A note on Jackson's theorems in Bayesian implementation

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1997

A NOTE ON JACKSON'S THEOREMS IN BAYESIAN IMPLEMENTATION

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This paper shows that in an incomplete information situation if the set of states of the society which occur with positive probability satisfies 'connection' condition, then closure condition will be satisfied by all social choice sets. It then follows from Jackson's (1991) two fundamental theorems that whenever 'connection' holds and there are at least three agents in the society, for the implementability of social choice sets in Bayesian equilibrium, incentive compatibility and Bayesian monotonicity conditions are both necessary and sufficient in economic environments wheras incentive compatibility and monotonicity-no-veto conditions are sufficient in noneconomic environments.

KEYWORDS: Bayesian implementation, incomplete information.

1. INTRODUCTION

IN HIS SEMINAL PAPER, Jackson (1991) examined the problem of implementing collections of social choice functions in situations where agents have incomplete information about the state of the society. His work has very important features; he characterized conditions for implementability not only in economic but also in noneconomic environments, both of which admit situations with externalities. The economic environments he considered is much more general than exchange economies, as the former cover any environment in which agents cannot be simultaneously satisfied. Moreover, the existence

[1]I am grateful to Matthew O. Jackson for his important suggestions which have greatly improved this paper. I thank Ennio Stacchetti and Tayfun Sönmez for a helpful conversation. This work has been done under the support of the Scientific and Technical Research Council of Turkey, the Economics Department of University of Michigan, the Economics Department of Bilkent University and the Center for Economic Design of Bosphorus University. Any errors are soleley mine.

of a worst outcome from the viewpoint of all agents in the society is not needed for his theorems characterizing implementable social choice sets. Regarding the distribution of information among the members of a society, he allowed for situations where agents possess exclusive information. Besides, the set of states which occur with positive probability is not necessarily required to coincide with the set of all possible states of the society.

His first theorem showed that a collection of social choice functions in an economic environment with at least three agents is Bayesian implementable if and only if closure (C), incentive compatibility (IC), and Bayesian monotonicity (BM) conditions are satisfied. As he stated, this result closed the gap between the necessary and sufficient conditions[2] of Palfrey and Srivastava (1989b), who examined[3] implementation for exchange economies in which agents may have exclusive information.

The second theorem of Jackson (1991) showed that closure, incentive compatibility, and a condition that combines Bayesian monotonicity and no-veto power (which he calls (MNV)) are sufficient for implementation in noneconomic environments with three or more agents.

This paper examines the situations in which the closure condition is satisfied by any collection of social choice rules in both economic and noneconomic environments. It is shown that when the set of states on whose occurance with positive probability all agents in the society agree satisfies a condition called 'connection', closure condition is automatically satisfied by all collection of social choice functions in both economic and noneconomic environments, leading to the result that under such circumstances the designer should pay attention to only (IC) and (BM) in economic environments and (IC) and (MNV) in noneconomic environments for implementation.

The paper is organized as follows: Section 2 reintroduces the environment and preliminary definitions of Jackson (1991). Section 3 presents Jackson's results in Bayesian implementation. Finally, Section 4 gathers the results of this paper and some concluding remarks.

[2]Palfrey and Srivastava (1989b) showed that a collection of social choice rules is implementable in Bayesian (Nash) equilibrium if it satisfies Bayesian monotonicity and incentive compatibility conditions. Moreover, they showed that Bayesian monotonicity and a stronger incetive compatibility condition (ε-IC) are sufficient for implementation.

[3]See also Palfrey and Srivastava (1987), and Postlewaite and Schmeidler (1986) for Bayesian implementation results in exchange economies where there are at least three agents and the information is nonexclusive.

2. BASIC STRUCTURES (JACKSON (1991))

Environments

There are a finite number, N, of agents. Let S^i describe the finite number of possible information sets of agent i. A state is a vector $S = (s^1, ..., s^N)$ and the set of states is $S = \prod_{i=1}^N S^i$.

Let A denote the set of feasible allocations. A social choice function is a function from states to allocations. The set of all social choice functions is $X = \{x | x : S \to A\}$.

Each agent has a probability measure q^i defined on S. It is assumed that if $q^i(s) > 0$ for some i and $s \in S$, then $q^j(s) > 0$ for all $j \neq i$. Let T denote the set of states which occur with positive probability, that is $T = \{s \in S | q^i(s) > 0,\ \forall i\}$.

Π^i are partitions of T defined by q^i. For a given information set $s^i \in S^i$, $\pi^i(s^i) = \{t \in S | t^i = s^i \text{ and } q^i(t) > 0\} \in \Pi^i$ denotes the set of states which i believes may be the true state. It is assumed that $\pi^i(s^i) \neq \emptyset$ for all i and $s^i \in S^i$. Let Π denote the finest partition which is coarser than each Π^i. For a given state $s \in S$, let $\pi(s)$ be the element of Π which contains s.

Each agent has preferences $U^i : A \times S \to \mathbb{R}_+$ over social choice functions which have a conditional expected utility representation. Given $x, y \in X$ and $s^i \in S^i$, agent i's weak preference relation R^i is such that

$$xR^i(s^i)y \Leftrightarrow \sum_{s \in \pi^i(s^i)} q^i(s)U^i[x(s), s] \geqslant \sum_{s \in \pi^i(s^i)} q^i(s)U^i[y(s), s].$$

Preferences are complete and transitive. The strict preference and indifference relations associated with R^i are P^i and I^i, respectively.

An environment is a collection $[N, S, A, \{q^i\}, \{U^i\}]$, whose structure is assumed to be common knowledge among the agents.

Definitions

DEFINITION 1: Given a vector or vector of functions $v = (v^1, ..., v^N)$, let $(v^{-i}, \tilde{v}^i)$ represent the vector $(v^1, ..., v^{i-1}, \tilde{v}^i, v^{i+1}, ..., v^N)$.

DEFINITION 2: A *social choice set* is a subset of X.

DEFINITION 3: The social choice functions x and y are *equivalent* if $x(s) = y(s)$ for all $s \in T$. The social choice sets F and $\hat{F}$ are *equivalent* if for each $x \in F$ there exists $\hat{x} \in \hat{F}$ which is equivalent to x, and for each $\hat{x} \in \hat{F}$ there exists $x \in F$ which is equivalent to $\hat{x}$.

DEFINITION 4: Let $x/_C z$ be a splicing of two social choice functions x and z along a set $C \in S$. The social choice function $x/_C z$ is defined by $[x/_C z](s) = x(s)\ \forall s \in C$, and $[x/_C z](s) = z(s)$ otherwise. An environment satisfies (E) if for any $z \in X$ and $s \in S$, there exist i and j $(i \neq j)$, $x \in X$ and $y \in X$ such that x and y are constant, $x/_C z P^i(s^i)z$ and $y/_C z P^j(s^j)z$ for all $C \subset S$ such that $s \in C$. Environments satisfying (E) are said to be *economic*.

DEFINITION 5: Let B and D be any disjoint sets of states such that $B \cup D = T$ and for any $\pi \in \Pi$ either $\pi \subset B$ or $\pi \subset D$. A social choice set F satisfies *closure* (C) if for any $x \in F$ and $y \in F$ there exists $z \in F$ such that $z(s) = x(s)\ \forall s \in B$ and $z(s) = y(s)\ \forall s \in D$.

DEFINITION 6: Given i, $x \in X$, and $t^i \in S^i$, define x_{t^i} by $x_{t^i}(s) = x(s^{-i}, t^i)$, $s \in S$. A social choice set F satisfies *incentive compatibility* (IC) if for all $x \in F$, i, and $t^i \in S^i$,

$$xR^i(s^i)x_{t^i} \quad \forall s^i \in S^i.$$

DEFINITION 7: A *deception* for i is a mapping $\alpha^i : S^i \to S^i$. Let $\alpha = (\alpha^1, \ldots, \alpha^N)$ and $\alpha(s) = [\alpha^1(s^1), \ldots, \alpha^N(s^N)]$. Let $x \circ \alpha$ represent the social choice function which results in $x[\alpha(s)]$ for each $s \in S$.

DEFINITION 8: Consider $x \in F$ and a deception α. A social choice set f satisfies *Bayesian montonicity* if whenever there is no social choice function in F which is equivalent to $x \circ \alpha$, there exists $i, s^i \in S^i$ and $y \in X$ such that

$$y \circ \alpha\, P^i(s^i)\, x \circ \alpha, \quad \text{while} \quad x\, R^i(t^i)\, y_{\alpha^i(s^i)} \quad \forall t^i \in S^i.$$

DEFINITION 9: A social choice function $z \in X$ satisfies the *no-veto hypothesis* (NVH) at $s \in T$ if there exists i such that $zR^j(s^j)b^j/_S z$ for all $j \neq i$.

DEFINITION 10: Consider the social choice set F, a deception α, and for each $x \in F$ and i consider a set $B_x^i \subset S^i$. Let $B_x = B_x^1 \times ... \times B_x^N$. Suppose that there exists z such that for each $x \in F$ and $s \in B_x$, $z(s) = x \circ \alpha(s)$. Furthermore, suppose that z satisfies (NVH) for each $s \in T - (U_{x \in F} B_x)$. F satisfies *monotonicity-no-veto* (MNV) if whenever there is no social choice function in F which is equivalent to z, there exists i, $y \in X$, $x \in F$, and $s^i \in B_x^i$ such that

$$y \circ \alpha /_{B_x} z \, P^i(s) \, z, \quad \text{while} \;\; x \, R^i(t^i) \, y_{\alpha^i(s^i)} \;\; \forall t^i \in S^i.$$

DEFINITION 11: An environment is said to have a "0" *outcome* if there exists a $0 \in A$ such that $U^i(0, s) = 0$ for all i and $s \in T$, and for each $s \in T$ and $a \neq 0$ there exists i such that $U^i(a, s) > 0$. In such an environment, given $x \in X$, let x^0 denote the allocation such that $x^0(s) = x(s)$ for $s \in T$ and $x^0(s) = 0$ otherwise. Given a social choice set F, let F^0 be the social choice set which is equivalent to F and such that $x = x^0$ for all $x \in F^0$.

Implementation

A mechanism is a pair consisting of an action space $M = \prod_{i=1}^N M^i$ and a function $g : M \to A$.

A strategy for agent i is a mapping $\sigma^i : S^i \to M^i$. Let $\sigma = [\sigma^1, ..., \sigma^N]$ and $\sigma(s) = (\sigma^1(s^1), ..., \sigma^N(s^N))$ and $g(\sigma)$ be the allocation which results when σ is played.

A vector of strategies σ is a Bayesian (Nash) equilibrium if $g(\sigma) R^i(s^i)$ $g(\sigma^{-i}, \tilde{\sigma}^i)$ for all i, s^i, and $\tilde{\sigma}^i$.

A mechanism (M, g) implements a social choice set F if:

(i) for any $x \in F$ there exists an equilibrium σ with $g[\sigma(s)] = x(s)$ for all $s \in T$, and

(ii) for any equilibrium σ there exists $x \in F$ with $g[\sigma(s)] = x(s)$ for all $s \in T$.

A social choice set F is implementable if there exists a mechanism (M, g) which implements F.

3. IMPLEMENTATION RESULTS OF JACKSON (1991)

THEOREM 1: (Jackson (1991)) *In an environment which satisfies (E) and* $N \geq 3$*, a social choice set* F *is implementable if and only if there exists a social choice set* $\hat{F}$ *which is equivalent to* F *and satisfies (C), (IC), and (BM).*

COROLLARY 1: (Jackson (1991)) *In an environment which satisfies (E),* $S = T$*, and* $N \geq 3$*, a social choice set* F *is implementable if and only if it satisfies (C), (IC), and (BM).*

COROLLARY 2: (Jackson (1991)) *In an environment which satisfies (E) and* $N \geq 3$*, and has a 0 outcome, a social choice set* F *is implementable if and only if* F^0 *satisfies (C), (IC), and (BM).*

THEOREM 2: (Jackson (1991)) *If* $N \geq 3$*, social choice set* F *which satisfies (C), (IC), and (MNV), is implementable.*

4. RESULTS

DEFINITION 12: A set of states $T \subseteq S$ satisfies *connection* (CO) condition if for all $s_a \in T$ and $s_b \in T$ there exists a string of states $s_a \equiv s_0, s_1, ..., s_r \equiv s_b$ such that for all $k \in \{0, ..., r-1\}$ there exists an agent $i(k)$ satisfying $s_k^{i(k)} = s_{k+1}^{i(k)}$.

LEMMA 1: *An environment satisfies (CO) if*[4] *and only if* $\Pi = \{T\}$.

PROOF : We will first show that $\Pi = \{T\}$ implies (CO). Take any environment such that $\Pi = \{T\}$. Suppose towards a contradiction that (CO) does not hold. Then there exists some $s_a \in T$ and $s_b \in T$ such that there exists no string of states $s_a \equiv s_0, s_1, ..., s_r \equiv s_b$ satisfying that for all $k \in \{0, ..., r-1\}$ there exists some agent $i(k)$ such that $s_k^{i(k)} = s_{k+1}^{i(k)}$. Now consider $\pi(s_a)$. We have $\pi(s_a) = T$ since $\Pi = \{T\}$. We also have $s_b \notin \pi(s_a)$ since (CO) does

[4]The 'if part' of the Lemma 1 as well as the need for (CO) condition for an iff statement were proposed by Matthew Jackson, for which the author is grateful. The previous version of Lemma 1 was just an 'only if' statement which stated that a condition called no-separation, which is stronger than (CO), implies $\Pi = \{T\}$.

not hold, contradicting that $s_b \in T$. Therefore in any environment where $\Pi = \{T\}$, (CO) must hold.

To show the sufficiency part, assume (CO) is satisfied. Take any $\tilde{s} \in T$ and $s \in T$. Since (CO) holds by assumption, there exists a string of states $\tilde{s} \equiv s_0, s_1, ..., s_r \equiv s$ such that for all $k \in \{0, ..., r-1\}$ there exists $i(k)$ such that $s_k^{i(k)} = s_{k+1}^{i(k)}$. Thus, $s \in \pi(\tilde{s})$. Since this is true for all $s \in T$, we have $T \subseteq \pi(\tilde{s})$. We also have $\pi(\tilde{s}) \subseteq T$ (by the suppositions that $\pi(s) \in \Pi$ and Π is a partition of T). It then follows that $\pi(\tilde{s}) = T$. Therefore, $\Pi = \{T\}$. Q.E.D.

Note that S satisfies (CO) condition since for all $s_a \in S$ and $s_b \in S$ the string $s_a \equiv s_0, s_1, s_2 \equiv s_b$ with $s_1 = (s_a^{-i}, s_b^i)$ for some agent i connects s_a to s_b. (Note s_1 is an element of S as $S = S^1 \times ... \times S^N$). See Example 1 of Jackson (1991) for an example of $T \subset S$ satisfying (CO) condition.

PROPOSITION 1: *In an environment which satisfies connection, all social choice sets satisfy closure.*

PROOF: Take any environment which satisfies (CO). Then $\Pi = \{T\}$ by Lemma 1. Let K be defined as

$$K = \{(B, D) | B \cap D = \emptyset,\ (B \cup D) = T \text{ and } \forall \pi \in \Pi, \pi \in B \text{ or } \pi \in D\}.$$

It is obvious that whenever $\Pi = \{T\}$, Π has the single element $\pi = T$. Thus we have

$$K = \{(T, \emptyset), (\emptyset, T)\}.$$

Any social choice set F then satisfies closure since for any $x \in F$ and $y \in F$, we have a social choice function $z \in F$ given by

$$z = \begin{cases} x & \text{if } B = T \\ y & \text{if } D = T \end{cases}$$

which guarantees that $z(s) = x(s)\ \forall s \in B$ and $z(s) = y(s)\ \forall s \in D$. Q.E.D.

We can now obtain a corollary for Theorem 1 of Jackson (1991).

COROLLARY 3: *In an environment which satisfies (E), (CO) and $N \geq 3$, a social choice set F is implementable if and only if there exists a social* *choice set $\hat{F}$ which is equivalent to F and satisfies (IC), and (BM).*

PROOF: Corollary 3 follows from Theorem 1 of Jackson (1991) and Proposition 1. Q.E.D.

Since S satisfies (CO) condition, we can restate the Corollary 1 of Jackson (1991) as follows:

COROLLARY 4 : *In an environment which satisfies (E), $T = S$, and $N \geq 3$, a social choice set F is implementable if and only if it satisfies (IC) and (BM).*

PROOF : When $T = S$, the collection of social choice sets which are equivalent to F coincides with F. From Corollary 3 it then follows that social choice set F is implementable if and only if it satisfies (IC) and (BM). Q.E.D.

As a special case of Corollary 2 of Jackson (1991), we obtain the following result.

COROLLARY 5: *In an environment which satisfies (E), (CO), $N \geq 3$, and has a 0 outcome, a social choice set F is implementable if and only if F^0 satisfies (IC), and (BM).*

PROOF: Corollary 5 follows from Corollary 2 of Jackson (1991) and Proposition 1. Q.E.D.

The (CO) condition has an implication on Theorem 2 of Jackson (1991), as well.

COROLLARY 6: *If $N \geq 3$, and (CO) holds, social choice set which satisfies (IC), and (MNV), is implementable.*

PROOF: Corollary 6 follows from Theorem 2 of Jackson (1991) and Proposition 1. Q.E.D.

It may be of an interest to know the ratio of the number of the sets of states which satisfies (CO) condition to the number of all possible sets of states in an environment with an infinite number of agents or states, as it allows one to better understand how restictive (CO) condition may be. For the simplification of this calculation, the following condition, which is much stronger than (CO), is quite helpful.

DEFINITION 13: A set of states $T \subseteq S$ satisfies *no-separation* (NS) condition if for all $s_a \in T$ and $s_b \in T$ there exists some $s_c \in T$ and some agents k, l such that $s_c^k = s_a^k$ and $s_c^l = s_b^l$.

We note that if an environment satisfies (NS) then it also satisfies (CO). But the converse is not necessarily true for all environments.[5]

PROPOSITION 2: *Let $|S_i|$ denote the cardinality of the set of states S_i and be*[6] *equal to $p \geq 2$ for all i. Let $r^{co}(p, N)$ denote the ratio of the number of all possible sets of states which does not satisfy (CO) condition to the number of all possible sets of states. Then*

$$\lim_{N\to\infty} r^{co}(p, N) = 0 \quad \textit{and} \quad \lim_{p\to\infty} r^{co}(p, N) = 0.$$

PROOF: Let $r^{ns}(N, p)$ denote the ratio of the number of all possible sets of states which does not satisfy (NS) to the number of all possible sets of states, given p and N. Since (NS) implies (CO), it is clear that $r^{co}(p, N) \leq r^{ns}(p, N)$ for all p and N.

Take any $\hat{s} \in S$. Let $D^{\hat{s}}$ represent the set $\{s \in S | s^i \neq \hat{s}^i, \ \forall i\}$. Note that $|D^{\hat{s}}| = (p-1)^N$ for all $\hat{s} \in S$. Let $E^{\hat{s}}$ denote the set $\{G \cup \{\hat{s}\} | G \subseteq D^{\hat{s}}$ and $G \neq \emptyset\}$. We have $|E^{\hat{s}}| = 2^{(p-1)^N} - 1$. Now consider the set $H = \{E^{\hat{s}} | \hat{s} \in S\}$. H includes all nonempty subsets of S which does not satisfy (NS) condition. We have $|H| = [2^{(p-1)^N} - 1]p^N$. The cardinality of the set of all possible nonempty subsets of S is equal to $2^{p^N} - 1$. Note that for all $s_a \in S$ and

[5] To see (CO) does not imply (NS) in all environments, consider the following example proposed by Matthew O. Jackson: $\Pi^1 = \{\{s_a\}, \{s_c, s_d\}, \{s_e, s_f\}, \{s_b\}\}$, $\Pi^2 = \{\{s_a, s_c\}, \{s_d, s_e\}, \{s_f, s_b\}\}$. One can connect s_a to s_d, s_d to s_f, and s_f to s_b; so (CO) is satisfied. But, there is not a single state which connects s_a to s_b, which implies that (NS) does not hold.

[6] Note when $|S_i| = 1$ for all i, that is when information is common knowledge, we have $|S| = 1$, and thus S satisfies (CO) condition regardless what the number of agents is.

$s_b \in S$ such that $s_a^i \neq s_b^i\ \forall i$, we have $\{s_a, s_b\} \in E^{s_a} \cap E^{s_b}$, therefore we must have

$$r^{co}(p,N) < \frac{[2^{(p-1)^N} - 1]p^N}{2^{p^N} - 1},$$

which implies that

$$r^{co}(p,N) < \frac{2^{(p-1)^N} p^N}{2^{p^N} - 1}.$$

Define $Y(p,N) = 2^{(p-1)^N} p^N / (2^{p^N} - 1)$. We note that $\lim_{N\to\infty} Y(p,N) = 0$ and $\lim_{p\to\infty} Y(p,N) = 0$, which completes the proof. Q.E.D.

Even though Proposition 2 does not cover situations where S_i may differ across agents, it, nevertheless, helps to make a conjecture that when either the number of possible states or the number of agents in the society is sufficiently high, the probability of the event that the set of states on whose occurence with a positive measure all agents in the society agree satisfies (CO) condition will be 'almost' one, provided, of course, that all sets of states are equally likely to occur. This observation together with the result that (CO) implies closure, strengthens the fundamental theorems of Jackson (1991) as in most environments where there exist sufficiently large number of agents or states, the closure may not be binding at all in Bayesian implementation.

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