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Abstract

We present a general model of n firms with differentiated production costs competing in a linear market within the framework of spatial price discrimination. We prove that the Nash equilibrium locations of firms are always socially optimal irrespective of the number of competitors, firm heterogeneity regarding marginal production costs, the level of privatization, the form of the transportation costs and the number and/or the varieties of the produced goods. An immediate implication of this result is that this form of competition is preferable from a welfare point of view. We also argue that (i) when firms are homogeneous regarding their marginal production costs, there always exists a unique Nash equilibrium, regardless of the form of the transportation cost function (ii) when firms are heterogeneous and transportation costs are linear, there is a unique Nash equilibrium which depends only on the relative mutual differences of the marginal production costs.

JEL classification: L13; L32; L33; R32
Keywords: Mixed oligopoly; Social optimality; Spatial competition; Differentiated goods

1 Introduction

Whenever we make online purchases from the web, we are witnessing a form of market segmentation due to discriminatory pricing dependent on geographical location. This pricing practice is called spatial price discrimination (Cabral, 2000). However, this is not the only market where this type of pricing is common. Spatial price discrimination manifests itself in markets in which firms are geographically differentiated[1] such as the markets of cement and steel or markets of customer-tailored goods. The wide application of this pricing strategy together with the fact that it is forbidden by some countries when it cannot be justified on the grounds of transportation/delivery costs (e.g., Robinson-Patman Act, 1936 in the US),[2] makes the investigation of spatial price discrimination of great interest for both academics and policymakers.

The main goal of the current paper is to examine the welfare properties of the equilibrium in a market where operating firms exercise spatial price discrimination. To this purpose, we develop an integrated model

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[1] Greenhut (1981) provides evidence that spatial price discrimination is apparent in cases where transportation cost represents at least 5% of total costs.

[2] For a review about the history of the enforcement of competition law against spatial discriminatory pricing, see Scherer and Ross (1980).

giving new insight into the structure of customer-specific pricing markets. The existing literature adopts at least one of the following five assumptions: (i) the number of firms in the market does not exceed two (ii) all firms are privately owned (iii) transportation costs are linear (iv) only one homogeneous good is traded and (v) firms have common marginal production costs. We relax all the above assumptions by assuming a market with an arbitrary number of heterogeneous competitors, an arbitrary level of privatization for each firm, a general transportation cost function and an arbitrary number and/or varieties of traded goods. Firm heterogeneity is reflected by assuming different marginal costs of production. It should be emphasized that this heterogeneity is firm-specific and not product-specific (i.e., the marginal cost of production differs across firms but remains the same for all goods/varieties produced by the same firm).

The effect of the above mentioned relaxation on the existence and the properties of the equilibrium is not clear. For example, d'Aspremont et al. (1979) showed that in the traditional Hotelling's model (Hotelling, 1929), the nature of travel costs is important for the existence of an equilibrium. Surprisingly enough, they showed that an equilibrium exists when transportation costs are proportional to the square of distance while it doesn't when the travel costs are linear. On the other hand, Cremer et al. (1991) highlighted the importance of the number of competing firms on the welfare properties of the equilibrium.

The present paper contributes to the existing literature in manifold ways. We show that in a model of spatial price discrimination, where the produced goods have the same reservation price for the buyers, the market outcome will be socially optimal, and this result is independent of the number of firms in the market, firm heterogeneity regarding marginal production costs, the level of privatization of each firm, the form of the transportation cost function and the number and/or the varieties of the goods offered by each competitor. We further argue that (i) when firms are homogeneous regarding their marginal production costs, there always exists a unique Nash equilibrium, regardless of the form of the transportation cost function extending a well-known similar result (e.g. Heywood and Ye, 2009a) restricted to linear transportation costs and (ii) when firms are heterogeneous and transportation costs are linear, there is a unique Nash equilibrium which does not depend on the distribution of the marginal production costs.

The driving force behind our welfare result is the same as in Lederer and Hurter (1986); a firm can increase its profit by opting for a production location so that the market is serviced with minimal total cost.[3] However, in Lederer and Hurter (1986) the discussion is restricted to only two exclusively privately owned firms offering the same good leaving untouched mixed markets with many competitors and multiple goods and the ensuing welfare questions. Moreover, our proof is completely different to the one found in Lederer and Hurter (1986) allowing for direct generalization.

[3] This implies the alignment of the social and private optima.

Our findings about the characterization of the equilibrium are also in line with Vogel (2011). Vogel (2011) proves the existence of a unique equilibrium for an arbitrary number of heterogeneous firms located in a circular market competing within the framework of spatial price discrimination bearing linear transportation costs, regardless of the distribution of their marginal production costs. We investigate the properties of the equilibrium, should it exist, when an arbitrary number of heterogeneous firms with various degrees of privatization offering multiple goods compete in a linear market under spatial price discrimination. In line with Vogel (2011), we prove the existence of a unique equilibrium when transportation costs are linear. We succeed, however, in establishing the social optimality of the Nash equilibrium in all cases.

Moreover, the fact that our model imposes no constraints on the level of privatization extends our contribution to the theory of mixed oligopoly under spatial price discrimination. The studies which are closest to ours are those of Heywood and Ye (2009a) and Heywood and Ye (2009b). Heywood and Ye (2009a) assume a market with an arbitrary number of homogeneous firms having binary ownership status (private or public) and focus on the role of the public firm in the Stackelberg equilibrium where the leader is a private or a public firm. They extend their model accounting for the existence of foreign firms in Heywood and Ye (2009b). The aforementioned papers impose too many restrictions in their modeling structure. Specifically, the framework of Heywood and Ye (2009a) and Heywood and Ye (2009b) imposes restrictions on the form of the transportation cost function, the degree of privatization and the attributes/number of goods in the market. In addition, a fundamental restriction of the aforementioned papers lies in the assumption of common marginal production costs.

Our paper is part of a wide literature on the welfare implications of spatial price discrimination; see, e.g., Greenhut and Ohta (1972), Holahan (1975), Thisse and Vives (1988), Hamilton et al. (1989), Hamilton et al. (1991), MacLeod et al. (1992), and Braid (2008). Building on this literature, we make inroads into the theory of mixed oligopoly when firms have differentiated marginal production costs.

The implications of our results can be summarized as follows: (i) A spatial price discriminatory market can serve as a typical example of how a 'laissez-faire' economy can lead to social optimality, (ii) the social optimality of the equilibrium is independent of firm heterogeneity and (iii) the dependence of the equilibrium locations on the relative difference of the marginal production costs has important policy implications for government intervention. In the example of subsection 3.2 the government can (pre)determine the locations of firms through intervention on the marginal production costs (e.g., tax incentives and subsidies for firms located in low-populated areas etc.) without affecting social optimality.

The rest of the paper is structured as follows. The next section presents the benchmark model where an arbitrary number of firms characterized by a different degree of privatization offer a homogeneous good. Two cases are examined: (a) firms have common marginal production costs and (b) firms are heterogeneous. The market is represented by a unit interval with the consumers uniformly distributed along it. A three-stage game of complete information is played by firms and consumers. More specifically, in the first stage firms simultaneously choose their locations. In the second stage, after observing their competitors' locations, firms engage in Bertrand competition à la Hoover (1937) and Lerner and Singer (1937). In other words, firms set their prices simultaneously and are allowed to price discriminate by charging a different price for different locations. Finally, in stage three, consumers make their purchasing choices to clear the market. After presenting our theoretical construct, we solve the game and characterize the Nash equilibrium. Section 3 generalizes the findings of section 2 for the case of multiple goods (or different varieties of the same good), section 4 concludes.

2 The benchmark model

We consider a market consisted of n private firms and a continuum of consumers uniformly distributed over the unit interval $[0, 1]$ representing a linear country.[4] Let x_i, $i = 1, ..., n$, denote the location of firm i in the interval $[0, 1]$ with $0 \leq x_1 < x_2 < ... < x_n \leq 1$. All firms produce and sell the same homogeneous good. Each consumer buys one unit of the good from the lowest price firm, providing that this price is lower or equal to her reservation price (i.e., the maximum price that the consumers are willing to pay for the good), $m > 0$. The marginal production cost of firm i is $c_i \geq 0$. Spatial price discrimination à la Hoover (1937) and Lerner and Singer (1937) is assumed. Specifically, the price charged for the good by the firm the consumer chooses to buy from, is equal to (or infinitesimally less than) the delivered cost of the remaining firms. Delivered costs are equal to the sum of transportation and production costs. Let $f(d(x_0, x_1)) := |F(x_1) - F(x_0)|$ evaluate the transportation cost between points x_0 and x_1, where d denotes the shipped distance and F be any function with F' positive and continuous on $[0, 1]$. Firms are located such that $f(d(x_{i+j}, x_i)) > |c_{i+j} - c_i|$ (non-negative profit condition)[5] for any $j > 0$ with $i, i + j \in \{1, ..., n\}$ and c_{i+j}, c_i the corresponding marginal costs of firms $i + j$ and i located at points x_{i+j} and x_i respectively. Let $s_{i,i+j}$ denote the locations of the indifferent consumer with respect to firms i and $i + j$.

Lemma 1. *(i)* $x_i < s_{i,i+j} < x_{i+j}$ *and* $F(s_{i,i+j}) = \frac{F(x_{i+j})+F(x_i)}{2} + \frac{c_{i+j}-c_i}{2}$ *and (ii) if* $j_1 < j_2$ *then (a)* $s_{i,i+j_1} < s_{i,i+j_2}$ *and (b) provided* $j_1 < j_2 < j$, $s_{i+j_1,i+j} < s_{i+j_2,i+j}$.

[4] By uniformly distributed, we mean that the proportion of consumers buying the good remains the same, regardless of the subinterval of $[0, 1]$.

[5] In the opposite case if, $f(d(x_{i+j}, x_i)) \leq |c_{i+j} - c_i|$, the total sales of either firm i or firm $i + j$ drop to zero.

Proof. By definition of $s_{i,i+j}$,

$$|F(s_{i,i+j}) - F(x_i)| + c_i = |F(x_{i+j}) - F(s_{i,i+j})| + c_{i+j}$$

By assumption F is an increasing function, so if $s_{i,i+j} \in [0, x_i] \cup [x_{i+j}, 1]$ then

$$f(d(x_{i+j}, x_i)) = F(x_{i+j}) - F(x_i) = |c_{i+j} - c_i|$$

a contradiction, thus, $x_i < s_{i,i+j} < x_{i+j}$ and $F(s_{i,i+j}) - F(x_i) + c_i = F(x_{i+j}) - F(s_{i,i+j}) + c_{i+j}$ i.e.,

$$F(s_{i,i+j}) = \frac{F(x_{i+j}) + F(x_i)}{2} + \frac{c_{i+j} - c_i}{2}$$

which proves *(i)*.

To prove *(ii) (a)* we argue by contradiction. Assume $s_{i,i+j_1} \geq s_{i,i+j_2}$, then

$$\begin{aligned}\frac{F(x_{i+j_1}) + F(x_i)}{2} + \frac{c_{i+j_1} - c_i}{2} &\geq \frac{F(x_{i+j_2}) + F(x_i)}{2} + \frac{c_{i+j_2} - c_i}{2} \Leftrightarrow \\ F(x_{i+j_2}) - F(x_{i+j_1}) &\leq c_{i+j_1} - c_{i+j_2}\end{aligned}$$

equivalently

$$f(d(x_{i+j_2}, x_{i+j_1})) \leq |c_{i+j_1} - c_{i+j_2}|$$

a contradiction. The proof of *(ii) (b)* follows similar lines. □

Consumers and firms engage in a three-stage game of complete information. In stage one, firms simultaneously decide their location. Having observed the location of their competitors, firms simultaneously choose delivered price schedules in the second stage. In the final stage, consumers take their purchasing decisions.

The rest of the section is structured as follows. Subsection 2.1 examines the properties of the equilibrium in the case of privately owned firms with common marginal costs. The case of mixed ownership with homogeneous firms is analyzed in subsection 2.2. The corresponding analyses for heterogeneous firms are presented in subsections 2.3 and 2.4, respectively.

2.1 Homogeneous firms

In this subsection, we examine the case where all firms have the same marginal cost of production, i.e. $c_1 = c_2 = ... = c_n$. The existence and social optimality of a symmetric equilibrium (see Proposition 2 below) provided that transportation costs are linear is known to exist and can be found in various references in the literature (e.g. Heywood and Ye, 2009a). We provide a different proof as an intermediate result, relaxing the assumption of linear transportation costs. In the case of linear transportation costs uniqueness follows from Proposition 3.

To simplify our analysis and without loss of generality, we set $c_i = 0$. The aggregate shipping cost[6] for all locations z of consumers who buy from any of the n firms is equal to

$$T^H(x_1, ..., x_n) = \sum_{i=1}^{n} T_i^H(x_1, ..., x_n) \tag{1}$$

where

$$T_i^H(x_1, ..., x_n) = \begin{cases} \begin{pmatrix} \int_0^{x_1}[F(x_1) - F(z)]dz \\ + \int_{x_1}^{\frac{x_1+x_2}{2}}[F(z) - F(x_1)]dz \end{pmatrix} & \text{for } i = 1 \\ \\ \begin{pmatrix} \int_{\frac{x_{i-1}+x_i}{2}}^{x_i}[F(x_i) - F(z)]dz \\ + \int_{x_i}^{\frac{x_i+x_{i+1}}{2}}[F(z) - F(x_i)]dz \end{pmatrix} & \text{for } 1 < i < n \\ \\ \begin{pmatrix} \int_{\frac{x_{n-1}+x_n}{2}}^{x_n}[F(x_n) - F(z)]dz \\ + \int_{x_n}^{1}[F(z) - F(x_n)]dz \end{pmatrix} & \text{for } i = n \end{cases} \tag{2}$$

is the total transportation cost for those consumers buying from firm i. As defined in Lederer and Hurter (1986) the social cost is the total supply cost when firms behave in a cooperative, cost minimizing manner. The fact that delivered costs coincide with transportation costs implies that the aggregate transportation cost represents the social cost. Hence, the socially optimal locations can be derived by minimizing the social cost with respect to each location x_i.[7]

Firm i is selling its product at a price matching (or which is infinitesimally less than) the delivery cost of its direct competitor which is the firm nearest to its location. The indifferent consumer between firms i

[6] The terms 'delivered cost' and 'shipping cost' are used interchangeably hereafter.

[7] In other words, social welfare is defined as the total consumer's willingness to pay less the aggregate transportation and production costs.

and $i+1$, according to Lemma 1, is located at $s_{i,i+1} = \frac{x_i + x_{i+1}}{2}$. Thus, the profit function of firm i is

$$\Pi_i^H(x_1,..,x_n) = \begin{cases} \begin{pmatrix} \int_0^{x_1} [F(x_2) - F(z) - F(x_1) + F(z)]\, dz \\ + \int_{x_1}^{\frac{x_1+x_2}{2}} [F(x_2) - F(z) - F(z) + F(x_1)]\, dz \end{pmatrix} \text{ if } i = 1 \\ \\ \begin{pmatrix} \int_{\frac{x_{i-1}+x_i}{2}}^{x_i} [F(z) - F(x_{i-1}) - F(x_i) + F(z)]\, dz \\ + \int_{x_i}^{\frac{x_{i-1}+x_{i+1}}{2}} [F(z) - F(x_{i-1}) - F(z) + F(x_i)]\, dz \\ + \int_{\frac{x_{i-1}+x_{i+1}}{2}}^{\frac{x_i+x_{i+1}}{2}} [F(x_{i+1}) - F(z) - F(z) + F(x_i)]\, dz \end{pmatrix} \\ \text{if } 1 < i < n \text{ and } x_i \leq \frac{x_{i-1}+x_{i+1}}{2} \\ \\ \begin{pmatrix} \int_{\frac{x_{i-1}+x_i}{2}}^{\frac{x_{i-1}+x_{i+1}}{2}} [F(z) - F(x_{i-1}) - F(x_i) + F(z)]\, dz \\ + \int_{\frac{x_{i-1}+x_{i+1}}{2}}^{x_i} [F(x_{i+1}) - F(z) - F(x_i) + F(z)]\, dz \\ + \int_{x_i}^{\frac{x_i+x_{i+1}}{2}} [F(x_{i+1}) - F(z) - F(z) + F(x_i)]\, dz \end{pmatrix} \\ \text{if } 1 < i < n \text{ and } \frac{x_{i-1}+x_{i+1}}{2} \leq x_i \\ \\ \begin{pmatrix} \int_{\frac{x_{n-1}+x_n}{2}}^{x_n} [F(z) - F(x_{n-1}) - F(x_n) + F(z)]\, dz \\ + \int_{x_n}^{1} [F(z) - F(x_{n-1}) - F(z) + F(x_n)]\, dz \end{pmatrix} \text{ if } i = n \end{cases} \quad (3)$$

.

Lemma 2. *When firms are homogeneous, the marginal aggregate transportation cost with respect to the location of firm i, $i = 1, ..., n$, is*

$$\partial T^H / \partial x_i = \begin{cases} F(x_1) - F(\frac{x_2 - x_1}{2}) \textit{ for } i = 1 \\ F(\frac{x_i - x_{i-1}}{2}) - F(\frac{x_{i+1} - x_i}{2}) \textit{ for } 1 < i < n \\ F(\frac{x_n - x_{n-1}}{2}) - F(1 - x_n) \textit{ for } i = n \end{cases} .$$

Proof. Let $G(y) := \int F(y)dy$, then for all i, $1 < i < n$, $T_i^H(x_1, ..., x_n) = [-G(x_i - z)]_{\frac{x_{i-1}+x_i}{2}}^{x_i} + [G(z - x_i)]_{x_i}^{\frac{x_i+x_{i+1}}{2}} = -2G(0) + G(\frac{x_i - x_{i-1}}{2}) + G(\frac{x_{i+1} - x_i}{2})$. x_i appears in the expression of the aggregate transportation cost only in T^H_{i-1}, T^H_i and T^H_{i+1}

$$T^H(x_1,...,x_n) = T^H_1(x_1,...,x_n) + ... + \underbrace{\left(\int_{\frac{x_{i-2}+x_{i-1}}{2}}^{x_{i-1}} [F(x_{i-1}) - F(z)]dz + \int_{x_{i-1}}^{\frac{x_{i-1}+x_i}{2}} [F(z) - F(x_{i-1})]dz\right)}_{T^H_{i-1}}$$

$$+T^H_i(x_1,...,x_n) + \underbrace{\left(\int_{\frac{x_i+x_{i+1}}{2}}^{x_{i+1}} [F(x_{i+1}) - F(z)]dz + \int_{x_{i+1}}^{\frac{x_{i+1}+x_{i+2}}{2}} [F(z) - F(x_{i+1})]dz\right)}_{T^H_{i+1}} + ... + T^H_n(x_1,...,x_n)$$

Hence, we get

$$\frac{\partial T^H}{\partial x_i} = \frac{\partial T^H_{i-1}}{\partial x_i} + \frac{\partial T^H_i}{\partial x_i} + \frac{\partial T^H_{i+1}}{\partial x_i} = \frac{1}{2}F(\frac{x_i - x_{i-1}}{2}) + \left[\frac{1}{2}F(\frac{x_i - x_{i-1}}{2}) - \frac{1}{2}F(\frac{x_{i+1} - x_i}{2})\right] - \frac{1}{2}F(\frac{x_{i+1} - x_i}{2}) = F(\frac{x_i - x_{i-1}}{2}) - F(\frac{x_{i+1} - x_i}{2})$$

for $1 < i < n$. The cases $i = 1$ and $i = n$ are treated similarly to yield $\partial T^H/\partial x_1 = F(x_1) - F(\frac{x_2 - x_1}{2})$ and $\partial T^H/\partial x_n = F(\frac{x_n - x_{n-1}}{2}) - F(1 - x_n)$ completing the proof of the Lemma. $\square$

Proposition 1. *When firms are homogeneous, the marginal aggregate transportation cost with respect to the location of firm i, $i = 1, ..., n$, is opposite to the marginal profit of firm i, i.e.*

$$\partial T^H/\partial x_i = -\partial \Pi^H_i/\partial x_i.$$

Proof. Letting $G(y) := \int F(y)dy$, for $1 < i < n$, (3) becomes

$$\Pi^H_i(x_1,...,x_n) = 2G(\frac{x_{i+1} - x_{i-1}}{2}) - 2G(\frac{x_i - x_{i-1}}{2}) - 2G(\frac{x_{i+1} - x_i}{2}) + 2G(0)$$

Differentiating the above expression, we get

$$\frac{\partial \Pi^H_i}{\partial x_i} = -F(\frac{x_i - x_{i-1}}{2}) + F(\frac{x_{i+1} - x_i}{2})$$

For $i = 1$ and $i = n$, we get respectively $\partial \Pi^H_1/\partial x_1 = -F(x_1) + F(\frac{x_2 - x_1}{2})$ and $\partial \Pi^H_n/\partial x_n = -F(\frac{x_n - x_{n-1}}{2}) + F(1 - x_n)$. Lemma 2 completes the proof of the proposition.

$\square$

Following our discussion above, the socially optimal locations are derived by minimizing (1) with respect to each firm's location. Hence, the socially optimal locations satisfy the system:

$$\partial T^H / \partial x_i = 0, \; i = 1, ..., n. \tag{4}$$

Proposition 1 shows that this is equivalent to:

$$\partial \Pi_i^H / \partial x_i = 0, \; i = 1, ..., n. \tag{5}$$

which proves

Proposition 2. *In models of spatial price discrimination, where firms are homogeneous, offer the same good and the market is represented by a uni-dimensional interval, the Nash equilibrium locations of firms are socially optimal.*

The next step is to check the existence and uniqueness of the Nash equilibrium.

Proposition 3. *In models of n homogeneous firms selling a good to consumers uniformly distributed along a linear country at prices conditional to consumer location, there always exist a unique Nash equilibrium which is socially optimal and does not depend on the form of the transportation cost function.*

Proof. According to Proposition 2, it suffices to exhibit either a socially optimum or a Nash equilibrium. We test fit of the unique Nash equilibrium when transportation costs are linear, i.e. $x_i^* = \frac{2i-1}{2n}$ $i = 1, ..., n.$[8] By Lemma 2, the above solution must satisfy the system $\nabla T^H(x_1, ..., x_n) = 0$. Indeed, for every $1 < i < n$, we must have $F(\frac{x_i - x_{i-1}}{2}) - F(\frac{x_{i+1} - x_i}{2}) = 0$. But this is equivalent to $F(\frac{\frac{1}{n}}{2}) - F(\frac{\frac{1}{n}}{2}) = 0$ which is true regardless of the transportation cost function f. The border cases are similarly verifiable. The uniqueness of the equilibrium can be proved as follows. The equilibrium satisfies (by Lemma 2)

$$\begin{array}{llllll}
F(x_1) & -F(\frac{x_2-x_1}{2}) & & & = & 0 \\
 & F(\frac{x_2-x_1}{2}) & -F(\frac{x_3-x_2}{2}) & & = & 0 \\
 & & \vdots & & & \vdots \\
 & & F(\frac{x_n-x_{n-1}}{2}) & -F(1-x_n) & = & 0
\end{array}$$

[8] For the derivation of the Nash equilibrium locations under linear transportation costs, see Heywood and Ye (2009a).

which implies $F(x_1) = ... = F(1 - x_n) = r$. Since F is $1-1$, there is a unique $D = F^{-1}(r)$ with

$$
\begin{gathered}
F(x_1) = q \Leftrightarrow x_1 = D \\
F(\tfrac{x_2 - x_1}{2}) = q \Longleftrightarrow \tfrac{x_2 - x_1}{2} = D \Leftrightarrow x_2 = 3D \\
F(\tfrac{x_3 - x_2}{2}) = q \Longleftrightarrow \tfrac{x_3 - x_2}{2} = D \Leftrightarrow x_3 = 5D \\
\vdots \\
F(\tfrac{x_n - x_{n-1}}{2}) = q \Leftrightarrow \tfrac{x_n - x_{n-1}}{2} = D \Leftrightarrow x_n = (2n-1)D \\
F(1 - x_n) = q \Leftrightarrow 1 - x_n = D \Leftrightarrow x_n = 1 - D
\end{gathered}
$$

The last two equations give $1 - D = 2nD - D \Longleftrightarrow D = \frac{1}{2n}$ completing the proof. □

2.2 Mixed oligopoly with homogeneous firms

In our analysis so far, all firms are privately owned. Let us now assume that single firm l, $l = \{1, ..., n\}$ is partly privately owned and partly publicly owned in proportions a_l and $1 - a_l$ (in other words a_l can be considered as the degree of privatization), respectively with $a_l \in [0, 1]$. In such a case, firm l will decide about its optimal location by maximizing the weighted average of its own profits and social welfare with weights a_l and $1 - a_l$, respectively. Social welfare is equal to the sum of the aggregate profits (the profit of all firms) and consumers' surplus. The consumers' surplus is given by

$$
CS^H(x_1, ..., x_n) = \sum_{i=1}^{n} CS_i^H(x_1, ..., x_n) \tag{6}
$$

where $CS_i^H(x_1, ..., x_n)$ is the consumer surplus generated for the consumers buying from firm i, therefore,

$$
CS_i^H(x_1,...,x_n) = \begin{cases}
\int_0^{x_1}[m - F(x_2) + F(z)]dz + \int_{x_1}^{\frac{x_1+x_2}{2}}[m - F(x_2) + F(z)]dz \text{ for } i = 1 \\
\\
\int_{\frac{x_{i-1}+x_i}{2}}^{x_i}[m - F(z) + F(x_{i-1})]dz + \int_{x_i}^{\frac{x_{i-1}+x_{i+1}}{2}}[m - F(z) + F(x_{i-1})]dz \\
\quad + \int_{\frac{x_{i-1}+x_{i+1}}{2}}^{\frac{x_i+x_{i+1}}{2}}[m - F(x_{i+1}) + F(z)]dz \\
\quad \text{for } x_i \leq \frac{x_{i-1}+x_{i+1}}{2} \text{ and } 1 < i < n \\
\\
\int_{\frac{x_{i-1}+x_i}{2}}^{\frac{x_{i-1}+x_{i+1}}{2}}[m - F(z) + F(x_{i-1})]dz + \int_{\frac{x_{i-1}+x_{i+1}}{2}}^{x_i}[m - F(x_{i+1}) + F(z)]dz \\
\quad + \int_{x_i}^{\frac{x_i+x_{i+1}}{2}}[m - F(x_{i+1}) + F(z)]dz \\
\quad \text{for } \frac{x_{i-1}+x_{i+1}}{2} \leq x_i \text{ and } 1 < i < n \\
\\
\int_{\frac{x_{n-1}+x_n}{2}}^{x_n}[m - f(z - x_{n-1})]dz + \int_{x_n}^{1}[m - f(z - x_{n-1})]dz \text{ for } i = n
\end{cases} \tag{7}
$$

Direct calculation proves

Lemma 3. $\Pi_i^H(x_1,...,x_n) + CS_i^H(x_1,...,x_n) = \begin{cases} \int_0^{\frac{x_1+x_2}{2}} mdz - T_1^H(x_1,...,x_n) \textit{ for } i = 1 \\ \\ \int_{\frac{x_{i-1}+x_i}{2}}^{\frac{x_i+x_{i+1}}{2}} mdz - T_i^H(x_1,...,x_n) \textit{ for } 1 < i < n \\ \\ \int_{\frac{x_{n-1}+x_n}{2}}^{1} mdz - T_n^H(x_1,...,x_n) \textit{ for } i = n \end{cases}$

Summing up over all firms one gets the following Proposition which could be viewed as the second main result of this subsection.

Proposition 4.

$$\sum_{i=1}^{n} \Pi_i^H(x_1,...,x_n) + CS^H(x_1,...,x_n) = m - T^H(x_1,...,x_n)$$

Proof. Straightforward calculations. □

The profit function of the partly publicly owned firm l will be

$$\breve{\Pi}_l^H(x_1,...,x_n) = \Pi_l^H(x_1,...,x_n) + (1 - a_l)\left[\sum_{i \neq l} \Pi_i^H(x_1,...,x_n) + CS^H(x_1,...,x_n)\right] \tag{8}$$

where Π_l^H would be the profit function of firm l if it was fully privately owned.

Proposition 5. *When firms are homogeneous, Nash equilibria remain socially optimal regardless of the degree of privatization of the individual firms* $l, 1 \leq l \leq n$.

Proof. Fix a random $l, 1 \leq l \leq n$. Using Proposition 4 and (8), we get

$$\breve{\Pi}_l^H(x_1, ..., x_n) = \Pi_l^H(x_1, ..., x_n) + (1 - a_l)\left[m - T^H(x_1, ..., x_n) - \Pi_l^H(x_1, ..., x_n)\right]$$

From Proposition 1

$$\partial T^H/\partial x_l = -\partial \Pi_l^H/\partial x_l \Longleftrightarrow -\partial T^H/\partial x_l - \partial \Pi_l^H/\partial x_l = 0$$

which implies that $\partial \breve{\Pi}_l^H/\partial x_l = \partial \Pi_l^H/\partial x_l$. Induction on i completes the proof. $\square$

2.3 Heterogeneous firms

In this subsection, we study the case of n firms producing with different marginal production costs.

In consistency with our notation above, let $s_{i,i+j}$ denote the locations of the indifferent consumer. Then the profit of firm i, $\Pi_i^N(x_1, ..., x_n)$, is given by

$$\Pi_1^N(x_1, ..., x_n) = \int_0^{s_{1,2}} [f(d(x_2, z)) + c_2 - f(d(x_1, z)) - c_1] dz$$

$$\begin{aligned} \Pi_i^N(x_1, ..., x_n) &= \int_{s_{i-1,i}}^{s_{i-1,i+1}} [f(d(z, x_{i-1})) + c_{i-1} - f(d(z, x_i)) - c_i] dz \\ &+ \int_{s_{i-1,i+1}}^{s_{i,i+1}} [f(d(x_{i+1}, z)) + c_{i+1} - f(d(z, x_i)) - c_i] dz \end{aligned}$$

$$\Pi_n^N(x_1, ..., x_n) = \int_{s_{n-1,n}}^{1} [f(d(z, x_{n-1})) + c_{n-1} - f(d(z, x_n)) - c_n] dz$$

The next Lemma relates the profit of firm i in this case, $\Pi_i^N(x_1, ..., x_n)$, to the corresponding profit of the same firm, $\Pi_i^H(x_1, ..., x_n)$, if all marginal costs were equal.

Lemma 4.

$$\begin{gathered} \Pi_1^N(x_1, ..., x_n) = \Pi_1^H(x_1, ..., x_n) + (c_2 - c_1)\tfrac{x_1 + x_2}{2} \\ + \int_{\frac{x_1+x_2}{2}}^{s_{1,2}} [F(x_1) + F(x_2) - 2F(z) + c_2 - c_1] dz \end{gathered}$$

$$\Pi_i^N(x_1,...,x_n) = \Pi_i^H(x_1,...,x_n)$$
$$-\int_{\frac{x_{i-1}+x_i}{2}}^{s_{i-1,i}} [F(z) - F(x_{i-1}) + c_{i-1} - F(x_i) + F(z) - c_i] dz$$
$$+\int_{\frac{x_i+x_{i+1}}{2}}^{s_{i,i+1}} [F(x_{i+1}) - F(z) + c_{i+1} - F(z) + F(x_i) - c_i] dz$$
$$-\int_{\frac{x_{i-1}+x_{i+1}}{2}}^{s_{i-1,i+1}} [F(x_{i+1}) - F(z) + c_{i+1} - F(z) + F(x_{i-1}) - c_{i-1}] dz$$
$$+(c_{i-1} - c_i)\tfrac{x_{i+1}-x_i}{2} + (c_{i+1} - c_i)\tfrac{x_i - x_{i-1}}{2}$$

$$\Pi_n^N(x_1,...,x_n) = \Pi_n^H(x_1,...,x_n)$$
$$-\int_{\frac{x_{n-1}+x_n}{2}}^{s_{n-1,n}} [2F(z) - F(x_{n-1}) - F(x_n) + c_{n-1} - c_n]\, dz$$
$$+(c_{n-1} - c_n)\tfrac{x_n - x_{n-1}}{2}$$
$$+(c_{n-1} - c_n)(1 - x_n)$$

Proof.

$$\Pi_1^N(x_1,...,x_n) = \int_0^{s_{1,2}} [f(d(x_2,z)) + c_2 - f(d(x_1,z)) - c_1]\, dz$$
$$= \int_0^{x_1} [F(x_2) - F(z) + c_2 - (F(x_1) - F(z) + c_1)]\, dz$$
$$+\int_{x_1}^{\frac{x_1+x_2}{2}} [F(x_2) - F(z) + c_2 - (F(z) - F(x_1) + c_1)]\, dz$$
$$+\int_{\frac{x_1+x_2}{2}}^{s_{1,2}} [F(x_2) - F(z) + c_2 - (F(z) - F(x_1) + c_1)]\, dz$$
$$= \int_0^{x_1} [F(x_2) - F(z) - F(x_1) + F(z)] dz$$
$$+(c_2 - c_1)x_1$$
$$+\int_{x_1}^{\frac{x_1+x_2}{2}} [F(x_2) - F(z) - F(z) + F(x_1)] dz$$
$$+(c_2 - c_1)\tfrac{x_2 - x_1}{2}$$
$$+\int_{\frac{x_1+x_2}{2}}^{s_{1,2}} [F(x_1) + F(x_2) - 2F(z) + c_2 - c_1] dz$$
$$= \Pi_1^H(x_1,...,x_n)$$
$$+(c_2 - c_1)\tfrac{x_2 + x_1}{2}$$
$$+\int_{\frac{x_1+x_2}{2}}^{s_{1,2}} [F(x_1) + F(x_2) - 2F(z) + c_2 - c_1] dz$$

which settles the proof for $\Pi_1^N(x_1,...,x_n)$. To prove the Lemma for the $\Pi_i^N(x_1,...,x_n)$, without any real loss of generality, we consider the case $x_i < s_{i-1,i} < \frac{x_{i-1}+x_i}{2} < x_i < \frac{x_{i-1}+x_{i+1}}{2} < \frac{x_i+x_{i+1}}{2} < s_{i-1,i+1} < s_{i,i+1} < x_{i+1}$.

Then,

$$
\begin{aligned}
\Pi_i^N(x_1,...,x_n) &= \int_{s_{i-1,i}}^{x_i}[f(d(x_{i-1},z))+c_{i-1}-f(d(x_i,z))-c_i]dz \\
&+\int_{x_i}^{s_{i-1,i+1}}[f(d(x_{i-1},z))+c_{i-1}-f(d(x_i,z))-c_i]dz \\
&+\int_{s_{i-1,i+1}}^{s_{i,i+1}}[f(d(x_{i+1},z))+c_{i+1}-f(d(x_i,z))-c_i]dz \\
&= \int_{s_{i-1,i}}^{\frac{x_{i-1}+x_i}{2}}[F(z)-F(x_{i-1})+c_{i-1}-(F(x_i)-F(z)+c_i)]dz \\
&+\int_{\frac{x_{i-1}+x_i}{2}}^{x_i}[F(z)-F(x_{i-1})+c_{i-1}-(F(x_i)-F(z)+c_i)]dz \\
&+\int_{x_i}^{\frac{x_{i-1}+x_{i+1}}{2}}[F(z)-F(x_{i-1})+c_{i-1}-(F(z)-F(x_i)+c_i)]dz \\
&+\int_{\frac{x_{i-1}+x_{i+1}}{2}}^{s_{i-1,i+1}}[F(z)-F(x_{i-1})+c_{i-1}-(F(z)-F(x_i)+c_i)]dz \\
&+\int_{\frac{x_{i-1}+x_{i+1}}{2}}^{\frac{x_i+x_{i+1}}{2}}[F(x_{i+1})-F(z)+c_{i+1}-(F(z)-F(x_i)+c_i)]dz \\
&+\int_{\frac{x_i+x_{i+1}}{2}}^{s_{i,i+1}}[F(x_{i+1})-F(z)+c_{i+1}-(F(z)-F(x_i)+c_i)]dz \\
&-\int_{\frac{x_{i-1}+x_{i+1}}{2}}^{s_{i-1,i+1}}[F(x_{i+1})-F(z)+c_{i+1}-(F(z)-F(x_i)+c_i)]dz \\
&= \Pi_i^H(x_1,...,x_n)-\int_{\frac{x_{i-1}+x_i}{2}}^{s_{i-1,i}}[2F(z)-F(x_{i-1})-F(x_i)+c_{i-1}-c_i]dz \\
&+\int_{\frac{x_i+x_{i+1}}{2}}^{s_{i,i+1}}[F(x_{i+1})+F(x_i)-2F(z)+c_{i+1}-c_i]dz \\
&-\int_{\frac{x_{i-1}+x_{i+1}}{2}}^{s_{i-1,i+1}}[F(x_{i+1})+F(x_{i-1})-2F(z)+c_{i+1}-c_{i-1}]dz \\
&+(c_{i-1}-c_i)\frac{x_{i+1}-x_i}{2}+(c_{i+1}-c_i)\frac{x_i-x_{i-1}}{2}.
\end{aligned}
$$

Finally, to prove the Lemma for $i=n$ consider

$$
\begin{aligned}
\Pi_n^N(x_1,...,x_n) &= \int_{s_{n-1,n}}^{1}[f(d(z,x_{n-1}))+c_{n-1}-f(d(z,x_n))-c_n]dz \\
&= \int_{s_{n-1,n}}^{x_n}[F(z)-F(x_{n-1})+c_{n-1}-(F(x_n)-F(z)+c_n)]dz \\
&+\int_{x_n}^{1}[F(z)-F(x_{n-1})+c_{n-1}-(F(z)-F(x_n)+c_n)]dz \\
&= -\int_{\frac{x_{n-1}+x_n}{2}}^{s_{n-1,n}}[2F(z)-F(x_{n-1})-F(x_n)+c_{n-1}-c_n]dz \\
&+\int_{\frac{x_{n-1}+x_n}{2}}^{x_n}[F(z)-F(x_{n-1})-F(x_n)+F(z)]dz+(c_{n-1}-c_n)\frac{x_n-x_{n-1}}{2} \\
&+\int_{x_n}^{1}[F(z)-F(x_{n-1})-F(z)+F(x_n)]dz+(c_{n-1}-c_n)(1-x_n) \\
&= \Pi_n^H(x_1,...,x_n)-\int_{\frac{x_{n-1}+x_n}{2}}^{s_{n-1,n}}[2F(z)-F(x_{n-1})-F(x_n)+c_{n-1}-c_n]dz \\
&+(c_{n-1}-c_n)\frac{x_n-x_{n-1}}{2}+(c_{n-1}-c_n)(1-x_n)
\end{aligned}
$$

which completes the proof of the Lemma. □

We now calculate the total shipping cost function $T^N(x_1,...,x_n)$.

$$
\begin{aligned}
T^N(x_1,...,x_n) &= \int_0^{s_{1,2}}[f(d(z,x_1))+c_1]dz+...+\int_{s_{i-1,i}}^{s_{i,i+1}}[f(d(z,x_i))+c_i]dz \\
&+...+\int_{s_{n-1,n}}^{1}[f(d(z,x_n))+c_n]dz
\end{aligned}
$$

The following Lemma holds true.

Lemma 5.

$$T^N(x_1,...,x_n) = T^H(x_1,...,x_n) + c_1x_1$$
$$+c_1\tfrac{x_2-x_1}{2} + ... + c_i\tfrac{x_i-x_{i-1}}{2} + c_i\tfrac{x_{i+1}-x_i}{2} + ... + c_n\tfrac{x_n-x_{n-1}}{2} + c_n(1-x_n)$$
$$+\textstyle\sum_{i=1}^{n-1}\int_{\frac{x_i+x_{i+1}}{2}}^{s_{i,i+1}}[2F(z) - F(x_i) - F(x_{i+1}) + c_i - c_{i+1}]dz$$

Proof.

$$
\begin{aligned}
T^N(x_1,...,x_n) = {} & \textstyle\int_0^{x_1}[F(x_1) - F(z) + c_1]dz \\
& +\textstyle\int_{x_1}^{\frac{x_1+x_2}{2}}[F(z) - F(x_1) + c_1]dz \\
& +\textstyle\int_{\frac{x_1+x_2}{2}}^{s_{1,2}}[F(z) - F(x_1) + c_1]dz + ... \\
& -\textstyle\int_{\frac{x_{i-1}+x_i}{2}}^{s_{i-1,i}}[F(x_i) - F(z) + c_i]dz \\
& +\textstyle\int_{\frac{x_{i-1}+x_i}{2}}^{x_i}[F(x_i) - F(z) + c_i]dz + \int_{x_i}^{\frac{x_i+x_{i+1}}{2}}[F(z) - F(x_i) + c_i]dz \\
& +\textstyle\int_{\frac{x_i+x_{i+1}}{2}}^{s_{i,i+1}}[F(z) - F(x_i) + c_i]dz - \int_{\frac{x_i+x_{i+1}}{2}}^{s_{i,i+1}}[F(x_{i+1}) - F(z) + c_{i+1}]dz \\
& +\textstyle\int_{\frac{x_i+x_{i+1}}{2}}^{x_{i+1}}[F(x_{i+1}) - F(z) + c_{i+1}]dz + \int_{x_{i+1}}^{\frac{x_{i+1}+x_{i+2}}{2}}[F(z) - F(x_{i+1}) + c_{i+1}]dz \\
& +\textstyle\int_{\frac{x_{i+1}+x_{i+2}}{2}}^{s_{i+1,i+2}}[F(z) - F(x_{i+1}) + c_{i+1}]dz + ... \\
& -\textstyle\int_{\frac{x_{n-1}+x_n}{2}}^{s_{n-1,n}}[F(x_n) - F(z) + c_n]dz \\
& +\textstyle\int_{\frac{x_{n-1}+x_n}{2}}^{x_n}[F(x_n) - F(z) + c_n]dz + \int_{x_n}^{1}[F(z) - F(x_n) + c_n]dz \\
= {} & \textstyle\int_0^{x_1}[F(x_1) - F(z)]dz + c_1x_1 + \int_{x_1}^{\frac{x_1+x_2}{2}}[F(z) - F(x_1)]dz + c_1\frac{x_2-x_1}{2} \\
& +\textstyle\int_{\frac{x_1+x_2}{2}}^{s_{1,2}}[2F(z) - F(x_1) - F(x_2) + c_1 - c_2]dz + ... + \int_{\frac{x_{i-1}+x_i}{2}}^{x_i}[F(x_i) - F(z)]dz \\
& +c_i\tfrac{x_i-x_{i-1}}{2} + \textstyle\int_{x_i}^{\frac{x_i+x_{i+1}}{2}}[F(z) - F(x_i)]dz + c_i\frac{x_{i+1}-x_i}{2} \\
& +\textstyle\int_{\frac{x_i+x_{i+1}}{2}}^{s_{i,i+1}}[2F(z) - F(x_i) - F(x_{i+1}) + c_i - c_{i+1}]dz + ... \\
& +\textstyle\int_{\frac{x_{n-1}+x_n}{2}}^{s_{n-1,n}}[2F(z) - F(x_{n-1}) - F(x_n) + c_{n-1} - c_n]dz \\
& +\textstyle\int_{\frac{x_{n-1}+x_n}{2}}^{x_n}[F(x_n) - F(z)]dz + c_n\frac{x_n-x_{n-1}}{2} + \int_{x_n}^{1}[F(z) - F(x_n)]dz + c_n(1-x_n)
\end{aligned}
$$

completing the proof of the Lemma. □

Proposition 6. *The marginal aggregate shipping cost with respect to the location of firm i, $i = 1,...,n$, is opposite to the marginal profit of firm i, i.e.*

$$\partial T^N(x_1,...,x_n)/\partial x_i = -\partial\Pi_i^N(x_1,...,x_n)/\partial x_i$$

Proof. We prove the Proposition for i, $1 < i < n$; the border cases, for $i = 1$ and $i = n$, being very similar.

According to Lemma 4

$$\begin{aligned}
\frac{\partial \Pi_i^N(x_1,\ldots,x_n)}{\partial x_i} &= \frac{\partial \Pi_i^H(x_1,\ldots,x_n)}{\partial x_i} - \tfrac{1}{2}(c_{i-1}-c_i) + \tfrac{1}{2}(c_{i+1}-c_i) \\
&+ \frac{\partial}{\partial x_i}\left[\int_{\frac{x_{i-1}+x_i}{2}}^{s_{i-1,i}} [F(x_{i-1})+F(x_i)-2F(z)+c_i-c_{i-1}]dz\right] \\
&+ \frac{\partial}{\partial x_i}\left[\int_{\frac{x_i+x_{i+1}}{2}}^{s_{i,i+1}} [F(x_{i+1})+F(x_i)-2F(z)+c_{i+1}-c_i]dz\right] \\
&+ \frac{\partial}{\partial x_i}\left[\int_{\frac{x_{i-1}+x_{i+1}}{2}}^{s_{i-1,i+1}} [2F(z)-F(x_{i+1})-F(x_{i-1})+c_{i-1}-c_{i+1}]dz\right]
\end{aligned} \tag{9a}$$

On the other hand, from Lemma 5, we get

$$\begin{aligned}
\frac{\partial T^N(x_1,\ldots,x_n)}{\partial x_i} &= \frac{\partial T^H(x_1,\ldots,x_n)}{\partial x_i} + \frac{c_{i-1}}{2} + \frac{c_i}{2} - \frac{c_i}{2} - \frac{c_{i+1}}{2} \\
&+ \frac{\partial}{\partial x_i}\left[\sum_{j=1}^{n-1}\int_{\frac{x_j+x_{j+1}}{2}}^{s_{j,j+1}} [2F(z)-F(x_j)-F(x_{j+1})+c_j-c_{j+1}]dz\right]
\end{aligned} \tag{9b}$$

For all $j \neq i-1, i$

$$\frac{\partial}{\partial x_i}\left[\int_{\frac{x_j+x_{j+1}}{2}}^{s_{j,j+1}} [2F(z)-F(x_j)-F(x_{j+1})+c_j-c_{j+1}]dz\right] = 0$$

turning (9a) into

$$\begin{aligned}
\frac{\partial \Pi_i^N(x_1,\ldots,x_n)}{\partial x_i} &= \frac{\partial \Pi_i^H(x_1,\ldots,x_n)}{\partial x_i} - \frac{c_{i-1}}{2} + \frac{c_{i+1}}{2} \\
&+ \frac{\partial}{\partial x_i}\left[\int_{\frac{x_{i-1}+x_i}{2}}^{s_{i-1,i}} [F(x_{i-1})+F(x_i)-2F(z)+c_i-c_{i-1}]dz\right] \\
&+ \frac{\partial}{\partial x_i}\left[\int_{\frac{x_i+x_{i+1}}{2}}^{s_{i,i+1}} [F(x_{i+1})+F(x_i)-2F(z)+c_{i+1}-c_i]dz\right]
\end{aligned} \tag{10a}$$

and (9b) into

$$\begin{aligned}
\frac{\partial T^N(x_1,\ldots,x_n)}{\partial x_i} &= \frac{\partial T^H(x_1,\ldots,x_n)}{\partial x_i} + \frac{c_{i-1}}{2} - \frac{c_{i+1}}{2} \\
&+ \frac{\partial}{\partial x_i}\left[\int_{\frac{x_{i-1}+x_i}{2}}^{s_{i-1,i}} [2F(z)-F(x_{i-1})-F(x_i)+c_{i-1}-c_i]dz\right] \\
&+ \frac{\partial}{\partial x_i}\left[\int_{\frac{x_i+x_{i+1}}{2}}^{s_{i,i+1}} [2F(z)-F(x_i)-F(x_{i+1})+c_i-c_{i+1}]dz\right]
\end{aligned} \tag{10b}$$

Proposition 1 of the homogeneous case ensures that $\frac{\partial T^H(x_1,\ldots,x_n)}{\partial x_i} = -\frac{\partial \Pi_i^H(x_1,\ldots,x_n)}{\partial x_i}$, therefore the right-hand sides of (10a) and (10b) are equal proving the Proposition.

□

From the analysis so far, we get that socially optimal locations satisfy the system

$$\partial T^N(x_1,\ldots,x_n)/\partial x_i = 0,\; i = 1,\ldots,n. \tag{11}$$

whereas Nash equilibrium locations satisfy

$$\partial\Pi_i^N(x_1,...,x_n)/\partial x_i = 0,\ i = 1,...,n. \tag{12}$$

Proposition 7. *In models of spatial price discrimination, where firms, offer the same good and the market is represented by a closed interval, the Nash equilibrium locations of firms are socially optimal.*

Proof. By a linear transformation any closed interval can be mapped in a one-to-one way onto $[0,1]$. The rest of the proof follows by Proposition 6 which ensures that the system of (11) and (12) are equivalent and therefore have the same solution. □

2.4 Mixed oligopoly with heterogeneous firms

The consumer surplus generated for the consumers buying from firm i is

$$CS_i^N(x_1,...,x_n) = \begin{cases} \int_0^{s_{1,2}}[m - f(d(x_2,z)) - c_2]dz \text{ for } i = 1 \\ \\ \begin{array}{c} \int_{s_{i-1,i}}^{s_{i-1,i+1}}[m - f(d(z,x_{i-1})) - c_{i-1}]dz \\ + \int_{s_{i-1,i+1}}^{s_{i,i+1}}[m - f(d(x_{i+1},z)) - c_{i+1}]dz \end{array} \quad \text{for } 1 < i < n \\ \\ \int_{s_{n-1,n}}^{1}[m - f(d(z,x_{n-1})) - c_{n-1}]dz \text{ for } i = n \end{cases}$$

Following a similar analytical reasoning with section 2.2, we get the following proposition

Proposition 8.
$$\sum_{i=1}^{n}\Pi_i^N(x_1,...,x_n) + CS^N(x_1,...,x_n) = m - T^N(x_1,...,x_n)$$

where $CS^N(x_1,...,x_n) = \sum\limits_{i=1}^{n} CS_i^N(x_1,...,x_n)$ *is the total consumers' surplus when firms are heterogeneous.*

Proof. Straightforward calculations. □

Similar to section 2.2, the profit function of the partly publicly owned firm l when marginal costs of production are different will be

$$\breve{\Pi}_l^N(x_1,...,x_n) = \Pi_l^N(x_1,...,x_n) + (1 - a_l)\left[\sum_{i\neq l}\Pi_i^N(x_1,...,x_n) + CS^N(x_1,...,x_n)\right] \tag{13}$$

where Π_l^N would be the profit function of firm l if it was fully privately owned.

Proposition 9. *When firms are heterogeneous, Nash equilibria remain socially optimal regardless of the degree of privatization of the individual firms $l, 1 \leq l \leq n$.*

Proof. Fix a random $l, 1 \leq l \leq n$. Using Proposition 8 and (13), we get

$$\breve{\Pi}_l^N(x_1, ..., x_n) = \Pi_l^N(x_1, ..., x_n) + (1 - a_l)\left[m - T^N(x_1, ..., x_n) - \Pi_l^N(x_1, ..., x_n)\right]$$

From Proposition 6

$$\partial T^N/\partial x_l = -\partial \Pi_l^N/\partial x_l \Longleftrightarrow -\partial T^N/\partial x_l - \partial \Pi_l^N/\partial x_l = 0$$

which implies that $\partial \breve{\Pi}_l^N/\partial x_l = \partial \Pi_l^N/\partial x_l$. Induction on i completes the proof. □

3 The case of multiple goods with heterogeneous firms

3.1 Private firms

We now assume the existence of L different goods or different varieties of the same good or both. Let k_j denote the number of firms producing good j, $j = 1, ..., L$ with $1 \leq k_j \leq n$. Let $T^{N,j}$ denote the aggregate transportation cost related to the provision of good j and $\Pi_i^{N,j}$ the corresponding profit per consumer of firm i from selling good j with $\Pi_i^{N,j} := 0$ if good j is not produced by firm i. The fraction of consumers buying product j is now denoted by $h_j \in (0, 1]$ uniformly spread over $[0, 1]$ with $\sum_{j=1}^{L} h_j = 1$; hence, there will be buyers for all available products. In the case where good j is produced by only one firm, then this firm enjoys monopoly privileges and charges a price equal to, or infinitesimally smaller than, the reservation price m_j, i.e. the maximum price the consumer is willing to pay for good j. A fundamental assumption in this multi-good setting is that $m_1 = ... = m_L = m$ (i.e. the reservation price of all goods is identical).[9] Let $\tilde{T}^N$ denote the aggregate shipping cost for all products and $\tilde{\Pi}_i^N$ the total profit of firm i for all products it produces.

Proposition 10. *The marginal aggregate shipping cost with respect to the location of firm i is opposite to the marginal profit of firm i, namely $\partial \tilde{T}^N/\partial x_i = -\, \partial \tilde{\Pi}_i^N/\partial x_i$.*

Proof. By definition $\tilde{T}^N = \sum_{j=1}^{L} h_j T^{N,j}$ and $\tilde{\Pi}_i^N = \sum_{j=1}^{L} h_j \Pi_i^{N,j}$. Applying Proposition 6 for every single traded product j we get

$$\partial \tilde{T}^N/\partial x_i = \sum_{j=1}^{L} h_j \partial T^{N,j}/\partial x_i = -\sum_{j=1}^{L} h_j \partial \Pi_i^{N,j}/\partial x_i = -\partial \tilde{\Pi}_i^N/\partial x_i$$

[9] It should be noted that this assumption is more realistic in the case of the different varieties of the same good and less in the case of different goods.

□

Theorem 1. *In models of spatial price discrimination, where firms have different marginal production costs, produce different combination of goods, consumers are distributed uniformly along a linear city of unit length and have the same reservation price for all goods, the Nash equilibrium locations of firms are socially optimal.*

Proof. To derive the socially optimal locations we have to minimize $\tilde{T}^N$ with respect to each firm's location. Hence, the socially optimal locations satisfy the following system of equations:

$$\partial \tilde{T}^N / \partial x_i = 0,\ i = 1, ..., n. \tag{14}$$

On the other hand, the Nash equilibrium locations are given by the solution of the following system:

$$\partial \tilde{\Pi}_i^N / \partial x_i = 0,\ i = 1, ..., n. \tag{15}$$

Because of Proposition 10, systems (14) and (15) are equivalent and hence they have the same set of solutions. □

3.2 Mixed oligopoly

Let's now turn to the case where some firm, say firm l is partly privately owned and partly publicly owned. Keeping the notation the same as in subsections 2.4 and 3.1, $\breve{\Pi}_l^N = \sum_{j=1}^{L} h_j \breve{\Pi}_l^{N,j}$ where $h_j \breve{\Pi}_l^{N,j}$ is the profit of the partially privatized firm l from selling good j. It is understood that $\breve{\Pi}_l^{N,j} = 0$ if good j is not produced by firm l.

Theorem 2. *The degree of privatization does not affect the socially optimal Nash equilibrium locations.*

Proof. From the proof of Proposition 9, we have that for every single product j

$$\partial \breve{\Pi}_l^{N,j} / \partial x_l = \partial \Pi_l^{N,j} / \partial x_l\,.$$

Therefore,

$$\partial \breve{\Pi}_l^N / \partial x_l = \sum_{j=1}^{L} h_j (\partial \breve{\Pi}_l^{N,j} / \partial x_l) = \sum_{j=1}^{L} h_j \partial \breve{\Pi}_l^{N,j} / \partial x_l = \sum_{j=1}^{L} \partial \Pi_l^{N,j} / \partial x_l = \partial \tilde{\Pi}_i^N / \partial x_i$$

$\square$

Theorem 3. *In a mixed oligopoly of n firms, with $n \geq 3$, producing different combinations of goods with differentiated marginal production costs and linear transportation costs, there exists a unique socially optimal Nash equilibrium of locations for any $(c_1, ..., c_n)$ in the non bounded subset $\mathbb{C}$, of the positive orthant $\mathbb{R}^n_+$, defined by the inequalities*

$$(n-2)c_i + (n-2)c_{i+1} - 2\sum_{j=1, j\neq i, i+1}^{n} c_j < 1 \tag{16}$$

$$2(n-1)c_1 - 2\sum_{j=2}^{n} c_j < 1 \tag{17}$$

and

$$-2\sum_{j=1}^{n-1} c_j + 2(n-1)c_n < 1 \tag{18}$$

for $i = 1, ..., n$. Further any two marginal cost vectors $(c_1, ..., c_n)$ and $(c_1^{'}, ..., c_n^{'})$ in the subset $\mathbb{C}$, such that $c_i' = c_i + u$ lead to the same equilibrium locations.

Proof. We prove Theorem 3 in the simplest possible setting that of private firms producing only one common good assuming the per distance transportation cost, t, equal to one. The general case for the mixed oligopoly with multiple goods and $t \neq 1$ can then be proved along similar to the analysis above lines.

According to Proposition 7, the optimal locations must satisfy the system

$$\begin{array}{cccccccccc}
3x_1 & -x_2 & & & & = & -c_1 & +c_2 & & \\
-x_1 & +2x_2 & -x_3 & & & = & -c_1 & & +c_3 & \\
 & -x_2 & +2x_3 & -x_4 & & = & & -c_2 & & +c_4 \\
 & & & \vdots & & & & & & \vdots \\
 & & & -x_{n-1} & +3x_n & = & & & & -c_{n-1}+c_n+2
\end{array}$$

It is straightforward to check that the above system is row equivalent to

$$\begin{array}{llllll}
3x_1 & -x_2 & & & = & -c_1+c_2 \\
& \frac{5}{3}x_2 & -x_3 & & = & -\frac{4}{3}c_1+\frac{1}{3}c_2+c_3 \\
& & \frac{7}{5}x_3 & -x_4 & = & -\frac{4}{5}c_1-\frac{4}{5}c_2+\frac{3}{5}c_3+c_4 \\
& & & \vdots & & \vdots \\
& & & \frac{4n}{2n-1}x_n & = & -\frac{4}{2n-1}c_1-\frac{4}{2n-1}c_2-\ldots-\frac{4}{2n-1}c_{n-1}+\frac{4(n-1)}{2n-1}c_n+2
\end{array}$$

where the i-line $1 < i < n$ is given by

$$\frac{2i+1}{2i-1}x_i - x_{i+1} = -\frac{4}{2i-1}c_1 - \frac{4}{2i-1}c_2 - \ldots - \frac{4}{2i-1}c_{i-1} + \frac{2i-3}{2i-1}c_i + c_{i+1}\,.$$

Solving for x_i we get

$$\begin{aligned}
x_i \quad = \quad & \frac{2i-1}{2i+1}[x_{i+1} + \frac{4}{2i-1}(c_i - c_1) + \ldots + \frac{4}{2i-1}(c_i - c_\rho) \\
& + \frac{\upsilon}{2i-1}(c_i - c_{\rho+1}) + \frac{4-\upsilon}{2i-1}(c_{i+1} - c_{\rho+1}) \\
& + \frac{4}{2i-1}(c_{i+1} - c_{\rho+2}) + \ldots + \frac{4}{2i-1}(c_{i+1} - c_{i-1})]
\end{aligned}$$

where $2i-3 = 4\rho + \upsilon$, $0 < \upsilon < 4$.

Inherent to the discussion leading to Proposition 7 was the assumption that $x_1 < x_2 < \ldots < x_n$. It is a straightforward, albeit tedious, calculation to show that

$$x_i < x_{i+1} \Longleftrightarrow (n-2)c_i + (n-2)c_{i+1} - 2\sum_{j=1, j\neq i,i+1}^{n} c_j < 1\,.$$

Further, we get

$$0 < x_1 \Longleftrightarrow 2(n-1)c_1 - 2\sum_{j=2}^{n} c_j < 1$$

and

$$x_n < 1 \Longleftrightarrow -2\sum_{j=1}^{n-1} c_j + 2(n-1)c_n < 1\,.$$

To prove that the domain, $\mathbb{C}$, defined by the above set of inequalities is not bounded it suffices to consider all n-tuples $(c_1, ..., c_n)$ with $c_1 = \ldots = c_n$ (homogeneous case) on the positive part of the main diagonal of $\mathbb{R}^n_+$. $\square$

Remark 1. *The case $n = 2$ is treated thoroughly in subsection 3.2.1*

3.2.1 A duopolistic model of heterogeneous firms - Policy implications

To highlight the policy implications of our findings in subsection 3.2, we present an application for a duopoly with linear transportation costs. Let C_i denote the marginal production cost of firm i. There are three varieties of a differentiated product offered to consumers, U and W from firm 1 and V and W from firm 2. Let also the fraction of consumers buying only good U equal the fraction of consumers buying good V, with both set equal to c. Product W is bought by a fraction b of consumers. Transportation costs are linear and equal to td, where t is a positive scalar and d is the distance shipped. The locations of firm 1 and 2 over the interval $[0, 1]$ are x_1 and x_2, respectively (without loss of generality $x_1 < x_2$). Keeping the structure of the game and the rest of the notation as above, the profit functions of firms 1 and 2 when both are privately owned are:

$$\begin{gathered}\tilde{\Pi}_1 = \left(c(m - C_1) - \tfrac{ct}{2}\left[x_1^2 + (1 - x_1)^2\right]\right) \\ + \left(\begin{array}{c}\int_0^{x_1} b[t(x_2 - x_1) + C_2 - C_1]dz \\ + \int_{x_1}^{\left(\frac{x_1+x_2}{2} + \frac{C_2-C_1}{2t}\right)} b[t(x_1 + x_2 - 2z) + C_2 - C_1]dz\end{array}\right)\end{gathered} \tag{19}$$

$$\begin{gathered}\tilde{\Pi}_2 = \left(c(m - C_2) - \tfrac{ct}{2}\left[x_2^2 + (1 - x_2)^2\right]\right) \\ + \left(\begin{array}{c}\int_{\left(\frac{x_1+x_2}{2} + \frac{C_2-C_1}{2t}\right)}^{x_2} b[t(2z - x_1 - x_2) + C_1 - C_2]dz \\ + \int_{x_2}^{1} b[t(x_2 - x_1) + C_1 - C_2]dz\end{array}\right)\end{gathered} \tag{20}$$

with $\frac{C_2-C_1}{2t} \leq \frac{x_2-x_1}{2}$.[10] The location s of the indifferent consumer for good W is determined by equating the two delivered costs in regard to the common good W: $t(x_2-s)+C_2 = t(s-x_1)+C_1 \Rightarrow s = \frac{x_1+x_2}{2} + \frac{C_2-C_1}{2t}$.

Having evaluated the integrals, (19) and (20) become

$$\begin{aligned}\tilde{\Pi}_1 = c(m - C_1) - \tfrac{ct}{2}\left[x_1^2 + (1 - x_1)^2\right] \\ +bx_1[t(x_2 - x_1) + C_2 - C_1] \\ +\tfrac{b}{4t}[t(x_2 - x_1) + C_2 - C_1]^2\end{aligned} \tag{19b}$$

$$\begin{aligned}\tilde{\Pi}_2 = c(m - C_2) - \tfrac{ct}{2}\left[x_2^2 + (1 - x_2)^2\right] \\ +b(1 - x_2)[t(x_2 - x_1) + C_1 - C_2] \\ +\tfrac{b}{4t}[t(x_2 - x_1) + C_1 - C_2]^2\end{aligned} \tag{20b}$$

[10] If $\frac{C_2-C_1}{2t} > \frac{x_2-x_1}{2}$, both firms are reduced to spatial-price discriminating monopolists where the common good W is now provided only by firm 1. We consider this case trivial and focus only on the case where $\frac{C_2-C_1}{2t} \leq \frac{x_2-x_1}{2}$.

Firm 1 chooses x_1 to maximize (19b), and firm 2 chooses x_2 to maximize (20b), leading to the following Nash equilibrium locations

$$(x_1, x_2) = \left(\frac{1}{2} - A + \omega, \frac{1}{2} + A + \omega\right) \tag{21}$$

where $\omega = \frac{b(C_2 - C_1)}{2t(b+2c)}$ and $A = \frac{b}{4(b+c)}$.

The total shipping cost will be equal to

$$\begin{gathered}\tilde{T} = \tfrac{ct}{2}\left[x_1^2 + (1-x_1)^2\right] + \tfrac{ct}{2}\left[x_2^2 + (1-x_2)^2\right] + cC_1 + cC_2 \\ + \left(\int_0^{x_1} b[t(x_1 - z) + C_1]dz + \int_{x_1}^{\left(\frac{x_1+x_2}{2} + \frac{C_2 - C_1}{2t}\right)} b[t(z - x_1) + C_1]dz\right) \\ + \left(\begin{array}{c}\int_{\left(\frac{x_1+x_2}{2} + \frac{C_2-C_1}{2t}\right)}^{x_2} b[t(x_2 - z) + C_2]dz \\ + \int_{x_2}^{1} b[t(z - x_2) + C_2]dz\end{array}\right)\end{gathered} \tag{22}$$

Maximizing (22) with respect to x_1 and x_2 gives the socially optimal locations

$$\left(\frac{1}{2} - A + \omega, \frac{1}{2} + A + \omega\right) \tag{23}$$

We now turn to the case where firm 2 is partly privately owned and partly publicly owned in proportions a_2 and $1 - a_2$, respectively with $a_2 \in [0, 1]$. In this case, the profits of firm 2 will be

$$\begin{gathered}\breve{\Pi}_2 = c(m - C_2) - \tfrac{ct}{2}\left[x_2^2 + (1-x_2)^2\right] \\ + b(1 - x_2)[t(x_2 - x_1) + C_1 - C_2] \\ + \tfrac{b}{4t}[t(x_2 - x_1) + C_1 - C_2]^2 + (1 - a_2)g(x_1, x_2)\end{gathered} \tag{24}$$

where

$$\begin{gathered}g(x_1, x_2) = \left(c(m - C_1) - \tfrac{ct}{2}\left[x_1^2 + (1-x_1)^2\right]\right) \\ + \left(\begin{array}{c}\int_0^{x_1} b[t(x_2 - x_1) + C_2 - C_1]dz \\ + \int_{x_1}^{\left(\frac{x_1+x_2}{2} + \frac{C_2-C_1}{2t}\right)} b[t(x_1 + x_2 - 2z) + C_2 - C_1]dz\end{array}\right) \\ + \int_0^{\left(\frac{x_1+x_2}{2} + \frac{C_2-C_1}{2t}\right)} b[m - t(x_2 - z) - C_2]dz \\ + \int_{\left(\frac{x_1+x_2}{2} + \frac{C_2-C_1}{2t}\right)}^{1} b[m - t(z - x_1) - C_1]dz \\ = \tfrac{(b+c)}{2}\left[2tx_1(1 - x_1) + 2m - t - 2C_1\right]\end{gathered} \tag{25}$$

It is straightforward to show that $\partial g(x_1, x_2)/\partial x_2 = 0$ showing that the equilibrium remains intact irrespective of the degree, a_2, of privatization.

Furthermore, the distance between the optimal locations, x_1 and x_2, is independent of marginal production costs and t and equals $2A$. It follows that anybody who wishes to influence the location x_1 of either firm 1, with $x_1 \in (0, 1/2)$,[11] or the location of firm 2, x_2, with $x_2 \in (1/2, 1)$, can do so by intervening on the marginal cost relative difference, $C_2 - C_1$. For example, given an a priori $X \in (0, 1/2)$, it suffices to choose $C_1 < C_2$ in such a way that $C_2 - C_1 = \frac{2t(b+2c)}{b}(X - \frac{1}{2} + A)$ for firm 1 to locate optimaly on the given X.

4 Conclusion

We have proved that when firms exercise spatial price discrimination, the equilibrium outcome is socially optimal and independent of the underlying assumptions on the number of firms, firm heterogeneity, the nature of transportation costs, the number or the varieties of the provided goods and the degree of privatization. Even though we expect our findings to hold when consumers are non-uniformly distributed, we have intentionally opted for the less technically demanding setting of consumer uniform distribution to showcase the essence of our ideas. The technical requirements in relation to the non-uniformly distributed case are thoroughly presented in Lederer and Hurter (1986). To the best of our knowledge, our analysis is the first attempt to present an 'holistic' view of models of spatial price discrimination. Moreover, our findings verify the robustness of the 'laissez-faire' doctrine and can be easily applied to the case of vertically related markets (see Eleftheriou and Michelacakis, 2016). It is also not hard to deduce the validity of our findings for 'discontinued' markets where specific locations are ruled out. Possible extensions could investigate strategic delegation effects and spatial two dimensional markets.

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[11] $x_1 \in (0, 1/2)$ is implied by the fact that $\frac{C_2 - C_1}{2t} \leq \frac{x_2 - x_1}{2}$ (non-negative profit condition) and $x_2 - x_1 = 2A$.

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