Parametric continuity from preferences when the topology is weak and actions are discrete

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Abstract

Empirical settings often involve discrete actions and rich parameter spaces where the notion of open set is constrained. This restricts the class of continuous functions from parameters to actions. Yet suitably continuous policies and value functions are necessary for many standard results in economic theory. We derive these tools from preferences when the parameter space is normal (disjoint closed sets can be separated). Whereas we use preferences to generate an endogenous pseudometric, existing results require metrizable parameter spaces. Still, weakly ordered parameters do not form a normal space. We provide a solution and close with an algorithm for eliciting preferences.

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1 Introduction

Consider a policy maker in a setting where actions are discrete. The optimal policy varies over a parameter space with a rich structure. Circumstances are such that ability to accurately observe or perturb parameters is constrained. That is to say, the topology on parameters is weak or coarse. In this setting, there is a shortage of continuous functions from parameters to actions.

Recall that Berge's theorem of the maximum provides conditions for the existence of an optimal policy correspondence with nonempty finite values that is upper hemicontinuous (u.h.c.). Berge's theorem requires (i) a jointly continuous function on on the product of actions and parameters that is a utility on actions at each parameter; (ii) a continuous feasibility correspondence with nonempty finite values. The composition of the resulting policy correspondence with the function of (i) immediately yields the maximum utility that is feasible given the parameter: the value function.

An optimal, nonempty-valued u.h.c. policy can also be derived from preferences. For this purpose, *pairwise stability* (strict preference for one action over another is robust to perturbations in the parameter) and the requirement that preferences form a weak ordering at each parameter are simple sufficient conditions. This result only requires that the parameter space is Hausdorff (distinct points can be separated by disjoint open sets).

We also show that, when the parameter space is *metrizable* (there is a continuous metric that generates the topology on the parameter space) and actions are discrete, pairwise stability and the ordering condition are in fact equivalent to (i). Nonmetrizable spaces arise when there is a shortage of basic open sets, so that the topology is weak. This is evident from the fact that the strongest topology (the discrete one) is always metrizable.

Our main contribution (theorem 2) is to provide necessary and sufficient conditions for (i) to hold when the parameter space is normal (disjoint closed sets can be separated by disjoint open sets). For this result, we require *perfect pairwise stability* (the open set on which strict preference for one action over another holds is a countable union of closed sets).

On a nonmetrizable parameter space, every metric is discontinuous. Perfect pairwise stability also allows us to construct an continuous pseudometric (in lemma 2) when the parameter space is normal. This pseudometric assigns distance zero to distinct parameters if and only if preferences coincide, yielding many benefits of metrizability (calculus, envelope theorems, etc.).

When the parameter space is normal, we can ask questions about preferences on closed neighbourhoods of a parameter as opposed to doing so pointwise. Perfect pairwise stability ensures that we can find an increasing sequence of closed neighbourhoods on which strict preference holds uniformly. Preferences are approximated by a finite number of questions and identified in the limit. In section 7, we provide a simple algorithm to this effect.

We adopt a broad definition of what it means for a topology to be weak. The first class of examples arises when the parameter space is large relative to the basic perturbations that are feasible, as in infinite-dimensional settings. Consider the product space $\mathbb{R}^I$ of functions f from the unit interval I to the set $\mathbb{R}$ of real numbers. Far from being metrizable, the space $\mathbb{R}^I$ fails to be normal, but a common way to deal with this issue is to consider to compactify each factor and obtain a product of compact Hausdorff spaces, which is normal. Products of compact Hausdorff spaces also feature in the literature on type spaces in Bayesian games (Mertens and Zamir [20]) and settings where sets of probability measures are endowed with the weak* topology.

The second class of examples are finite-dimensional. In this setting, weak topologies arise when the parameter space is weakly ordered, so that the ordering permits ties (equivalence classes with two or more elements). Such parameter spaces are not even Hausdorff, precisely because the open order intervals do not separate points that are tied in the order.

The final class of examples we consider arises when the parameter space is connected (the only partition into open sets is the trivial one). First, every feasibility correspondence that satisfies condition (ii) of Berge's theorem is constant. This means that standard budget correspondences that vary continuously with prices and wealth are precluded when actions (or commodities) are discrete. Second, suppose some pair of actions are mutually exclusive (e.g. accept vs reject) and each is strictly preferred to the other at some parameter. Then there exists a parameter such that the policy maker is indifferent between these actions. The trouble is that any u.h.c. policy correspondence chooses both actions at this parameter.

In section 5, we provide a simple procedure for handling spaces that are weakly ordered and connected. We do so via a simple policy example where parameter-dependent claims to "right of asylum" are accepted or rejected by the policy maker. A simple and subtle strengthening of the topology accommodates the above concerns and yields a normal parameter space. Once preferences are known, a pseudometric restores much of the original topological structure. Section 6 considers products of ordered spaces, where parameters are trajectories relating to house prices and a recession variable.

1.1 Relation to the literature

Metrizability of the parameter space is a key assumption for results that prove the existence of a representation that is jointly continuous on actions and parameters (Hildenbrand [13], Mas-Colell [18], and Levin [17]). Recently, Caterino, Ceppitelli, and Maccarino [2, Theorem 4.1] extend to submetrizable spaces. This requires that the original topology can be weakened (some open sets can be excluded) in such a way that every remaining open set is a union of the balls of that metric. For obvious reasons, this result does not help when the topology is already weak. We hold that in empirical settings, circumstances will often constrain the supply of basic open sets.

For the case where the action space is discrete, proposition 2 proves that pairwise stability is essentially equivalent to the closed graph axiom of the literature on jointly continuous representations. Proposition 3 provides an useful upper bound for the parameter space when preferences satisfy pairwise stability (but not perfect pairwise stability). Together, these facts prove that the submetrizable parameter spaces of [2, Theorem 4.1] are normal and *perfect* (every closed set is a countable intersection of open sets).

In proposition 4, we show that whenever the product of the action space and the parameter space fails to be perfectly normal, there exist preferences that fail to have a jointly continuous representation even though they satisfy the closed graph axiom. These results confirm that any extension beyond the perfectly normal case requires a "perfectly closed graph" axiom (analogous to perfect pairwise stability). Our method of pseudometrization provides a potential template for future work in this direction.

Whilst perfect pairwise stability appears to be a novel requirement for preferences, pairwise stability is due to Gilboa and Schmeidler [10, 9]. There the authors impose further conditions and derive a representation that is linear in the parameter (which represents a memory or a belief). Since linearity does not imply continuity in infinite dimensional spaces, these results do not yield continuity in the parameter in nonmetrizable settings.

Since we capture nonlinear and continuous dependence on the parameter, we provide a stepping stone to (or partial foundation for) models that appeal to the envelope theorems of Milgrom and Segal [23] and Sah and Zhao [28] or the nonsmooth calculus of Heinonen [12]. Indeed, in the same spirit as Milgrom and Segal, theorems 1 and 2 are purely about parametric continuity, requiring no topology on actions. The foundation is partial because we do not deliver cardinality (uniqueness upto a common positive affine transformation) of the utility representation and pseudometric we derive. Uniqueness is useful because derivatives and integrals are then unambiguously defined.

We view this lack of uniqueness as a feature of the present model: preferences over actions provide the first step towards a more refined setup. The optimal utility representation and pseudometric might be calibrated through simulations or learnt through trial and error in the case-based spirit of Gilboa and Schmeidler [10].[†] Such extensions are left to future work.

2 Parameters, pairwise stability and choice

Let A denote a nonempty set of *actions* and let Θ denote a nonempty set of *parameters*. Recall that a space X becomes a topological space provided it is endowed with a certain collection τ of subsets. Every set $G \in \tau$ is then *open* in X as is an arbitrary union or a finite intersection of members of τ. As usual, reference to τ is typically suppressed. For example, when A is *discrete*, we understand that it has the strongest or finest possible topology (every singleton set $\{a\}$ is open, so that in fact every subset is open).

The weakest or coarsest possible topology is the trivial topology, where the only open sets are Θ the empty set $\varnothing$. A basic topological requirement

[†]Alternatively, the policy maker may exploit the convex nature of the set of utility representations of the preferences we study. As a distinct step, a von Neumann–Morgenstern representation [32] on the set of such functions would yield the desired uniqueness.

that we take for granted (unless stated otherwise) is that every singleton $\{\theta\}$ is *closed* (the complement of an open set in Θ). This is the $\mathcal{T}_1$ separation axiom of topology. In the present section, we require the stronger, Hausdorff, assumption: if $\theta \neq \theta'$, then there exist disjoint open neighbourhoods N and N' of θ and θ' respectively. Section 5 provides a method for handling ordered parameter spaces where these assumptions fail to hold. An ordered space that is Hausdorff is the standard Euclidean topology generated by the open intervals of the strict ordering $<$ on the real numbers $\mathbb{R}$.

2.1 Ordering of actions given the parameter

The data on preferences comes in the form of statements such as "at θ, action b is strictly preferred to action a". Such statements are summarised by a family of binary relations $\prec_\theta$ on A, one for each $\theta \in \Theta$. $\prec_\theta$ is a subset of $A^2 = A \times A$ and is referred to as *preferences at* θ or *given* θ. The collection $\{\prec_\Theta\} \stackrel{\text{def}}{=} \{\prec_\theta : \theta \in \Theta\}$ is the object we refer to as *preferences*.

For the case where Θ is a singleton and strict preference is primitive, the following axiom is standard. It is the requirement that $\prec_\theta$ is an *ordering* of A, a condition that is necessary for a utility function to exist at θ.[†]

Axiom $\mathcal{O}$. *Both* asymmetry *and* negative transitivity *at each* θ :

$\mathcal{O}_1$ *For every* $(a, b, \theta) \in A^2 \times \Theta$, *if* $a \prec_\theta b$, *then not* $b \prec_\theta a$;

$\mathcal{O}_2$ *For every* $(a, b, c, \theta) \in A^3 \times \Theta$, *if* $a \prec_\theta b$, *then* $a \prec_\theta c$ *or* $c \prec_\theta b$.

For a and b that $\prec_\theta$ finds incomparable (neither $a \prec_\theta b$ nor $b \prec_\theta a$) we write $a \sim_\theta b$. $\mathcal{O}_2$ ensures that $\{\sim_\Theta\}$ is a collection of *indifference* (transitive

[†]Fishburn [7, 6] uses the term *asymmetric weak ordering*. The term *strict weak ordering* is also common. Here the term *weak* is compatible with the topological notion for it means that distinct pairs may be indistinguishable or tied according to the order.

incomparability) relations on A.[†] Weak preference $\precsim_\theta$ is then the union of $\prec_\theta$ and $\sim_\theta$, so that by construction it is complete and $\mathcal{O}$ ensures it is transitive at each θ.[‡] The description "weak preference" is usually intended to provide an analogy with the term "weak inequality" of the Euclidean ordering $\leqslant$ on $\mathbb{R}$. In the topological sense, $\leqslant$ is not weak because, in addition to being complete and transitive, it is *antisymmetric* ($r \leqslant s$ and $s \leqslant r$ implies $r = s$).

2.2 Pairwise stability of strict preference

The following *closed graph* axiom provides the traditional route to parametric continuity from preferences. This simple formulation is due to Levin [17].

Axiom $\mathcal{CG}$. *The set $\{(\theta, a, b) : a \precsim_\theta b\}$ is closed in $\Theta \times A^2$.*

Strict preference is *pairwise stable at* θ provided that, for every $a, b \in A$ such that $a \prec_\theta b$, there exists an open neighbourhood N of θ in Θ such that $a \prec_\eta b$ for every η in N. Since the arbitrary union of open sets is open, the following axiom captures (global) *pairwise stability* of strict preference.

Axiom $\mathcal{PS}$. *For every $a, b \in A$, the set $\{\theta : a \prec_\theta b\}$ is open in Θ.*

Note that, with the trivial topology on Θ, $\mathcal{PS}$ implies that strict preference is constant on Θ. Yet, regardless of the topology on A or Θ, $\mathcal{PS}$ implies that the strict preference correspondence $\theta \mapsto \prec_\theta$ is *lower hemicontinuous* (l.h.c.), so that $\{\theta : \prec_\theta \cap B \neq \varnothing\}$ is open for every open $B \subseteq A^2$.

Proposition 1. [§] *If preferences satisfy $\mathcal{PS}$, then the strict preference correspondence is lower hemicontinuous. The converse is true when A is discrete.*

[†]Recall $\sim_\theta$ is *transitive* provided that $a \sim_\theta b \sim_\theta c$ implies $a \sim_\theta c$ for every $(a, b, c) \in A^3$.
[‡]Recall $\precsim_\theta$ is *complete* provided that $a \precsim_\theta b$ or $b \precsim_\theta a$ holds for every $(a, b) \in A^2$.
[§]See page 28 for proof.

Even in the presence of $\mathcal{O}$, $\mathcal{PS}$ does not imply that the weak preference correspondence $\theta \mapsto \precsim_\theta$ is *upper hemicontinuous*: $\{\theta : \precsim_\theta \cap F \neq \emptyset\}$ is closed for every closed $F \subseteq A^2$. Indeed, let F be any closed, infinite subset of A^2, then although $\mathcal{O}$ and $\mathcal{PS}$ together ensure the set $\{\theta : a \precsim_\theta b\}$ is closed, the union over the pairs $a \times b \in F$, need not be closed. On the other hand:

Proposition 2. † *Let A be discrete, let Θ be Hausdorff and let preferences satisfy $\mathcal{O}_1$. Then $\mathcal{PS}$ holds if and only if $\mathcal{CG}$ holds.*

Experiments in the field and the laboratory frequently deal with discrete action sets. As such, proposition 2 provides a simple way to check if the hypothesis that weak preferences define a closed correspondence is reasonable.

2.3 Pairwise stability and discrete choice

When A is discrete, $\mathcal{PS}$ alone is enough to yield upper hemicontinuity of a choice or policy correspondence C on Θ that selects the set of undominated actions that are feasible at each θ. As well as being a requirement for many fixed point theorems, upper hemicontinuity of choice allows us to derive a continuous value function in theorem 3. In what follows, 2^X denotes the set of closed and nonempty subsets of a space X, as in [22].

Lemma 1. ‡ *Let A be discrete and let Θ be Hausdorff. For any continuous and compact-valued feasibility correspondence $\Phi : \Theta \to 2^A$, if preferences satisfy $\mathcal{PS}$, then the policy $C(\theta) \stackrel{\text{def}}{=} \{a \in \Phi(\theta) :$ there is no $b \in \Phi(\theta)$ with $a \prec_\theta b\}$ is compact-valued and u.h.c. on Θ. If $\mathcal{O}$ holds, then C is nonempty-valued.*

When A is discrete and Θ is connected, the requirement that the feasibility correspondence is continuous (both u.h.c. and l.h.c.) is severe, for it

†See page 28 for proof.
‡See page 28 for proof.

implies that Φ is constant.† In Sah and Zhao [28], for discrete A and Θ equal to the unit interval I, envelope theorems are derived. Similarly, in Milgrom and Segal [23], $\Theta = I$ in the canonical case, though no structure on A is assumed. In both these papers, the feasibility correspondence is absent. A minimal extension of the unit interval that accommodates a continuous and variable feasibility correspondence is the following.

2.4 The split interval

The space we now describe also serves to clarify many topological definitions of the sequel. In section 5 it allows us to model a simple tie-breaking rule.

Definition. *Consider the Euclidean order* $<$ *on the unit interval* I*. Split each* $r \in I$ *into a pair of elements* $r \times 0$ *and* $r \times 1$*. On the resulting set* $I \times \{0, 1\}$*,* $<$ *is an (asymmetric weak) ordering such that* $r \times 0$ *and* $r \times 1$ *are tied for each* r*. Let* $<^*$ *be the ordering that contains* $<$ *and satisfies* $r \times 0 <^* r \times 1$ *for each* r*. Then* $<^*$ *is the usual lexicographic ordering and the open order intervals* $(r \times l, r' \times l')$ *such that* $r \times l <^* r' \times l'$ *generate a topological space* $\mathbb{I}$ *known as the* split interval.

Like the unit interval, $\mathbb{I}$ is linearly ordered, so that no two distinct pairs are incomparable under $<^*$ (so that $\leqslant^*$ is antisymmetric). This property makes $\mathbb{I}$ a Hausdorff space. The "clopen" intervals of fig. 1 make this space well-suited to modelling a feasibility correspondence that is both continuous and variable when the action space is discrete. Such sets also demonstrate that the topology τ^* of $\mathbb{I}$ is a not a weakening of the usual Euclidean topology on $\mathbb{R}^2$. Since every neighbourhood of the point $s \times 0$ in fig. 1 contains some

†Recall that if Θ is connected, the only subsets that are both open and closed are $\varnothing$ and Θ itself; thus $\{\theta : \Phi(\theta) \subseteq B\}$ is either empty or equal to Θ for every $B \in 2^A$.

element $s' \times 1$ such that $s' < s$ confirms that τ^* is not a strengthening of the Euclidean topology on $\mathbb{R}^2$ (see section 5). Thus, Θ is far from discrete.

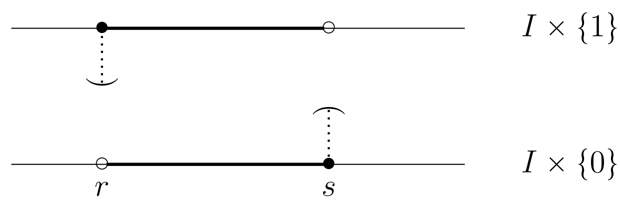


Figure 1: The closed interval $[r \times 1, s \times 0]$ is equal to the open interval $(r \times 0, s \times 1)$ because both $(r \times 0, r \times 1)$ and $(s \times 0, s \times 1)$ are empty intervals.

$\mathbb{I}$ is the leading example of a separable and compact space that is perfectly normal, but nonmetrizable. *Separability* follows because every nonempty open interval contains an element with rational first dimension. If $\mathbb{I}$ were (sub)metrizable, then separability implies *second countability* (there are countably many basic open sets). The latter (and hence metrizability) fails because of the uncountable collection of empty intervals $(r \times 0, r \times 1)$ such that $r \in I$.

Interestingly, compactness follows because $\mathbb{I}$ is *homeomorphic* to a closed subset of the compact product space $\{0,1\}^I$. In particular, Johnson [14, Theorem 2.3] shows that there is a continuous bijection with continuous inverse from $\mathbb{I}$ to set of weakly increasing functions $f : I \to \{0,1\}$. The fact that $\mathbb{I}$ is compact Hausdorff ensures that it is *normal* (for every pair F, F' of closed and disjoint subsets, there exists a disjoint pair N, N' of open neighbourhoods of F and F' respectively). Separability, compactness and normality can be combined for an elementary proof of the fact that $\mathbb{I}$ is *perfect* (every closed subset is a countable intersection of open sets).

3 Parametric continuity of utility

In the present section, no topology (discrete or otherwise) is imposed on A. Our first result holds for perfectly normal spaces. We then extend this to normal spaces. $U(\cdot, \theta) : A \to \mathbb{R}$ is a *utility* function for $\prec_\theta$ (or a *representation* at θ) if it satisfies the following property: for every $a, b \in A$, $a \prec_\theta b$ if and only if $U(a, \theta) < U(b, \theta)$. U is a *parametrically continuous* (utility) representation if it satisfies conditions 1 and 2 of the following theorem.

Theorem 1. *Let A be countable and let Θ be perfectly normal. $\mathcal{O}$ and $\mathcal{PS}$ hold for $\{\prec_\Theta\}$ if and only if there exists a function $U : A \times \Theta \to \mathbb{R}$ such that*

1. *for every $\theta \in \Theta$, $U(\cdot, \theta)$ is a utility function for $\prec_\theta$;*

2. *for every $a \in A$, $U(a, \cdot)$ is continuous on Θ.*

Proof of theorem 1. We proceed by induction on A. Since the initial case (step 1) is useful for the discussion that follows we present it next. In the appendix, the inductive step appeals to Michael's selection theorem.

Step 1. Let $A = \{a, b\}$. By $\mathcal{O}_1$ and $\mathcal{PS}$, $F = \{\theta : a \sim_\theta b\}$ is closed in Θ. Since Θ is perfect, there exists $\{G_n : n \in \mathbb{N}\}$ of open sets satisfying $\bigcap_1^\infty G_n = F$. For each n, note that F and $\Theta - G_n$ are disjoint and the latter is also closed. Since Θ is normal, the Urysohn lemma guarantees the existence of a continuous, real-valued function on Θ such that $f_n(\theta) = 0$ on F, $f_n(\theta) = 1$ on $\Theta - G_n$, and $0 \leqslant f_n(\theta) \leqslant 1$ otherwise.

Let $f = \sum_1^\infty 2^{-n} f_n$ and note that $f : \Theta \to I$ is the continuous and uniform limit of the partial sums $\sum_1^m 2^{-n} f_n$. Moreover, since every $\theta \in \Theta - F$ belongs to some $\Theta - G_n$, $f(\theta) = 0$ if and only if $a \sim_\theta b$. Let $U(a, \cdot)$ be the zero function on Θ. We obtain a utility function for each $\prec_\theta$ by taking

$$U(b, \theta) \stackrel{\text{def}}{=} \begin{cases} f(\theta) & \text{if } a \prec_\theta b, \\ -f(\theta) & \text{otherwise.} \end{cases}$$

For continuity of $U(b, \cdot)$ note that: by $\mathcal{PS}$, for every θ such that $b \prec_\theta a$, there is an open neighbourhood N_θ such that $b \prec_\eta a$ for every $\eta \in N_\theta$; moreover $U(b, \cdot) = -f(\cdot)$ on N_θ; and finally, f is continuous on N_θ.

The proof of theorem 1 continues on page 29. □

Step 1 in the proof of theorem 1 yields the "only if" part of the following well-established, alternative route to perfect normality. (For the converse, note that if $f : X \to \mathbb{R}$ is a continuous function, then $G_n = \{x : |f(x)| < 1/n\}$ is an open neighbourhood of $F = f^{-1}(0)$ for each $n \in \mathbb{N}$ and $\bigcap_1^\infty G_n = F$.)

Definition 1. *X is perfectly normal if and only if every closed subset F of X is a* zero set*. That is, for some continuous $f : X \to \mathbb{R}$, $f^{-1}(0) = F$.*

Definition 1 tells us that if a space is not perfectly normal, then it contains a closed subset that is not a zero set. This leads to the following justification of our claim that perfect normality is a minimal requirement for parametric continuity without further restrictions on preferences.

Proposition 3. † *If Θ is not perfectly normal, then there are preferences that have no parametrically continuous representation and satisfy both $\mathcal{O}$ and $\mathcal{PS}$.*

A surprising example of a space that is not perfectly normal is the extension of the split interval $\mathbb{I}$ to three or more elements in the second dimension. Indeed, the lexicographically ordered space $I \times \{0, \frac{1}{2}, 1\}$ is normal, but it is not

†See page 32 for proof.

perfect. (Any ordered space with singleton indifference sets is normal, and $\mathbb{I}$ is a closed subset of the latter space that is not a zero set.[†]) So if $A = \{a, b\}$, $\Theta = I \times \left\{0, \frac{1}{2}, 1\right\}$ and $\{\theta : a \sim_\theta b\} = \mathbb{I}$, then, by definition 1 and proposition 3, preferences have no parametrically continuous representation.

Similarly, whilst the product $\mathbb{I}^2$ of two (or more) copies of the split interval is normal (as a product of compact Hausdorff spaces), any such product fails to be perfect. This follows because the product contains a subspace that is homeomorphic to the Sorgenfrey plane of proposition 4. In essence, in the absence of metrizability, perfect normality is a fragile property.

3.1 Extension to normal parameter spaces

When Θ is not perfect, preferences may still be "perfectly pairwise stable".

Axiom $\mathcal{PS}^*$. *For every $a, b \in A$, $\{\theta : a \prec_\theta b\}$ is the open union of an increasing and countable collection of sets that are closed in Θ.*

When preferences satisfy $\mathcal{PS}^*$, there exists a countable collection $\{F_n\}$ of closed subsets of $\{\theta : a \prec_\theta b\}$, such that, for each $\eta \in \{\theta : a \prec_\theta b\}$, there exists $m \in \mathbb{N}$ such that $\eta \in F_m$. For instance, if Θ were levels of wealth, F_m would be a finite union of closed intervals. This is the basis for the algorithm of section 7, where we also provide simple sufficient conditions for $\mathcal{PS}^*$ in certain settings. Finally, $\mathcal{PS}^*$ extends our model to normal parameter spaces.

[†]A sketch proof of this claim is as follows. $\mathbb{I}$ is closed (and therefore compact) because its complement $I \times \left\{\frac{1}{2}\right\}$ is the union of open intervals $(r \times 0, r \times 1)$ such that $r \in I$. Then if $\mathbb{I} \subset G$, where G is open in $I \times \left\{0, \frac{1}{2}, 1\right\}$, then G is a union of basic open order intervals. Since $\mathbb{I}$ is compact, we may take G to be a finite union. Finally, if $\mathbb{I}$ equals the intersection of a countable collection of such sets G, then $\mathbb{I}$ is the union of a countable collection of intervals. (Since the intersection of intervals is an interval.) But any interval that is not a singleton contains elements of $I \times \left\{\frac{1}{2}\right\}$. This is a contradiction because $\mathbb{I}$ is uncountable.

Theorem 2. *Let A be countable and let Θ be normal. Preferences have a parametrically continuous representation if and only if $\mathcal{O}$ and $\mathcal{PS}^*$ hold.*

Proof of theorem 2 of page 14. In the next subsection, we show that $\mathcal{PS}^*$, allows us to construct a pseudometric p on Θ. Like a metric, the pseudometric $p : \Theta^2 \to \mathbb{R}$ is continuous, nonnegative, symmetric and satisfies the triangle inequality. In contrast with a metric, the pseudometric p may satisfy $p(\zeta, \eta) = 0$ for $\zeta \neq \eta$. Such pairs are incomparable under p and the collection of such pairs forms a binary relation $\bowtie$ on Θ.

In lemma 2 we show that $\bowtie$ is an equivalence relation. This ensures that $\bowtie$ partitions Θ and that we may pass to the quotient space $\Theta_p \stackrel{\text{def}}{=} \Theta_{/\bowtie}$ that identifies points that are incomparable under p. This latter identification ensures that, with the open sets generated by p, Θ_p is a $\mathcal{T}_1$ space. In fact, Θ_p is perfectly normal because every pseudometrizable $\mathcal{T}_1$ space is metrizable.

In lemma 2 we also show that p has the property $\zeta \bowtie \eta$ implies [for every $a, b \in A$, $a \prec_\zeta b$ if and only if $a \prec_\eta b$]. Theorem 1 then ensures the existence of a parametrically continuous representation $U_p : A \times \Theta_p \to \mathbb{R}$. Finally, the extension from Θ_p to Θ is simple: take $U : A \times \Theta \to \mathbb{R}$ to be constant on each equivalence class of $\bowtie$. The fact that p is continuous ensures that each equivalence class is closed in Θ, so that U is a suitable representation. □

3.2 Pseudometrics for nonmetrizable spaces

Whilst this subsection is essential to the proof of theorem 2, it is also motivated by the needs of standard tools in the analysis of policy and in particular the envelope theorems of Milgrom and Segal [23] and Sah and Zhao [28] and nonsmooth calculus of Heinonen [12] which require a metric space.

Lemma 2. [†] *Let A be countable and let Θ be normal. If preferences satisfy $\mathcal{O}$ and $\mathcal{PS}^*$, then there exists a continuous pseudometric $p : \Theta^2 \to \mathbb{R}_+$ such that $p(\zeta, \eta) = 0$ implies $\prec_\zeta$ is equal to $\prec_\eta$.*

In the special case where Θ is the split interval and $A = \{a, b\}$, p of lemma 2 may even coincide with the standard metric on I. For example suppose there exists $0 < \tilde{r} < 1$ such that $\{\theta \in \mathbb{I} : a \sim_\theta b\} = \{\tilde{r} \times d : d = 0, 1\}$ and $\theta <^* \tilde{r} \times 0$ if and only if $a \prec_\theta b$. In this case, a suitable pseudometric is

$$p\left(r \times d, r' \times d'\right) \stackrel{\text{def}}{=} |r - r'| \qquad \text{for every } r, r' \in I.$$

Uncountably many pairs of elements in $\mathbb{I}$ are incomparable under any such a pseudometric. As elaborate in 5, the pseudometric will in general contain jumps that are compatible with continuity on $\mathbb{I}$. Derivatives and the nonsmooth calculus techniques of Heinonen [12] may then be applied on the quotient space that treats equivalence classes of the pseudometric as points.

4 Joint continuity of utility

When A is discrete, the next result shows that the conditions (on preferences and the parameter space) of the last section suffice for a utility representation that is continuous on $A \times \Theta$ (and hence called jointly continuous).

Theorem 3. [‡] *Let A be discrete and let Θ be normal. Preferences have a jointly continuous representation $U : A \times \Theta \to \mathbb{R}$ if and only if $\mathcal{O}$ and $\mathcal{PS}^*$ hold and A is countable. Moreover, for $\Phi : \Theta \to 2^A$ continuous and*

[†] See page 32 for proof.
[‡] See page 33 for proof.

compact-valued, the following value function is continuous on Θ

$$V(\cdot) \stackrel{\text{def}}{=} \max\{U(a,\cdot) : a \in \Phi(\cdot)\}.$$

When A is discrete, the requirement that it is countable is clearly necessary for a real-valued representation. This fact highlights a natural source of applications for the results of the preceding section. Theorem 3 provides a partial foundation in preferences for models that appeal to the envelope theorem of Sah and Zhao [28]. There A is also discrete and in addition to continuity, each $U(a,\cdot)$ is assumed to be concave on Θ [28, p.628].

4.1 Extension to uncountable A

In general, axiom $\mathcal{CG}$ is necessary for joint continuity. Conversely, a strengthening of $\mathcal{CG}$ to "preferences have a perfectly closed graph", ought to allow an extension to uncountable actions and normal parameter spaces. However, the uncountable collection of pseudometrics does not combine to form a single pseudometric (see Kelley [15, Theorem 13] and surrounding discussion).

When $A \times \Theta$ fails to be perfectly normal, we have the following result which, for the jointly continuous case, is analogous to proposition 3.

Proposition 4. † *If* $A \times \Theta$ *is not perfectly normal, then there are preferences that have no jointly continuous representation and satisfy both* $\mathcal{O}$ *and* $\mathcal{CG}$*.*

Levin [17, Theorem 1] yields a jointly continuous representation for A second countable (defined in section 2.4) and *locally compact* (each point in A has a compact neighbourhood). Levin's restriction to metrizable parameter

† See page 34 for proof.

spaces is unnecessary because $A \times \Theta$ is perfectly normal when A is second countable and Θ is perfectly normal (Tkachuk [31, p.249]).

Theorem 4. † *Let A be second countable and locally compact and let Θ be perfectly normal. Preferences have a jointly continuous representation if and only if $\mathcal{O}$ and $\mathcal{CG}$ hold.*

It may be possible to extend Levin's theorem to allow for $A \times \Theta$ that is not perfectly normal without strengthening $\mathcal{CG}$. However, the best one can hope for is a *separately continuous* representation ($U(a, \cdot)$ is continuous for each a and $U(\cdot, \theta)$ is continuous for each θ). Maximum theorems given separate continuity are derived in Dutta, Majumdar, and Sundaram [5].

5 A procedure for weakly ordered spaces

Consider a policy maker, such as Angela Merkel (the present chancellor of Germany), formulating one particular aspect of her policy towards the recent refugee crisis. Angela's parameter space describes the range of possible types of claim for political asylum in Germany. For simplicity, we assume that claims are either lodged in the locality $l = 0$ of Syria or in Germany $l = 1$. (Specifications that include more countries are easily accommodated.)

The primary criterion for ranking claims is a parameter r that varies within the unit interval I. We assume that $r = 0$ represents cases where there is no evidence of a valid claim and $r = 1$ represents the case where the individual has comprehensive proof. Most cases will lie somewhere inbetween.

†The proof of sufficiency follows directly from that of Levin. In particular, at the bottom of p.717, Levin only uses the fact that metrizable spaces are perfectly normal. Finally, our definition of a utility representation is equivalent to that of Levin when $\mathcal{O}$ holds, and $\mathcal{CG}$ is necessary for joint continuity by a simple proof by contradiction.

The set of parameters is then $I \times \{0,1\}$. Suppose an ordering $<^\tau$ of this set is given to Angela by her constituency or perhaps by her own "moral compass". In any case, Angela takes this as given when she formulates her policy.[†] Initially, we suppose location has no bearing on $<^\tau$, so that $r \times l <^\tau r' \times l'$ if and only if $r < r'$. (We relax this assumption in the next subsection.) The ordered topological space $S \stackrel{\text{def}}{=} (I \times \{0,1\}, \tau)$ that $<^\tau$ generates fails to be Hausdorff simply because, for each $r \in I$, $r \times 0$ and $r \times 1$ are not separated by a pair of disjoint open order intervals of $<^\tau$.

In fact, S is homeomorphic to I and therefore connected. It is therefore impossible to specify a continuous and variable feasibility correspondence Φ on S when actions are discrete. Consider the case where her policy only involves the choice between a = "accept" and b = "reject". For sufficiently severe cases θ (r close to 1), rejecting an application may be politically untenable. This is a question of feasibility rather than choice. For such θ, our model should allow for the possibility that $\Phi(\theta) = \{a\}$.

Strengthening the topology Whilst there are potentially many ways to strengthen the topology τ, the simplest one that preserves the topologial structure is to use location to break ties in such a way that claims from Germany take precedence over those from Syria.[‡ §] The resulting ordering $<^*$ is compatible with $<^\tau$ in the sense that the latter is a subset of the former. Since $r \times 0 <^* r \times 1$ for each $r \in I$, $<^*$ is a strict linear ordering on $I \times \{0,1\}$. Clearly, $<^*$ generates the topology τ^* of the split interval $\mathbb{I}$ of section 2.4.

[†] To emphasise this point, unlike preferences, the notation for orderings on the parameter space comes with a superscript and noncalligraphic font.

[‡] Repatriating individuals frequently makes headlines (see [3] for example).

[§] Tossing a coin generates more complicated parameter space that is unmeasurable.

Preferences Let $\Theta = \mathbb{I}$ and $A = \{a, b\}$. Then lemma 1 and theorems 1 to 3 all apply provided Angela's preferences $\{\prec_\Theta\}$ satisfy $\mathcal{O}_1$ and $\mathcal{PS}^*$. Suppose there exists a unique cutoff r^c such that a claim $r \times l$ is rejected whenever $r \times l \leqslant^* r^c \times 0$ and accepted otherwise. The resulting sets $\{\theta : a \prec_\theta b\}$ and $\{\theta : b \prec_\theta a\}$ are then disjoint and open with union equal to $\mathbb{I}$. Angela's preferences then satisfy $\mathcal{PS}^*$ and her policy is u.h.c. The fact that $\{\theta : a \sim_\theta b\}$ is empty is plausible given that accept and reject are mutually exclusive actions.[†] We may now recover almost all the structure of τ.

Pseudometrizing the parameter space The natural pseudometric with which to reimpose the ties that are irrelevant to Angela's preferences is

$$p(r \times l, r' \times l') \overset{\text{def}}{=} \begin{cases} |r - r'| & \text{if } \prec_{r \times l} \text{ equals } \prec_{r' \times l'} \\ |r - r'| + 1 & \text{otherwise.} \end{cases}$$

Clearly, p assigns distance zero to every pair $r \times 0$ and $r \times 1$ such that $r \neq r^c$. It is also clear that p is the minimal modification that generates $<^\tau$ whilst allowing for the discrete change in policy between $r^c \times 0$ and $r^c \times 1$. Let $S_p = (I \times \{0, 1\}, \tau_p)$ denote the topological space generated by the open balls of p. Continuity of p on $\mathbb{I}$ follows from the next proposition.

Proposition 5. [‡] *The topologies of the present section satisfy* $\tau \subsetneq \tau_p \subsetneq \tau^*$ *and* S_p *is (pseudo)isometric to the subset* $[0, r^c] \cup [r^c + 1, 2]$ *of* $\mathbb{R}$.[§]

The isometry of proposition 5 gives rise to an alternative specification for the parameter space. One that allows the straightforward application of the envelope theorems in Milgrom and Segal [23] and Sah and Zhao [28].

[†] $\{\theta : a \sim_\theta b\}$ is necessarily nonempty when Θ is connected and the axioms hold.
[‡] See page 35 for proof.
[§] For some $f : \mathbb{I} \to [0, r^c] \cup [r^c + 1, 2]$, $|f(\theta) - f(\theta')| = p(\theta, \theta')$ for every $\theta, \theta' \in \Theta$.

(Indeed a smooth utility representation is: $U(b,r) = 0$ for every $r \in [0, r^c] \cup [r^c, 2]$; $U(a,r) = -1$ for every $r \leqslant r^c$ and $U(a,r) = 1$ otherwise.) Yet this seemingly obvious specification is only available after observing $\{\prec_{\mathbb{I}}\}$. Observing preferences tells us where to disconnect the parameter space.

Contrast with the Euclidean metric Consider the Euclidean metric $d : \mathbb{R}^2 \to \mathbb{R}_+$ and corresponding topology τ_d. First, $d(r \times 0, r \times 1) = 1$ for every $r \in I$, further apart than any pair of claims from the same country. This suggests that p is a better measure of distance than d. Second, d is discontinuous relative to τ^* because $\tau_d \nsubseteq \tau^*$. (Indeed, for any $\epsilon < 1$ and $r \in I$, the ball $\{\theta : d(\theta, r \times 0) < \epsilon\}$ is a subset of $I \times \{0\}$, whereas every open interval of $<^*$ that contains $r \times 0$ contains elements of $I \times \{1\}$.) Finally, every utility representation of $\{\prec_{\mathbb{I}}\}$, is discontinuous relative to τ_d. Such arguments are not specific to d because S, S_p and $\mathbb{I}$ are all nonmetrizable.

5.1 Alternative specifications for Θ

Now suppose that a claim lodged in Germany has a strictly lower burden of proof relative to one that is lodged in Syria. Then, for some $0 < \epsilon < 1$, $<^\tau$ satisfies $(r + \epsilon) \times 0 \simeq^\tau r \times 1$ for each feasible r. Formally, $r \times l <^\tau r' \times l'$ if one of the following mutually exclusive conditions hold:

1. $0 < r' - r$ and $l = l'$ (stronger evidence is ranked higher);
2. $\epsilon < r' - r$ and $l \neq l'$ (significantly higher evidence dominates);
3. $|r' - r| < \epsilon$ and $l < l'$ (location dominates for otherwise similar cases).

As before, $<^\tau$ fails to yield a topology that is Hausdorff. The obvious tie-breaking rule yields a linear order $<^\epsilon$ and space $\mathbb{I}_\epsilon$ such that $(r + \epsilon) \times 0 <^\epsilon$

$r \times 1$ for each feasible r. (This strengthens condition 2 so that it holds for $\epsilon \leqslant r' - r$.) As ϵ tends to 0, $<^\epsilon$ tends to $<^*$ and fig. 2 shows that $\mathbb{I}_\epsilon$ contains a copy of the split interval and is thereby nonmetrizable whenever $\epsilon < 1$.

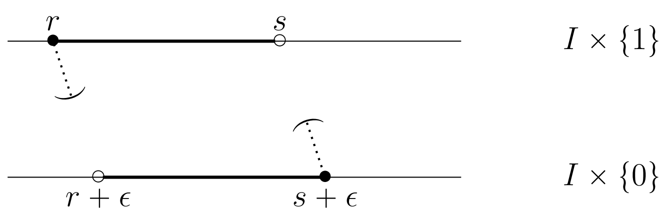


Figure 2: The ϵ-lexicographic ordering finds the closed order interval $[r \times 1, (s + \epsilon) \times 0]$ equal to the open order interval $((r + \epsilon) \times 0, s \times 1)$.

The results of this paper also apply if we take the natural step of allowing for uncertainty. Since $\Theta \overset{\text{def}}{=} \mathbb{I}_\epsilon$ is compact Hausdorff, the set Δ of probability measures on the σ-algebra generated by the open sets of Θ is compact Hausdorff in the usual weak* topology.† Since the weak* topology on Θ is an infinite-dimensional dimensional product of ordered spaces, the discussion of section 6 is related to this kind of extension.

Finally, suppose Angela wishes to take into account the fact that the refugees may migrate. That is, she may instead take parameters to be continuous feasibility correspondences $\Phi : \Theta \to 2^\Theta$ and her parameter space to be a suitable subspace Θ'. In this case, Gale [8, Theorem 1] provides reasonably simple conditions for Θ' to be compact Hausdorff and hence normal.‡

†Recall this is the weakest topology that ensures the maps $\int_\Theta f \, \mathrm{d}\mu$ are continuous in $\mu \in \Delta$ for each continuous and real-valued function f on Θ.

‡This holds for the compact-open, weak or strong topology all of which coincide when Θ is compact Hausdorff (see McLennan [19, Chapter 5]). Indeed, Michael [22, Theorem 4.9] together with the fact that Θ is compact Hausdorff ensures that 2^Θ is compact Hausdorff, so that the latter satisfies the conditions for the set Y in Gale [8, Theorem 1].

6 Cylinder sets

Building on the discussion of section 1 we recall Stone [29, Theorem 4]: a product of metric spaces is normal if and only if all but countably many factors in the product are compact. This means that the product space $\mathbb{R}_+^I$ fails to be normal, let alone metrizable. Many authors such as Taylor [30] and Parthasarathy [27] address this issue by considering the one-point compactification $\dot{\mathbb{R}}_+ \stackrel{\text{def}}{=} \mathbb{R}_+ \cup \infty$. Since $\dot{\mathbb{R}}_+$ is homeomorphic to the closed unit interval, $\dot{\mathbb{R}}_+^I$ is homeomorphic to I^I. This means that it is compact Hausdorff and hence normal, but nonmetrizable.

An important reason why the product topology is important is the following. A function $f : \Omega \to \mathbb{R}_+^I$ (such as a random variable, for abstract Ω) is continuous in the product topology if and only if $f(\cdot, r)$ is continuous for each $r \in I$ (see [24, Theorem 19.6]). Another is the fact that the basic open sets of the product topology (the open cylinder sets) reflect real-life restrictions on our ability to store information, observe stochastic processes, perturb trajectories or take limits. The uniform metric $|f - g|_\infty \stackrel{\text{def}}{=} \sup \{|f(r) - g(r)| : r \in I\}$ is a much finer or stronger topology than the product topology on $\mathbb{R}^I$.

Such considerations motivate the Kolmogorov extension theorem that allows us to define a stochastic process in terms of its "finite-dimensional distributions". This theorem may be expressed in terms of cylinder sets (see for instance Dhrymes [4, Proposition 8.6]). The importance of this theorem suggests that cylinder sets provide a good starting point when dealing with an infinite dimensional problem, especially if the goal is to elicit preferences.

Parametric continuity in this setting The usual Borel σ-algebra of measure theory is generated by the open sets of a topology. It is common therefore to seek a continuous utility and value function when the parameter space is some subspace of trajectories (realisations in $\dot{\mathbb{R}}^I_+$ of the stochastic process). Consider for instance the Malliavin calculus where "weak" directional derivatives in $\mathbb{R}^I_+$ are considered (see Nualart [26]).

The fact that theorem 2 holds on normal spaces means that the policy maker's preferences can be defined for $\Theta = \dot{\mathbb{R}}^I$ directly. This avoids the need to extend from the metrizable product $\dot{\mathbb{R}}^Q_+$, for Q equal to the rational numbers in I, to Θ. Although such an extension is a useful simplification for measure theoretic purposes, it may result in a discontinuous representation of preferences that satisfy $\mathcal{PS}$.†

Discrete actions As $\dot{\mathbb{R}}^I_+$ is connected, it does not admit a continuous and variable feasibility correspondence or mutually exclusive actions.‡

Suppose Angela is a potential buyer of an indivisible asset such as a house. The actions are $a =$"abstain from buying" and $b =$"buy" and that a trajectory $p : I \mapsto \dot{\mathbb{R}}_+$ of prices $p(t)$ such that $t \in I$ is the primary determinant of Angela's preferences. Prices are typically not the only determinant however. For instance, the economy may or may not be in a recession or the house may or may not belong to a desirable school catchment area.

Consider recessions and note that $\dot{\mathbb{R}}^I_+$ accommodates the possibility that they affect the price trajectory p discontinuously: this space admits every possible real-valued trajectory. We model recessions through a separate function $l : I \to \{0, 1\}$ such that $l(t) = 1$ if and only if there is a recession. The set of parameters is now $\left(\dot{\mathbb{R}}_+ \times \{0, 1\}\right)^I$.

†For a direct approach to the construction of the measure of a Brownian motion on this space see Nelson [25, Appendix].

‡The proof that $\dot{\mathbb{R}}^I$ is connected is as follows. If G and G' are nonempty, open, disjoint and with union equal to $\dot{\mathbb{R}}^I_+$, then for some $r \in I$, the image sets $\pi_r(G)$ and $\pi_r(G')$ of the natural projection are nonempty, open and disjoint with union equal to $\dot{\mathbb{R}}_+$. Since the latter is connected, we have a contradiction.

Suppose that Angela perceives the price impact of a recession to be ϵ. To her, if the house is worth r in a recession, then it is worth $r+\epsilon$ in the absence of a recession. Effectively, Angela evaluates the price of the house on the basis of a recession-adjusted ordering $<^{\tau}$ of $\dot{\mathbb{R}}_+ \times \{0,1\}$ that satisfies conditions 1, 2 and 3 of section 5.1. $<^{\tau}$ does not distinguish between $(r+\epsilon)\times 0$ and $r \times 1$. With the order topology of $<^{\tau}$, $\dot{\mathbb{R}}_+ \times \{0,1\}$ is connected, but not Hausdorff.

As in section 5, a simple tie breaking rule extends $<^{\tau}$ to the relation $<^{\epsilon}$ that satisfies $(r+\epsilon)\times 0 <^{\epsilon} r \times 1$ for each $r \in \dot{\mathbb{R}}_+$. The open intervals of $<^{\epsilon}$ generate a space S_{ϵ} that is homeomorphic to $\mathbb{I}_{\epsilon}$. The parameter space $\Theta \stackrel{\text{def}}{=} S_{\epsilon}^{I}$ of trajectories $\theta \stackrel{\text{def}}{=} p^{\epsilon} : I \to S_{\epsilon}$ that we adopt is then homeomorphic to $\mathbb{I}_{\epsilon}^{I}$. It is compact, Hausdorff and hence normal.

Preferences that satisfy $\mathcal{PS}^*$ Suppose, for some cut-off price r^c, Angela's preferences $\left\{\prec_{S_{\epsilon}^{I}}\right\}$ have the property : there is a finite set of times $t_1, t_2, \ldots t_m$ such that, if $p^{\epsilon}(t_n) < r^c \times 1$ for some $n \leqslant m$, then $a \prec_{\theta} b$; otherwise $b \prec_{\theta} a$. This means that, during the time interval I, Angela buys the house if and only if the recession-adjusted price is below $r^c \times 1$ at one or more of the times $t_1, \ldots, t_m$. Such preferences satisfy $\mathcal{PS}^*$ because the set $\{p^{\epsilon} : p^{\epsilon}(t_n) <^{\epsilon} r^c \times 1\}$ is a clopen cylinder set and the finite union of clopen sets is clopen.

This example is well-motivated if Angela only checks the house price at finitely many points in time. This may arise due to the costs of assigning unlimited attention to the price movements. Indeed, what happens at other times than $t_1, \ldots, t_m$ is irrelevant to Angela's policy.

Generalising this example leads to a sufficient condition for $\mathcal{PS}^*$. This simple case is potentially useful for the algorithm of section 7. On a product space, preferences are *convex-cylindrical* if, for each $a, b \in A$, $\{\theta : a \prec_{\theta} b\}$ is a countable union of cylinder sets such that each factor is an order interval.

Proposition 6. † *If Θ is a product of first countable linearly ordered spaces and preferences are convex-cylindrical, then $\mathcal{PS}$ implies $\mathcal{PS}^*$.*

Preferences that fail to satisfy $\mathcal{PS}$ The next example of preferences shows that it is not the case that anything goes in S_ϵ^I. Suppose, that instead, $a \prec_\theta b$ if and only if $\theta(t) <^\epsilon r^c \times 1$ for some $t \in I$; moreover, suppose $b \prec_\theta a$ otherwise. Then, during the time interval I, Angela buys the house if and only if the recession-adjusted price is below $r^c \times 1$ for some t. Then $\{\theta : b \prec_\theta a\} = F$, where $F \stackrel{\text{def}}{=} \{\theta : r^c \times 1 \leqslant^\epsilon \theta(t) \text{ for every } t \in I\}$. The following proposition shows that such preferences fail to satisfy $\mathcal{PS}$.

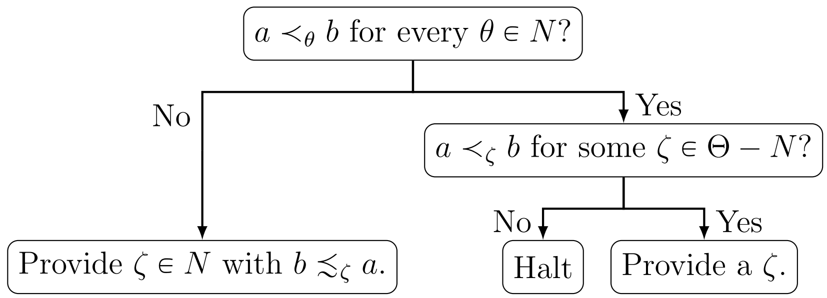
Proposition 7. *The set F is closed, but not open in S_ϵ^I.*

Proof. The interval $[0 \times 0, r^c \times 1)$ is clopen in S_ϵ. Thus, the cylinder set $G_t \stackrel{\text{def}}{=} \{p^\epsilon : p^\epsilon(t) <^\epsilon r^c \times 1\}$ is clopen in $\mathbb{I}_\epsilon^I$ for each t. Hence, the union $G \stackrel{\text{def}}{=} \bigcup \{G_t : t \in I\}$ is open. The complement $F = \Theta - G$ is the closed set of paths p^ϵ such that $r^c \times 1 \leqslant^\epsilon p^\epsilon(t)$ for every $t \in I$. F is precisely equal to $\{\theta : b \prec_\theta a\}$. Let C be any finite intersection of open cylinder sets. Since I is uncountable, C has infinitely many factors equal to S_ϵ and $C \nsubseteq F$. □

7 An algorithm for eliciting preferences

When the parameter space has a continuum dimension it is natural to question the value to the present results, since preferences can never be elicited at each point. The algorithm uses normality of Θ and $\mathcal{PS}^*$ to approximate preferences through a finite number of questions of the form of fig. 3. We will make use of the closure and interior operators: $\operatorname{cl} N$ and $\operatorname{int} N$ (respectively closed and open, and both nonempty for every neighbourhood N of a point).

† See page 35 for proof.



(∗)

Figure 3: We refer to this decision tree as question (∗).

Let $A = \{a, b\}$. The algorithm is characterised by states s_1, s_2 and s_3. We initiate the algorithm in state s_1 with $N'' \stackrel{\text{def}}{=} \Theta$.

s_1 Ask question (∗) with $N \stackrel{\text{def}}{=} N''$. If the answer is No, then let $N' \stackrel{\text{def}}{=} \{\zeta\}$, $N''' \stackrel{\text{def}}{=} N''$ and go to s_2. If Yes×Yes, then let $N' \stackrel{\text{def}}{=} N''$ and go to s_3.

s_2 Choose an open set N_{-1} such that $N' \subsetneq N_{-1}$ and $\operatorname{cl} N_{-1} \subsetneq \operatorname{int} N''$ and ask question (∗) with $N \stackrel{\text{def}}{=} \operatorname{cl} N_{-1}$. If the answer is No, then let $N'' \stackrel{\text{def}}{=} \operatorname{cl} N_{-1}$, $N''' \stackrel{\text{def}}{=} \operatorname{cl} N_{-1}$ and remain in s_2. If Yes×Yes, then let $N' \stackrel{\text{def}}{=} \operatorname{cl} N_{-1}$, $N'' \stackrel{\text{def}}{=} \operatorname{cl} N_{-1}$ and go to s_3.

s_3 Choose a open set N_{+1} such that $N'' \subsetneq N_{+1}$ and $\operatorname{cl} N_{+1} \subsetneq \operatorname{int} N'''$ and ask question (∗) with $N \stackrel{\text{def}}{=} \operatorname{cl} N_{+1}$. If the answer is No, then let $N'' \stackrel{\text{def}}{=} \operatorname{cl} N_{+1}$, $N''' \stackrel{\text{def}}{=} \operatorname{cl} N_{+1}$ and go to s_2. If Yes×Yes, then let $N' \stackrel{\text{def}}{=} \operatorname{cl} N_{+1}$, $N'' \stackrel{\text{def}}{=} \operatorname{cl} N_{+1}$ and remain in s_3.

The existence of N_{-1} and N_{+1} in s_2 and s_3 is entirely due to normality of Θ (Munkres [24, Lemma 31.1]). $\mathcal{PS}^*$ then guarantees that we can generate a convergent sequence of closed neighbourhoods in Θ using this algorithm. The quality of the approximation may be measured endogenously in absence of a metric. For instance, it may be suitable to halt when the policy maker no longer finds it worthwhile to produce examples of ζ in question (∗).

A Proofs

Proof of proposition 1 of page 8. The map $\theta \mapsto \prec_\theta$ defines a correspondence on Θ with values in the power set of $A \times A$. This is l.h.c. provided that, for every open $G \subseteq A \times A$, the set $\{\theta : \prec_\theta \cap G \neq \varnothing\}$ is open. This latter set is just the union of $\{\theta : a \prec_\theta b\}$ such that $a \times b \in G$. Thus, $\mathcal{PS}$ implies l.h.c. of $\theta \mapsto \prec_\theta$.

When A is discrete, every $B \subseteq A \times A$ is both open and closed. If $\theta \mapsto \prec_\theta$ is l.h.c., take $B = \{a \times b\}$. Then $\{\theta : \prec_\theta \cap B \neq \varnothing\}$ is open and equal to $\{\theta : a \prec_\theta b\}$. Thus, when A is discrete the converse holds, as required. □

Proof of proposition 2 of page 9. Suppose $\mathcal{PS}$ holds. Let D be a directed set and let $\eta_\nu \times a_\nu \times b_\nu \to \eta \times a \times b$ be a net such that $a_\nu \precsim_{\eta_\nu} b$ for each $\nu \in D$. Then there exists $\mu \in D$ such that for every $\nu \geqslant \mu$, $\eta_\nu \times a_\nu \times b_\nu = \eta_\nu \times a \times b$. Now suppose that $b \prec_\eta a$, so that the graph of $\theta \mapsto \precsim_\theta$ is not closed. Then by $\mathcal{PS}$, there exists a neighbourhood N of θ such that $a \prec_\theta b$ for every $\theta \in N$. But since the net converges to $\eta \times a \times b$, there exists m' such that $\eta_\nu \in N$ for every $\nu \geqslant \mu'$. But then for every $\nu \geqslant \max\{\mu, \mu'\}$, we have both $a \precsim_{\eta_\nu} b$ (because $\eta_\nu \times a \times b$ belongs to the graph of $\precsim_\Theta$) and $b \prec_{\eta_\nu} a$ (because $\eta_\nu \in N$): a contradiction of $\mathcal{O}_1$.

Now suppose that the graph of $\theta \mapsto \precsim_\theta$ is closed. Then for fixed $a, b \in A$, the set $\{(\theta, a, b) : a \precsim_\theta b\}$ is closed. By $\mathcal{O}_1$, this is equivalent to $\mathcal{PS}$. □

Proof of lemma 1 of page 9. u.h.c. of C follows from Aliprantis and Border [1, Theorem 17.25]: the intersection of a closed correspondence and a compact-valued u.h.c. correspondence is u.h.c. By assumption, Φ is u.h.c. and compact-valued, and since it is a feasibility constraint, at each θ, $C(\theta)$ is indeed equal to the intersection $C(\theta) \cap \Phi(\theta)$. Therefore, C is u.h.c. provided it is a closed correspondence: that is, provided the graph $\operatorname{gr} C \stackrel{\text{def}}{=} \{(\theta, a) : a \in C(\theta)\}$ is closed.

First note that Φ is itself a closed correspondence by Aliprantis and Border [1, Theorem 17.10]. This latter theorem requires A Hausdorff (which every discrete set is) and that Φ is u.h.c. and compact-valued. Let $(\theta_n, a_n)_{n \in D}$ be a net with values in $\operatorname{gr} C$ and limit equal to (η, a). This limit is well-defined because $\Theta \times A$ is Hausdorff. Since $C(\theta) \subseteq \Phi(\theta)$ for every $\theta \in \Theta$, $a \in \Phi(\theta_n)$ for every n. Since Φ is a closed correspondence, a is feasible at θ. Since A is discrete, the singleton set $\{a\}$ is the smallest open neighbourhood of any $a \in A$. Since $(\theta_n, a_n) \to (\eta, a)$, there exists $m \in D$ such that $(\theta_n, a_n) = (\theta_n, a)$ for every $n \geqslant m$.

By way of contradiction, suppose $a \prec_\eta b$ for some $b \in \Phi(\eta)$. (So that $a \notin C(\eta)$ and $\operatorname{gr} C$ is not closed.) Since Φ is l.h.c., $\Phi^-(b) \stackrel{\text{def}}{=} \{\theta : \Phi(\theta) \cap \{b\} \neq \varnothing\}$ is open. Since $\eta \in \Phi^-(b)$, there is a neighbourhood N of η such that b is feasible on N. By $\mathcal{PS}$, there exists a neighbourhood N' of η such that $a \prec_\theta b$ for every $\theta \in N'$. Let $N'' = N \cap N'$. Since $\theta_n \to \eta$, there exists m' such that $\theta_n \in N''$ for every $n \geqslant m'$. Let $m'' = \max\{m, m'\}$. Then for every $n \geqslant m''$ both $a, b \in \Phi(\theta_n)$ and $a \prec_{\theta_n} b$. But then we arrive at a contradiction, for every $n \geqslant m''$, $a_n = a$ is suboptimal. That is, contrary to our assumption, we have shown that $(\theta_n, a_n) \notin \operatorname{gr} C$ for every $n \geqslant m''$.

Finally, the fact that C is compact-valued follows because $C(\theta)$ is a closed subset of the compact set $\Phi(\theta) \in 2^A$ for each θ. In turn, compactness in a discrete space is equivalent to finiteness. As such, $\mathcal{O}$ guarantees C is nonempty-valued. □

Remaining steps in the proof of theorem 1 of page 12.

Step 1. See page 12 for the initial step in the induction on A.

Step 2 (Inductive step). Let $\{1, 2, 3 \dots\}$ be an arbitrary enumeration of A, and let $[j] \subseteq A$ denote the first j elements of the enumeration. Fix $j \in A$. The induction hypothesis ensures the existence of a function $U^{j-1} : [j-1] \times \Theta \to [-1, 1]$ that satisfies (1) and (2) of theorem 1. For each $a \in [j-1]$ take $U^j(a, \cdot) \stackrel{\text{def}}{=} U^{j-1}(a, \cdot)$. It remains to show that we can find an extension of U^j to $[j]$ that satisfies (1) and (2) of theorem 1.

The required function $U^j(j, \cdot)$ will coincide with f in the following version of Michael's selection theorem [21, Theorem 3.1"'].

Theorem (Good and Stares [11]). *Θ is perfectly normal if and only if, whenever $g, h : \Theta \to \mathbb{R}$ are respectively upper and lower semi-continuous functions and $g \leqslant h$, there is a continuous $f : \Theta \to \mathbb{R}$ such that $g \leqslant f \leqslant h$ and $g(\theta) < f(\theta) < h(\theta)$ whenever $g(\theta) < h(\theta)$.*

In our setting, g and h will be envelope functions. To ensure they are well-defined, we introduce two fictional actions $\underline{a}$ and $\overline{a}$. These satisfy the property: $\underline{a} \precsim_\theta k \precsim_\theta \overline{a}$ for all $(k, \theta) \in [j] \times \Theta$. Accordingly, we define $[j-1]' = [j-1] \cup \{\underline{a}, \overline{a}\}$, and let $U^j(\underline{a}, \cdot) \equiv -1$ and $U^j(\overline{a}, \cdot) \equiv +1$. Both are clearly continuous functions on Θ. Moreover, for all $\theta \in \Theta$, the following functions are well-defined.

$$g(\theta) \stackrel{\text{def}}{=} \max \left\{ U^j(k, \theta) : k \precsim_\theta j \text{ and } k \in [j-1]' \right\},$$
$$h(\theta) \stackrel{\text{def}}{=} \min \left\{ U^j(k, \theta) : j \precsim_\theta k \text{ and } k \in [j-1]' \right\}.$$

In the three claims that follow, we prove that g and h satisfy the conditions for Michael's selection theorem. In particular $g \leqslant h$; $g(\theta) < h(\theta)$ whenever $j \not\sim_\theta k$ for every $k \in [j-1]'$; g is upper semicontinuous and h is lower semicontinuous. The inductive step is then completed by letting $U^j(j, \cdot) = f$, where f satisfies the conditions of Michael's selection theorem. Clearly, U^j

satisfies 1 and 2 of theorem 1. Moreover, U^j takes values in $[-1, 1]$.

CLAIM 1. *For all $\theta \in \Theta$, $g(\theta) \leqslant h(\theta)$.*

PROOF OF CLAIM 1 OF PAGE 31. Fix θ. By construction, there exist $k, l \in [j-1]'$ satisfying $g(\theta) = U^j(k, \theta)$ and $h(\theta) = U^j(l, \theta)$. By definition, $k \precsim_\theta j$ and $j \precsim_\theta l$. By $\mathcal{O}_2$, $k \precsim_\theta l$ and the inductive hypothesis then ensures that $g(\theta) \leqslant h(\theta)$. □

CLAIM 2. *For all $\theta \in \Theta : g(\theta) = h(\theta)$ iff $k \sim_\theta j$ for some $k \in [j-1]$.*

PROOF OF CLAIM 2 OF PAGE 31. If $g(\theta) = h(\theta)$, then, by construction, there is some $k \in [j-1]' \cap \{l : l \precsim_\theta j\} \cap \{l : j \precsim_\theta l\}$. By $\mathcal{O}_1$, for every such k, $k \sim_\theta j$. Conversely, if $k \sim_\theta j$, then both $k \precsim_\theta j$ and $j \precsim_\theta k$. □

CLAIM 3. *$g : \Theta \to \mathbb{R}$ is upper semicontinuous.*

A symmetric argument to the one that follows, but with inequalities and direction of weak preference reversed, shows that h is lower semicontinuous.

PROOF OF CLAIM 3 OF PAGE 31. Recall (or see [15, p.101]) that g is upper semicontinuous provided the set $\{\theta : r \leqslant g(\theta)\}$ is closed for each $r \in \mathbb{R}$. Note that by the construction of g,

$$\{\theta : r \leqslant g(\theta)\} = \bigcup_{k \in [j-1]'} \left(\{\theta : r \leqslant U^j(k, \theta)\} \cap \{\theta : k \precsim_\theta j\}\right) .$$

Recall that the finite union of closed sets is closed. Moreover, since $U^j(k, \cdot)$ is continuous, $\{\theta : r \leqslant U^j(k, \theta)\}$ is closed (preimage of a closed set is closed); and $\{\theta : k \precsim_\theta j\}$ is closed by $\mathcal{O}_1$ and $\mathcal{PS}$. □

Step 3 (The countably infinite case). The above argument holds for each j in $\mathbb{N}$.[†] For countably infinite A, we choose $U : A \times \Theta \to \mathbb{R}$ such that its graph satisfies $\operatorname{gr} U = \bigcup_{j \in \mathbb{N}} \operatorname{gr} U^j$. Since Michael's selection theorem is used at each j, for this step we appeal to the axiom of dependent choice. Alternatively, following [16, p.23], let $U(j, \cdot) = U^j(j, \cdot)$ for each $j \in \mathbb{N}$, and again appeal to the axiom of (dependent) choice.

Step 4 (Necessity of the axioms). The necessity of $\mathcal{O}_1$ and $\mathcal{O}_2$ is well-known and the following argument confirms that $\mathcal{PS}$ is necessary.

Take any $U : A \times \Theta \to \mathbb{R}$ satisfying (1) and (2) of theorem 1. Fix $a, b \in A$. Let $G = \{\theta : U(a, \theta) - U(b, \theta) < 0\}$. Since the difference of two continuous functions is continuous, G is open. Moreover, $G = \{\theta : a \prec_\theta b\}$.

This completes the proof of theorem 1. □

Proof of proposition 3 of page 13. Let $A = \{a, b\}$ and suppose that $F = \{\theta : a \sim_\theta b\}$ for some closed set F that is not a zero set. Such an F exists whenever Θ fails to be perfectly normal. Since preferences satisfy $\mathcal{O}_1$ and there are only two actions, there exists a representation of preferences. Take $U : A \times \Theta \to \mathbb{R}$ to be any such representation and define $f : \Theta \to \mathbb{R}$ to be the map $\theta \mapsto U(a, \theta) - U(b, \theta)$. Since U is a representation, $f(\theta) = 0$ if and only if $\theta \in F$. Thus $f^{-1}(0) = F$ and, since F is not a zero set, $f = U(a, \cdot) - U(b, \cdot)$ is discontinuous. By the algebra of continuous functions, at least one of $U(a, \cdot)$ and $U(b, \cdot)$ is discontinuous. □

Proof of lemma 2 of page 16. For any given $a, b \in A$, the set $F_{ab} = \{\theta : a \sim_\theta b\}$ is closed by $\mathcal{PS}^*$. Moreover, $\mathcal{O}_1$ and $\mathcal{PS}^*$ ensure the existence of a countable and decreasing sequence of open sets with intersection equal to F_{ab}. Since Θ is normal, the argument of step 1 of theorem 1 ensures the

[†]I thank Atsushi Kajii for bringing this subtle issue to my attention.

existence of a continuous function $f_{ab} : \Theta \to [-1,1]$ such that $f_{ab}^{-1}(0) = F_{ab}$ and $0 < f_{ab}(\theta)$ if and only if $a \prec_\theta b$. Let $p_{ab}(\zeta,\eta) \stackrel{\text{def}}{=} |f_{ab}(\zeta) - f_{ab}(\eta)|$ for each $\zeta, \eta \in \Theta$.

Clearly $p_{ab} : \Theta^2 \to \mathbb{R}$ inherits positivity, symmetry and the triangle inequality from $|\cdot|$ on $\mathbb{R}$. Moreover, $p_{ab}(\zeta,\eta) = 0$ implies [$a \prec_\zeta b$ if and only if $a \prec_\eta b$]. (The latter holds because whenever $b \precsim_\zeta a$ and $a \prec_\eta b$, we have $f_{ab}(\zeta) \leqslant 0 < f_{ab}(\eta)$, so that $p_{ab}(\zeta,\eta) \neq 0$.)

The above argument generates a collection of continuous pseudometrics $\Pi \stackrel{\text{def}}{=} \{p_{ab} : a, b \in A\}$ on Θ. Crucially for the next step, A is countable: the collection of pseudometrics is then countable; only countable intersections of perfect sets are countable. For an arbitrary enumeration $\{p_1, p_2, \dots\}$ of Π, take $p \stackrel{\text{def}}{=} \sum_1^\infty 2^{-n} p_n$. Clearly, if $p(\zeta,\eta) = 0$, then $p_{ab}(\zeta,\eta) = 0$ for every $a, b \in A$. By the preceding paragraph therefore, it only remains to check that p is indeed a continuous pseudometric. Since each p_n is nonnegative and symmetric with values in $[0,2]$, so is p. Moreover, for each m, the partial sum $\sum_1^m 2^{-n} p_n(\theta,\eta)$ satisfies the triangle inequality by induction: the sum of two pseudometrics preserves this inequality. The sandwich or squeeze lemma for sequences then ensures $p(\theta,\eta)$ also satisfies the triangle inequality. Continuity follows by uniform convergence of the continuous partial sums to p. This completes the proof of the lemma. □

Proof of theorem 3 of page 16. U is jointly continuous by the following argument. Fix $(a,\theta) \in A \times \Theta$ and consider, for some directed set D, a net $E = ((a_\nu, \theta_\nu))_{\nu \in D}$ in $A \times \Theta$ with limit (a,θ). We show that $U(a_\nu, \theta_\nu) \to U(a,\theta)$. Recall that (a,θ) is the limit of E if and only if, for every neighborhood N of (a,θ), there exists $\mu \in D$ such that for every $\nu \geqslant \mu$, $(a_\nu, \theta_\nu) \in N$. Since A is discrete, $\{a\}$ is open and for some N_θ open in Θ, the set $\{a\} \times N_\theta$ is an (open) neighborhood of (a,θ) in the product topology on

$A \times \Theta$. Thus, there exists μ such that for every $\nu \geqslant \mu$, $U(a_\nu, \theta_\nu) = U(a, \theta_\nu)$. Finally, part 2 of theorem 1 ensures that $U(a, \theta_\nu) \to U(a, \theta)$.

For continuity of V, let $U^* : \Theta \times A \to \mathbb{R}$ satisfy $U^*(\theta, a) \stackrel{\text{def}}{=} U(a, \theta)$ for every $(\theta, a) \in \Theta \times A$. By the preceding paragraph, U^* is continuous on $\Theta \times A$. In lemma 1, we derived a u.h.c. choice correspondence C that coincides with $\operatorname{argmax}\{U(a, \cdot) : a \in \Phi(\cdot)\}$. Finally, note that $V = U^* \circ \operatorname{gr} C$. V is then u.h.c. as the continuous composition of u.h.c. correspondences [1, Theorem 17.23], and since it is single-valued, it in fact continuous. □

Proof. [PROOF OF PROPOSITION 4 OF PAGE 17] By assumption, there exists a closed, nonzero subset F of $A \times \Theta$. Let $\{(a, \theta) : a \sim_\theta b\} = F$ and let preferences satisfy $\mathcal{O}$ and $\mathcal{CG}$ on $A - \{b\}$. Then every representation has $U(a, \theta) - U(b, \theta) = 0$ for every $(a, \theta) \in F$. Let U' be the following transformation of U. For every $a \in A$, $U'(a, \cdot) = U(a, \cdot) - U(b, \cdot)$. Then $U' : A \times \Theta \to \mathbb{R}$ satisfies $U'(F) = 0$. That is, $(a, \theta) \in F$ implies $U'(a, \theta) = 0$. Let $b \prec_\theta a$ for every (a, θ) in the open set $(A \times \Theta) - F$. Since F is closed and $b \precsim_\theta a$ for every $(a, \theta) \in A \times \Theta$, preferences satisfy $\mathcal{O}$ and $\mathcal{CG}$ on all of A. Since $U'(b, \cdot)$ is identically equal to zero, $0 < U'(a, \theta)$ for every $(a, \theta) \notin F$. Since $F = (U')^{-1}(0)$ is not a zero set, U' is discontinuous on $A \times \Theta$.

For an explicit example consider the Sorgenfrey line $\mathbb{L}$. This is the unit interval I where the basic open sets are half-open intervals $[r, s)$ such that $r < s$ in I. $\mathbb{L}$ is a well-known example of a perfectly normal, separable space that is not second countable and such that the Sorgenfrey plane $\mathbb{L}^2$ is not normal. Take A to be the discrete union of $\mathbb{L}$ and $\{b\}$ for some $b \notin \mathbb{L}$ and take $\Theta = \mathbb{L}$. Finally, take F to be the anti-diagonal of $\mathbb{L}^2$ and let $\{\prec_\Theta\}$ be such that for each $-r \in \Theta$, r is the worst element in $A - \{b\}$; $\prec_{-r}$ assigns higher order to elements that are further from r according to the standard metric on $\mathbb{R}$; and, moreover, for each feasible $\epsilon > 0$, $-\epsilon + r \sim_{-r} \epsilon + r$. Finally, for every rational number $q \in \mathbb{L}$, let $b \sim_{-q} q$; and for every irrational number $s \in \mathbb{L}$, suppose that $b \prec_{-s} s$.

Clearly $\{\prec_\Theta\}$ satisfies $\mathcal{O}$. To check $\mathcal{CG}$, suppose otherwise that $a_\nu \sim_{\theta_\nu} b$ for every ν and $(a_\nu, \theta_\nu) \to (a, \eta)$ such that $b \prec_\eta a$. Then by construction, each θ_ν is a rational number and $a_\nu = -\theta_\nu$. Moreover, since $a_\nu \to a$ and $\theta_\nu \to \eta$, we have $a = -\eta$. Since the anti-diagonal of $\mathbb{L}^2$ is a discrete, there exists a finite number μ such that $(a_\nu, \theta_\nu) = (a, \eta)$ for every $\nu \geqslant \mu$, a contradiction of the assumptions regarding the sequence. $\square$

Proof of proposition 5 of page 20. Let $<^p$ on S_p satisfy $r \times l <^p r' \times l'$ if and only if $r < r'$ or $[r = r' = r^c$ and $l < l']$. We claim that the topology generated by the open intervals of $<^p$ coincides with τ_p. This suffices for the proof of $\tau \subsetneq \tau_p \subsetneq \tau^*$ since $<^\tau \subsetneq <^p \subsetneq <^*$ holds by construction.

Let $(r \times l, r' \times l')_p$ be a nonempty open interval of $<^p$. Then $r < r'$ and, for some ϵ, $|r - r'| = \epsilon$. Consider the case $r \leqslant r^c < r'$. Then for every θ such that $r^c \times 1 <^p \theta <^p r' \times l'$, there exists $\epsilon_\theta < \epsilon$ such that $B_p(\theta, \epsilon_\theta) \subseteq (r^c \times 1, r' \times l')_p$. The open interval $[r^c \times 1, r' \times l')_p$ is equal to the union of such balls. Similarly, $(r \times l, r^c \times 0]_p$ is a union of open balls of p whenever $r < r^c$. The remaining cases are similar. Thus, τ_p contains every open interval of $<^p$.

Conversely, let $B_p(\theta, \epsilon)$ be an open ball of p. For ϵ sufficiently small, there exists $r \times l <^p \theta <^p r' \times l'$ such that $p(r \times l, \theta) = \epsilon = p(\theta, r' \times l')$. In this case, $B_p(\theta, \epsilon) = (r \times l, r' \times l')_p$. The case where ϵ is large is similar and omitted.

For the pseudoisometry, let f be the map $\theta \mapsto f(\theta) \stackrel{\text{def}}{=} p(\theta, 0 \times 0)$. Clearly, the image of f is $[0, r^c] \cup [r^c + 1, 2] \subset \mathbb{R}$. Take any $r \times l, r' \times l'$ in S_p. If $r, r' \leqslant r^c$ or $r^c \leqslant r, r'$, then straightforward substitution confirms that $|f(r \times l) - f(r' \times l')| = |r - r'|$, as required. If $r \leqslant r^c < r'$, then $|f(r \times l) - f(r' \times l')| = |r - (r' + 1)|$, which equals $|r - r'| + 1$, as required. $\square$

Proof of proposition 6 of page 26. Consider the case where Θ has just a single factor. Since preferences are convex-cylindrical, $\{\theta : a \prec_\theta b\}$ is a finite union of nonempty order intervals. Let G be one such order interval, so that G is of the form $[\theta', \theta'']$, $[\theta', \theta'')$, $(\theta', \theta'']$ or (θ', θ''). Since Θ is first countable, there exists a countable collection $\{N_n\}$ of open neighbourhoods of θ' such that, if N is a neighbourhood of θ', then $N_n \subseteq N$ for some n. Since Θ is linearly ordered, it is Hausdorff and $\{\theta'\} = \bigcap_1^\infty N_n$. Let $F_n = G \cap (\Theta - N_n)$ for each n. Since N_n is open, F_n is a closed relative to G and the union over the F_n generates an increasing sequence of sets with union G. Clearly, the argument can be repeated for θ'' and for each interval in the union $\{\theta : a \prec_\theta b\} = \bigcup_{k=1}^m G_k$. Finally, $\mathcal{PS}^*$ follows from $\mathcal{PS}$.

The case where Θ has two factors is similar. F_n is then an intersection of inverse projections $\pi_i^{-1}(F_n^i) \cap \pi_j^{-1}(F_n^j)$, where $\{(F_n^i, F_n^j) : n \in \mathbb{N}\}$ are the sequences of closed subsets of $\pi_i(G)$ and $\pi_j(G)$ respectively. The extension to finitely many factors is similar and omitted. Since the finite union of cylinder sets is a cylinder set, the extension to arbitrarily many factors follows. The extension to a countable union of cylinder sets follows by a diagonalisation argument (similar to the one we use for the proof of theorem 1). □

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