the methodological differences between these detection methods, they provided the same outcome: the oil price experienced a statistically significant negative financial bubble in the last months of 2014 and at the beginning of 2015. These results also hold after several robustness checks which consider the effect of conditional heteroskedasticity, model set-ups with additional restrictions, longer data samples, tests with lower frequency data and with an alternative proxy variable to measure the fundamental value of oil.

Keywords: Oil, WTI, Brent, Generalized sup ADF test, LPPL, Bubble.

JEL classification: C15, C22, C51, C53, G17, O13, Q47.

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### 1 Introduction

The Brent and WTI prices of crude oil fell by 60% between June 2014 and January 2015, marking one of the quickest and largest declines in oil history. This fall in oil prices is large but it is not an unprecedented event: the oil price fell more than 30% in a seven-month sample already five times in the last three decades (1985-1986, 1990-1991, 1997-1998, 2001, 2008). Of these five episodes, the price slide in 1985-86 has some similarities with the fall in 2014/2015, because it followed a period of strong expansion of oil supply from non-OPEC countries and Saudi-Arabia decided to increase production and to stop defending prices. Several factors have been proposed to explain this latest price crash: Arezki and Blanchard (2014) suggested an important contribution of positive oil supply shocks after June 2014. For example, there was a faster than expected recovery of Libyan oil production due to a lull in the local civil war, as it is visible from the EIA estimated historical unplanned OPEC crude oil production outages:

Moreover, Iraq oil production was not affected by the civil war enraging in the west and in the north of the country, as initially feared. The success of US shale oil production (+0.9 million b/d in 2014) and the OPEC decision in November 2014 to maintain its production level of 30 mb/d, signalling a shift in the cartel’s policy from oil price targeting to maintaining market share, put additional pressure on oil prices. Oil demand seems to have played a minor role compared to supply shocks: Arezki and Blanchard (2014) suggested that unexpected lower demand between June and December 2014 could account for only 20 to 35 percent of the price decline, while Hamilton (2014) found that that only two-fifths of the fall in oil prices was due to weak global demand. Baumeister and Kilian (2016) used the reduced-form representation of the structural oil market model developed in Kilian and Murphy (2014) and argued that, out of a $49 fall in the Brent oil price, $11 of this decline was due to adverse demand shocks in the first half of 2014, $16 to (positive) oil supply shocks that occurred prior to July 2014, while the remaining part was due to a “shock to oil price expectations in July 2014 that lowered the demand for oil inventories and a shock to the demand for oil associated with
an unexpectedly weakening economy in December 2014, which lowered the price of oil by an additional $9 and $13, respectively”.

These and other potential factors which could have influenced the oil price decline are discussed in an extensive World Bank policy research note by Baffes, Kose, Ohnsorge, and Stocker (2015). Similarly to previous works, they also found out that supply shocks roughly accounted for twice as much as demand shocks in explaining the fall in oil prices. An alternative explanation is put forward by Tokic (2015) who suggested that the 2014 oil price collapse was partially an irrational over-reaction to the falling Euro versus the dollar. This seems to be consistent with a Bank of International Settlements report (Domanski, Kearns, Lombardi, and Shin, 2015), which shows that production and consumption alone are not sufficient for a fully satisfactory explanation of the collapse in oil prices. In this regard, Domanski, Kearns, Lombardi, and Shin (2015) advanced the idea that “if financial constraints keep production levels high and result in increased hedging of future production, the addition to oil sales would magnify price declines. In the extreme, a downward-sloping supply response of increased current and future sales of oil could amplify the initial decline in the oil price and force further deleveraging”.

Given this background, we want to propose a potential explanation for the part of the oil price decline which can not be explained using supply and demand alone, particularly in the last months of 2014, as highlighted by Baumeister and Kilian (2016). More specifically, we suggest that there was a negative financial bubble which decreased oil prices beyond the level justified by economic fundamentals. A negative financial bubble is a situation where the increasing pessimism fuelled by short positions lead investors to run away from the market, which spirals downwards in a self-fulfilling process, see Yan, Woodard, and Sornette (2012) for a discussion.

We employ two approaches to corroborate this proposition: the first approach consists of tests for financial bubbles proposed by Phillips, Shi, and Yu (2015) (hereafter PSY) and Phillips and Shi (2014) (hereafter PS). These tests are based on recursive and rolling right-tailed Augmented Dickey-Fuller unit root test, wherein the null hypothesis is of a unit root and the alternative is of a mildly explosive process. They can identify periods of statistically significant explosive price behavior and date-stamp their occurrence. The
second approach consists of the log-periodic power law (LPPL) model for negative financial bubbles developed by Yan, Woodard, and Sornette (2012). This model adapts the the Johansen-Ledoit-Sornette (JLS) model of rational expectation bubbles developed by Sornette, Ledoit, and Johansen (1999), Johansen, Sornette, and Ledoit (1999) and Johansen, Ledoit, and Sornette (2000) to the case of a price fall occurring during a transient negative bubble, which they interpret as an effective random down payment that rational agents accept to pay in the hope of profiting from the expected occurrence of a possible rally. Despite the methodological differences between these bubble detection methods, they provide the same result: the oil price experienced a statistically significant negative financial bubble in the last months of 2014 and at the beginning of 2015. A set of robustness checks finally show that our results also hold with different tests, different model set-ups and alternative datasets.

The paper is organized as follows: the bubble detection methods are presented in Section 2, while the data employed in the empirical analysis are discussed in Section 3. The main results are described in Section 4, while robustness checks are reported in Section 5. Conclusions and policy implications are presented in Section 6.

2 Methods - Testing for Financial Bubbles

We wanted to verify the presence of a negative financial bubble in oil prices at the end of 2014 using a set of tests for financial bubbles. We first employed the test by Phillips, Shi, and Yu (PSY, 2015) which builds on the previous work by Phillips, Wu, and Yu (2011, hereafter PWY) and it is designed to identify periods of statistically significant explosive price behavior. Strictly related to this, we also employed the test by Phillips and Shi (PS, 2014) for detecting a potential bubble implosion and estimating the date of market recovery. We then used the log-periodic power law (LPPL) model by Yan et al. (2012) which is specifically designed for negative financial bubbles. Differently from the approach by PSY and PS, the LPPL model does not require the formation of a bubble as a pre-requisite for a price crash.
2.1 Econometric Tests for Explosive Behavior

The generalized-supremum ADF test (GSADF) proposed by Phillips, Shi, and Yu (2015) builds upon the work by Phillips and Yu (2011) and Phillips, Wu, and Yu (2011). This is a test procedure based on ADF-type regressions using rolling estimation windows of different size, which is able to consistently identify and date-stamp multiple bubble episodes even in small sample sizes. It was recently used by Caspi, Katzke, and Gupta (2015) to date stamp historical periods of oil price explosivity using a sample of yearly data ranging between 1876 and 2014.

The first step is to consider an ADF regression for a rolling sample, where the starting point is given by the fraction \( r_1 \) of the total number of observations, the ending point by the fraction \( r_2 \), while the window size by \( r_w = r_2 - r_1 \). The ADF regression is given by

\[
x_t = \mu + \rho x_{t-1} + \sum_{i=1}^{p} \phi_{r_w}^i \Delta x_{t-i} + \varepsilon_t
\]

where \( \mu, \rho \), and \( \phi_{r_w}^i \) are estimated by OLS, and the null hypothesis is of a unit root \( \rho = 1 \) vs an alternative of a mildly explosive autoregressive coefficient \( \rho > 1 \). Then, PSY (2015) proposed a backward sup ADF test where the endpoint is fixed at \( r_2 \) and the window size is expanded from an initial fraction \( r_0 \) to \( r_2 \). The test statistic is then given by:

\[
BSADF_{r_2}(r_0) = \sup_{r_1 \in [0, r_2 - r_0]} ADF_{r_1}^{r_2}
\]

We remark that the PWY (2011) procedure for bubble identification is a special case of the backward sup ADF test where \( r_1 = 0 \), so that the sup operation is superfluous.

The generalized sup ADF (GSADF) test is computed by repeatedly performing the BSADF test for each \( r_2 \in [r_0, 1] \):

\[
GSADF(r_0) = \sup_{r_2 \in [r_0, 1]} BSADF_{r_2}(r_0)
\]

PSY (2015, Theorem 1) provides the limiting distribution of (3) under the null of a random

\footnote{A detailed analysis of model specification sensitivity in right-tailed unit root testing for explosive behavior was performed by Phillips, Shi, and Yu (2014).}
walk with asymptotically negligible drift, while critical values are obtained by numerical simulation.

If the null hypothesis of no bubbles is rejected, it is then possible to date-stamp the starting and ending points of one (or more) bubble(s) in a second step. More specifically, the starting point is given by the date -denoted as \( T_{re} \) - when the sequence of BSADF test statistics crosses the critical value from below, whereas the ending point -denoted as \( T_{rf} \) - when the BSADF sequence crosses the corresponding critical value from above:

\[
\hat{r}_e = \inf_{r_2 \in [r_0, 1]} \left\{ r_2 : BSADF_{r_2}(r_0) > cv_{r_2}^{\beta_T} \right\} \\
\hat{r}_f = \inf_{r_2 \in [\hat{r}_e + \delta \log(T)/T, 1]} \left\{ r_2 : BSADF_{r_2}(r_0) < cv_{r_2}^{\beta_T} \right\}
\]

where \( cv_{r_2}^{\beta_T} \) is the \( 100(1 - \beta_T)\% \) right-sided critical value of the BSADF statistic based on \( \lfloor T_{r_2} \rfloor \) observations, and \( \lfloor \cdot \rfloor \) is the integer function. \( \beta_T \) was set to 5%. \( \delta \) is a tuning parameter which determines the minimum duration for a bubble: this value is set to 1 in PWY (2011), PSY (2015) and most of previous applied work, thus implying a minimum bubble-duration condition of \( \log(T) \) observations (i.e. a sample fraction of \( \log(T)/T \)). In this regard, Figuerola-Ferretti, Gilbert, and Mccrorie (2016) reported results for weekly non-ferrous metals prices with different choices of the tuning parameter \( \delta = 1, 2, 4 \), and they found that while the imposition of larger minimum length criterion eliminates some cases of mildly exploding periods, the main results did not change.

Homm and Breitung (2012) compared several tests for detecting financial bubbles and found that the PWY strategy has higher power than the other procedures in detecting periodically collapsing bubbles and in real time monitoring. However, Phillips et al. (2015) showed that the PSY strategy outperforms the PWY strategy in the presence of multiple bubbles.

Phillips, Shi, and Yu (2015) and Phillips, Shi, and Yu (2016) examined the power of the previous test under alternative hypotheses where bubbles collapse instantaneously. However, Yiu, Yu, and Lu (2013) and Figuerola-Ferretti, Gilbert, and Mccrorie (2016) suggest that the PSY procedure might have some efficacy in detecting bubble implosion and market crashes in general. Strictly speaking, the test proposed by PSY (2015) is for
explosive behaviour, so that a situation of upward explosive behavior can be interpreted as bubbles, while downward explosive behavior can be interpreted as crashes or panic-selling. In this regard, Phillips and Shi (2014) discussed alternative bubble collapse models where the collapse can be “sudden”, “disturbing” or “smooth”, and they proposed a reverse sample use of the PSY test procedure for detecting crises and estimating the date of market recovery. More specifically, they propose to use the BSDF test to data $x_t^*$ arranged in reverse order to the original series $x_t$, so that $x_t^* = x_{T+1-t}$, for $t = 1, 2, \ldots T$. The BSDF statistic for detecting a bubble implosion/market crash is then defined as $BSDF_g^*(g_0)$, where the recursion (in reverse direction) initiates with a minimum window size $g_0$, and the test is repeatedly computed for each fraction $g \in [g_0, 1]$ of $X_t^*$. The market recovery date $(f_r)$ and the crisis origination date $(f_c)$, both expressed in fractions of the original series sequence, are then computed as follows:

\[
\hat{f}_r = 1 - \hat{g}_e, \quad \text{where} \quad \hat{g}_e = \inf_{g \in [g_0, 1]} \left\{ g : BSADF_g^*(g_0) > cv_g^{\beta r} \right\} \\
\hat{f}_c = 1 - \hat{g}_f, \quad \text{where} \quad \hat{g}_f = \inf_{g \in [\hat{g}_e, 1]} \left\{ g : BSADF_g^*(g_0) < cv_g^{\beta r} \right\}
\]

(6) \hspace{1cm} (7)

where $cv_g^{\beta r}$ is the $100(1 - \beta_T)\%$ critical value of the $BSADF_g^*(g_0)$ statistic. Phillips and Shi (2014) pointed out that the slowly varying function $\delta \log(T)/T$ in this case is not needed, given the interest in identifying abrupt market crashes. Similarly to eq. (3), a generalized sup ADF test $(GSADF_g^*(g_0))$ can be computed by repeatedly performing the $BSADF_g^*(g_0)$ test for each $g \in [g_0, 1]$. We will also use this second test to verify the presence of a downward market bubble in the oil price in 2014/2015.

2.2 Log-Periodic Power Law (LPPL) models for negative financial bubble detection

PSY (2015) and PS (2014) consider a model where asset prices follow a random walk during normal periods, a mildly explosive process during the bubble period, and then a bubble implosion which can be abrupt -as in PSY (2015)- or modelled by a stationary integrated process -as in PS (2014)-. Even though the PSY procedure is formally to test for explosive behavior, which can be positive or negative, PSY (2015) focus only on
upward trending bubbles. Moreover, the model by PS(2014) requires the formation of a bubble as a pre-requisite for the following price crash. Therefore, we employed also the log-periodic power law (LPPL) model by Yan, Woodard, and Sornette (2012) which is specifically designed for negative financial bubbles and does not require the formation of a bubble as a pre-requisite for a price crash. This model is an extension of the LPPL model proposed by Sornette, Ledoit, and Johansen (1999), Johansen, Sornette, and Ledoit (1999) and Johansen, Ledoit, and Sornette (2000), which posits the presence of two types of agents in the market (traders with rational expectations and irrational “noise” traders with herding behavior), and assumes that they are organized into networks and can have only two states, buy or sell. Moreover, their trading behavior is influenced by the decisions of other traders and by external shocks. A bubble can then emerge when traders form groups with self-similar behavior, which is regarded as a situation of “order”, differently from the “disorder” which takes place during normal market conditions, see Geraskin and Fantazzini (2013) for a recent extensive review and Sornette (2003) for a discussion at the textbook level. Several ex-ante forecasts of bubble episodes were discussed by Zhou and Sornette (2003), Zhou and Sornette (2006), Sornette and Zhou (2006), Zhou and Sornette (2008), Sornette, Woodard, and Zhou (2009) and Zhou and Sornette (2009).

The expected value of the asset log price in an upward trending bubble (before a crash) according to the LPPL equation is given by,

$$E[\ln p(t)] = A + B(t_c - t)^\beta + C(t_c - t)^\beta \cos[\omega \ln(t_c - t) - \phi]$$

where $\beta$ quantifies the power law acceleration of prices, $\omega$ represents the frequency of the price oscillations during the bubble, $t_c$ is the so-called ‘critical time’ that corresponds to the end of the bubble, while $A$, $B$, $C$ and $\phi$ are simply units distributions of betas and omegas and do not have any structural information, see Sornette, Johansen, et al. (2001), Johansen (2003), Sornette (2003), Geraskin and Fantazzini (2013) and Lin, Ren, and Sornette (2014) for more details.

The first major condition for a bubble to occur within the JLS framework is $0 < \beta < 1$, which guarantees that the crash hazard rate accelerates. The second major condition is that the crash rate should be non-negative, as highlighted by Bothmer and Meister (2003),
which imposes that
\[ b = -B\beta - |C|\sqrt{\beta^2 + \omega^2} \geq 0. \]

Financial bubbles are defined in the LPPL model as transient regimes of faster-than-exponential price growth resulting from positive feedbacks, and these regimes represent “positive bubbles”. Positive feedbacks can also occur in a downward price regime with faster-than-exponential downward acceleration: Yan, Woodard, and Sornette (2012) refer to these regimes as “negative bubbles”. In the latter case, the smaller the price, the larger is the decrease of future price. Moreover, the increasing pessimism fuelled by short positions leads investors to run away from the market which falls down in a self-fulfilling process. The JLS model can be easily modified to accommodate for negative bubbles, requiring only that both the expected excess return and the crash amplitude become negative, see Yan, Woodard, and Sornette (2012) for details. It is possible to show that equation (8) remains the same, with the inequalities \( B > 0, \ b < 0 \) being the opposite to those corresponding to a positive bubble, while the first major condition \( 0 < \beta < 1 \) does not change.

The estimation of LPPL models can be rather difficult and several algorithms were recently reviewed by Geraskin and Fantazzini (2013). In this regard, Filimonov and Sornette (2013) proposed a stable and robust calibration scheme of the log-periodic power law model by rewriting the formula (8) as follows:

\[
E[ln p(t)] = A + B(t_c - t)^\beta + C_1(t_c - t)^\beta \cos[\omega \ln(t_c - t)] + C_2(t_c - t)^\beta \sin[\omega \ln(t_c - t)] (9)
\]

where \( C_1 = C \cos \phi, \ C_2 = C \sin \phi \), and which can be derived from (8) by expanding the cosine term. Similarly to Filimonov and Sornette (2013), we estimated (9) with nonlinear least-squares, but differently from them we employed a variant of the multi-stage procedure proposed in Geraskin and Fantazzini (2013) and Fantazzini (2010) to improve the numerical convergence in small-to-medium sized samples, see the Appendix A for details.

3 Data

3.1 Which oil price to use?

Some studies tried to identify speculative bubbles in the oil market using the standard present-value model for stocks adapted to commodity markets by Pindyck (1992). In this
framework, the fundamental value of oil is defined as the sum of discounted oil dividends which are approximated by the convenience yield, see Lammerding, Stephan, Trede, and Wilfling (2013), Areal, Balcombe, and Rapsomanikis (2013) and Shi and Arora (2012)). Unfortunately, as shown -inter alia- by Figuerola-Ferretti and Gonzalo (2010), Lammerding, Stephan, Trede, and Wilfling (2013) and Figuerola-Ferretti, Gilbert, and McCorrie (2016), the estimated convenience yield can become negative so that the ratio between the commodity price and the measured convenience yield becomes uninterpretable and cannot be used for testing bubbles, as done for equity prices by PWY (2011). Moreover, in case of daily data, the estimates are rather volatile and has to be smoothed. Given these issues, we preferred to employ the previous tests with nominal oil prices, as done by Gilbert (2010) and Homm and Breitung (2012) and with real oil prices, as done by Caspi, Katzke, and Gupta (2015) and Phillips and Yu (2011). To compute the daily real oil prices, we built a daily consumer price index (CPI) series using the methodology used by the US and UK governments for the indexation of Treasury Inflation-Protected Securities (TIPS) and of Index-Linked Gilts, respectively2. We considered both nominal and real oil prices also due to the current debate about which price series is better suited for analyzing the relationship between the price of oil and the level of economic activity (see the *Macroeconomic Dynamics* special issue on “Oil Price Shocks” published in 2011 for a detailed discussion): we decided to take a neutral stance on this issue and examined both type of prices.

3.2 Time Sample for Model Estimation

We analyzed the daily nominal and real WTI and Brent oil prices from January 2013 to April 2015. The nominal prices are the spot prices as provided by the US Energy Information Administration (EIA), while the real prices are computed using the US and UK CPIs, using the methodology described in section 3.1. We chose this time span because we focus on the price crash at the end of 2014. Moreover, Sornette (2003) and Jiang, Zhou, Sornette, Woodard, Bastiaansen, and Cauwels (2010) remarked that a bubble cannot be

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2Both the US and the UK governments calculate the daily CPI using a linear interpolation between the CPI applicable to the first day of the month and the CPI applicable to the first day of the following month. We used a cubic-spline interpolation because of the better mathematical properties. However, the differences with linearly interpolated data were very small and did not change the outcome of the tests.
diagnosed more than 1 year in advance, so that a statistical test for detecting a bubble at the end of 2014 can be computed using data starting from the year 2013 at the latest. Furthermore, a recent literature examined the interaction between market prices and media coverage and suggested that media hype can be a potential source of speculation and financial bubbles, see (among many) Shiller (2000), Shiller (2002), Dyck and Zingales (2003), Case and Shiller (2003), Veldkamp (2006), Bhattacharya, Galpin, Ray, and Yu (2009). In this regard, Geraskin and Fantazzini (2013) suggested to use the *Search Volume Index* (SVI) by Google Trends to get some insights as to when a potential bubble may have started: this index computes how many searches have been done for a term on Google over time.\(^3\) If a keyword has both a large number of searches and several potential meanings, Google Trends offers the possibility to choose the SVI related to a specific topic, so that unrelated searches are filtered out: we report in Figure 2 the SVIs for the topics “West Texas Intermediate” and “Brent Crude”.

[INSERT FIGURE 2 ABOUT HERE]

Figure 2 shows that a large interest about these oil prices started to build at the beginning of 2014, so that a time sample from January 2013 to April 2015 seems appropriate. We will verify in section 5.3 whether our results continue to hold with longer samples that start before 2013.

4 Results

4.1 Econometric Tests for Explosive Behavior

Table 1 reports the GSADF and GSADF\(^*\) statistics with the 95% critical values obtained by Monte Carlo simulation using 1000 replications, with minimum estimation windows \(r_0 = g_0 = 0.01 + 1.8/\sqrt{T}\), as suggested by PSY (2015). The start and end dates for weakly explosive behaviour as identified using the PSY (2015) procedure, as well as the crisis origination date and market recovery date as identified using the PS (2014) procedure are also reported. The sequences of BSADF and BSADF\(^*\) statistics (with 95% critical values)\(^3\)

\(^3\)See [https://support.google.com/trends](https://support.google.com/trends) for more details. The time span starts from 2004, which is the first year available for this service.
for nominal and real oil prices are reported in Figures 3 and 4, respectively.

[INSERT TABLE 1 ABOUT HERE]

[INSERT FIGURES 3 AND 4 ABOUT HERE]

The GSADF tests identify a period of explosive behavior in Brent prices between October 2014 and February 2015, whereas between December 2014 and March 2015 in WTI prices. There is also a very short spike of the BSADF statistic in February 2014 associated with a mild increase of the WTI price, but it seems more a computationally anomaly rather than a period of explosive behavior. In this regard, PS(2014) and PSY (2015) warned that the BSADF statistic may exceed its critical value for a small number of observations and give a false signal, so that they suggested to use a minimum length criterion. The imposition of a minimum length requirement of $\log(T) \approx 7$ days does not change the results, but if we consider a tuning parameter $\delta = 4$ (i.e. 1 month), as suggested by Figuerola-Ferretti, Gilbert, and McCrorie (2016), the mild exploding period in February 2014 is eliminated. Instead, the GSADF* tests in Table 1 fail to identify significant period of market implosion for all oil prices considered. In this regard, some insights are given by the BSADF* statistics in the second row of Figures 3 and 4, which show an erratic behavior and are unable to cross the 95% critical values for sustained periods of time and with high values. These latter results may be due to the relatively short period of time considered for estimation: we will see in section 5.4 that longer estimation samples make the oil price implosion at the end of 2014 strongly significant.

4.2 LPPL model for negative financial bubble detection

Sornette, Woodard, and Zhou (2009), Jiang, Zhou, Sornette, Woodard, Bastiaensen, and Cauwels (2010) and Geraskin and Fantazzini (2013) suggested to use estimation samples of varying size to deal with potential parameter instability. Following their example, we fit the logarithm of the examined oil price by using the LPPL eq. (9) in shrinking windows and in expanding windows. More specifically, for each end date $t_2 = 02/01/2014, \ldots, 30/04/2015$ the starting date $t_1$ ranged from $t_2 - 120$ to $t_2 - 250$ in steps of one (trading) day. Fol-
ollowing Jiang, Zhou, Sornette, Woodard, Bastiaensen, and Cauwels (2010) and Geraskin and Fantazzini (2013), we then used the set of parameters estimated with all samples $t_2 - j, j = 120, \ldots, 250$ to compute the moving 20%/80% and 5%/95% quantile range of the parameters of interest. The 20%/80% and 5%/95% quantile ranges of the LPPL parameters $B$ and $\beta$ for nominal and real oil prices are reported in Figures 5 and 6, respectively.

[INSERT FIGURES 5 AND 6 ABOUT HERE]

The crash hazard rate $b$ was negative over all time sample (as required by a negative bubble) and therefore was not reported. In general, the LPPL parameters $B$ and $\beta$ satisfy jointly the conditions for a negative financial bubble between October 2014 and March 2015 for the Brent and between December 2014 and March 2015 for the WTI. However, the evidence for the latter is somewhat weaker. It is interesting to note that despite the methodological differences between the LPPL approach and the econometric tests by PSY (2015), they provide substantially the same result: oil prices experienced a statistically significant negative financial bubble in the last months of 2014 and at the beginning of 2015.

## 5 Robustness checks

We wanted to verify that our previous results hold also with different tests and alternative datasets. Therefore, we performed the following robustness checks: a) we performed the FTS-GARCH test for financial bubbles by Corsi and Sornette (2014) which takes conditional heteroskedasticity into account; b) we employed the ‘volatility-confined’ LPPL model by Lin, Ren, and Sornette (2014), which is a generalization of the previous LPPL model; c) we used an alternative longer daily data sample; d) we verified that our results hold also with a weekly dataset. All checks confirmed that the oil price experienced a statistically significant negative financial bubble from the end of 2014 till the beginning of 2015.
5.1 Accounting for Heteroskedasticity: The FTS-GARCH test for financial bubbles

Corsi and Sornette (2014) proposed a reduced form model for the joint dynamics of liquidity and asset prices, where the self-reinforcing feedback between credit creation and the market value of the financial assets employed as collateral in the bank loans (i.e. the financial accelerator) is modelled as a multivariate non-linear stochastic process. They showed that such model can produce explosive dynamics in the financial variables which can lead to a market crash in finite time. Exploiting the implications of their model for asset returns, they proposed an extension of the GARCH process which can provide an early warning identification of financial bubbles. More specifically, they showed that the positive feedbacks of price on money and money on price leads to a finite time singular (FTS) dynamics where these two variables follow a self-reinforcing dynamics of the type $dX/dt \simeq X^{1+\delta}$, with $\delta > 1$, so that the conditional mean of asset log-returns $r_t = \ln(P_t) - \ln(P_{t-1})$ depends on price levels $P_t$. The resulting FTS-GARCH model proposed by Corsi and Sornette (2014) is given by:

$$r_t = \mu + \gamma P_{t-1} + \varepsilon_t, \quad \varepsilon_t = \sigma_t z_t, \quad z_t \sim N(0, 1)$$

$$\sigma_t^2 = \omega + \alpha \varepsilon_{t-1}^2 + \beta \sigma_{t-1}^2$$

where $\omega, \alpha, \beta$ are positive parameters and the rejection of the null hypothesis of $\gamma = 0$ is interpreted as evidence of a bubble. This test is a type of right-tailed Dickey-Fuller test with GARCH errors: given the moderate sample size, we employed bootstrap methods to compute the test distribution, following the suggestion by Harvey, Leybourne, Sollis, and Taylor (2016) who performed a comprehensive analysis of the impact of different volatility structures on the size of the SADF test by PWY (2011). The sequences of t-statistics of the FTS $\gamma$ parameter (with 95% critical values) for nominal and real oil prices are reported in Figure 7.

[INSERT FIGURE 7 ABOUT HERE]

1The sequences start on the 2nd of January 2014, given the need to have a minimum time sample for the estimation of the GARCH models.
The null hypothesis of $\gamma = 0$ is rejected between December 2014 and February 2015 for the Brent, similarly to previous tests, whereas it is not rejected for the WTI. Therefore, the FTS-GARCH approach seems to be more restrictive than the previous tests and the evidence of a potential bubble is confirmed only for Brent oil prices.

5.2 Diagnostic Tests based on the LPPL fitting residuals

Lin, Ren, and Sornette (2014) proposed a generalization of the LPPL model for financial bubbles where the log-prices fluctuate around the LPPL trajectory and the fitting residuals follow a mean-reverting Ornstein-Uhlenbeck process. The main advantage of the “volatility-confined LPPL model” proposed by Lin, Ren, and Sornette (2014) is to guarantee the consistency of direct estimation with prices, which was not possible with the original LPPL model due to the presence of a random walk component with increasing variance.

Lin, Ren, and Sornette (2014) used the Phillips-Perron (PP) and the Augmented Dickey-Fuller (ADF) to test the stationarity of the LPPL fitting residuals, whereas we used here the test by Kwiatkowski, Phillips, Schmidt and Shin (Kwiatkowski, Phillips, Schmidt, and Shin, 1992), where the null hypothesis is a stationary process. We employed the latter test because it has higher power when the underlying data-generating process is an AR(1) process with a coefficient close to one, see Geraskin and Fantazzini (2013). Substantially, the model by Lin, Ren, and Sornette (2014) adds an additional restriction to the original LPPL model.

Following Geraskin and Fantazzini (2013) and Lin, Ren, and Sornette (2014), we first computed the fraction $P_{LPPL}$ of the previous estimation windows $[t_2 - j; t_2]$, $j = 120, \ldots, 250$ that met the LPPL conditions for a negative bubble. Then, we computed the conditional probability $P_{Stat.Res.|LPPL}$ that, out of the fraction $P_{LPPL}$ of windows that satisfied the LPPL conditions, the null hypothesis of stationarity was not rejected for the residuals. The sequences of the probabilities $P_{LPPL}$ and $P_{Stat.Res.|LPPL}$ for nominal and real oil prices are reported in Figure 8.

[INSERT FIGURE 8 ABOUT HERE]
The probabilities $P_{Stat.Res.\mid LPPL}$ are almost always higher than 50% and often close to 100%, thus confirming the previous evidence in the baseline case. These results are similar to those reported by Jiang, Zhou, Sornette, Woodard, Bastiaensen, and Cauwels (2010), Geraskin and Fantazzini (2013) and Lin, Ren, and Sornette (2014).

5.3 Longer Time Sample

The estimation sample used in the baseline case range from January 2013 till May 2015. We wanted to verify that our results continue to hold with a longer sample. In this regard, we used the range January 2005 - June 2015, which is the time span used by Baffes, Kose, Ohnsorge, and Stocker (2015) for their empirical analysis. Table 2 reports the GSADF and GSADF* statistics with the 95% critical values, while the sequences of BSADF and BSADF* statistics (with 95% critical values) for nominal and real oil prices are reported in Figures 9 and 10, respectively. Similarly to the baseline case, we imposed a minimum length requirement of $4 \cdot \log(T) \approx 32$ days.

The results in Table 2 and in Figures 9 and 10 not only confirm what we found in the baseline case, but also show that the evidence of explosive behavior in oil prices is stronger for the sample 2014-2015 than for the 2008 oil crash. Moreover, differently from the baseline case, the GSADF* tests are now significant at the 95% level and the identified periods of significant market implosion are June 2008-September 2008 and June 2014-November 2014. The GSADF* test for the Brent real oil price is not significant at the 95% level but only at the 90% level. In this regard, the BSADF* date-stamping procedure seems to anticipate the real market recovery date by a couple of months for both episodes of price declines (2008 and 2014/15). We remark that a large body of the literature examined the oil price crash in 2008 and the underlying factors to the price build-up before this crash: see (among many), Sornette, Woodard, and Zhou (2009), Khan (2009), Tokic (2010), Lombardi and Van Robays (2011), Areal, Balcombe, and Rapsomanikis

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5 The LPPL approach was not considered here because it is not intended for detecting multiple bubbles over a long time span.
Therefore, we refer the interested reader to these references for more details. In general, this evidence strengthens the case of a negative bubble in oil prices at the end of 2014 - beginning of 2015, which decreased the prices beyond the level justified by economic fundamentals.

5.4 Tests with lower frequency data

The analysis has so far considered only daily data because we could estimate the competing tests using the most recent observations, an advantage highlighted by the World Bank in the work by Baffes, Kose, Ohnsorge, and Stocker (2015). However, for sake of generality, we considered also a particular weekly dataset which could give additional insights about the 2014/2015 oil price crash.

The US EIA has published weekly the total amount of crude oil stocks in the US since January 1986. We used this data to compute the weekly supply ratio, that is the ratio of the WTI nominal price relative to the US inventory supply stock. This ratio was used by Phillips and Yu (2011) and Caspi, Katzke, and Gupta (2015) as an alternative proxy variable to measure the fundamental value of oil, using a measure of the oil supply based on the inventory of crude oil in the United States. Table 3 reports the GSADF statistic with the 95% critical values, while the sequence of BSADF statistics (with 95% critical values) for the supply ratio is reported in Figure 11. The results for the weekly nominal WTI price as well as for the weekly real WTI price are also reported for comparison purposes. Given the use of weekly data, we imposed a minimum length requirement of $log(T) \approx 7$ weeks.

Table 3 and Figure 11 provide some evidence of explosive behavior in oil prices from the

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*The BSADF* statistic to test for significant bubble implosion was not considered because the initial minimum window size $g_0$ on the reversed time series eliminates the last year and half of data up to the beginning of 2014, so that it is not useful for this analysis. Similarly to section 5.3, the LPPL approach was also not considered because it is not intended for detecting multiple bubbles over a long time span.
end of 1999 till March 2000: after reaching a minimum close to 10$ in December 1998, the WTI rose nearly threefold by March 2000, as world petroleum consumption strongly increased. This was followed by another decline in 2001, following the DotCom bubble and the subsequent recession in the US. However, while the sequences of BSADF statistics for the supply ratio and the nominal WTI price agree on a potential bubble at the beginning of 2000, this is not confirmed by the BSADF statistics for the real WTI price, which is in line with the evidence reported by Caspi, Katzke, and Gupta (2015) who did not find any price explosivity for this time span using monthly data and the same test procedure. Instead, all three GSADF tests show a period of price explosivity between the end of 2007 and August/September 2008, a range close to those reported by Phillips and Yu (2011) and Caspi, Katzke, and Gupta (2015) with monthly data. Finally, all three tests identify a period of price explosivity between December 2014 and March 2015, thus confirming our previous evidence. It is interesting to note that the BSADF statistics for nominal and real WTI prices reach a value of 2 or higher in the latter time span, whereas they are much lower (but still significant) for the supply ratio. This may be due to the strong build-up in US oil inventories since January 2015 due to shale oil: the supply ratio is clearly more sensitive to the oil excess supply, which is definitely one of the main factors behind the price crash, as highlighted by Arezki and Blanchard (2014), Baumeister and Kilian (2016) and Baffes, Kose, Ohnsorge, and Stocker (2015).

We also considered two monthly datasets: 1) the US refiners’ acquisition cost for imported crude oil, as reported by the EIA, extrapolated from 1974M1 back to 1973M1 as in Barsky and Kilian (2002). Kilian and Murphy (2014) suggest to use this oil price since it is a better proxy for the price of oil in global markets than the US price of domestic crude oil, which was regulated during the 1970s and early 1980s; 2) the monthly Brent oil prices as provided by the IMF since January 1980. The GSADF tests were strongly significant in both cases and identified a period of price explosivity from November/December 2014 till February/March 2015, depending on the type of oil price selected and whether nominal or real prices are considered.\(^7\) However, the imposition of a minimum bubble-duration length in this case would eliminate this evidence, even considering the smallest length

\(^7\)Other periods of price explosivity were also detected, but are not of interest for the current analysis.
requirement possible of $1 \cdot \log(T) \approx 6$ months. Given that all previous results point out to a (negative) bubble of $4/5$ months, probably the tuning parameter $\delta$ discussed above should be smaller than 1. However, this technical issue goes beyond the scope of this paper and we leave it as an avenue of further research. This is why we do not report here the results with monthly data, but they are available from the authors upon request.

5.5 The price fall in 2015/2016: preliminary evidence

At the time of finishing writing this work (May 2016), the oil price experienced a new fall during the winter period in 2015/2016. While a full analysis of this event will be discussed in a separate work -due to the computational efforts needed and the lack of data-, we nevertheless present some preliminary evidence using the GSADF test and the LPPL model with the most recent data of the real WTI oil price till April 2016\(^8\).

The GSADF and GSADF\(^*\) statistics and the sequences of BSADF and BSADF\(^*\) statistics (with 95\% critical values) for the real WTI price are reported in Figure 12 (left column), while the 20\%/80\% and 5\%/95\% quantile ranges of the LPPL parameters $B$ and $\beta$ are reported in Figure 12 (right column).

The results in Figure 12 not only confirmed again the presence of a negative bubble in oil prices at the end of 2014 - beginning of 2015, but the evidence in this case is even stronger than in the baseline case. Interestingly, both the GSADF test and the LPPL model did not find any significant evidence of a negative bubble during the winter period in 2015/2016. However, the full analysis of this event will be developed in a separate work.

6 Conclusions and Policy Implications

The aim of this paper is to propose a potential explanation for the part of the oil price decline in 2014/15 which can not be explained using supply and demand alone. More specifically, we suggest that there was a negative financial bubble which decreased oil prices beyond the level justified by economic fundamentals.

\(^8\)The author wants to thank an anonymous referee for pointing out this issue.
We employed two sets of bubble detection strategies to corroborate this proposition: the first set consisted of tests for financial bubbles proposed by Phillips, Shi, and Yu (2015) and Phillips and Shi (2014). These tests are based on recursive and rolling right-tailed Augmented Dickey-Fuller unit root test, wherein the null hypothesis is of a unit root and the alternative is of a mildly explosive process. They can identify periods of statistically significant explosive price behavior and date-stamp their occurrence. The second set consisted of the log-periodic power law (LPPL) model for negative financial bubbles developed by Yan, Woodard, and Sornette (2012). This model adapts the Johansen-Ledoit-Sornette (JLS) model of rational expectation bubbles developed by Sornette, Ledoit, and Johansen (1999), Johansen, Sornette, and Ledoit (1999) and Johansen, Ledoit, and Sornette (2000) to the case of a price fall occurring during a transient negative bubble. Despite the methodological differences between these bubble detection methods, they provided the same result: the oil price experienced a statistically significant negative financial bubble in the last months of 2014 and at the beginning of 2015.

A set of robustness checks showed that our results also hold with different tests, model set-ups and alternative datasets: all checks confirmed that the oil price experienced a statistically significant negative financial bubble from the end of 2014 till the beginning of 2015, thus supporting the idea put forward by Domanski, Kearns, Lombardi, and Shin (2015) and Tokic (2015) that this price collapse cannot be explained by supply and demand alone.

These results can be important for regulatory purposes, since it is clear that the enhanced regulations imposed after the 2008 oil bubble (see Collins (2010) and Cosgrove (2009)) cannot ensure the oil price efficiency. In this regard, Tokic (2015) and Domanski, Kearns, Lombardi, and Shin (2015) suggested that the oil price collapse 2014/2015 could have been caused by the increased leverage of oil firms (the debt of oil and gas sector increased from $1 trillion in 2006 to $2.5 trillion in 2014): the increasing need to keep high production levels and to hedge future production to satisfy financial constraints could have easily amplified the initial price decline due to economic fundamentals. Therefore, a revised and more effective regulatory framework should include not only oil traders/speculators, but all market participants including oil producers. The design of this revised framework is
definitively an important avenue of future research.

Another implication of the evidence found in this work is that market regulators should be concerned not only about positive price bubbles, but also about negative bubbles. In this regard, it is well known that the oil supply shows cyclical boom and bust cycles in prices and production, see Maugeri (2010) for a large historical review. Extremely low prices are not necessarily beneficial, even for countries which are (mainly) oil consumers: for example, Kilian (2008) showed that the large fall in investment in the oil and gas industry following the oil price crash in 1985/1986 was one of the main causes why real consumption in the US did not grow as expected. In general, there is a large literature which tried to find if and why economic activity responds asymmetrically to oil price shocks -i.e. high oil prices decrease economic activity much more than low oil prices stimulate it-, see the *Macroeconomic Dynamics* special issue on “Oil Price Shocks” published in 2011 for more details. Moreover, several authors have recently investigated the linkages between the oil market and other markets, focusing particularly on the volatility transmission across financial markets. Diebold and Yilmaz (2012) found that cross-market volatility spillovers across US stock, bond, foreign exchange and commodities markets were quite limited until 2007, but have increased since then: particularly, they found that the commodity market was a net recipient of small levels of volatility shocks from the other markets till 2007, but it has become a net transmitter after the beginning of the global financial crisis. Similar evidence was found by Ji and Fan (2012) who found that the crude oil market has significant volatility spillover effects on non-energy commodity markets and they have strengthened after the crisis. A similar result was also reported by Creti, Joëts, and Mignon (2013) who showed increased links between stock and commodity markets, and by Gomes and Chaibi (2014) who highlighted that shock and volatility spillovers tend to go more often from oil to stock markets than vice versa, see also Arouri and Nguyen (2010), Filis, Degiannakis, and Floros (2011), Kumar, Managi, and Matsuda (2012), Awartani and Maghyereh (2013) and Khalfaoui, Boutahar, and Boubaker (2015). Given this increased influence of the oil market on the other markets, regulators should consider a regulatory framework able to mitigate an oil price crash due to panic selling and/or market manipulation: a potential starting point could be the model developed by Dutt and Harris (2005), which can be used
to set position limits for cash-settled derivative contracts.

References


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**Figures**

**Figure 1:** Estimated Historical Unplanned OPEC Crude Oil Production Outages (million barrels per day). Source: EIA (2015).

**Figure 2:** Google Trends SVIs for “West Texas Intermediate” and “Brent Crude”
Figure 3: BSADF statistics (first row) and BSADF* statistics (second row) for date-
stamping periods of explosive behavior and market implosions, respectively. Nominal oil prices: Brent (first column) and WTI (second column).

Figure 4: BSADF statistics (first row) and BSADF* statistics (second row) for date-
stamping periods of explosive behavior and market implosions, respectively. Real oil prices: Brent (first column) and WTI (second column).
Figure 5: 20%/80% and 5%/95% quantile range of the LPPL parameters $B$ (first row) and $\beta$ (second row): shaded areas highlight the time samples when $B > 0$ and $0 < \beta < 1$. 

Nominal oil prices: Brent (first column) and WTI (second column).

Figure 6: 20%/80% and 5%/95% quantile range of the LPPL parameters $B$ (first row) and $\beta$ (second row): shaded areas highlight the time samples when $B > 0$ and $0 < \beta < 1$. 

Real oil prices: Brent (first column) and WTI (second column).
Figure 7: T-statistics of the FTS parameter $\gamma$ computed with an expanding window: Brent (first column) and WTI (second column). Nominal oil prices are in the first row, while real price in the second row.

Figure 8: The fraction $P_{LPPL}$ of estimation windows that met the LPPL conditions (first row). The conditional probability $P_{Stat.Res.|LPPL}$ that, out of the fraction $P_{LPPL}$ of windows that satisfied the LPPL conditions, the null hypothesis of stationarity was not rejected for the residuals (second row). Brent (first column) and WTI (second column). If $P_{LPPL} = 0$, then $P_{Stat.Res.|LPPL} = NA$. 

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Figure 9: BSADF statistics (first row) and BSADF* statistics (second row) for date-stamping periods of explosive behavior and market implosions, respectively. *Nominal oil prices*: Brent (first column) and WTI (second column).

Figure 10: BSADF statistics (first row) and BSADF* statistics (second row) for date-stamping periods of explosive behavior and market implosions, respectively. *Real oil prices*: Brent (first column) and WTI (second column).
Figure 11: BSADF statistics for date-stamping periods of explosive behavior. Supply ratio (first plot), nominal WTI prices (second plot) and real WTI prices (third plot).
Figure 12: Real WTI: BSADF statistics (left-first row) and BSADF* statistics (left-second row), respectively. The GSADF statistic was equal to 3.31 ($CV_{95} = 2.39$), whereas the GSADF* statistic was 2.10 ($CV_{95} = 2.39$). 20%/80% and 5%/95% quantile ranges of the LPPL parameters $B$ (right-first row) and $\beta$ (right-second row): shaded areas highlight the time samples when $B > 0$ and $0 < \beta < 1$. 
### Tables

<table>
<thead>
<tr>
<th>Oil price</th>
<th>GSADF (PSY-2015)</th>
<th>c.v.</th>
<th>Start (explosive beh.)</th>
<th>End (explosive beh.)</th>
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<tr>
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<td>23.03.2015</td>
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<th>Oil price</th>
<th>GSADF* (PSY-2015)</th>
<th>c.v.</th>
<th>Crisis origination date</th>
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<td>/</td>
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<td>/</td>
<td>/</td>
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<td>Real WTI</td>
<td>1.97</td>
<td>2.28</td>
<td>/</td>
<td>/</td>
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Table 1: GSADF statistics (upper part) and GSADF* statistics (lower part), 95% critical values, start and end dates for explosive behaviour (GSADF test), crisis origination date and market recovery date (GSADF* test). * Significant at the 95% level.

<table>
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<th>Oil price</th>
<th>GSADF (PSY-2015)</th>
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<th>End (explosive beh.)</th>
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<td>Nominal WTI</td>
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<td>2.34</td>
<td>03.06.2014</td>
<td>12.11.2014</td>
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<td>Real Brent</td>
<td>2.29</td>
<td>2.40</td>
<td>/</td>
<td>/</td>
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<td>Real WTI</td>
<td>3.48*</td>
<td>2.38</td>
<td>09.06.2014</td>
<td>26.11.2014</td>
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Table 2: GSADF statistics (upper part) and GSADF* statistics (lower part), 95% critical values, start and end dates for explosive behaviour (GSADF test), crisis origination date and market recovery date (GSADF* test). * Significant at the 95% level.

<table>
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<th>Oil price</th>
<th>GSADF (PSY-2015)</th>
<th>c.v.</th>
<th>Start (explosive beh.)</th>
<th>End (explosive beh.)</th>
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<td>05.11.1999</td>
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<td>2.31</td>
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<td>Real WTI</td>
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<td>08.02.2008</td>
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<td>27.03.2015</td>
</tr>
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Table 3: GSADF statistics, 95% critical values, start and end dates for explosive behaviour (GSADF test). * Significant at the 95% level.
Appendix A

We estimated (9) with nonlinear least-squares, using a variant of the 3-step procedure proposed in Geraskin and Fantazzini (2013) and Fantazzini (2010):

1. Set $t_c = t_2 + 0.1 \cdot (t_2 - t_1)$, where $t_2$ and $t_1$ are the last and the first observation of the estimation sample, respectively. Estimate the remaining LPPL parameters $[A, B, C_1, C_2, \beta, \omega]$ by using the BFGS (Broyden, Fletcher, Goldfarb, Shanno) algorithm.

2. Keeping fixed the LPPL parameters $\hat{\Phi} = [\hat{A}, \hat{B}, \hat{C}_1, \hat{C}_2, \hat{\beta}, \hat{\omega}]$ computed in the first stage, estimate the critical time $t_c$.

3. Use the estimated parameters in the first and second stages as starting values for estimating all the LPPL parameters.

Similarly to Geraskin and Fantazzini (2013), we found that this multi-step procedure improves considerably the numerical convergence and the estimation efficiency in small-to-medium sized samples.