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15 January 2016

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*I am thankful to Pedro Gomis, Cathy Zhang, Ryan Baranowski, Guilleaume Rocheteau and Randy Wright for valuable suggestion and comments. I am especially indebted with Alberto Trejos for his unceasing encouragement and guidance through the whole process of writing this paper. I have also benefited from the comments of audiences at The Chicago Fed Workshop and University of Costa Rica. All the errors are my own.

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Abstract

There have been many episodes in history where low-denomination money holdings have been exchanged with a premium over its face value. The most recent occurred in Panama only twenty five years ago, under a modern banking system. In such episodes, even where there is an entity capable to provide convertibility of money holdings at a fixed rate, and when agents expect this rate to prevail in the long run, arbitrage possibilities in the denomination of money arise as a consequence of a shortage of liquid assets and the presence of low prices in the economy. Despite of its relevance and recurrence, this phenomenon cannot be explained by current models of fiat money. To explain it we need a model where: (i) fiat money comes in different denominations which are used as a medium of exchange, (ii) there is an entity that provides convertibility of denominations at fixed rate, (iii) the natural rate is a feasible equilibrium of the model, and (iv) there are parameterizations where low denomination money holdings are given an extrinsic value. In this paper we build a money search model with all this characteristics and determine theoretically the specific conditions under which such equilibrium naturally arises.

Keywords: Convertibility, Extrinsic value, Money holdings, Poisson technology.
JEL: E50 E52

1 Introduction

Throughout history, the imbalances in the supply of low-denomination coins (bills) have entailed serious problems to trade in different regions, and for that reason they have been exchanged with a premium over its face value, or in other words, at a different rate from their "natural rate." To illustrate this situation, observe that in United States bills of $100 and $10 circulates, and are accepted as medium of exchange. At the same time, buyers and sellers know the Federal Reserve will freely exchange a bill of $100 for 10 bills of $10, and they expect this exchange rate to prevail in any trade. Any departure from such "natural rate" in favor of the low-denomination bills would grant them an extrinsic value. Such monetary phenomenon has been recurrent along time, and its most recent episode occurred only twenty five years ago in the Panama of 1987-1989, under a modern banking system[1]. There, as a result of a major political and economic crises, $300 million were withdrawn from the banking system, which obligated banks to borrow abroad and reduce their liquid assets to compensate for the loss of local resources. Along with this palliative measure, banks were closed and reopened later under several restrictions. In particular, individuals could withdraw up to 50% of current accounts and up to 5% from saving accounts, while certificates of deposits remain frozen. Banks were authorized to issue notes known as Investment Certificates (CEDIS) used by citizens as a method of payment

[1] Another episode that exhibited the same phenomenon was the one occurred in United States at the beginning of XIXth century. There, with the purpose of controlling inflation, the Treasury only permitted issuing bank notes for an amount greater than $5. The argument was due to Adam Smith under the logic that if banks only issue notes of high denominations, individuals would have an incentive to redeem them more frequently, and then, banks would stay away from "over printing" money. However, and despite the goodness of such macroeconomic policy, people still needing small coins (or bank notes) to make daily transactions. In fact, to have an idea of the scale of the problem, at that time a newspaper had a price of a penny, a laborer typically earned $5 per week, and a family paid about six cents for a pound of bacon, while stores faced big problems to give the change to their customers (Champ, 2007). These difficulties caused the emergence of private coinage of low denominations among population, since stores paid a premium when they traded with small-coin customers. In addition, small stores did not use to have large inventories to satisfy the demand of high-coin buyers (i.e. they had capacity constraints), and therefore buyers incurred in losses when traded. Another example is found in Rolnick et al. (1996) who document episodes in which low coins (made up of an inferior metallic or having less metallic content) were traded with premium over its content in specie, in recognition of its flexibility to trade with sellers. Specifically, they cite the situation occurred in the Spain of centuries XVI and XVII, in which copper vellons were more valued than their counterparts of gold and silver coins.

but were trade with a discount of 15 to 25 percent. At the same time, prices and wages declined as much as 20% in many sectors (Moreno-Villalaz, 1999). In this environment characterized by the scarcity of liquid assets and low prices (wages), low-denomination coins and bills were traded at premium over its face value. This situation propitiated the arbitrage in low-denomination money holdings through the exchange of $100 and $50 bills for $5 and $1 bills, when Panamanians traded with foreign visitors. In that sense, even though foreign visitors provided convertibility of portfolio at the "natural rate," and Panamanians expected it to be the exchange rate in the long run, there was an equilibrium in which low-denomination money holdings were bestowed an extrinsic value in virtue of its *enhanced liquidity.* This situation continued even when the banking system returned to normal operations, all restrictions to withdrawals and deposits were removed, and banks could provide convertibility of portfolios.

This paper builds a theoretical model capable to explain under which conditions a *bubble* of this nature can arise even when money holders can freely reconvert their portfolios at the natural rate. Here, it is important to highlight that even though the *premium* over low-denomination money holdings has existed in many episodes under a shortage of liquid assets, it has not been always the case. The understanding of this phenomenon results crucial for monetary policy-makers, especially nowadays when Greek banks prepare for a banking closure like the one happened in Panama twenty five years ago. However, despite of its recurrence and importance, such kind of phenomenon cannot be explained with the current models of indivisible money. Therefore, we consider that the importance of counting with such a model is twofold. On one hand, it has value in that it permits to explain what are the economic forces underlying such puzzling equilibria which are not so rare in history; and on the other hand, permits to explore other associated monetary phenomena, such as: speculative trading strategies, counterfeit coins, and capacity constraints. Moreover, it provides a theoretical foundation for some results elaborated by authors of economic history. For instance, it validates the findings in Reddish and Weber (2011a, 2011b) where they showed that for the case of mid-XIIth century England, the inclusion of small coins where sought by the sovereign to increase the population's welfare through an increase in the frequency of trade.

The model constructed is a money search model of divisible product and indivisible money holdings as in Trejos and Wright (1995) and Shi (1995), that shares four fundamental characteristics. First, it is a model where fiat money serves the purpose of medium of exchange, but where agents are equipped with low and high denomination coins, and therefore where the *denomination indifference* of standard models is removed[2]. Second, there is a natural rate at which high coins can be converted into low coins. Such convertibility is provided exogenously by a Poisson technology which with agents overlaps in the economy. Third, to value money holdings at the natural rate constitute a feasible equilibrium of the model. Fourth, for a specific parameterization of the model, there exists an equilibrium where the convertibility of money holdings departs from the natural rate, and low coins are traded with a premium. Specifically, we show that if the arrival rate of such convertibility opportunities is positive but bounded above, an equilibrium when low-denominations have an extrinsic value exists for any positive interest rate. Moreover, we find there exists an equilibrium in which agents will generically split a two-low-coins portfolio in a trade when they have the opportunity to do so. On the other hand, when the arrival rate of the exogenous technology is sufficiently high, this equilibrium is no longer guaranteed since a high-coin buyer would prefer to wait, convert their portfolio, and then consume the alleged higher quantity that a two-low-coins portfolio would give him.

Finally, as mentioned before, this theoretical construction also allows us to explore some issues related with the presence (or lack of) small denominations. In particular, we can relax one important feature of indivisible/divisible monetary search models: that the quantity traded only depends on the kind of the monetary array in exchange, since it is assumed sellers always are able to produce the quantity requested by buyers. This assumption ignores the fact that sellers regularly have shortage of inventories or capacity constraints -recall the situation in the US of XIXth century, but which is worthy to investigate in our environment since it could make low-denomination and convertible-high-denomination portfolios welfare-improving. Precisely, as an extension of the baseline model we impose capacity restrictions to assess the extent at which the portfolio composition can remediate such friction. The main result shows that under a proper discounting rate, divisibility can eliminate a sort of "no-trade" inefficiency.

[2]It is important to highlight that our interest resides in exploring the extrinsic value of low denomination coins over their counterpart, and hence our model has to be one of *indivisible money*, since precisely a premium in exchange emerges from the fact that the spectrum of prices in the bargaining process of exchange is greater when small coins are present. According to Wallace (2003) "...it is hard to imagine that a small change problem could exist if the stuff out of which large coins were made were divisible in the sense in which goods in standard models are divisible."

Organization of the paper. —The paper is organized as follows. Section 2 introduces the environment and stresses the main distinctive features of the model. Section 3 defines the different classes of equilibria proposed and show their existence. Section 4 presents the extended model with capacity constraints and reformulate the different equilibria. Section 5 discusses how the presence of fixed costs and inflation can affect the sustained equilibria. Finally, section 6 concludes. All proofs are relegated to the appendix.

2 The environment

The economy consists of a continuum of infinitely lived agents, whose population is normalized to unity and grows at an exogenous rate $\gamma > 0$. There are $K \geq 3$ types of goods -which are divisible but not storable- and K types of agents, who are uniformly distributed among types. So, for a given type k the measure of agents that share it is $\eta = 1/K$.

Agents produce and consume goods in discrete points of continuous time, using $r > 0$ as their preference discount rate. In her production facet, an agent of type k produces a good of type k, incurring in a disutility cost of $c(q) = q$. The unique input in the production function is the costly participation of the agent, which precludes her to look for another peer to make a transaction once exchange has been agreed. Similarly, in her consumption facet an agent of type k has specialized preferences toward the good of type $k + 1$ (mod K); which means that the demand of any agent can be satisfied only by a fraction η of the population. Agents derive satisfaction from consumption according to the function $u(q) : \mathbb{R}^+ \to \mathbb{R}^+$, which is strictly increasing, strictly concave, satisfies $u(0) = 0$ and fulfills Inada conditions: $\lim_{q\to\infty} u'(q) = 0$ and $\lim_{q\to 0} u'(q) = \infty$. Additionally, Q^* and $\bar{Q}$ denote, respectively, the points where $u(Q^*) = Q^*$ and $u'(\bar{Q}) = 1$.

In the search process, agents meet randomly and pair wise according to a Poisson matching technology with arrival rate $\lambda \in \mathbb{R}^+$. For convenience, the rate $\lambda\eta$ at which a buyer meets an agent of her consumption-type (i.e. of the type located at the right of her own) is normalized to unit without loss of generality.

All these assumptions make the exchange process a necessary event for consumption to take place. Nonetheless, barter, reputation and commodity monies are not viable mechanisms to generate trade, and hence it is necessary to introduce an object that plays the role of medium of exchange in order to achieve a distinct equilibrium from autarchy[3]. Precisely,

[3] Since agents meet randomly, it is not possible for them to be engaged in a long-lasting relationship in

to serve this role it is assumed the existence of an intrinsically worthless and storable object called *fiat money*, which is available in coins of high and low denominations.

Initially, the fractions of the population who are endowed with one high coin, one low coin, and two low coins are m_h, m_1, and m_2, respectively. Such sub-populations are disjoint in pairs with the property that $0 < \sum_{h,1,2} m_i < 1$. Their union is called buyers, while the remaining $m_0 \equiv 1 - m_h - m_1 - m_2$ are called sellers. Buyers attempt to exchange money for goods and sellers attempt to produce for money. Consequently, an agent carrying money cannot be a seller.

No agent can produce these coins at any cost, nor can they derive utility from direct consumption of them. Permanently, a fraction M_2 of the newborn agents are endowed with two low coins, whereas a proportion M_h is endowed with one high coin. The rest $1 - M_h - M_2$ participate in the economy as sellers[4]. To keep the distribution on money holdings tractable, it is assumed that there exist limitations in storage capacity, which imposes a restriction to keep a maximum stock of one high coin or two low coins at any time (i.e. low-coins are more portable).

Institutionally, there is not a centralized market where a walrasian auctioneer set prices and enforces long term agreements; therefore prices are determined through bilateral bargaining. Moreover, all actions in the economy are considered anonymous in the sense that the record of transactions for each agent is private information. All these conditions imply that every exchange shall be made on a *quid pro quo* basis, meaning that the amount of product delivered by the seller should equalize the value of the money given in exchange by the buyer at current prices (i.e. financial markets are not available). Because goods cannot be stored, their production, exchange and consumption must be simultaneous.

In addition, agents are assumed to have access to a Poisson technology[5] with exogenous

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which reputation can act as a signal of the commitment of one agent to deliver a promised good in the future, given that she receives a good of her consumption-type in the present.

[4]Notice no newborn agents incorporate into the economy as low-coin buyers, and hence, its steady state value is determined by the strategies of high and two-low coins buyers with respect to the conversion and splitting of their money holdings. Therefore, a higher measure of low-coin buyers in steady-state would reflect the extrinsic value of counting with convertible and divisible portfolios.

[5]Think for example in an automaton with infinite storage capacity in each type of coin, which with agents overlaps in the market. Its natural analogy in actual times is automatic telling machines which provide convertibility up to some denomination, whose availability rate is governed by its distribution on a determined region. However, unlike the present environment, people have to pay an annual fee for using them and so, their decision of splitting money holdings shall take this variable in the correspondent

arrival rate θ, that allows them to change -without any cost and at any time- a high coin for two low coins and *vice versa*. This is the only way buyers can convert their portfolios since they are not allowed to swap notes among themselves.

Finally, following Trejos and Wright (1995), when a buyer meets a seller of her consumption-type, the amount of output traded obeys a bilateral bargaining of "take it or leave it" repeated offers, that resembles a standard bargaining game as in Rubinstein (1982). In general, the outcome of the game would depend on some exogenous variables such as the bargaining power, the impatience of the agents, and the intensity of the preferences over the good traded; however, for the sake of isolating the analysis of the problem at hand, we assume buyers retain the entire bargaining power and thus, that they extract the whole surplus in any trade with a seller.

3 Equilibrium analysis

The environment proposed in the previous section produces a large set of equilibria, but the analysis will be restricted only to the stationary class[6]. Let Q_h, Q_1 and Q_2 be the quantities of consumption traded in exchange for a high coin, one low coin, and two low coins, respectively. It is possible to observe that such quantities depend solely on the type of coins involved in the exchange and not on the valuation assigned by the seller to the good traded, due to the fact that negotiations correspond to a "take it or leave it" offer, in which the buyer retains all the bargaining power. Likewise, we denote V_h, V_1 and V_2 buyers' value functions associated to the states of holding such combinations of coins, and V_0 the life-time value of being a seller.

To complete the description, define ν_h, ν_1 and ν_2 as the probabilities that a buyer with such money holdings concretes a deal when an opportunity comes to her; ρ as the probability that a buyer changes a high coin for two low coins when she has access to the Poisson technology; and δ as the probability that a buyer who holds two low coins spends only one of them when she has the opportunity to concrete an exchange with a seller (i.e. the probability that she divides her two-low-coin portfolio).

benefit-cost analysis.

[6] The path for the value of money is given exogenously at a constant level $Q_t = Q$ for each type of coin, which allows to solve explicitly for the value functions of the stationary equilibria.

According to this, Bellman's equation for a high-coin buyer yields

$$rV_h = m_0 \max_{\nu_h \in [0,1]} \{\nu_h [u(Q_h) + V_0 - V_h]\} + \theta \max_{\rho \in [0,1]} \{\rho [V_2 - V_h]\} \tag{1}$$

This equation describes the life-time value of being a high-coin buyer, rV_h, as the sum of two terms. The first one corresponds to the rate at which a buyer meets an agent of her consumption-type (that is normalized to the unit), times the probability that such agent be a seller in the moment, m_0, times the probability that the buyer concretes a deal, ν_h, times the profit of such transaction -which equalize the utility obtained from consuming the correspondent quantity, plus the net profit of switching the status from buyer to seller: $u(Q_h)+V_0-V_h$. Likewise, the second term corresponds to the profit realized when changing a high coin for two low coins, $V_2 - V_h$, times the probability that the agent optimally uses the mentioned technology when the opportunity comes to her. Both terms appear in an additively separable form, since they are governed by two independent Poisson stochastic processes, and thus the probability that an agent faces both opportunities at the same time has zero measure.

Analogously, the value function for two-low-coins buyer is given by the following Bellman equation

$$rV_2 = m_0 \max_{\nu_2 \in [0,1]} \left\{ \max_{\delta \in [0,1]} \{(1-\delta)[u(Q_2) + V_0 - V_2] + \delta [u(Q_1) + V_1 - V_2]\} \right\} + \theta \max_{\rho \in [0,1]} \{(1-\rho)[V_h - V_2]\} \tag{2}$$

Unlike the former equation, the latter presents the difference that when a buyer meets a seller of her consumption-type, she must decide whether or not to spend her whole portfolio. If she does it, she consumes as much as her money holdings allow, and then switches status from buyer to seller, deriving a net profit of $u(Q_2)+V_0-V_2$. If on the contrary she splits her portfolio, then the correspondent quantity is consumed, and the remaining coin is kept by the agent, deriving a utility of $u(Q_1)+V_1-V_2$. Thus, in a trade with a seller, a two-low-coin buyer earns a profit that is computed as an average of the previous expressions weighted by the corresponding factors: $(1-\delta)$ and δ. By symmetry with the previous case, a buyer of this kind can also access the same exogenous technology to recompose her money holdings which, when utilized, yields to her an expected profit of $\theta[V_h - V_2]$[7].

$$rV_1 = m_0 \max_{\nu_1 \in [0,1]} \{\nu_1 [u(Q_1) + V_0 - V_1]\} \tag{3}$$

[7]So far it has been used the word type to differentiate an agent by her production-type, and now -to

Bellman's equation for one-low-coin buyer satisfies (3), which shall be interpreted in a similar fashion of the other two. Finally, seller's Bellman equation satisfies

$$rV_0 = m_h\nu_h[V_h - V_0 - c(Q_h)] + m_2\nu_2[V_2 - V_0 - c(Q_2)] + m_1\nu_1[V_1 - V_0 - c(Q_1)] \qquad (4)$$

Here, seller's life-time value corresponds to the sum of the expected profits obtained by trading with each kind of buyer. Such profits are equal to the product of two factors: the probability of trading with each kind of buyer, $m_i\nu_i$, and the net profit of changing status, $V_i - V_0 - c(Q_i)$. Nonetheless, using the fact that buyers retain the entire bargaining power and seller's cost function is linear, the equilibrium quantities system can be simplified as follows[8]:

$$rQ_h = m_0 \max_{\nu_h \in [0,1]} \{\nu_h[u(Q_h) - Q_h]\} + \theta \max_{\rho \in [0,1]} \{\rho[Q_2 - Q_h]\} \qquad (5)$$

$$rQ_2 = m_0 \max_{\nu_2 \in [0,1]} \left\{ \nu_2 \{ \max_{\delta \in [0,1]} \{(1-\delta)[u(Q_2) - Q_2] + \delta[u(Q_1) + Q_1 - Q_2]\} \right\} \qquad (6)$$
$$+\theta \max_{\rho \in [0,1]} \{(1-\rho)[Q_h - Q_2]\}$$

$$rQ_1 = m_0 \max_{\nu_1 \in [0,1]} \{\nu_1[u(Q_1) - Q_1]\} \qquad (7)$$

It is not hard to observe that buyers of each kind will trade with certainty if the argument of the following indicator functions is satisfied

$$\nu_h = \mathcal{I}\{u(Q_h) \geq Q_h\} \qquad (8)$$

$$\nu_2 = \mathcal{I}\{\delta[u(Q_1) + Q_1] + (1-\delta)u(Q_2) \geq Q_2\} \qquad (9)$$

$$\nu_1 = \mathcal{I}\{u(Q_1) \geq Q_1\} \qquad (10)$$

From (8) and (10), a buyer who holds either a high or low coin will always trade if the utility of consuming the quantity negotiated with the seller is at least equal to the discounted value of preserving her monetary array. In the same sense, from (9) a two-low-coin buyer will agree to trade if the linear combination of the discounted utilities obtained for exchanging one and two coins respectively, is at least equal to the discounted utility of preserving her monetary endowment intact.

avoid reader's confusion- it will be adopted the convention of using the word kind to differentiate a buyer by her money holdings.

[8]It in turn implies that $Q_i = V_i$ for all $i \in \mathbb{I} \equiv \{1, 2, h\}$, and consequently that $Q_0 = 0$.

Likewise, buyer's optimal strategy of using the Poisson technology to convert their portfolios must satisfy,

$$\rho = \begin{cases} 0 & \text{if} \quad Q_h > Q_2 \\ \in [0,1] & \text{if} \quad Q_h = Q_2 \\ 1 & \text{if} \quad Q_h < Q_2 \end{cases} \tag{11}$$

Such pattern indicates that high-coin buyers will always convert their unitary portfolio into one composed by two low coins, if a low denomination has an extrinsic value in an exchange with a seller.

On the other hand, the decision of splitting a monetary array composed by two low coins depends on whether the present utility of consuming the quantity given in exchange for one low coin $u(Q_1)$, plus the continuation value of holding a low coin Q_1, exceeds the present utility of consuming the equilibrium quantity for two low coins $u(Q_2)$. In summary,

$$\delta = \begin{cases} 0 & \text{if} \quad u(Q_1) + Q_1 < u(Q_2) \\ \in [0,1] & \text{if} \quad u(Q_1) + Q_1 = u(Q_2) \\ 1 & \text{if} \quad u(Q_1) + Q_1 > u(Q_2) \end{cases} \tag{12}$$

Law of motion. —The law of motion for the correspondent sub-population of buyers are presented below. Furthermore, they are equated to zero to solve for steady state values:

$$\dot{m_h} = m_2\theta(1-\rho) - m_h\theta\rho + \gamma(M_h - m_h) = 0 \tag{13}$$

$$\dot{m_2} = m_h\theta\rho - m_2\theta(1-\rho) - (1 - m_h - m_2 - m_1)m_2\nu_2\delta + \gamma(M_2 - m_2) = 0 \tag{14}$$

$$\dot{m_1} = 2(1 - m_h - m_2 - m_1)m_2\nu_2\delta - \gamma m_1 = 0 \tag{15}$$

To understand how these expressions are derived, first notice that an exchange realized, either between a seller and a one-high-coin buyer or between a seller and a two-low-coin buyer who decides to spend all her cash, does not impact equilibrium sub-populations since the coin just changes hands leaving the distribution unaltered. However, high-coin and two-low-coin buyers can eventually modify the distribution by accessing the exogenous technology to recompose their money holdings, or in the case of the latter by splitting their portfolio in a trade with a seller.

Particularly, first term in (13) indicates the fraction of high-coin buyers increases when two-low-coin buyers use the exogenous technology to "aggregate" their portfolios. On the contrary, second term implies it diminishes when there are high-coin buyers who use the Poisson technology to divide their money holdings. In turn, the third term captures the effect that the constant flow of new agents -who join as high-coin buyers- produces to the system.

In the same line of reasoning, the first two terms in (14) are the counterpart analogous of (13) applied to the two-low-coin case. However, here a third term shall be added to account for the reduction made up when two-low-coin buyers decide to divide their portfolio in a particular meeting with a seller. By undertaking such action, the fraction of two-low-coin buyers diminishes because the original buyer remains with a low coin for a future purchase, while the seller switches to a buyer with a low coin. As before, the fourth term stands for the adjustment made by the inflow of newborn agents endowed with such money holdings. With respect to the fraction of one-low-coin buyers, it is precisely duplicated when two-low coin buyers split their money holdings in a trade (by virtue of the previous analysis), and decreases at population's relative rate of growth. Naturally, the proportion of sellers is computed as the complement of buyers' union.

Once the whole environment is specified, the definition of the equilibrium constitutes the next step, which, precisely, is introduced below.

Definition 1 *A stationary equilibrium is a collection of a vector of quantities* (Q_h, Q_2, Q_1)*, a vector of buyers' strategies* $(\nu_h, \nu_2, \nu_1, \rho, \delta)$*, and a vector of sub-populations* (m_h, m_2, m_1)*, that satisfy equations (5) - (15).*

There is always a degenerate nonmonetary equilibrium which satisfies all the conditions stated before. Clearly, in such an equilibrium agents do not grant any value to money (i.e. $Q_h = Q_1 = Q_2 = 0$) and hence autarky is maintained. Nevertheless, this equilibrium is trivial and its existence does not serve the aim of deriving any type of conditions that could justify the extrinsic value of low-denomination money holdings; therefore, other classes of equilibria that emerge under the proposed environment must be explored.

Definition 2 *Let* $\mathbf{Q} \equiv \{Q_h, Q_2, Q_1\}$ *be the set of equilibrium quantities, and let* $\mathcal{P}(\mathbf{Q})$ *the power set of* $\mathbf{Q}$*. Each subset* $\mathcal{Q} \subset \mathcal{P}(\mathbf{Q})$ *is considered an equilibrium configuration, such that* $Q_i > 0$ *for all* $Q_i \in \mathcal{Q}$*, and* $Q_j = 0$ *for all* $Q_j \in \mathbf{Q}/\mathcal{Q}$*. Hence, it is possible to define three classes of equilibria (i.e. three selections of* $\mathcal{P}(\mathbf{Q})$*), equilibria of class one:* $\mathcal{Q}_1 \equiv \{\{Q_1, Q_2, Q_3\}\}$*, equilibria of class two:* $\mathcal{Q}_2 \equiv \{\{Q_h, Q_2\}, \{Q_2, Q_1\}, \{Q_h, Q_1\}\}$*, and* *equilibria of class three:* $\mathcal{Q}_3 \equiv \{\{Q_h\}, \{Q_2\}, \{Q_1\}\}$. *Moreover, a degenerate equilibrium corresponds to* $\emptyset$.

Nonetheless, there are some restrictions over the set of equilibria that can emerge. For instance, there can be no equilibrium in which $Q_1 > 0$ and $Q_2 = 0$, because agents always have the option of disposing or storing the additional coin with no cost. Moreover, if agents encounter the automaton with positive probability, then equilibrium quantities exchanged for two low coins and one high coin should be either zero or both positive. The feasible equilibria under this environment are summarized in the following proposition.

Proposition 3 *For all* $r > 0$, *there exist equilibria* $\mathcal{Q}_1$ *and* $\mathcal{Q}_2^1 \equiv \{Q_h, Q_2\}$. *Furthermore, if* $\theta = 0$ *(i.e. if convertibility of money holdings is absent) there will be three additional equilibria:* $\mathcal{Q}_2^2 \equiv \{Q_2, Q_1\}$, $\mathcal{Q}_3^1 \equiv \{Q_h\}$, *and* $\mathcal{Q}_3^2 \equiv \{Q_2\}$.

From the previous proposition we conclude that if there exists an exogenous technology that allows the re-composition of money holdings, it is possible to sustain equilibria where both denominations have value.[9] In other words, the absence of a mechanism that provides convertibility of the currency is a necessary condition to support equilibria in which only one configuration of the money holdings has positive value.

Observe that in equilibria $\mathcal{Q}_2^1$ and $\mathcal{Q}_3^2$ divisibility is worthless, and therefore the buyer will never split her money holdings in a particular trade. Such situation is plausible, for instance under "high inflation," where the proportion of newborn buyers who injects money in the economy is very high, and so, agents would like to spend the highest amount of fiat money when they have the opportunity to do so -a *hot potato* effect. Similarly, when sellers have to incur in a fixed cost to produce a positive quantity, and hence, there exists a minimum threshold under which it is not profitable to produce for a particular buyer, a $\mathcal{Q}_2^1$ equilibrium could emerge. In addition, it is interesting to examine whether in such equilibrium Q_2 has an extrinsic value over Q_h. As it is demonstrated in the following proposition, in equilibrium both portfolios should be traded at par.

Proposition 4 *In any equilibrium* $\mathcal{Q}_2^1$, *it must be the case that* $Q_2 = Q_h$.

[9] An identical result would be obtained if the exogenous technology is replaced by a monetary authority that controls the amount of fiat money in the economy and provides the convertibility of money holdings.

A situation of this nature, where low coins are traded only when they come in full capacity portfolios, resembles an economy with two kind of coins (e.g. white and black) where the denomination role is irrelevant. According to this result, if both money holdings are traded in equilibrium, they must do it at the same value, otherwise the "strong" coin would displace the "weak" one, as it has happened in many episodes of the history.

At first glance, this result may be surprising because, if for instance, agents believe that one unit of high-denomination money values less than two units of low denomination, then the latter should command more value in exchange, despite of whether one unit of low denomination money has value or not. A potential explanation for this *puzzle* would come from models where money has some small albeit positive intrinsic value, as in Wallace and Zhu (2004). Their argument dictates that if goods are divisible and marginal utility at zero consumption is bounded but sufficiently large, then a buyer can always extract some small but positive output from a seller due to the fact that money has some small but positive intrinsic value. Anticipating such situation, a seller agrees to produce for more than the face value of the coin used in the exchane , since she reasons the same will apply in a future meeting with a buyer. The authors show that under this environment a unique production level arise in equilibrium. Now, suppose in the present setting an equilibrium where $Q_1 = 0$ and $Q_2 > Q_h > 0$ exists. If here the value of convertibility $\theta(Q_2 - Q_h)$ were identified with the intrinsic value of low-denomination money holdings in the model of Wallace and Zhu (2004), then a unique production level would be expected[10]. However, this argument suffers from two weaknesses. The first is that convertibility does not correspond *vis-a-vis* to the intrinsic value property of a monetary unit, since *per se* it does not render any utility from direct consumption. Moreover, the service of convertibility varies with respect to θ, and so, the application of the Wallace-Zhu result only holds if θ is sufficiently low, but proposition 4 is stated for all $\theta > 0$.

From the mosaic of equilibria analyzed up to this point, we have found justification for the endogenous value of fiat money, however, we still have to show that low-denomination money holdings have an extrinsic value in exchange. It is, that some sellers will be willing to deliver a major quantity of output when they deal with a buyer holding two low coins, than the quantity they would render when dealing with a buyer holding a high coin (i.e. $Q_2 > Q_h$). In general, it could be possible to have $Q_h < Q_1$, but in this case a class-one

[10]The same analysis will hold in an equilibrium where $Q_1 = 0$ and $Q_h > Q_2 > 0$.

equilibrium in which $Q_2 > Q_1 > Q_h > 0$ would arise. In such situation, the low coin strictly dominates the high one, and the option of splitting a two-low-coin portfolio does not play any role in the determination of the quantities traded. So, the analysis will be focused on finding conditions that support a configuration where $Q_2 > Q_h > Q_1 > 0$ (i.e. the only class-one equilibrium in which low-coin portfolios have an extrinsic value). In fact, its existence is stated in the proposition below.

Proposition 5 *If $\theta < \bar{\theta}$ then for all $r > 0$ there exists an equilibrium where $Q_2 > Q_h > Q_1$ and buyers generically split their portfolios. Moreover, for $r = \bar{r}_{(0,1)}$ buyers are indifferent with respect to the decision of splitting their portfolio.*

In this context two-low-coin portfolios are traded with a premium over their high-single-unit counterparts if the arrival rate of the exogenous technology is not too high, because otherwise a high-coin buyer would prefer to wait to convert the portfolio to a two-low-coins portfolio and consume a higher quantity, even when they are sufficiently patient.

4 Capacity constraints in the production

One important feature of the environment presented in the last section is that a seller is always able to produce the equilibrium quantity requested by any kind of buyer in a particular trade. However, it might be the case that sellers experience capacity constraints that make it impossible to produce a quantity greater than a certain level. Under such a scenario, a buyer who meets a seller of this class[11], shall decide whether to instantaneously consume a quantity lesser than the maximum feasible, or waiting for the event to meet either an unconstrained or a "less constrained" seller of her consumption-type.

A priori, an environment of this nature could magnify the potential benefits of having a portfolio with low denominations, because if agents can split their money holdings, then

[11] To further clarify the notation utilized, the word *class* is reserved exclusively to differentiate an agent by her seller's nature: constrained or unconstrained. Thus, a buyer can be totally determined by her type of good, by the kind of her money holdings, and by her class when acting as seller. In particular, it is assumed that at $t = 0$ nature (or God) selects the class of the agent, which never changes but only manifests when the agent is a seller. In other words, the class of an agent is private information that is revealed until a meet is realized, but before the negotiation is carried over. A natural generalization would make seller's type private, even during the bargaining process, as in Trejos (1999).

they could partially remediate the curse of facing a constrained seller by smoothing their consumption path.

Notice that even if lotteries over the quantities and denominations traded are allowed in the present environment, as in Berentsen et al (2002), it does not solve the difficulties entailed by the capacity constraints. In their paper, they show that when buyer's bargaining power tends to one, the unit of indivisible money is exchanged with a probability less than one, and the socially efficient quantity is always produced. However, in the present environment albeit buyers retain the whole bargaining power, there might be cases in which they do not get the socially efficient quantity if they meet a sufficiently constrained seller. Therefore, low denominations have a role to play in an economy with capacity constraints, like one formed by small and unconnected producers[12].

Modifying the baseline model to serve this purpose, we assume now the existence of two classes of sellers: constrained and unconstrained, as cited before. Moreover, to assure symmetry on the types of goods among the classes of sellers, it is assumed the latter are uniformly distributed among the former. In other words, we assume that the probability of being a constrained seller (denoted by α) is the same for any type of agent.

Recall that Q_h, Q_2 and Q_1 are the equilibrium quantities given in exchange for the respective money holdings, which for our purposes are designed as the maximum quantities attainable in an exchange with a seller. Let the random variable $\tilde{Q}$, with distribution $F(\cdot)$, be the maximum quantity that a constrained seller is able to produce; and let $[Q_1^{\max}, Q_2^{\max}]$ be the support of $\tilde{Q}$, where $Q_1^{\max}$ and $Q_2^{\max}$ are the maximum quantities produced in exchange for those money holdings, among all equilibria defined in proposition 1.

4.1 When only one denomination is available

In both equilibria of class three, the environment resembles an economy in which there is just one denomination of fiat money, and therefore, there is just one kind of buyers. Since we assume they can throw fiat money without any cost at any time, in such equilibria the buyers holding the worthless monetary array become sellers instantaneously.

Thus, when only a denomination is available, the value function for buyers of class j

[12]For instance, the economy of United States in the mid-XIXth century.

holding an i monetary array[13] (within the configuration where i has value) is given by

$$rQ_i^j = m_0 \left(\alpha \int_{Q_1^{\max}}^{Q_2^{\max}} \max\{u(\min\{\tilde{Q}, Q_i\}) - Q_i^j + Q_0^j, 0\} \cdot \mathcal{I}_{\{Q_i^c - Q_0^c \geq \tilde{Q}\}}\, dF(\tilde{Q}) \right.$$
$$\left. +(1-\alpha)[u(Q_i) - Q_i^j + Q_0^j] \cdot \mathcal{I}_{\{Q_i^u - Q_0^u \geq \tilde{Q}\}} \right) \tag{16}$$

Unlike Bellman's equation in the case without capacity constraints, buyer's life-time value now depends on whether she meets a constrained or an unconstrained seller. Specifically, it is composed as the probability of meeting a seller of her consumption-type, m_0, times the expected gain of a match -which now is linear in the probability of finding a constrained seller α. However, notice that such gain depends on the willingness of the seller to produce for money (which is summarized in the argument of the indicator function above). Hence, when a buyer meets an unconstrained seller, the gain of the trade is $[u(Q_i) - Q_i^j + Q_0^j]$: the utility of consuming the maximum quantity attainable in that economy (or equilibrium configuration), Q_i, minus the life-time value of delivering the corresponding money array Q_i^j, plus the flow return of being a seller, Q_0^j. On the other hand, when a buyer meets a constrained seller, the gain depends on the realization of the random variable $\tilde{Q}$, and is equal to $\int_{Q_1^{\max}}^{Q_2^{\max}} \max\{u(\min\{\tilde{Q}, Q_i\}) - Q_i^j + Q_0^j, 0\}$. In a single coincidence-meeting either the buyer spends her money holding, which yields $u(\min\{\tilde{Q}, Q_i\}) - Q_i^j + Q_0^j$, or remains a buyer, which yields no surplus. The argument in the utility function reflects the fact that although the maximum quantity feasible to produce in the economy is Q_2^{max}, no seller would deliver a quantity greater than Q_i for an i monetary array. So, the quantity exchanged when a buyer trades with a constrained seller of her consumption-type is $\min\{\tilde{Q}, Q_i\}$. From this fact follows that the probability of reaching the maximum quantity attainable in the economy when meeting a constrained seller is $1 - F(Q_i)$.

Because utility function $u(\cdot)$ is strictly increasing, buyers' value function satisfies a reservation property. That is, there exists a minimum threshold under which it is not profitable for a buyer to accept a deal. Let $\mathcal{X}_i^j = [\bar{Q}_i^j, Q_i]$ denote the set of acceptable quantities for a buyer of kind i and class j. Here, $\bar{Q}_i^j$ (the reservation value) satisfies

$$\bar{Q}_i^j = u^{-1}(Q_i^j - Q_0^j) \tag{17}$$

[13] Here the subscript differentiates the agent by her monetary array (i.e. $i \in \{2, h\}$) and the superscript does it by her class of seller (i.e., $j \in \{c, u\}$ where c denotes constrained and u unconstrained).

Clearly, if $\tilde{Q} > \bar{Q}_i^j$, buyer (i, j) is willing to trade with the seller. Hence, seller's value function becomes

$$rQ_0^c = m_i \alpha \int_{\mathcal{X}_i^c} \max\{-\tilde{Q} + Q_i^j - Q_0^j, 0\}\, dF(\tilde{Q}) \tag{18}$$

Her flow return is now equal to the probability of having a meeting-coincidence with a constrained buyer, αm_i, times the expected gain in that trade. It turns out the value function of an unconstrained seller is always zero, since in that case $F(\cdot)$ places all the probability in the maximum quantity attainable in the economy. On the other hand, the value function for a constrained seller, Q_0^c is not always zero, because she could trade for less than the maximum attainable, and then materialize a profit later in her buyer's facet. An implication of this result is that the reservation quantity for constrained agents is less than for unconstrained ones, and consequently, $\mathcal{X}_i^c \subset \mathcal{X}_i^u \equiv Q_i$, and therefore that unconstrained buyers only trade with unconstrained sellers.

In virtue of proposition 4, the analysis of equilibrium $\mathcal{Q}_1^2$ under capacity constraints is totally determined by the set of equations established above. Therefore, up to this point there exist all the required elements to define a refinement of equilibria, when there are constrained sellers in the economy and only one denomination is available.

Definition 6 *A refinement of equilibria $\mathcal{Q}_2^2$ and $\mathcal{Q}_1$, under the capacity constraints assumption, is a collection of a vector of quantities $\{Q_0^c, Q_1^j, Q_2^j, Q_h^j\}_{j\in\{c,u\}}$, and a vector of reservation values $\{\bar{Q}_1^j, \bar{Q}_2^j, \bar{Q}_h^j\}_{j\in\{c,u\}}$ that satisfies equations (16) - (18), in the configurations where $i \in \mathbb{I}$ has a positive value.*

4.2 When low denominations have an extrinsic value

In the configurations where portfolios are divisible (i.e. $\mathcal{Q}_2^2$ and $\mathcal{Q}_1$) agents have an additional instrument to deal with the frictions introduced by sellers' capacity constraints. In fact, given that all sellers have the capacity to produce Q_1, all two-low-coin buyers can expect a return of at least $u(Q_1) + Q_1$ in any single coincidence-meeting.

In general, value functions for sellers and buyers are similar in structure to those in the last subsection, but have slight differences that arise from the presence of low denominations in money holdings. Since for practical purposes equilibrium $\mathcal{Q}_2^2$ is embedded in equilibrium $\mathcal{Q}_1$, the former is considered as the general framework[14]. Bellman's equations

[14]Under the awareness that value function for high-coin buyers is zero in $\mathcal{Q}_2^2$.

then corresponds to,

$$rQ_h^j = m_0 \left(\alpha \int_{Q_1^{\max}}^{Q_2^{\max}} \max\{u(\min\{\tilde{Q}, Q_h\}) - Q_h^j + Q_0^j, 0\} \cdot \mathcal{I}_{\{Q_h^c - Q_0^c \geq \tilde{Q}\}} \, dF(\tilde{Q}) \right.$$
$$\left. +(1-\alpha)[u(Q_h) - Q_h^j + Q_0^j] \cdot \mathcal{I}_{\{Q_h^u - Q_0^u \geq \tilde{Q}\}} \right) + \theta(Q_2^j - Q_h^j) \qquad (19)$$

Observe that the quantity a high-coin buyer can extract from a seller in a trade has Q_1 as its minimum lower bound. So, if in equilibrium $\mathcal{Q}_1$, $Q_h \to^{(-)} Q_1$, the burden of capacity constraints is negligible for those buyers, but is maximized when $Q_h \to^{(+)} Q_2$. Nonetheless, and despite this fact, high-coin buyers count with an "indirect" mechanism to avoid the curse of facing a constrained seller by means of the exogenous technology. Hence, if the frequency to meet the Poisson automaton is high enough and agents are patient, it would be profitable to delay a trade, given the distribution of the random variable that governs the constraint.

Life-time value for one-low-coin buyers is the same as in (7),

$$rQ_1^j = m_0(u(Q_1) - Q_1) = rQ_1 \qquad (20)$$

Two-low-coin buyers have additionally the option of splitting their money holding when facing a constrained seller. Recall that for any single coincidence-meeting, the lower bound of the flow return -if they trade- is given by $u(Q_1) + Q_1$. Likewise, if they attempt to trade the whole portfolio, they will face seller's constraint, and hence, a reservation value can be defined as before. Bellman's equation then yields,

$$rQ_2^j = m_0 \left\{ (1-\delta) \left[\alpha \left(\int_{Q_1^{max}}^{Q_2^{max}} \max\{u(\tilde{Q}) - Q_2^j + Q_0^j, 0\} \cdot \mathcal{I}_{\{Q_2^j - Q_0^j \geq \tilde{Q}\}} \, dF(\tilde{Q}) \right) \right. \right. \qquad (21)$$
$$\left. \left. +(1-\alpha)(u(Q_2) - Q_2^j + Q_0^j) \right] + \delta[u(Q_1) + Q_1 - Q_2^j] \right\}$$

To complete the framework, the bellman equation for sellers is defined as in the former case, with the only particularity that they can now trade with buyers holding any array of coins. Nonetheless, the summation is just taken over $\{2, h\}$ because for any kind of seller the surplus of producing for a low coin is always zero.

$$rQ_0^c = \sum_{h,2} m_i \alpha \int_{\mathcal{X}_c} \max\{-\tilde{Q} + Q_i^j - Q_0^j, 0\} \, dF(\tilde{Q}) 0 \qquad (22)$$

Since the Bellman equation for high coin buyers is independent of the analysis conducted above, the definition of the equilibrium $\mathcal{Q}_2^2$ does not require any additional assumption. So, the following definition formalizes the refinement for the remaining equilibria in that model.

Definition 7 *A refinement of equilibria $\mathcal{Q}_2^2$ and $\mathcal{Q}_1$, under the capacity constraints assumption, is a collection of a vector of quantities $\{Q_0^c, Q_1^j, Q_2^j, Q_h^j\}_{j\in\{c,u\}}$, and a vector of reservation values $\{\bar{Q}_1^j, \bar{Q}_2^j, \bar{Q}_h^j\}_{j\in\{c,u\}}$ that satisfies equations (19) - (22) , in the configurations where $i \in \mathbb{I}$ has a positive value.*

When there exist capacity constraints in the economy, the criterion defined early to explain how agents divide their money holdings shall be modified slightly to add -in the right hand side of the inequalities in (12), seller's continuation value, which in the case of constrained agents is not always zero. In such formulation, various scenarios should be addressed to analyze the strategies of splitting in relation with the capacity constraints.

Case 1: $u(Q_1) + Q_1 > u(Q_2) + Q_0^c$. Notice that, for both classes of agents the flow return of trading the two-low-coin portfolio is lesser than the life-time value of splitting their money holdings, even if the seller is able to produce the maximum quantity feasible in the economy. So, two-low-coin buyers will always split their portfolio, and therefore, both classes of sellers will always produce for money when meeting them.

Case 2: $u(Q_1) + Q_1 = u(Q_2) + Q_0^c$. Here, constrained buyers are indifferent between splitting their monetary array and consuming Q_2, provided that seller produce such quantity. Observe that the unconstrained sellers are the unique class able to produce the maximum attainable, and they will agree to do so, because after trading and becoming buyers they can get -with certainty- $u(Q_1) + Q_1$ in any single coincidence-meeting. Moreover, when a constrained buyer meets a constrained seller, she will always split her money holding because Q_2 is not attainable. On the other hand, unconstrained buyers will always split their two-low-coin portfolio, rather than being indifferent, because for them $Q_0^u = 0$, and so Q_2 is not high enough.

Case 3: $u(Q_1) + Q_1 < u(Q_2)$. In this case the analysis of divisibility is not straightforward. Let's first analyze constrained buyers. Since the utility function $u(\cdot)$ is strictly increasing, there exists a $\hat{Q}_2^c$ such that $u(Q_1) + Q_1 - Q_0^c = u(\hat{Q}_2^j)$. Comparing this threshold with the reservation quantity introduced in equation (17), it is possible to obtain a pattern for splitting portfolios of money in a particular trade. First of all, look that $\hat{Q}_2^c \geq \bar{Q}_2^c$ because splitting acts as an insurance or a lower bound for consumption. Inequality is strict if a two-low-coin buyer expects to get a higher output than such "insurance-consumption." However, if the buyer is impatient enough, then there exists an interest rate $\tilde{r}$ at which $\hat{Q}_2^c = \bar{Q}_2^c$, or equivalently where $u(Q_1) + Q_1 = Q_2^c$. In such a case, a buyer and seller will trade with certainty.

Summarizing, if buyers' time preference is sufficiently high, then two-low-coin buyers will split their money holdings more frequently to avoid the constraint friction. In that scenario, both classes of agents would produce for money and a sort of "no-trade friction" is ameliorated. As a corollary, the quantity produced under divisible equilibria is higher and so is the welfare attained. The result is analogous to the one in Berentsen and Rocheteau (2002) under divisible money. Specifically, they show that divisibility -allowing agents to spend a small amount on low-valued varieties- eliminates the so-called "no- trade" and "too-much-trade" inefficiencies, because buyers always have the option to split their money holding and procrastinate consumption. However, in the case of indivisible money the result is even stronger because total quantities and not varieties are involved.

5 Discussion: Fixed costs and inflation

So far, it has been analyzed just one specific characteristic in sellers' production function: their constraint to produce beyond a certain threshold defined by a random variable. However, technologies can display many other irregularities such as non-convexities and discontinuities that are reflected in their correspondent cost functions. One example of the former is the presence of fixed costs, which force firms to pay a fixed amount of money to produce any positive quantity.

In the realm of the model developed here, fixed costs could reduce the potential gains in welfare produced by the presence of divisible money, and in the limit might justify the presence of an equilibrium like $\mathcal{Q}_2^1$. Specifically, if the environment is modified to allow sellers retain part of the bargaining power, and it is introduced a random variable that governs the amount that each seller shall pay to produce, then, under some conditions of time preference, there could be equilibria in which low coins would have no value, or in which its value is substantially lesser than the exhibited in an economy without such friction.

As it is known, an economy with fixed costs privileges the scale of production, because the higher the quantity, the lesser the average cost. Hence, the surplus of sellers should be sufficiently high to make it profitable the production of small quantities. However, even if seller's bargaining power is high, those might have an incentive to reject the trade and waiting either for a one-high-coin or a two-low-coin buyers.

In a framework of that nature, two-low-coin buyers now shall assess the probability of facing a high-fixed cost seller (who is the analogous counterpart of a low capacity seller), to determine whether splitting the money holding is worthy. Depending on how patient buyers are, and what is the distribution of the fixed costs, it might be the case that divisibility does not bestow a premium, in virtue proposition 4, that the only equilibrium that would emerge is one in which a high-coin and two-low-coins money holdings are traded at par.

Nevertheless, if an inflation mechanism is introduced á la Li (1995), in which (for any kind of buyers) a proportion of money holdings is confiscated and reallocated to sellers, then buyers would have an incentive to spend more quickly because the value of their money holdings is decreasing in time. Under a fixed cost environment -and given a low time preference- such situation would set a tradeoff in the decision of splitting a two-low-coin portfolio, since even the "hot potato" effect of inflation calls for a quick spending (which could be attained partially through splitting), the presence of fixed cost acts in the other direction.

With respect to an economy with capacity constraints, the tradeoff is produced because even inflation urges for a quick spend, there are potential losses in which buyers could incur due to the capacity constraints of some sellers. Therefore, depending on the preference rate, inflation could reinforce o deter trading strategies under fixed costs and capacity constraints.

6 Concluding remarks

The model presented here is a search model of indivisible money -available in two denominations: low and high- and divisible product. Buyers cannot swap notes among themselves, but are capable to exchange two low coins for one high coin -and *viceversa*, by means of an exogenous Poisson technology. This exchange rate is exogenously determined and fixed, and therefore it is consider the natural rate to prevail in any trade. Under this environment we show analytically the existence of three classes of equilibria in which different denomination configurations exhibit positive value. Moreover, we prove that if only two-low-coin and high-coin portfolios have positive value in equilibrium, they should be traded at par, without any deviation from its natural rate. This can be seen as an extension of the result stated in the refinement of Wallace and Zhou (2004), where they showed that if money has a small intrinsic value, only one quantity of equilibrium could exist. Furthemore -and most important- the model is able to prove the existence of an equilibrium where low-denomination coins are traded with a premium if the arrival rate of the Poisson technology is positive but bounded above. In that sense, even though the exogenous technology converts portfolios at the natural rate, sellers are willing to exchange a higher quantity when trade with low-coin buyers in virtue of the enhance liquidity their assets offer. Therefore, buyers will generically split a two-low coins portfolio when they have the opportunity to do so. Using these results we showed in the second part of the paper that if there exist capacity constraints in the supply side, the existence of a convertibility technology plus the presence of low denomination money holdings could remediate (partially or totally) the "no-trade" inefficiencies produced by such restriction. A result that cannot be trivially obtained by introducing lotteries in the exchange process. Specifically, for proper discount rates, two-low-coin buyers will always split their money holdings, and hence constrained and unconstrained sellers will produce for money when meeting them. As a consequence, the quantity produced is greater and so is the welfare attained. In the final comments, we discussed the approach to incorporate fixed costs and inflation in the economy, which in the absence of a profound analysis would have an ambiguous effect on the decision of dividing money holdings.

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Appendix: Proofs

A useful result for proving the propositions presented in the main text, is stated in the following lemma.

Lemma 8 *If $u(\cdot)$ is a utility function that satisfies all the properties stated in the text, then, $a + bu(\cdot)$, is concave and has a unique positive fixed point for all $a, b \in \mathbb{R}$.*

Proof. Concavity follows immediately. For uniqueness of the fixed point, let $h(\cdot) : \mathbb{R} \to \mathbb{R}$ defined by $h(q) = a + bu(q) - q$. From the properties of $u(q)$, it is possible to observe that $\lim_{q\to\infty} h'(q) = \lim_{q\to\infty}(bu'(q) - 1) = -1$, and so there exists $q_0 \in \mathbb{R}^+$, such that $h'(q) < -\frac{1}{2}$ for all $q > q_0$. From the latter it follows that $h(q) \leq h(q_0) - \frac{1}{2}(q - q_0)$ for all $q > q_0$; which in turn implies that $\lim_{q\to\infty} h(q) = -\infty$. That is, $h(q) = 0$ for some $q > q_0$. Then it is clear that there exists some $\bar{q} > 0$, such that $a + bu(\bar{q}) = \bar{q}$. ■

Proof of Proposition 3. Solving the system of equations (5) - (7), equilibrium quantities can be expressed as a function of buyers' strategies δ, ρ, seller's steady-state sub-population m_0, the arrival rate of the exogenous technology, θ, and the interest rate r,

$$Q_h = \frac{m_0(m_0 + r + \theta(1-\rho))}{(m_0 + r)(m_0 + r + \theta)} u(Q_h) + \frac{m_0\theta\rho(1-\delta)}{(m_0 + r)(m_0 + r + \theta)} u(Q_2) + \frac{m_0\delta\theta\rho(2m_0 + r)}{(m_0 + r)^2(m_0 + r + \theta)} u(Q_1) \tag{23}$$

$$Q_2 = \frac{m_0\theta(1-\rho)}{(m_0 + r)(m_0 + r + \theta)} u(Q_h) + \frac{m_0(1-\delta)(1+\theta\rho)}{(m_0 + r)(m_0 + r + \theta)} u(Q_2) + \frac{m_0\delta(2m_0 + r)(m_0 + r + \theta\rho)}{(m_0 + r)^2(m_0 + r + \theta)} u(Q_1) \tag{24}$$

$$Q_1 = \frac{m_0}{(m_0 + r)} u(Q_1) \tag{25}$$

In order to prove equilibria $\{\mathcal{Q}_2^2, \mathcal{Q}_3^1, \mathcal{Q}_3^2\}$, substitute $\theta = 0$, in (23)-(24), to obtain the following simplified expressions:

$$Q_h = \frac{m_0}{m_0 + r} u(Q_h) \tag{26}$$

$$Q_2 = \frac{m_0(1-\delta)}{(m_0 + r)^2} u(Q_2) + \frac{\delta(2m_0 + r)}{m_0 + r} Q_1 \tag{27}$$

Clearly, $Q_1 = 0$ and $Q_h = 0$ are respective solutions of (25) and (26). Hence, by plugging $Q_1 = 0$ into (27) and by invoking the result stated in lemma 1, there exists $Q_2 > 0$ solution of (27). Thus, the existence of equilibrium $\mathcal{Q}_3^2$ is proved. By the same argument, existence of equilibrium $\mathcal{Q}_3^1$ can also be assured.

Likewise, to prove the existence of equilibrium $\mathcal{Q}_2^2$, take the positive solution of (25), and substitute it in (27), which by the result stated in lemma 1 has a unique positive solution. Finally, the proof of equilibria $\mathcal{Q}_2^1$ and $\mathcal{Q}_1$ is embedded in the proof of propositions **??** and **??**. ■

Proof of Proposition 4.

To prove the claim substitute $\delta = 0$ in (23)-(25) since in equilibrium $\mathcal{Q}_2^1$ low coins only have value when they are traded in portfolios of full storage capacity. Considering this, and the fact that $Q_1 = 0$ is a solution of (25), the expressions obtained in (23) and (24) can be rearranged as below:

$$Q_h - \frac{m_0(m_0 + r + \theta)u(Q_h) + m_0\theta\rho[u(Q_2) - u(Q_h)]}{(m_0 + r)(m_0 + r + \theta)} = 0 \tag{28}$$

$$Q_2 - \frac{m_0\{(m_0 + r)u(Q_2) + \theta u(Q_h) + \theta\rho[u(Q_2) - u(Q_h)]\}}{(m_0 + r)(m_0 + r + \theta)} = 0 \tag{29}$$

Now, suppose by contradiction that $Q_h < Q_2$. In such case, ρ must be equal to one. Then, (28)-(29) can be re-arranged as

$$Q_h - \Psi(Q_h) = 0 \tag{30}$$

$$Q_2 - \Gamma(Q_2) = 0 \tag{31}$$

where $\Psi(Q_h) = \frac{m_0}{m_0+r+\theta}u(Q_h) + \frac{\theta}{m_0+r+\theta}\Gamma(Q_2)$, and $\Gamma(Q_2) = \frac{m_0}{m_0+r}u(Q_2)$ as before. Clearly, Q_h^f and Q_2^f, the unique positive fixed points of $\Psi(\cdot)$ and $\Gamma(\cdot)$ are the solutions of (30)-(31). Nonetheless, observe that evaluating $\Psi(\cdot)$ at the fixed point of $\Gamma(\cdot)$, it follows that $\Psi(Q_2^f) = Q_2^f$ Thus, Q_2^f is also a fixed point of $\Psi(\cdot)$, but due to the uniqueness of the fixed point, it must be satisfied that $Q_2^f = Q_h^f$. It contradicts the assumption that $Q_2 > Q_h$.

For the other direction, suppose that $Q_2 < Q_h$, which requires that ρ must be equal to zero. In analogy with the previous case, (28)-(29) can be re-arranged as

$$Q_h - \Gamma(Q_h) = 0 \tag{32}$$

$$Q_2 - \Psi(Q_2) = 0 \quad (33)$$

where Q_h and Q_2 interchange roles as the arguments of $\Gamma(\cdot)$ and $\Psi(\cdot)$. After substituting the fixed point of $\Gamma(\cdot)$ in $\Psi(\cdot)$ we get a solution in which $Q_2^f = Q_h^f$, contradicting the assumption that $Q_2 < Q_h$. Therefore, in all equilibria of class $\mathcal{Q}_2^1$, equilibrium quantities must satisfy $Q_h = Q_2$. ■

Proof of Proposition 5. We will guess and verify the existence of an equilibrium where $Q_2 > Q_h > Q_1 > 0$. First, we assume that trade is desirable and so, $u(Q_i) > Q_i$ for all i, which in turn implies that $Q_i < Q^*$ for all i. Second, following the optimal strategy in (17) $\rho = 1$ which permits to simplify the system (5)-(7) to set conditions on r and θ, in order to validate the inequalities $Q_2 > Q_h$ and $Q_h > Q_1$.

Case 1: $\delta = 1$. —Inequality $Q_h > Q_1$ implies that

$$m_0[u(Q_1) + Q_1 - Q_2] > m_0[u(Q_h) - Q_h] + \theta[Q_2 - Q_h]$$

Since $\delta = 1$ and hence $u(Q_1) + Q_1 > u(Q_2)$, it is sufficient to show that

$$m_0[u(Q_2) - Q_2] > m_0[u(Q_h) - Q_h] + \theta[Q_2 - Q_h]$$

Equivalently,

$$\frac{u(Q_2) - u(Q_h)}{Q_2 - Q_h} > 1 + \frac{\theta}{m_0}$$

Taking the limit $Q_2 \to Q_h$ the latter expression becomes

$$u'(Q_h) > 1 + \frac{\theta}{m_0} \quad (34)$$

Since $u(Q)$ satisfies the Inada condition, the existence of Q_h is guaranteed. Moreover, it has to satisfy $Q_h < \bar{Q}$.

Now, since $Q_2 > Q_h$ and $\theta > 0$, inequality $Q_h > Q_1$ is satisfied for all $r > 0$.

Then, using the simplified version of (7) and (6) to obtain $u(Q_1) + Q_1$ and Q_2 respectively, the condition to validate $\delta = 1$ implies that

$$\frac{m_0}{m_0 + r} u(Q_1) + Q_1 < u(Q_1) + Q_1$$

which is satisfied for all $r > 0$.

Case $\delta \in (0,1)$. —As before, the equilibrium sought has to satisfy (34) but since $\delta \in (0,1)$ the interest rate has to satisfy the following equality

$$r = m_0 \frac{Q_2 - 2Q_1}{Q_1 - Q_2}$$

To have a positive interest rate $Q_2 < 2Q_1$.

Case $\delta = 0$. —If the $\delta = 0$ from the reduced Bellman equations it is possible to see that the only admissible solution for $Q_1 = 0$ since $u(Q)$ has a unique fixed point greater than zero. Therefore, it is not possible to have an equilibrium where $Q_2 > Q_h > Q_1 > 0$.

Finally, recalling that $Q_2 \to Q_h$ from the right, it is immediate to inspect that for all $\delta \in (0,1]$, $u(Q_i) > Q_i$ for all i. In summary, if $\theta < \bar{\theta}$ then, (i) for all $r > 0$ there exists an equilibrium $Q_2 > Q_h > Q_1$ where buyers always split their portfolios and (ii) for $r = \bar{r}_{(0,1)}$ there exists an equilibrium $Q_2 > Q_h > Q_1$ where buyers are indifferent with respect to the decision of splitting their portfolio.

■