Utilitarianism with and without expected utility

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UTILITARIANISM WITH AND WITHOUT EXPECTED UTILITY

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Abstract. We give two social aggregation theorems under conditions of risk, one for constant population cases, the other an extension to variable populations. Intra and interpersonal comparisons are encoded in a single 'individual preorder'. The individual preorder then uniquely determines the social preorder. The theorems have features that may be considered characteristic of Harsanyi-style utilitarianism, such as indifference to *ex ante* and *ex post* equality. If in addition the individual preorder satisfies expected utility, the social preorder must be represented by expected total utility. In the constant population case, this is the conclusion of the social aggregation theorem of Harsanyi [63] under anonymity, but *contra* Harsanyi, it is derived without assuming expected utility at the social level. However, the theorems are also consistent with the rejection of all of the expected utility axioms, at both the individual and social levels. Thus expected utility is inessential to Harsanyi's approach under anonymity. In fact, the variable population theorem imposes only a mild constraint on the individual preorder, while the constant population theorem imposes no constraint at all. We therefore give further results related to additional constraints on the individual preorder. First, stronger utilitarian-friendly assumptions, like Pareto or strong separability, are essentially equivalent to the main expected utility axiom of strong independence. Second, the individual preorder satisfies strong independence if and only if the social preorder has a mixture-preserving total utility representation; here the utility values can be taken as vectors in a preordered vector space, or more concretely as lexicographically ordered matrices of real numbers. Third, if the individual preorder satisfies a 'local expected utility' condition popular in nonexpected utility theory, then the social preorder is 'locally utilitarian'.

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Introduction

The central results of this article are two social aggregation theorems under conditions of risk, one for the constant population case, the other an extension to variable populations. These were prompted by a puzzle about Harsanyi's [63] celebrated social aggregation theorem. Following Harsanyi [64, 65] and many others, we take interpersonal and intrapersonal comparisons to be encoded in a single preorder, which we will call the individual preorder. What we will call Harsanyi's anonymous theorem shows that in the constant population case, if the individual and social preorders satisfy the expected utility axioms,[1] strong Pareto is satisfied, and anonymity is imposed, then the social preorder is uniquely determined by

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[1]For notational convenience, we take the expected utility axioms to apply to preorders, so that preordering is not itself taken to be an expected utility axiom.

the individual preorder.[2] In particular, it is represented by the expected sum of individual utilities.

It is easy to get the impression from Harsanyi that his theorem is somehow all about expected utility. But the axioms of expected utility theory are quite limiting, and their significance for his theory is not entirely clear. They are most controversial for the social preorder, but even for the individual preorder, and even at the normative level, each of the axioms has been denied. The completeness axiom implies that all individual goods are comparable, but this has frequently been challenged. Loosely speaking, continuity axioms deny that some individual goods are infinitely more valuable than others, but this is sometimes said to be a merely technical assumption, and can be explicitly rejected in cases involving extreme outcomes such as torture or death. The independence axiom also has its critics at the normative level; some, for example, maintain that the paradoxical preference structure in the Allais paradox is rational.[3]

The puzzle therefore arises as to what extent the limitations of expected utility theory can be avoided while following an approach to social aggregation that is somehow still in the spirit of Harsanyi. In particular, to what extent can one leave out the premise of expected utility theory while retaining the conclusion that the individual preorder uniquely determines the social preorder?

Each of the expected utility axioms has also been challenged at the positive level. There are well-documented violations of completeness and independence, and continuity has often been regarded as difficult to test for, raising a doubt about including it with other axioms in positive theories.[4] This prompts the question of to what extent subjects' social judgments reflect their expected-utility-violating judgments about individual risk. But to approach this question, we first need a conceptual connection between the two levels of judgment, providing another reason to ask whether Harsanyi's approach can be preserved while relaxing expected utility.

Finally, even if one is confident in expected utility, it is still of considerable theoretical interest to ask to what extent Harsanyi's approach has to rely upon it, particularly given the central role contrasts with utilitarianism have played in the development of other distributive theories.[5]

Section 1 presents our constant population aggregation theorem, Theorem 1.3.1. It shows that three plausible and relatively weak premises, which we call Anteriority, Two-Stage Anonymity, and Reduction to Prospects, are sufficient for the individual preorder to uniquely determine the social preorder, with no restrictions at all on the individual preorder. In particular, the expected utility axioms can *all* fail at both the individual and the social levels.

Section 2 extends this result to the variable population case in Theorem 2.3.1. Here, the individual preorder is extended to deal with comparisons between lotteries involving nonexistence as possible states for an individual. Our constant population axioms extend to the variable case more or less trivially. When combined with a mild restriction on welfare comparisons involving nonexistence which we call Omega

[2]Harsanyi often stated this result informally, but for an explicit statement, see e.g. Harsanyi [65, §4.1]. For a relatively simple proof, use Proposition 1 of De Meyer and Mongin [41] to obtain the conclusion of Harsanyi's original theorem, then impose anonymity to obtain the result.

[3]For entries into a vast literature, see e.g. Pivato [89] and Dubra, Maccheroni and Ok [47] (completeness as deniable); Richter [92] and Fishburn [51] (continuity as a technical assumption); Luce and Raiffa [73] and Kreps [72] (continuity as deniable); Allais [2] and Buchak [28] (independence as deniable).

[4]See, for example, Starmer [103], Schmidt [97] and Wakker [106].

[5]We understand 'utilitarianism' in the spirit of a utilitarian interpretation of Harsanyi's anonymous theorem. But we offer no argument for this. Those who dispute this interpretation, like Sen [99, 100] and Weymark [108], can replace 'utilitarianism' with 'Harsanyi-style utilitarianism'.

Independence, the individual preorder still uniquely determines the social preorder, and the expected utility axioms can still all fail at both levels.

Given that the individual preorder is completely unconstrained in the constant population case, and almost unconstrained in the variable case, much of the paper focuses on consequences for the social preorder of imposing normatively or positively natural conditions on the individual preorder. Section 1.5 illustrates how standard utilitarianism, leximin, incomplete and rank-dependent social preorders emerge from natural choices for the individual preorder. In the variable population case, Section 2.5 and 2.6 show how critical level utilitarianism, different forms of average utilitarianism, and various ways of avoiding the Repugnant Conclusion of Parfit [88] emerge from natural choices for the individual preorder, including different philosophical treatments of the value of existence. These examples foreshadow general results under the assumptions of our aggregation theorems.

Section 3 focuses on imposing expected utility conditions on the individual preorder, and gives four types of results. First, Proposition 3.1.1 shows that if the individual preorder satisfies any of a wide range of axioms associated with expected utility theory, then so does the social preorder.

Second, Proposition 3.2.1 shows that various Pareto, separability, and independence axioms are essentially equivalent. For example, although it seems to have gone unmentioned in the literature, what appears to be the strongest plausible Pareto condition turns out to be essentially equivalent to the strongest and best known independence condition, strong independence, as well as to a strong separability condition. Once one has imposed strong independence on the individual preorder, it is therefore redundant to impose any of strong independence or the corresponding Pareto and separability conditions on the social preorder.

Third, Proposition 3.3.1 shows that if the individual preorder has an expected utility integral representation, then the social preorder has a total expected utility or expected total utility representation. In its simplest form, this provides a derivation of the conclusion of Harsanyi's anonymous theorem from premises which are vastly weaker than Harsanyi's. But the general form of expected utility representation we consider takes values in (typically infinite-dimensional) preordered vector spaces. As we explain, preordered vector spaces are particularly suitable for understanding generalized total utility representations. The advantage here is that we can provide total expected utility representations while allowing for failures of both continuity and completeness.

Fourth, Proposition 3.4.2, the main result of section 3, is that the individual preorder is strongly independent if and only if the social preorder has a Harsanyi-like total utility mixture-preserving representation into a preordered vector space. Proposition 3.4.4 then shows that this representation can be expressed as a Harsanyi-like total utility lexicographic representation involving a matrix of mixture-preserving real-valued functions. The rows and columns of this matrix are generally infinite. But when in addition the individual preorder is complete, the matrix can be taken to be a single row matrix. When the individual preorder is mixture continuous, it can be taken to be a single column matrix. When the individual preorder is complete and mixture continuous, it can be given a single entry.

Section 4 explores analogous issues for nonexpected utility theory, again under the assumptions of our aggregation theorems. We first note that if the individual preorder satisfies any of a wide range of nonexpected utility axioms, then so does the social preorder. But this is not universal. For example, monotonicity, or respect for stochastic dominance, has been seen as a very weak and plausible condition, and some popular nonexpected utility theories have been built around it. Nevertheless, Example 4.1.2 shows that imposing many of these theories on the individual preorder leads to the forced rejection of monotonicity for the social preorder. In fact, monotonicity is a demanding condition at the social level. Propositions 4.2.2 and 4.2.4 then show that when the individual preorder satisfies a 'local expected utility' condition in the style of Machina [74], (specifically: representability by a Gâteaux differentiable function), then the social preorder is a form of 'local utilitarianism'. This forms part of our case that while social preorders generated by strongly independent individual preorders should be seen as utilitarian, social preorders generated by any individual preorder under the conditions of our aggregation theorems are appropriately described as 'generalized utilitarian', although we use this term differently from some of the literature. We call those which violate strong independence 'nonexpected utilitarian'.

Section 5 relates our aggregation theorems to some standard topics. Nonexpected utilitarian social preorders form an important but apparently undiscussed class, and section 5.1 contrasts them with egalitarian social preorders. At the level of outcomes, nonexpected utilitarian preorders can coincide with all kinds of egalitarian preorders. Nevertheless, there is a sharp contrast at the level of risk. All generalized utilitarian preorders are in a precise sense indifferent to both *ex ante* and *ex post* equality. But the fact that nonexpected utilitarian preorders can appear egalitarian at the level of outcomes suggests that many apparently egalitarian social judgments may actually be expressions of independence-violating attitudes to individual risk. This suggests directions for empirical work. Section 5.2 reconsiders the *ex ante* versus *ex post* distinction (at least in the framework of risk) and suggest a simplification which is detached from expected utility theory. Generalized utilitarian social preorders are then those which, in a natural and precise sense, are weakly *ex ante* and anonymously *ex post*. Section 5.3 discusses the disputed role of interpersonal comparisons in Harsanyi's approach to social aggregation. By comparing our results with those of Pivato [89], we argue that our approach solves a crucial difficulty in Harsanyi's. Section 5.4 ends by concluding that while the present work amplifies several of Harsanyi's insights, the core of his anonymous theorem turns out to have nothing to do with expected utility theory. Most proofs are in the appendix.

From a technical point of view, we make several contributions to utility theory, expected utility theory in particular. First, the general form of expected utility representations we consider in section 3.3 uses the weak integral rather than the ordinary (Lebesgue) integral. As far as we know, the weak integral has been little used in expected utility theory. But this approach allows us to state expected-utility-style integral representations with a single utility function into a preordered vector space without having to assume either continuity or completeness. It allows us to subsume as a special case the common representation of incomplete but strongly independent and continuous preorders on probability measures via families of real-valued expected utility functions. Second, Theorem 2.4.2 shows that any preorder on a convex set, and *a fortiori* a convex set of probability measures, can be represented by a single function into a preordered vector space. Third, Theorem 3.4.1 shows that any strongly independent preorder on a convex set can be represented by a single mixture-preserving function into a preordered vector space. Fourth, Theorem 3.4.3 shows that in the latter case, the preorder can also be given a lexicographic mixture-preserving representation involving a matrix of real-valued functions. This last result rests on a fundamental structure theorem for preordered vector spaces. To situate this, the best known result for complete, strongly independent preorders of probability measures in expected utility theory is Hausner [68]. This rests on a structure theorem for ordered vector spaces given in Hausner and Wendel [69]. By contrast, our Theorem 3.4.5 applies to more general preordered vector spaces, allowing for incompleteness, and unlike theirs, our representation is essentially unique and constructively defined.

1. A Constant Population Aggregation Theorem

1.1. **Framework.** We are ultimately concerned with the social ranking of lotteries over outcomes in which individuals have particular lives. But given the view that social welfare depends only on information about individual welfare (see e.g. Sen [101, 102]), we need only individuate outcomes in terms of each individual's welfare level. Given a set of lives and an indifference relation 'exactly as good a life as', we could think of the set of welfare levels as the partition of the set of lives under the indifference relation. We take no view about what constitutes welfare, so our discussion will compatible with all the usual views, including preference satisfaction, hedonistic, and more objective accounts.

Formally, however, our basic framework starts with a set $\mathbb{W}$ of welfare levels, and a finite, nonempty set $\mathbb{I}$ of individuals. We also assume given a set $\mathbb{H}$ of *histories*. A 'history' (or welfare distribution) here is an assignment of welfare levels to individuals; we write $\mathcal{W}_i(h)$ for the welfare level that individual i has in history h. We could take $\mathbb{H}$ to contain all logically possible histories, and then $\mathbb{H}$ would equal $\mathbb{W}^{\mathbb{I}}$, the product of copies of $\mathbb{W}$ indexed by $\mathbb{I}$. But we can allow $\mathbb{H}$ to be any subset of $\mathbb{W}^{\mathbb{I}}$ that satisfies certain conditions shortly to be announced.

Besides welfare levels and histories *per se*, we consider probability measures over them. Thus we assume that $\mathbb{W}$ and $\mathbb{H}$ are measurable spaces. We call probability measures over $\mathbb{W}$ *prospects*, and those over $\mathbb{H}$ *lotteries*. Notationally, if P is (say) a prospect and U is a measurable subset of $\mathbb{W}$, then we write $P(U)$ for the probability that P assigns to U. Instead of just considering all prospects and all lotteries, we will, for generality, focus on arbitrary non-empty convex sets $\mathbb{P}$ and $\mathbb{L}$ of prospects and lotteries respectively.

Here is what we will assume about the finite set $\mathbb{I}$, the measurable spaces $\mathbb{W}$ and $\mathbb{H} \subset \mathbb{W}^{\mathbb{I}}$, and the convex sets of probability measures $\mathbb{P}$ and $\mathbb{L}$.

(A). First, we assume that for each individual $i \in \mathbb{I}$ the projection $\mathcal{W}_i \colon \mathbb{H} \to \mathbb{W}$ is a measurable function. This allows us to define a prospect $\mathcal{P}_i(L)$ for each lottery L. Explicitly, if U is a measurable subset of $\mathbb{W}$, then

$$\mathcal{P}_i(L)(U) = L(\mathcal{W}_i^{-1}(U)).$$

We further assume that $\mathcal{P}_i(\mathbb{L}) \subset \mathbb{P}$.

(B). Second, for each $w \in \mathbb{W}$, we assume that that $\mathbb{H}$ contains the history $\mathcal{H}(w)$ in which every individual $i \in \mathbb{I}$ has welfare w. Thus

$$\mathcal{W}_i(\mathcal{H}(w)) = w.$$

We further assume that the function $\mathcal{H} \colon \mathbb{W} \to \mathbb{H}$ is measurable. This allows us to define a lottery $\mathcal{L}(P)$ for each prospect P. Explicitly, if V is a measurable subset of $\mathbb{H}$, then

$$\mathcal{L}(P)(V) = P(\mathcal{H}^{-1}(V)).$$

In $\mathcal{L}(P)$, every individual $i \in \mathbb{I}$ faces prospect P ($\mathcal{P}_i(\mathcal{L}(P)) = P$), and it is certain that all individuals have the same welfare.[6]

We further assume that $\mathcal{L}(\mathbb{P}) \subset \mathbb{L}$.

[6] This second statement just means that any measurable subset of $\mathbb{H}$ containing the image of $\mathcal{H}$ has probability 1 according to $\mathcal{L}(P)$. We do not assume that the image of $\mathcal{H}$ is itself measurable. It may not be, even when $\mathbb{H}$ has the product sigma algebra, without modest further assumptions (see Dravecký [45]).

(C). Third, we assume that $\mathbb{H}$ is invariant under permutations of individuals. Formally, let Σ be the group of permutations of $\mathbb{I}$. For each $\sigma \in \Sigma$ and $h \in \mathbb{H}$, the assumption is that $\mathbb{H}$ contains the history σh such that

$$\mathcal{W}_i(\sigma h) = \mathcal{W}_{\sigma^{-1}i}(h)$$

for all $i \in \mathbb{I}$.

We further assume that the action of Σ on $\mathbb{H}$ is measurable. That is: if $V \subset \mathbb{H}$ is measurable, then σV is measurable, for any $\sigma \in \Sigma$. This allows us to define an action of Σ on lotteries L:

$$(\sigma L)(V) := L(\sigma^{-1}V)$$

for any $\sigma \in \Sigma$, lottery L and measurable $V \subset \mathbb{H}$.

We further assume that $\mathbb{L}$ is invariant under this action.

Remark 1.1.1. The various measurability conditions are automatically met if $\mathbb{H}$ has the product sigma algebra, i.e. the *smallest* one for which the functions $\mathcal{W}_i$ are measurable. For example, to check that $\mathcal{H}$ is measurable with respect to that sigma algebra, it suffices to check that $\mathcal{H}^{-1}(\mathcal{W}_i^{-1}(U))$ is measurable whenever U is a measurable subset of $\mathbb{W}$. But, in fact, $\mathcal{H}^{-1}(\mathcal{W}_i^{-1}(U)) = U$. It is worth remarking that the product sigma algebra may not be the most natural one. For example, if $\mathbb{W} = \mathbb{R}$ with the Lebesgue sigma algebra, then the Lebesgue sigma algebra on $\mathbb{H}$ is *not* the product one – but it would also suffice for our purposes.

Remark 1.1.2. The assumption that $\mathcal{L}(\mathbb{P}) \subset \mathbb{L}$ may seem unrealistic: if some are to have the welfare levels of kings, others may have to have the welfare levels of paupers. Thus general applications of our results will have to assume the ethical relevance of logically possible but non-feasible domains. But we do not regard this as a particular difficulty for our approach, partly as all results in this area we know of need such a principle to handle some feasible sets. For example, they typically assume that the feasible set of lotteries is convex, but convexity can fail; there may be a cost to randomization.

1.2. **Axioms for Aggregation.** Now we assume that $\mathbb{P}$ and $\mathbb{L}$ are each preordered. The preorder $\succsim_{\mathbb{P}}$ on $\mathbb{P}$ is the individual preorder; the preorder $\succsim$ on $\mathbb{L}$ is the social preorder. As already mentioned, the individual preorder encodes interpersonal and intrapersonal comparisons. Thus for any individuals i and j, not necessarily distinct, $\mathcal{P}_i(L) \succsim_{\mathbb{P}} \mathcal{P}_j(L')$ if and only if L is at least as good for i as L' is for j. We will use obvious notation, e.g. writing $P \sim_{\mathbb{P}} P'$ to mean the conjunction of $P \succsim_{\mathbb{P}} P'$ and $P' \succsim_{\mathbb{P}} P$. Since $\succsim_{\mathbb{P}}$ is allowed to be incomplete, we will also write $P \curlywedge_{\mathbb{P}} P'$ to mean neither $P \succsim_{\mathbb{P}} P'$ nor $P' \succsim_{\mathbb{P}} P$.

We will sometimes informally treat the individual and social preorders as ranking welfare levels and histories respectively. Strictly speaking, this presupposes that we can identify welfare levels and histories with the corresponding degenerate prospects and lotteries. This does not always make sense in our framework: singletons may not be measurable, and degenerate probability measures may not be in the convex sets of probability measures under consideration. But we often ignore this detail.

Our first principle of aggregation says that the social preorder only depends on which prospect each individual faces.

Anteriority. If $\mathcal{P}_i(L) = \mathcal{P}_i(L')$ for every $i \in \mathbb{I}$, then $L \sim L'$.

Second, we need a principle which captures the idea that individual welfare contributes positively towards social welfare.

Reduction to Prospects. For any $P, P' \in \mathbb{P}$, $\mathcal{L}(P) \succsim \mathcal{L}(P')$ if and only if $P \succsim_{\mathbb{P}} P'$.

These two principles are relatively weak. We discuss their relation to stronger and more familiar Pareto principles in section 5.2.1,[7] where we argue that they express a weak sense in which the social preorder is *ex ante*.

Third, we need a principle of impartiality or permutation-invariance. The simplest such principle is

Anonymity. Given $L \in \mathbb{L}$ and $\sigma \in \Sigma$, we have $L \sim \sigma L$.

We will in fact use the following slightly stronger condition.

Two-Stage Anonymity. Given $L, M \in \mathbb{L}$, $\sigma \in \Sigma$, and $\alpha \in [0,1]$,

$$\alpha L + (1-\alpha)M \sim \alpha(\sigma L) + (1-\alpha)M.$$

Two-Stage Anonymity can be motivated in at least two ways. First, define an 'anonymous history' to be an element of the quotient $\mathbb{H}/\Sigma$. One natural principle says that L and L' are equally good if they define the same probability distribution over anonymous histories. Here is a convenient reformulation:

Posterior Anonymity. Given $L, L' \in \mathbb{L}$, suppose that $L(U) = L'(U)$ whenever U is a measurable, permutation-invariant subset of $\mathbb{H}$. Then $L \sim L'$.

In section 5.2.2 we will argue that this principle expresses a weak sense in which the social preorder is *ex post*. Posterior Anonymity is easily seen to logically entail Two-Stage Anonymity, and that is our preferred motivation for accepting the latter as an axiom.[8] However, a second motivation is available: Two-Stage Anonymity follows from the combination of Anonymity and the central axiom of expected utility theory, Strong Independence, or even its restriction to the indifference relation. The fact that Two-Stage Anonymity is much weaker than the conjunction of Anonymity and Strong Independence means that our aggregation theorems will be compatible with many non-expected utility theories at both the individual and social levels.

1.3. **The Aggregation Theorem.**

Theorem 1.3.1. *Given an arbitrary preorder $\succsim_{\mathbb{P}}$ on $\mathbb{P}$, there is a unique preorder $\succsim$ on $\mathbb{L}$ satisfying Anteriority, Reduction to Prospects, and Two-Stage Anonymity. Namely,*

$$L \succsim L' \iff p_L \succsim_{\mathbb{P}} p_{L'} \tag{1}$$

where p_L (similarly $p_{L'}$) is the prospect

$$p_L = \frac{1}{\#\,\mathbb{I}} \sum_{i \in \mathbb{I}} \mathcal{P}_i(L).$$

Given that we favour Posterior Anonymity over and above Two-Stage Anonymity, the following answers a natural question.

Proposition 1.3.2. *The social preorders given by* (1) *satisfy Posterior Anonymity.*

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[7] The standard Pareto and expected utility axioms mentioned in this section are formally defined in sections 3.1 and 3.2.

[8] Posterior Anonymity itself follows from Anonymity and the widely accepted principle of Monotonicity, provided the social preorder is upper-measurable, a common domain assumption needed for Monotonicity to apply (see section 4.1). Anonymity is the case of Two-Stage Anonymity where $\alpha = 1$.

1.4. **Terminology.**

Definition 1.4.1. We will say that a social preorder $\succsim$ is *generated* by the individual preorder $\succsim_{\mathbb{P}}$ whenever the constant population domain conditions (A)–(C) hold and $\succsim$ satisfies (1).

Definition 1.4.2. Given two preordered spaces $(X, \succsim_X)$ and $(Y, \succsim_Y)$, a function $f\colon X \to Y$ *represents* $\succsim_X$ (or is a *representative* of $\succsim_X$) when for all $x_1, x_2 \in X$, $x_1 \succsim_X x_2 \iff f(x_1) \succsim_Y f(x_2)$.

The mere existence of a representative is trivial; let $X = Y$ and f be the identity mapping. The interesting case is where $(Y, \succsim_Y)$ is better behaved or easier to understand or more fundamental than $(X, \succsim_X)$. For example, Y may be $\mathbb{R}^n$ with the natural order, the case with $n = 1$ being especially common. For another example, the conclusion of Theorem 1.3.1 can be put by saying that the function $\mathbb{L} \to \mathbb{P}$ given by $L \mapsto p_L$ represents $\succsim$. Alternatively, we will be much concerned with the case where $(Y, \succsim_Y)$ is a preordered vector space $(\mathbb{V}, \succsim_{\mathbb{V}})$. By definition, this consists of a vector space $\mathbb{V}$ with a preorder $\succsim_{\mathbb{V}}$ that is compatible with the linear structure in the sense that, for any $v, v', w \in \mathbb{V}$ and $\lambda > 0$, $v \succsim_{\mathbb{V}} v' \iff \lambda v + w \succsim_{\mathbb{V}} \lambda v' + w$. Preorders on vector spaces do not in general respect linear structure, so we will say that a preorder $\succsim_{\mathbb{V}}$ on a vector space $\mathbb{V}$ is a *vector preorder* precisely when $(\mathbb{V}, \succsim_{\mathbb{V}})$ is a preordered vector space. As we will explain in sections 3.3 and 3.4, preordered vector spaces are particularly useful for making sense of very general forms of utilitarian 'total utility' representations in the absence of continuity or completeness assumptions.

When f is injective, we say that it is an *embedding* (of $(X, \succsim_X)$ in $(Y, \succsim_Y)$). When f is especially natural (for example, an inclusion), we say that $\succsim_X$ is *embedded* or *included* in $\succsim_Y$.

1.5. **Examples.** Now let us give some examples of corresponding individual and social preorders. For concreteness and simplicity, we will take $\mathbb{W}$ to be the interval $[-10, 10]$ and take $\mathbb{P}$ to be the set of all finitely-supported probability measures on $\mathbb{W}$.

Example 1.5.1 (Utilitarianism). Suppose that $\succsim_{\mathbb{P}}$ orders $\mathbb{P}$ by expected value. (More generally: suppose that $\succsim_{\mathbb{P}}$ is represented by the expected value of a utility function on $\mathbb{W}$.) The corresponding social preorder has *utilitarian* form. That is, $L \succsim L'$ if and only if L has greater expected total welfare. (Since we are here talking about a fixed population $\mathbb{I}$, there is no effective difference between total and average utilitarianism.) As we explain in section 3.3, the social preorder will always have a broadly utilitarian form whenever the individual preorder satisfies Strong Independence.

Example 1.5.2 (Leximin). Here the individual preorder satisfies Strong Independence, but not the Archimedean continuity axiom. Let $\succsim_{\mathbb{P}}$ order $\mathbb{P}$ so that $P \succsim_{\mathbb{P}} P'$ if and only if either $P = P'$ or the smallest $x \in \mathbb{W}$ at which $P(\{x\}) \neq P'(\{x\})$ is such that $P(\{x\}) < P'(\{x\})$. When restricted to histories, the corresponding social preorder is leximin: $h \succ h'$ if and only if the worst off individual in h is better off than the worst off in h'; if they are tied, turn to the next worst off. This social preorder is still 'broadly utilitarian' in the sense just mentioned, and which we elaborate in section 3.3: it turns out to be represented by a function which assigns to each lottery its expected total utility in an infinite-dimensional preordered vector space.

Example 1.5.3 (Incompleteness). Here the individual preorder satisfies Strong Independence and the Archimedean axiom, but it is not in general complete. Let $\mathcal{U}$ be a set of real-valued functions on $\mathbb{W}$. Let $\succsim_{\mathbb{P}}$ preorder $\mathbb{P}$ so that $P \succsim_{\mathbb{P}} P'$ if and only if, for all u in $\mathcal{U}$, the expected value of u is at least as great under P as under P'. This individual preorder is not in general complete. The corresponding social preorder ranks $L \succsim L'$ if and only if, for each u in $\mathcal{U}$, the expected total value of u is at least as great under L as under L'. This social preorder is also not in general complete.

Example 1.5.4 (Non-Separability). Finally, here is an example in which the individual preorder violates Strong Independence, even though it is complete and satisfies the Archimedean axiom. This has interesting consequences for the social preorder.

Say that $\succsim_{\mathbb{P}}$ is a 'rank-dependent' individual preorder (RDI) if it has a 'rank-dependent utility' representation.[9] In other words, there is an increasing function $r\colon [0,1] \to [0,1]$, with $r(0) = 0$ and $r(1) = 1$, which we will call the 'risk function', such that $\succsim_{\mathbb{P}}$ is represented by $U\colon \mathbb{P} \to \mathbb{R}$ where

$$U(P) := -10 + \int_{-10}^{10} r(P([x, 10]))\, dx.$$

If in addition r is convex, we will say that $\succsim_{\mathbb{P}}$ is 'risk-avoidant'.[10]

Although $U(1_w) = w$ holds in general, and ordinary expected utility theory is satisfied when $r(x) = x$, $U(P)$ is not in general simply the expected value of P. To see the deviation from ordinary expected utility, assume for concreteness $r(x) = x^2$. In the following prospects, each listed welfare level has a probability of one quarter, while the histories contain four people.

$$P_A = [1,1,1,1], P_B = [5,0,1,1], \quad h_A = [1,1,1,1], h_B = [5,0,1,1]$$
$$P_C = [1,1,0,0], P_D = [5,0,0,0], \quad h_C = [1,1,0,0], h_D = [5,0,0,0].$$

Computing the value of U for each prospect yields $P_A \succ_{\mathbb{P}} P_B$ and $P_D \succ_{\mathbb{P}} P_C$. This has the 'paradoxical' preference structure of the Allais paradox, violating Strong Independence. For the corresponding social preorder, our aggregation theorem then implies that $h_A \succ h_B$ and $h_D \succ h_C$, violating Strong Separability.

Such violations of Strong Separability have been seen as expressions of egalitarianism. Thus it might be said that while the perfect equality in h_A outweighs the greater total welfare in h_B, there is there is not much difference in inequality between h_C and h_D, so the greater total in h_D is decisive (Sen [98, p. 41], Broome [23]).

Returning to the general case, assume a population of size n. Say that a preorder $\succsim$ on histories is a rank-dependent social preorder (RDS) when for some $a_1, \dots, a_n \geq 0$ and $\sum_k a_k = 1$, $\succsim$ ranks a history h with welfare levels $w_1 \leq w_2 \leq \cdots \leq w_n$ according to the aggregate score

$$V(h) := a_1 w_1 + a_2 w_2 + \cdots + a_n w_n$$

If in addition $a_1 \geq a_2 \geq \cdots \geq a_n$, we will say that $\succsim$ is 'downwards increasing'.

Downward increasing RDSs are called 'generalized Gini' by Blackorby and Donaldson [12] and Weymark [107], who take them to be natural examples of egalitarian preorders. We will say more about the relationship between apparently egalitarian preorders and our aggregation theorems in section 5.1. But for now, by setting $a_k = r(\frac{n-k+1}{n}) - r(\frac{n-k}{n})$, we see that $\succsim$ is a [downwards increasing] RDS if and only if it is generated by a [risk-avoidant] RDI. Thus what has been taken to be a canonical form of egalitarianism at the social level emerges from what has been characterized as 'pessimism about risk' at the individual level. For example, by

[9] For an entry into the literature on this popular nonexpected utility theory, see e.g. Wakker [105], Schmidt [97, §4.2], and Buchak [28, Ch. 2]. The function r is often required to be continuous and strictly increasing, but the weaker definition will be useful.

[10] This term is from Buchak [28, p. 66]; Yaari [109] and Chateauneuf and Cohen [31] use 'pessimistic'.

setting $r(x) = 1$ if $x = 1$, 0 otherwise, we obtain the social preorder on histories given by the Rawlsian maximin rule.

The examples illustrate how distributive views which are traditionally seen as very different can be obtained within our model simply by adjusting welfare comparisons at the individual level.[11] General results corresponding to such possibilities will be given in section 3.

2. A Variable Population Aggregation Theorem

In this section we present a version of the aggregation theorem in which the population is allowed to vary from one history to another.

2.1. **Framework.** At a basic level, the generalisation to variable populations is straightforward: we simply introduce a new element Ω representing nonexistence, and set $\mathbb{W}^* := \mathbb{W} \cup \{\Omega\}$. This allows each history to represent some individuals as nonexistent and, otherwise, Theorem 1.3.1 remains unchanged. To be sure, there are some questions of interpretation. For example, we will speak of Ω as a welfare level, but one need not take this literally. We will say more about comparisons involving Ω in section 2.2 below.

The shortcoming of the approach just mentioned is there is only a finite set $\mathbb{I}$ of possible individuals. The interesting generalisation is to allow the population size to be unbounded. We will, however, insist that any given lottery involves only finitely many individuals. We spell this out as assumption (D) below.[12] In comparing two lotteries, then, only a finite population will be relevant, and we can apply the ideas of section 1. Only a little more work is required to ensure that these pairwise comparisons combine into a well-defined social preorder. That is what we now explain.

Thus let $\mathbb{I}^*$ be an infinite set of possible individuals. Assume that $\mathbb{W}^*$ and $\mathbb{H}^* \subset (\mathbb{W}^*)^{\mathbb{I}^*}$ are measurable spaces, with $\Omega \in \mathbb{W}^*$, and that $\mathbb{P}^*$ and $\mathbb{L}^*$ are non-empty convex sets of probability measures. We make the following domain assumptions, in parallel to those of section 1.1.

(A). First, we assume that, for each $i \in \mathbb{I}^*$, the projection $\mathcal{W}_i^* \colon \mathbb{H}^* \to \mathbb{W}^*$ is measurable. This again allows us to define the function $\mathcal{P}_i^*$ from lotteries to prospects, so that $\mathcal{P}_i^*(L)(U) = L((\mathcal{W}_i^*)^{-1}(U))$ for measurable $U \subset \mathbb{W}^*$.

We further assume that $\mathcal{P}_i^*(\mathbb{L}^*) \subset \mathbb{P}^*$.

(B). Second, for each $w \in \mathbb{W}^*$ and each finite population $\mathbb{I} \subset \mathbb{I}^*$, we assume that that our set $\mathbb{H}^*$ of histories contains the history $\mathcal{H}_{\mathbb{I}}^*(w)$ such that

$$\mathcal{W}_i^*(\mathcal{H}_{\mathbb{I}}^*(w)) = \begin{cases} w & \text{if } i \in \mathbb{I} \\ \Omega & \text{if not.} \end{cases}$$

We further assume that $\mathcal{H}_{\mathbb{I}}^* \colon \mathbb{W}^* \to \mathbb{H}^*$ is measurable. We can then define a corresponding function $\mathcal{L}_{\mathbb{I}}^*$ from prospects to lotteries. Thus if V is a measurable subset of $\mathbb{H}^*$, $\mathcal{L}_{\mathbb{I}}^*(P)(V) = P((\mathcal{H}_{\mathbb{I}}^*)^{-1}(V))$.

[11]This echoes social choice theory, where, for example, classical utilitarianism and leximin can be derived from common axioms except for different assumptions about the measurability of welfare; see d'Aspremont and Gevers [5].

[12]Aggregating the welfare of infinitely many individuals raises quite general and formidable problems which we thus set aside. For example, full Anonymity is inconsistent with Strong Pareto. See Bostrom [22] for an overview of such problems; Pivato [90] for a careful study of separable aggregation in the infinite case with applications to the present setting of risk; Zhou [110] for an infinite population version of Harsanyi's original theorem (i.e. without interpersonal comparisons and Anonymity); and McCarthy, Mikkola and Thomas [77] for an infinite population version of Harsanyi's original theorem, but dispensing with continuity and completeness.

We further assume that $\mathcal{L}^*_{\mathbb{I}}(\mathbb{P}^*) \subset \mathbb{L}^*$.

(C). Third, we assume that $\mathbb{H}^*$ is invariant under permutations of $\mathbb{I}^*$. We write Σ^* for the group of all such permutations.

We further assume that the action of Σ^* on $\mathbb{H}^*$ is measurable. This allows us to define the action of Σ^* on lotteries.

We further assume that $\mathbb{L}^*$ is Σ^*-invariant.

(D). Finally, we assume that each history in $\mathbb{H}^*$ and each lottery in $\mathbb{L}^*$ involves only finitely many individuals.

Let us explain what this means. For a history h, the assumption is that $\mathcal{W}^*_i(h) = \Omega$ for all but finitely many $i \in \mathbb{I}^*$. One might guess that for a lottery L to 'involve only finitely many individuals', it would suffice that $\mathcal{P}_i(L) = 1_\Omega$ for all but finitely many $i \in \mathbb{I}^*$. But this is not conceptually the right criterion, as the following example shows.

Example 2.1.1. Suppose that $\mathbb{I}^* = [0, 1]$, and let h_i be the history in which only individual i exists, with welfare level w. Let L be the uniform distribution over these h_i. Then each person i is certain not to exist – each has prospect 1_Ω – yet there is a clear sense in which L involves infinitely many individuals, rather than no individuals. Namely, for any finite population $\mathbb{I} \subset \mathbb{I}^*$, it is certain that someone not in $\mathbb{I}$ exists. One reason that this is problematic is that it would be natural to reject Anteriority in this example. Anteriority would say that L is just as good as no one existing at all, but intuitively it is rather as good as having one person who is certain to exist at level w.

To state a better criterion, given finite $\mathbb{I} \subset \mathbb{I}^*$, let $\mathbb{H}^*_{\mathbb{I}}$ be the subset of $\mathbb{H}^*$ consisting of histories h such that $\mathcal{W}^*_i(h) = \Omega$ for all $i \notin \mathbb{I}$. The assumption we make for lotteries is that every $L \in \mathbb{L}^*$ is supported on some $\mathbb{H}^*_{\mathbb{I}}$. In other words, there must exist some finite $\mathbb{I} \subset \mathbb{I}^*$ such that $L(V) = 0$ whenever measurable V is disjoint from $\mathbb{H}^*_{\mathbb{I}}$. We write $\mathbb{L}^*_{\mathbb{I}}$ for the subset of $\mathbb{L}^*$ consisting of lotteries which are supported on $\mathbb{H}^*_{\mathbb{I}}$.

Note that, if $\mathbb{I}$ is contained in some larger population $\mathbb{I}'$, then $\mathbb{H}^*_{\mathbb{I}} \subset \mathbb{H}^*_{\mathbb{I}'}$, and any lottery supported on $\mathbb{H}^*_{\mathbb{I}}$ is also a lottery supported on $\mathbb{H}^*_{\mathbb{I}'}$. Because of this, any two lotteries in $\mathbb{L}^*$ are members of some common $\mathbb{L}^*_{\mathbb{I}}$, with $\mathbb{I} \subset \mathbb{I}^*$ finite.

Remark 2.1.2. The various measurability assumptions are again guaranteed if $\mathbb{H}^*$ has the product sigma algebra with respect to the projections $\mathcal{W}^*_i$. However, it would be natural to consider a finer-grained sigma algebra by including the sets $\mathbb{H}^*_{\mathbb{I}}$. Then $\mathbb{L}^*_{\mathbb{I}}$ would be the subset of $\mathbb{L}^*$ containing lotteries L such that $L(\mathbb{H}^*_{\mathbb{I}}) = 1$. But we do not need this assumption.

The implications of the domain assumptions are illustrated by the following lemma.

Lemma 2.1.3. *Assume the domain conditions (A)–(D).*

(i) Given $L \in \mathbb{L}^*_{\mathbb{I}}$, we have $\mathcal{P}^*_i(L) = 1_\Omega$ for any $i \in \mathbb{I}^* \setminus \mathbb{I}$. In particular, $1_\Omega \in \mathbb{P}^*$.

(ii) $\mathbb{H}^*$ contains the 'empty history' h_Ω such that $\mathcal{W}^*_i(h_\Omega) = \Omega$ for all $i \in \mathbb{I}^*$.

(iii) $\mathbb{L}^*$ contains the 'empty lottery' 1_{h_Ω}, and $\mathcal{P}^*_i(1_{h_\Omega}) = 1_\Omega$ for all $i \in \mathbb{I}^*$.

(iv) Suppose $\{\Omega\}$ is measurable in $\mathbb{W}^*$. If $\mathcal{P}^*_i(L) = 1_\Omega$ for all $i \in \mathbb{I}^*$, then $L = 1_{h_\Omega}$.

2.2. **Axioms for Aggregation.** We let $\succsim_{\mathbb{P}^*}$ be the individual preorder on $\mathbb{P}^*$ and $\succsim^*$ be the social preorder on $\mathbb{L}^*$.

The key axioms for the social preorder are much as before, with one minor change due to the infinity of $\mathbb{I}^*$. Anteriority works as before, but 'Reduction to Prospects' must be understood relative to each given finite population $\mathbb{I}$:

Reduction to Prospects (Variable Population). For any $P, P' \in \mathbb{P}^*$, $\mathcal{L}^*_{\mathbb{I}}(P) \succsim \mathcal{L}^*_{\mathbb{I}}(P')$ if and only if $P \succsim_{\mathbb{P}^*} P'$.

Posterior Anonymity and Two-Stage Anonymity work as before, except that they are understood in terms of Σ^*, the group of permutations of $\mathbb{I}^*$.

In line with Theorem 1.3.1, those three axioms will turn out to be satisfied by at most one social preorder. However, for such a social preorder to exist, we will need a new condition on the individual preorder.[13]

Omega Independence. For any $P, P' \in \mathbb{P}^*$ and rational number $\alpha \in (0, 1)$,

$$P \succsim_{\mathbb{P}^*} P' \iff \alpha P + (1-\alpha)1_\Omega \succsim_{\mathbb{P}^*} \alpha P' + (1-\alpha)1_\Omega.$$

We will present a defence of this condition, and discuss its relation to other independence axioms, in section 2.4.

A conceptual challenge for our approach is to provide an interpretation of $\succsim_{\mathbb{P}^*}$. As in the constant population case, we assume that

(†) $\mathcal{P}^*_i(L) \succsim_{\mathbb{P}^*} \mathcal{P}^*_j(L')$ if and only if L is at least as good for i as L' is for j

on the qualification that the individuals i and j are certain to exist under L and L' respectively. But in the presence of Reduction to Prospects (Variable), it would be controversial to maintain that interpretation without the qualification. Two examples will illustrate.

First, it has often been argued that existing at a given welfare level cannot be better or worse for an individual than not existing at all. [14] A similar view arises for comparisons between prospects in which Ω has positive probability. This is not by itself a problem for us, since our framework allows Ω to be incomparable under $\succsim_{\mathbb{P}^*}$ with all, some, or no members of $\mathbb{W}$. However, suppose that w is a very low welfare level, corresponding to a life full of terrible suffering. One can maintain that a history containing a single person at w is worse than the empty history, while accepting that it is not worse for the sole person.

Second, it has also often been argued that there are lives which are worth living which are nevertheless not worth creating (see e.g. Blackorby and Donaldson [13, p. 21], and section 2.6 below). One possible version of this view would claim that for some welfare level w, having w is better for someone than nonexistence even though a history containing a single person at w is worse than the empty history.

Each of these positions clashes with Reduction to Prospects (Variable), if we accept (†) without qualification. In response, it could be said that neither of the positions is uncontroversial. For example, against the first view, the claim that having welfare level w is worse for someone than nonexistence might be taken to mean that having w harms the individual in an absolute rather than comparative sense (see Bykvist [29, §2] for the relevant distinctions). Nevertheless, the existence of such controversies suggests caution, and two comments will enable us to sidestep these issues. First, we do not rely on any particular interpretation of $\succsim_{\mathbb{P}^*}$, as long it makes plausible the axioms of our aggregation theorems. Second, at least one such interpretation is available: one could identify $\succsim_{\mathbb{P}^*}$ with the one-person social preorder. Thus for any population $\mathbb{I}$ of size one, this position takes $\mathcal{L}^*_{\mathbb{I}}(P) \succsim^* \mathcal{L}^*_{\mathbb{I}}(P') \iff P \succsim_{\mathbb{P}^*} P'$ to be a conceptual equivalence. This would leave open

[13]We can in fact apply Theorem 1.3.1 to determine a unique social preorder on each $\mathbb{L}^*_{\mathbb{I}}$ separately. The issue is whether these are compatible, in the sense of defining a social preorder on $\mathbb{L}^*$ as a whole.

[14]See Broome [25, p.168] for one classic statement of this view.

the question of how $\succsim_{\mathbb{P}^*}$ relates to betterness for individuals, and turn (†) into a substantive claim which could be debated.[15]

2.3. **The Aggregation Theorem.**

Theorem 2.3.1. *Given an individual preorder $\succsim_{\mathbb{P}^*}$, there is at most one social preorder satisfying Anteriority, Reduction to Prospects (Variable Population), and Two-Stage Anonymity. When it exists, it is given by*

$$L \succsim^* L' \iff p_L^{\mathbb{I}} \succsim_{\mathbb{P}^*} p_{L'}^{\mathbb{I}} \tag{2}$$

for finite non-empty $\mathbb{I} \subset \mathbb{I}^*$ such that L and L' are lotteries in $\mathbb{L}_{\mathbb{I}}^*$ and where $p_L^{\mathbb{I}}$ (similarly $p_{L'}^{\mathbb{I}}$) is the prospect

$$p_L^{\mathbb{I}} = \frac{1}{\#\mathbb{I}} \sum_{i \in \mathbb{I}} \mathcal{P}_i^*(L).$$

It exists if and only if the individual preorder satisfies Omega Independence.

Definition 2.3.2. We will say that a social preorder $\succsim^*$ is *generated* by the individual preorder $\succsim_{\mathbb{P}^*}$ whenever the variable population domain conditions (A)–(D) hold and $\succsim^*$ satisfies (2).

The social preorders described by Theorem 1.3.1 turned out to automatically satisfy Posterior Anonymity. We can prove a similar result here, but we need a technical assumption. Say that the sigma algebra on $\mathbb{H}^*$ is *coherent* if the following holds: $U \subset \mathbb{H}^*$ is measurable if and only if $U \cap \mathbb{H}_{\mathbb{I}}^*$ is measurable for every finite $\mathbb{I} \subset \mathbb{I}^*$. Note that coherence is a trivial assumption, in the sense that one can always expand the sigma algebra on $\mathbb{H}^*$ to make it coherent without changing $\mathbb{L}^*$ or invalidating any of the domain conditions.

Proposition 2.3.3. *Suppose that the sigma algebra on $\mathbb{H}^*$ is coherent. Then the social preorders given by* (2) *satisfy Posterior Anonymity.*

2.4. **Constant vs Variable Population.** We now show that the constant population theorem is a special case of the variable population theorem, and offer several reasons to think that Omega Independence is a fairly undemanding requirement.

To do this we need to be able to identify members of $\mathbb{P}$ with members of $\mathbb{P}^*$. Thus we assume that $\mathbb{P}$ is a (non-empty) convex set of probability measures on a measurable space $\mathbb{W}$, that $\mathbb{W}^* = \mathbb{W} \cup \{\Omega\}$, and that $\mathbb{W}^*$ has the sigma algebra generated by the one on $\mathbb{W}$. This enables us to identify members of $\mathbb{P}$ with probability measures on $\mathbb{W}^*$ by the natural inclusion $P \mapsto P^*$, where $P^*(U) := P(U \cap \mathbb{W})$ for all measurable U in $\mathbb{W}^*$. We then identify $\mathbb{P}^*$ with the convex hull of $\mathbb{P}_\Omega := \mathbb{P}^* \cup \{1_\Omega\}$. We summarize these standing assumptions by saying that $\mathbb{P}^*$ *includes* $\mathbb{P}$.

We understand embeddings or inclusions of relevant preorders in terms of this identification (see section 1.4). For example, $\succsim_{\mathbb{P}^*}$ includes $\succsim_{\mathbb{P}}$ when for all P, $P' \in \mathbb{P}$, $P \succsim_{\mathbb{P}} P' \iff P \succsim_{\mathbb{P}^*} P'$.

A *constant population model* is any $\mathbb{M} = \langle \mathbb{I}, \mathbb{W}, \mathbb{P}, \succsim_{\mathbb{P}}, \mathbb{H}, \mathbb{L}, \succsim \rangle$ satisfying the constant population domain conditions (A)–(C). A *variable population model* is any $\mathbb{M}^* = \langle \mathbb{I}^*, \mathbb{W}^*, \mathbb{P}^*, \succsim_{\mathbb{P}^*}, \mathbb{H}^*, \mathbb{L}^*, \succsim^* \rangle$ satisfying the variable population domain conditions (A)–(D).

Assume that $\mathbb{P}^*$ includes $\mathbb{P}$. Given a variable population model $\mathbb{M}^*$ and finite, nonempty $\mathbb{I} \subset \mathbb{I}^*$, we define the *restriction* $\mathbb{M}_{\mathbb{I}} = \langle \mathbb{I}, \mathbb{W}, \succsim_{\mathbb{P}}, \mathbb{H}_{\mathbb{I}}, \mathbb{L}_{\mathbb{I}}, \succsim_{\mathbb{I}} \rangle$ as follows. First, $\succsim_{\mathbb{P}}$ is the restriction of $\succsim_{\mathbb{P}^*}$ to $\mathbb{P}$. Second, $\mathbb{H} \subset \mathbb{W}^{\mathbb{I}}$ is identified with the set

[15] Hammond [59] essentially accepts the equivalence *and* (†) on conceptual grounds, and views this as defining interpersonal and intrapersonal comparisons in terms of the one-person social preorder. This proposal stipulates away the two positions sketched in the text, and is consistent with our approach, but not mandated by it.

of histories h in $\mathbb{H}^*_{\mathbb{I}}$ such that $\mathcal{W}^*_i(h) \neq \Omega$ for all $i \in \mathbb{I}$, and equipped with the sigma algebra induced by $\mathbb{H}^*$. Third, the set of lotteries $\mathbb{L}_{\mathbb{I}}$ on $\mathbb{H}_{\mathbb{I}}$ is identified with the set of lotteries L in $\mathbb{L}^*_{\mathbb{I}}$ such that $\mathcal{P}^*_i(L) \in \mathbb{P}$ for all $i \in \mathbb{I}$. Finally, $\succsim_{\mathbb{I}}$ is the restriction of $\succsim^*$ to $\mathbb{L}$. It is easy to show that $\mathbb{M}_{\mathbb{I}}$ is a constant population model. Conversely, we say that $\mathbb{M}^*$ *includes* a constant population model $\mathbb{M}$ just in case $\mathbb{M}$ is a restriction of $\mathbb{M}^*$.

Proposition 2.4.1. *Assume that $\mathbb{P}^*$ includes $\mathbb{P}$.*
EMBEDDING $\succsim_{\mathbb{P}}$ IN $\succsim_{\mathbb{P}^*}$. *Suppose given a preorder $\succsim_{\mathbb{P}}$. Let $\succsim_{\mathbb{P}_\Omega}$ be any preorder on $\mathbb{P}_\Omega$ that includes $\succsim_{\mathbb{P}}$.*

- *(i) There is a preorder $\succsim_{\mathbb{P}^*}$ on $\mathbb{P}^*$ that includes $\succsim_{\mathbb{P}_\Omega}$ (and hence $\succsim_{\mathbb{P}}$) and satisfies Omega Independence.*
- *(ii) There is a preorder $\succsim_{\mathbb{P}^*}$ on $\mathbb{P}^*$ that includes $\succsim_{\mathbb{P}_\Omega}$ (and hence $\succsim_{\mathbb{P}}$) and violates Omega Independence.*

EMBEDDING $\mathbb{M}$ IN $\mathbb{M}^*$.

- *(iii) For any variable population model $\mathbb{M}^*$ satisfying* (2) *and finite $\mathbb{I} \subset \mathbb{I}^*$, the restriction $\mathbb{M}_{\mathbb{I}}$ is a constant population model that satisfies* (1).
- *(iv) For any constant population model $\mathbb{M}$ containing $\succsim_{\mathbb{P}}$ and satisfying* (1), *and any Omega Independent $\succsim_{\mathbb{P}^*}$ that includes $\succsim_{\mathbb{P}}$, there is a variable population model $\mathbb{M}^*$ that includes $\mathbb{M}$, contains $\succsim_{\mathbb{P}^*}$, and satisfies* (2).

Part (i) is a consequence of the following very general result.

Theorem 2.4.2. *Every preorder has a representation with values in a preordered vector space.*[16]

It shows that certain nonexistence can be slotted into any position in a given preorder $\succsim_{\mathbb{P}}$, and the result can always be embedded in an Omega Independent $\succsim_{\mathbb{P}^*}$. For example, Omega Independent $\succsim_{\mathbb{P}^*}$ can be chosen so that for a given $P \in \mathbb{P}$, $1_\Omega \sim_{\mathbb{P}^*} P$; alternatively, Omega Independent $\succsim_{\mathbb{P}^*}$ can be chosen so that $1_\Omega \curlywedge_{\mathbb{P}^*} P$ for all $P \in \mathbb{P}$. In permitting such a wide range of comparisons between nonexistence and other welfare levels or prospects over such levels, this provides the first sense in which Omega Independence is fairly undemanding.

Combining (i) with (iv) shows that any constant population model which satisfies (1) can be extended to a variable population model which satisfies (2) in a correspondingly wide range of ways. In conjunction with (iii), this articulates the sense in which the constant population theorem is a special case of the variable population theorem.

Parts (i) and (ii) show that no matter how nonexistence is compared with other welfare levels, Omega Independence of $\succsim_{\mathbb{P}^*}$ is logically independent of Strong Independence of $\succsim_{\mathbb{P}}$, despite the formal resemblance between these principles. In particular, because of the qualitative distinction between nonexistence and other welfare levels, anyone who is moved by something like the Allais paradox to reject Strong Independence for $\succsim_{\mathbb{P}}$ might well accept Omega Independence for $\succsim_{\mathbb{P}^*}$. This provides a second sense in which Omega Independence is fairly undemanding.

In fact, our variable population Theorem 2.3.1 gives strong reasons to accept Omega Independence, and a third sense in which Omega Independence is reasonably modest. If the social preorder satisfies Anteriority, Reduction to Prospects (Variable Population) and Two-Stage Anonymity, then the theorem tells us that the individual preorder must satisfy Omega Independence. Since the first three premises are modest extensions of their constant population analogues, it is arguable that the assumptions of Theorem 2.3.1 are hardly more difficult to defend

[16]This was defined in section 1.4. We will consider the structure of preordered vector spaces in section 3.4, especially Theorem 3.4.5.

than those of Theorem 1.3.1, despite the fact that variable population problems have been seen as challenging.

2.5. **Examples.** Let us show how some of the examples from section 1.5 can be extended to variable populations. Among other things this will illustrate part (a)(i) of Proposition 2.4.1. Although we present some general constructions, we focus on the extension of constant population utilitarianism (Example 1.5.1) to the variable case. As in section 1.5, we take $\mathbb{W} = [-10, 10]$ and let $\mathbb{P}^*$ consist of all finitely supported probability measures on $\mathbb{W}^* = \mathbb{W} \cup \{\Omega\}$.

Example 2.5.1 (Critical Level Utilitarianism). A 'critical level for a history h' is a welfare level $c \in \mathbb{W}$ such that adding one extra person to h at welfare level c is a matter of social indifference. For example, if $1_c \sim_{\mathbb{P}^*} 1_\Omega$, then c is a critical level for the empty history. It does not follow that c is a 'critical level'; that is, a welfare level such that adding someone at c to *any* history is a matter of social indifference. But c would be a critical level if we assumed that it is equivalent to Ω in a stronger sense. That is, given P, $P' \in \mathbb{P}$ and α, $\alpha' \in [0, 1]$,

$$\begin{aligned} &\alpha P + (1-\alpha)1_\Omega \succsim_{\mathbb{P}^*} \alpha' P' + (1-\alpha')1_\Omega \\ \iff &\alpha P + (1-\alpha)1_c \succsim_{\mathbb{P}} \alpha' P' + (1-\alpha')1_c. \end{aligned}$$

Suppose that $\succsim_{\mathbb{P}}$ satisfies Strong Independence. Then $\succsim_{\mathbb{P}^*}$ satisfies Omega Independence includes $\succsim_{\mathbb{P}}$, and c is a critical level under the assumptions of Theorem 2.3.1.

When we apply this definition of $\succsim_{\mathbb{P}^*}$ in the case of constant population utilitarianism, the corresponding variable population social preorder has the form of *critical level utilitarianism*. That is, define a utility function u on $\mathbb{W}$ by $u(w) = w - c$. Then $L \succsim^* L'$ if and only if L has greater expected total u.[17] The case of standard total utilitarianism is when c coincides with the welfare level of lives that are neutral (neither good nor bad) for the individual living them. If one is skeptical of the notion of a neutral life, then one may also be skeptical that there is such a canonical version of total utilitarianism.

Example 2.5.2 (Average Utilitarianism and Value Conditional on Existence). Here is a second way to embed $\succsim_{\mathbb{P}}$ in a preorder $\succsim_{\mathbb{P}^*}$ satisfying Omega Independence, whether or not $\succsim_{\mathbb{P}}$ satisfies Strong Independence. The idea is that certain nonexistence is incomparable to any other prospect, while in other cases the value of a prospect P is to be identified with its value conditional on the existence of the individual.[18] So define $\succsim_{\mathbb{P}^*}$ by the rule that, given $P, P' \in \mathbb{P}$ and $\alpha, \alpha' \in [0, 1]$,

$$\alpha P + (1-\alpha)1_\Omega \succsim_{\mathbb{P}^*} \alpha' P' + (1-\alpha')1_\Omega \iff \begin{cases} \alpha, \alpha' > 0 \text{ and } P \succsim_{\mathbb{P}} P', \text{ or} \\ \alpha = \alpha' = 0. \end{cases}$$

In the case of constant population utilitarianism, the resulting variable population social preorder is a version of *average utilitarianism*. It ranks lotteries by expected total welfare divided by expected population size. (The 'empty' lottery in which it is certain that no one exists is incomparable to the others.)

Two other ways of modeling the idea that value is conditional on existence lead to more obvious versions of average utilitarianism, but these are less well behaved. Ranking lotteries by expected average welfare violates Anteriority. Alternatively, let expected welfare be understood as expected welfare conditional upon existence, and take the average of expected welfare over individuals with a non-zero chance of

[17] This position is defended by Blackorby, Bossert and Donaldson in a number of places, such as Blackorby *et al* [17], and under the name 'the standardized total principle', by Broome [26].

[18] Such an idea is emphasised, for example, by Fleurbaey and Voorhoeve [54], and also seemingly endorsed by Harsanyi in his correspondence with Ng [86].

existence. Ranking lotteries by average expected welfare then violates Two-Stage Anonymity.

Example 2.5.3 (Incomparability of Nonexistence). A third method of defining $\succsim_{\mathbb{P}^*}$ may appeal to those who take to heart the view mentioned in section 2.2 that nonexistence is incomparable to other welfare levels. For $P, P' \in \mathbb{P}$ and $\alpha, \alpha' \in [0,1]$, they may define

$$\alpha P + (1-\alpha)1_\Omega \succsim_{\mathbb{P}^*} \alpha' P' + (1-\alpha')1_\Omega \iff \begin{cases} \alpha = \alpha' > 0 \text{ and } P \succsim_{\mathbb{P}} P', \text{ or} \\ \alpha = \alpha' = 0. \end{cases}$$

This invariably produces an individual preorder satisfying Omega Independence. However, it leads to widespread social incomparability: we will have $L \curlywedge^* L'$ unless the expected population size under L equals that under L'. If we extend constant population utilitarianism in this way, then the social preorder ranks lotteries of a given expected population size by their expected total utility – or, equivalently, by expected total utility divided by expected population size, as in Example 2.5.2.

2.6. **The Repugnant Conclusion.** A great deal of discussion of variable-population aggregation has centred around the 'Repugnant Conclusion' of Parfit [88]. This is the statement that for any history in which every individual has the same very high welfare level, there is a better history containing more people in which every individual has the same very low welfare level, corresponding to a life barely worth living. For example, this is a consequence of standard total utilitarianism, with the understanding that 'barely worth living' lives have positive utility. Many people find the Repugnant Conclusion, or variations on it, as repugnant as the name suggests (see e.g. Parfit [88]; Hammond [58]; Blackorby, Bossert and Donaldson [14]). In what follows, let w_0 be the welfare level of a life that is barely worth living, and W a much higher welfare level, representing an excellent quality of life. Let P_α be the prospect $\alpha 1_W + (1-\alpha)1_\Omega$, for $\alpha \in [0,1]$.

Under the conditions of our aggregation theorem, the Repugnant Conclusion amounts to the claim that $1_{w_0} \succ_{\mathbb{P}^*} P_\alpha$, for some rational probability $\alpha > 0$. There are, at least formally, many ways in which this claim about prospects can be denied. This illustrates the extent to which social preorders satisfying the aggregation theorem can deviate from total utilitarianism. We will study such deviations systematically in sections 3 and 4, but it may be useful to give a less formal preview from the point of view of the Repugnant Conclusion.

As we explain in section 3.3, especially Proposition 3.4.2, the social preorder has an additive representation like that of total utilitarianism as long as the individual preorder satisfies the Strong Independence axiom. However, even accepting Strong Independence, there are still several natural approaches to avoiding the Repugnant Conclusion.

First, we can hold that $P_\alpha \succsim_{\mathbb{P}^*} 1_{w_0}$ for all $\alpha \in [0,1]$. In particular, for $\alpha = 0$, $1_\Omega \succsim_{\mathbb{P}^*} 1_{w_0}$. Thus if $1_c \sim_{\mathbb{P}^*} 1_\Omega$, we have a version of critical level utilitarianism in which the critical level c is at least w_0. Note, though, that there is some tension between accepting this comparison and claiming that w_0 is 'worth living'.

Second, it is possible to maintain that $P_\alpha \succsim_{\mathbb{P}^*} 1_{w_0}$ for every $\alpha > 0$, but, corresponding to $\alpha = 0$, $1_{w_0} \succ_{\mathbb{P}^*} 1_\Omega$. This requires that the individual preorder violate the Archimedean axiom, and naturally suggests that the value difference between W and Ω is 'infinitely greater' than the one between w_0 and Ω. On this kind of theory, lives at level w_0 contribute positively to the social value of a population, but no number of lives at level w_0 can contribute more than even one life at level W.

Finally, we can continue to deny that $1_{w_0} \succ_{\mathbb{P}^*} P_\alpha$ for any $\alpha > 0$, but allow that $1_{w_0} \curlywedge_{\mathbb{P}^*} W_\alpha$ for some α. Thus we allow the individual preorder to be incomplete.

On the most natural version of this view, the incomparability holds for all α small enough, and, in particular, $1_{w_0} \curlywedge_{\mathbb{P}^*} 1_\Omega$. This is the kind of theory developed by Broome [26] and Blackorby, Bossert and Donaldson [17]. In terms of the Repugnant Conclusion, the theory holds that some population at level W is not worse than any population at level w_0, although such populations may be incomparable if the latter is sufficiently large.

The second and third approaches just discussed involve violations of expected utility theory for the individual and social preorders. Although we will not make a broader evaluation of these approaches, the Repugnant Conclusion and similar issues provide a reason to consider such violations, as we do in section 3. In particular, section 3.4 provides a general characterization of the situation in which the individual preorder satisfies Strong Independence, but is allowed to violate completeness and the Archimedean axiom.

3. Expected utility

Given a view about the social preorder, one might ask what it implies about the individual preorder. To use examples already given, one might ask which classes of individual preorder generate given classes of apparently egalitarian social preorders, or avoid the Repugnant conclusion. Answering such questions could be seen as a kind of reverse engineering. But we now focus on the opposite direction: what can be said about the social preorder, given that the individual preorder satisfies various standard conditions? This project is prompted by the view, often expressed in social choice theory, that it is better to impose conditions on the individual preorder and then derive them for the social preorder, rather than impose them directly on the social preorder. Section 5.3 will give a concrete example to illustrate why; first we give general results.

One of the key features of our approach is that it is compatible with a wide variety of both expected and non-expected utility theories. The next section looks at the latter, while this section examines the former. Section 3.1 shows that the social preorder inherits (properties expressed by) the most normatively central expected utility axioms from the individual preorder, in the sense that under the conditions of our aggregation theorems, if the individual preorder satisfies a given axiom, then so does the social preorder. Perhaps more strikingly, section 3.2 shows the equivalence (under the aggregation theorems) of various Pareto, independence, and separability axioms. This illustrates how conditions on the individual preorder restrain how the value of lotteries depends aggregatively on the value of prospects. Section 3.3 shows that the social preorder inherits a variety of expected-utility-style integral representations from the individual preorder. One of these is fairly general, allowing for failures of both continuity and completeness. Section 3.4 shows that the recognizably 'Harsanyi-like' flavour of these results can be generalized still further. In particular, the main result of this section is that a social preorder which satisfies the conditions of our aggregation theorems has a Harsanyi-like 'total utility' mixture-preserving representation in a preordered vector space if and only if it is generated by an individual preorder that satisfies Strong Independence. We then show that this result can be expressed in terms of a lexicographic mixture-preserving representation involving a family of real-valued utility functions. This result rests on what appears to be a new fundamental structure theorem for preordered vector spaces. Section 3.5 sums up the initial case for defining utilitarian social preorders to be precisely those social preorders generated by individual preorders which satisfy Strong Independence. We mostly state results in the more general variable population setting only. In such cases, analogous results for the constant population case are easily obtained.

3.1. **Axioms.** At the heart of expected utility theory is the notion of independence. Several different independence axioms are possible, and, like other axioms from expected utility theory, they can be posited separately for either the individual or the social preorder. Thus we state them generically for a preorder $\succsim_X$ on a convex set X.

Independence axioms. Suppose given p, p', $q \in X$ and $\alpha \in (0,1)$.
($\mathrm{I_a}$) $p \sim_X p' \implies \alpha p + (1-\alpha)q \sim_X \alpha p' + (1-\alpha)q$.
($\mathrm{I_b}$) $p \succ_X p' \implies \alpha p + (1-\alpha)q \succ_X \alpha p' + (1-\alpha)q$.
($\mathrm{I_c}$) $p \curlywedge_X p' \implies \alpha p + (1-\alpha)q \curlywedge_X \alpha p' + (1-\alpha)q$.

Let ($\mathrm{I_1}$) := ($\mathrm{I_a}$), ($\mathrm{I_2}$) := ($\mathrm{I_a}$) $\wedge$ ($\mathrm{I_b}$), and ($\mathrm{I_3}$) := ($\mathrm{I_a}$) $\wedge$ ($\mathrm{I_b}$) $\wedge$ ($\mathrm{I_c}$). These seem to be the reasonable packages of independence axioms. In particular, ($\mathrm{I_3}$) is equivalent to perhaps the best known independence axiom, Strong Independence, that is, $p \succsim_X p' \iff \alpha p + (1-\alpha)q \succsim_X \alpha p' + (1-\alpha)q$.

Just as Omega Independence only quantified over scalars in $(0,1) \cap \mathbb{Q}$, we similarly define the *Rational Independence* axioms ($\mathrm{I_i^{\mathbb{Q}}}$) for $i = 1, \ldots, 3$ as the corresponding independence axioms, but with α restricted to $(0,1) \cap \mathbb{Q}$. Contrary to what has sometimes been claimed, ($\mathrm{I_i}$) and ($\mathrm{I_i^{\mathbb{Q}}}$) strictly increase in strength with i.

The following complete the main expected utility axioms.

Ordering (O). $\succsim_X$ is an ordering (a complete preorder).

Archimedean axiom (Ar). For all $p, q, r \in X$, $p \succ_X q \succ_X r$ implies that there exist $\alpha, \beta \in (0,1)$ such that $\alpha p + (1-\alpha)r \succ_X q$ and $q \succ_X \beta p + (1-\beta)r$.

Given ($\mathrm{I_3}$) and (O), (Ar) is equivalent to the following axiom, which may, however, be more natural than (Ar) when $\succsim_X$ is incomplete.[19]

Mixture Continuity (MC). For all $p, q, r \in X$, the set $\{\alpha \in [0,1] : \alpha p + (1-\alpha)r \succsim_X q\}$ is closed in $[0,1]$, as is the set $\{\alpha \in [0,1] : q \succsim_X \alpha p + (1-\alpha)r\}$.

When X is equipped with a topology, many continuity conditions typically stronger than (MC) have been considered. The following is the most popular.

Continuity (C). $\{p \in X : p \succsim_X q\}$ and $\{p \in X : q \succsim_X p\}$ are closed for all $q \in X$.

One can only expect nice results about (C) if the basic operations on prospects and lotteries are themselves continuous. Say that *mixing is continuous* on X if for any $\lambda \in (0,1)$, $\lambda p + (1-\lambda)q$ is a continuous function of $p, q \in X$. In the constant population case, the basic assumption is as follows.

Topology (Top). $\mathbb{P}$ and $\mathbb{L}$ have topologies such that $\mathcal{L}$ and all the maps $\mathcal{P}_i$ are continuous, and mixing is continuous on $\mathbb{P}$.

In the variable population case, we need a further condition on the topology of $\mathbb{L}^*$ that allows us to pass from continuity on each $\mathbb{L}^*_{\mathbb{I}}$ to continuity on $\mathbb{L}^*$ itself. Say that $\mathbb{L}^*$ is *topologically coherent* if it satisfies the following condition: $U \subset \mathbb{L}^*$ is closed if and only if $U \cap \mathbb{L}^*_{\mathbb{I}}$ is closed in $\mathbb{L}^*_{\mathbb{I}}$ for every $\mathbb{I}$. (Here $\mathbb{L}^*_{\mathbb{I}}$ has a topology as a subspace of $\mathbb{L}^*$.) As a standard example, suppose that $\mathbb{W}^*$ is a topological space, and each $\mathbb{H}^*_{\mathbb{I}}$ has the product topology. Using the Borel sigma algebras, we can give $\mathbb{P}^*$ the weak topology, and define the topology on $\mathbb{L}^*$ by the condition that U

[19]See Dubra [46] for a discussion of the relationship between (Ar) and (MC) (or 'Herstein-Milnor continuity').

is closed if and only if $U \cap \mathbb{L}^*_{\mathbb{I}}$ is closed in the weak topology on $\mathbb{L}^*_{\mathbb{I}}$, for every $\mathbb{I}$.[20] Thus in the variable population case we use

> **Topology (Variable Population)** (Top*). $\mathbb{P}^*$ and $\mathbb{L}^*$ have topologies such that all the maps $\mathcal{L}_{\mathbb{I}}$ and $\mathcal{P}^*_i$ are continuous, mixing is continuous on $\mathbb{P}^*$, and $\mathbb{L}^*$ is topologically coherent.

Proposition 3.1.1 (Inheritance)**.**
CONSTANT POPULATION. *Suppose that* $\succsim$ *is generated by* $\succsim_{\mathbb{P}}$*. Then*

(i) Each of (O), ($\mathrm{I_i}$), ($\mathrm{I_i^{\mathbb{Q}}}$), (Ar), *and* (MC), $i = 1, 2, 3$, *is satisfied by* $\succsim$ *if and only if it is satisfied by* $\succsim_{\mathbb{P}}$ *.*
(ii) Assuming (Top), $\succsim$ *satisfies* (C) *if and only* $\succsim_{\mathbb{P}}$ *does.*

VARIABLE POPULATION. *Suppose that* $\succsim^*$ *is generated by* $\succsim_{\mathbb{P}^*}$*. Then*

(iii) Each of (O), ($\mathrm{I_i}$), ($\mathrm{I_i^{\mathbb{Q}}}$), (Ar), *and* (MC), $i = 1, 2, 3$, *is satisfied by* $\succsim^*$ *if and only if it is satisfied by* $\succsim_{\mathbb{P}^*}$*.*
(iv) Assuming (Top*), $\succsim^*$ *satisfies* (C) *if and only* $\succsim_{\mathbb{P}^*}$ *does.*

Thus the most normatively central expected utility axioms are all inherited by the social preorder. Similar results hold for many other normatively natural expected utility axioms.[21]

3.2. **Pareto, separability, and independence.** We will continue to consider both constant and variable population settings. However, even in the constant population setting, the next result is most striking, and easiest to state, if we consider a family of constant population models with populations of different sizes. If one accepts the assumptions of Theorem 1.3.1 for one constant population, it is natural to accept them for every constant population. The same goes for various conditions like Pareto or Separability (to be formulated below).

Formally, suppose given a set of welfare levels $\mathbb{W}^g$, a convex set of prospects $\mathbb{P}^g$, and an individual preorder $\succsim_{\mathbb{P}^g}$. (The superscript g stands for 'generic', for reasons to appear momentarily.) Also assume given an infinite population $\mathbb{I}^*$. Then a *family $\mathcal{F}$ of constant population models* consists of a constant population model $\langle \mathbb{I}, \mathbb{W}^g, \mathbb{P}^g, \succsim_{\mathbb{P}^g}, \mathbb{H}_{\mathbb{I}}, \mathbb{L}_{\mathbb{I}}, \succsim_{\mathbb{I}} \rangle$ for each finite $\mathbb{I} \subset \mathbb{I}^*$. Note that $\mathbb{W}^g$, $\mathbb{P}^g$, and $\succsim_{\mathbb{P}^g}$ are independent of $\mathbb{I}$.

In fact, this framework is generic between constant and variable population settings, in the following way. Given a variable population model, we can define $\mathbb{W}^g = \mathbb{W}^*$, $\mathbb{P}^g = \mathbb{P}^*$, $\succsim_{\mathbb{P}^g} = \succsim_{\mathbb{P}^*}$, $\mathbb{H}_{\mathbb{I}} = \mathbb{H}^*_{\mathbb{I}}$, $\mathbb{L}_{\mathbb{I}} = \mathbb{L}^*_{\mathbb{I}}$, and let $\succsim_{\mathbb{I}}$ be the restriction of $\succsim^*$ to $\mathbb{L}^*_{\mathbb{I}}$. The result is formally a 'family of constant population models', even though the set $\mathbb{W}^g$ of welfare levels includes Ω. In this way, results about families of constant population models immediately imply results about variable population models.

With this background, let us consider *Pareto* and *Separability* conditions on a family $\mathcal{F}$. Because the individual preorder can be incomplete, Pareto conditions need to be defined with some care.

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[20] It does not follow automatically that the topology on $\mathbb{L}^*_{\mathbb{I}}$ as a subspace of $\mathbb{L}^*$ equals the weak topology, as one might wish. But this does follow if $\{1_\Omega\}$ is closed in $\mathbb{P}^*$, which is guaranteed e.g. if $\mathbb{W}^*$ is metrizable Bogachev [20, Corollary 8.2.4].

[21] For example, the social preorder also inherits various finite dominance axioms, and suitably restricted countable dominance axioms (cf. Fishburn [50] and Hammond [61]); the restriction arises because $\mathbb{L}$ or $\mathbb{L}^*$ may not be closed under countable mixing operations. In the topological setting, other continuity conditions may be of interest, such as the one Dubra, Maccheroni and Ok [47] use to obtain an EUT representation for incomplete preorders. It is again easy to see that this condition is inherited in the constant population case, but we do not know whether it is inherited in by $\succsim^*$. (Note, though, that various forms of EUT representation *are* inherited, as in Proposition 3.3.1 below.)

We first define $\approx^{\mathbb{J}}_{\mathbb{P}^g}$, $\rhd^{\mathbb{J}}_{\mathbb{P}^g}$ and $\bowtie^{\mathbb{J}}_{\mathbb{P}^g}$. For any lotteries L and L' in $\mathbb{L}_{\mathbb{I}}$ and $\mathbb{J} \subset \mathbb{I}$:

$$\begin{aligned} L \approx^{\mathbb{J}}_{\mathbb{P}^g} L' &\iff \mathcal{P}_i(L) \sim_{\mathbb{P}^g} \mathcal{P}_i(L') \text{ for all } i \in \mathbb{J} \\ L \rhd^{\mathbb{J}}_{\mathbb{P}^g} L' &\iff \mathcal{P}_i(L) \succ_{\mathbb{P}^g} \mathcal{P}_i(L') \text{ for all } i \in \mathbb{J} \\ L \bowtie^{\mathbb{J}}_{\mathbb{P}^g} L' &\iff \mathcal{P}_i(L) \curlywedge_{\mathbb{P}^g} \mathcal{P}_i(L'), \mathcal{P}_i(L) \sim_{\mathbb{P}^g} \mathcal{P}_j(L), \text{ and} \\ &\qquad \mathcal{P}_i(L') \sim_{\mathbb{P}^g} \mathcal{P}_j(L') \text{ for all } i, j \in \mathbb{J} \end{aligned}$$

We might read $\approx^{\mathbb{J}}_{\mathbb{P}^g}$, $\rhd^{\mathbb{J}}_{\mathbb{P}^g}$ and $\bowtie^{\mathbb{J}}_{\mathbb{P}^g}$ as, respectively, equally good, better, and equi-incomparable for all members of $\mathbb{J}$. To explain the last of these, suppose $\mathbb{I} = \{1, 2\}$ and consider the inference: $\mathcal{P}_i(L) \curlywedge_{\mathbb{P}^g} \mathcal{P}_i(L')$ for $i = 1, 2 \implies L \curlywedge_{\mathbb{I}} L'$. This may seem natural: if L and L' are incomparable for both 1 and 2, they are incomparable. But suppose $\mathbb{W}^g$ includes welfare levels a and b, and define histories $h = [a, b]$ and $h' = [b, a]$. Treating welfare levels and histories as degenerate prospects and lotteries, suppose $a \curlywedge_{\mathbb{P}^g} b$. Then the inference just considered implies $h \curlywedge_{\mathbb{I}} h'$. But this violates any standard formulation of anonymity (in our framework, Two-Stage Anonymity). The use of $\bowtie^{\mathbb{J}}_{\mathbb{P}^g}$ in the following blocks this kind of inference.

Pareto axioms. Suppose given $\mathbb{I} \subset \mathbb{I}^*$, $L, L' \in \mathbb{L}_{\mathbb{I}}$, and a partition $\mathbb{I} = \mathbb{J} \sqcup \mathbb{K}$ with $\mathbb{J} \neq \emptyset$.
(P_a) $L \approx^{\mathbb{I}}_{\mathbb{P}^g} L' \implies L \sim_{\mathbb{I}} L'$.
(P_b) $L \rhd^{\mathbb{J}}_{\mathbb{P}^g} L'$ and $L \approx^{\mathbb{K}}_{\mathbb{P}^g} L' \implies L \succ_{\mathbb{I}} L'$.
(P_c) $L \bowtie^{\mathbb{J}}_{\mathbb{P}^g} L'$ and $L \approx^{\mathbb{K}}_{\mathbb{P}^g} L' \implies L \curlywedge_{\mathbb{I}} L'$.

We will focus on the natural packages $(P_1) \coloneqq (P_a)$, $(P_2) \coloneqq (P_a) \wedge (P_b)$, and $(P_3) \coloneqq (P_a) \wedge (P_b) \wedge (P_c)$. Some of these have familiar names. (P_1) is Pareto Indifference; (P_2) is known as Strong Pareto, though we will shortly question this label; but (P_3) appears to be novel.

Separability assumptions only make sense under some further domain conditions. Suppose that we have finite populations $\mathbb{J} \subset \mathbb{I}$. We want to be able to 'restrict' lotteries in $\mathbb{L}_{\mathbb{I}}$ to the subpopulation $\mathbb{J}$. There is a natural projection $(\mathbb{W})^{\mathbb{I}} \to (\mathbb{W})^{\mathbb{J}}$. We have to assume that this restricts to a measurable function $\mathbb{H}_{\mathbb{I}} \to \mathbb{H}_{\mathbb{J}}$, resulting in a function $\mathbb{L}_{\mathbb{I}} \to \mathbb{L}_{\mathbb{J}}$. This maps a lottery $L \in \mathbb{L}_{\mathbb{I}}$ to its *restriction* $L|_{\mathbb{J}}$. Thus the following axioms presuppose that, in this sense, *restrictions exist*.

Separability axioms. Suppose given $\mathbb{I} \subset \mathbb{I}^*$, $L, L' \in \mathbb{L}_{\mathbb{I}}$, and a partition $\mathbb{I} = \mathbb{J} \sqcup \mathbb{K}$ with $\mathbb{J} \neq \emptyset$.
(S_a) $L|_{\mathbb{M}} \sim_{\mathbb{J}} L'|_{\mathbb{M}}$ and $L|_{\mathbb{K}} \sim_{\mathbb{K}} L'|_{\mathbb{K}} \implies L \sim_{\mathbb{I}} L'$.
(S_b) $L|_{\mathbb{M}} \succ_{\mathbb{J}} L'|_{\mathbb{M}}$ and $L|_{\mathbb{K}} \sim_{\mathbb{K}} L'|_{\mathbb{K}} \implies L \succ_{\mathbb{I}} L'$.
(S_c) $L|_{\mathbb{M}} \curlywedge_{\mathbb{J}} L'|_{\mathbb{M}}$ and $L|_{\mathbb{K}} \sim_{\mathbb{K}} L'|_{\mathbb{K}} \implies L \curlywedge_{\mathbb{I}} L'$.

We consider the natural combinations $(S_1) \coloneqq (S_a)$, $(S_2) \coloneqq (S_a) \wedge (S_b)$, and $(S_3) \coloneqq (S_a) \wedge (S_b) \wedge (S_c)$. When $L|_{\mathbb{K}} \sim_{\mathbb{K}} L'|_{\mathbb{K}}$, (S_3) says that the members of $\mathbb{K}$ can be ignored in the comparison between L and L'. That is to say, $L \succsim_{\mathbb{I}} L' \iff L|_{\mathbb{J}} \succsim_{\mathbb{J}} L'|_{\mathbb{J}}$. Thus (S_3) can be seen as an axiom of strong separability across individuals.

Separability is most interesting when the lotteries faced by $\mathbb{J}$ and $\mathbb{K}$ can vary independently. As a weak form of this independence, say that the family of models is *compositional* if, for any partition $\mathbb{I} = \mathbb{J} \sqcup \mathbb{K}$, and any $P, Q \in \mathbb{P}^g$, there exists $L \in \mathbb{L}_{\mathbb{I}}$ such that $\mathcal{P}_j(L) = P$ for all $j \in \mathbb{J}$ and $\mathcal{P}_k(L) = Q$ for all $k \in \mathbb{K}$. For example, the family is compositional if each $\mathbb{H}^*_{\mathbb{I}} = (\mathbb{W}^*)^{\mathbb{I}}$ is equipped with the product sigma algebra and $\mathbb{L}_{\mathbb{I}}$ is the set of all lotteries on $\mathbb{H}_{\mathbb{I}}$ (Bogachev [20, Theorem 3.3.1]).

Proposition 3.2.1 (Equivalence of Pareto, Separability, and Independence)**.** *Suppose that each social preorder $\succsim_{\mathbb{I}}$ in the family $\mathcal{F}$ is generated by $\succsim_{\mathbb{P}^g}$. Suppose that restrictions exist and that $\mathcal{F}$ is compositional. Then, for $i = 1, 2, 3$:*

$$\mathcal{F} \textit{ satisfies } (\mathrm{S_i}) \iff \mathcal{F} \textit{ satisfies } (\mathrm{P_i}) \iff$$
$$\textit{every } \succsim_{\mathbb{I}} \textit{ satisfies } (\mathrm{I_i^{\mathbb{Q}}}) \iff \succsim_{\mathbb{P}^g} \textit{ satisfies } (\mathrm{I_i^{\mathbb{Q}}}).$$

Moreover, if $\mathcal{F}$ is obtained from a variable population model, we can omit the assumption that $\mathcal{F}$ is compositional.

This result has several lessons. First, given our aggregation theorems, there is no real normative difference between Pareto, separability, and independence. This more or less follows even from the basic constant population case. For example, fix a population of size n, assume that $\succsim_{\mathbb{P}}$ generates $\succsim$, and restrict the domain assumptions of the proposition accordingly. Then the proof of the proposition shows that if $\succsim$ satisfies the Pareto condition ($\mathrm{P_3}$), then $\succsim_{\mathbb{P}}$ satisfies the the independence condition ($\mathrm{I_3}$) for all $\alpha = \frac{m}{n} \in (0,1)$. When n is large, it is difficult to believe that there could be a normative case for accepting ($\mathrm{P_3}$) but rejecting the unrestricted version of ($\mathrm{I_3}$). From a technical point of view, however, ($\mathrm{I_3}$) is a more appealing condition. It only takes a very weak continuity assumption to pass from ($\mathrm{I_3^{\mathbb{Q}}}$) to ($\mathrm{I_3}$), but on the other hand, the right to left direction of the equivalences in the proposition hold without assuming $\mathcal{F}$ is compositional. In the sequel we present results which assume that the individual preorder satisfies ($\mathrm{I_3}$). If one preferred, one could instead use ($\mathrm{S_3}$) or ($\mathrm{P_3}$), along with the needed domain conditions and the weak continuity assumption. To anticipate, this could be used to provide an derivation of the conclusion of Harsanyi's anonymous theorem without assuming independence at either the individual or the social level.[22]

Second, the complications concerning incomparability and Pareto discussed in section 3.2 suggest caution in claiming that ($\mathrm{P_3}$) is the right way of extending the usual ($\mathrm{P_2}$) Pareto condition to say something 'Pareto-style' about incomparability. One question is whether ($\mathrm{P_3}$) is plausible, and the crucial issue is the status of its component ($\mathrm{P_c}$). Suppose first that $\mathbb{K}$ in the statement of ($\mathrm{P_c}$) is empty. Then ($\mathrm{P_c}$) is entailed by the conjunction of ($\mathrm{P_1}$) and the following plausible principle: $P \curlywedge_{\mathbb{P}^*} P' \implies \mathcal{L}^*_{\mathbb{I}}(P) \curlywedge^* \mathcal{L}^*_{\mathbb{I}}(P')$. In the general case where $\mathbb{K}$ can be non-empty, ($\mathrm{P_c}$) is then motivated by the kind of separability principle which underlies ($\mathrm{P_b}$), that of ignoring groups of indifferent individuals. The next question is whether ($\mathrm{P_3}$) is sufficiently strong. But ignoring technicalities, Proposition 3.2.1 shows that under the conditions of our aggregation theorems, each ($\mathrm{P_i}$) is equivalent to the corresponding ($\mathrm{I_i}$) for $i = 1, 2, 3$. Since ($\mathrm{I_3}$) is so well-established, this suggests that ($\mathrm{P_3}$) is both natural and appropriately strong. Terminology is unfortunate, however; while ($\mathrm{I_3}$) is Strong Independence, and ($\mathrm{S_3}$) is a principle of strong separability, it is ($\mathrm{P_2}$) which is customarily referred to as Strong Pareto. We suggest, rather, that it is the strictly stronger ($\mathrm{P_3}$) which should be called Strong Pareto.

Third, our aggregation theorems are compatible with the adoption of any non-expected utility theory for the individual preorder, provided only that Omega Independence is satisfied in the variable population case. This allows nonexpected utility theory to be easily inserted into a mainstream approach to aggregation. But Proposition 3.2.1 reveals a potential cost. Nonexpected utility theories typically reject every independence axiom. But given the assumptions of Theorem 2.3.1, rejecting any independence axiom requires rejecting the dual Pareto axiom. To its critics, this may be a further strike against nonexpected utility theory; to its defenders, it may be evidence for a hidden problem with Pareto.

3.3. **Expected utility representations.** Our aggregation theorems are neutral about whether the individual preorder satisfies ($\mathrm{I_3}$), that is, Strong Independence.

[22]Compare Mongin and Pivato [84, p. 159] for a similar observation in a different informational framework, and see also Pivato [90, pp.39–40].

But (I_3) is widely regarded as plausible, so we now begin further investigation of social preorders generated by individual preorders that satisfy (I_3).

This section focuses on expected-utility-style integral representations. We state these in terms of the generic preorder $\succsim_X$ on a convex set X, but we further assume $X = \mathcal{P}(Y)$ for some convex set of probability measures $\mathcal{P}(Y)$ on a set Y equipped with a sigma algebra. We say that $u\colon Y \to \mathbb{R}$ is *$\mathcal{P}(Y)$-integrable* just in case it is measurable and Lebesgue integrable with respect to all $p \in \mathcal{P}(Y)$. The basic form of an expected utility representation is as follows.

> **EUT** There is a $\mathcal{P}(Y)$-integrable function u such that $U\colon \mathcal{P}(Y) \to \mathbb{R}$, defined by $U(p) = \int_Y u\, dp$, represents $\succsim_X$.

But there are a number of ways in which $\succsim_X$ can violate EUT. First, if $\succsim_X$ violates the relatively weak continuity conditions (Ar) or (MC), it cannot satisfy EUT, but it may have a vector-valued expected-utility-style representation. The vector space can be preordered, allowing for the possibility that $\succsim_X$ violates (O) as well as (MC).

Here is the general set-up. First, recall from section 1.4 that a *preordered vector space* is a vector space $\mathbb{V}$ with a preorder $\succsim_{\mathbb{V}}$ such that $v \succsim_{\mathbb{V}} v' \iff \lambda v + w \succsim_{\mathbb{V}} \lambda v' + w$, for all $V, v', w \in \mathbb{V}$ and $\lambda > 0$. So $\mathbb{R}$ with the standard ordering is an example. Second, we need a way of integrating $\mathbb{V}$-valued functions. Suppose we have a set $\mathcal{A}$ of linear functionals on $\mathbb{V}$ that separates the points of $\mathbb{V}$. A function $u\colon Y \to \mathbb{V}$ is *weakly $\mathcal{P}(Y)$-integrable* (with respect to $\mathcal{A}$) if there exists $U\colon \mathcal{P}(Y) \to \mathbb{V}$ such that $\int_Y \Lambda \circ u\, dp = \Lambda \circ U(p)$ for all $\Lambda \in \mathcal{A}, p \in \mathcal{P}(Y)$. (In particular, every $\Lambda \circ u$ must be $\mathcal{P}(Y)$-integrable.) In this case we write $U(p) = \int_Y u\, dp$.[23]

> **Vector-EUT** For some preordered vector space $(\mathbb{V}, \succsim_{\mathbb{V}})$ and $\mathcal{A}$ a separating set of linear functionals on $\mathbb{V}$, there is a weakly $\mathcal{P}(Y)$-integrable function $u\colon Y \to \mathbb{V}$ such that $U(p) = \int_Y u\, dp$ represents $\succsim_X$.

Ordinary EUT is the special case where $\mathbb{V} = \mathbb{R}$, $\succsim_{\mathbb{V}}$ is the standard ordering, and $\mathcal{A}$ contains only the identity map $\mathbb{R} \to \mathbb{R}$.

The following propositions say that the social preorder inherits both EUT and vector-EUT representations from the individual preorder. Moreover, the social preorder is represented by *expected total utility.*

Proposition 3.3.1 (EUT Inheritance)**.**
CONSTANT POPULATION. *Suppose the social preorder $\succsim$ is generated by $\succsim_{\mathbb{P}}$.*

(i) Then $\succsim_{\mathbb{P}}$ satisfies EUT or Vector EUT if and only if $\succsim$ does too.
(ii) In either case, if $\succsim_{\mathbb{P}}$ is represented by $U(p) = \int_{\mathbb{W}} u\, dp$, then $\succsim$ is represented by $V(L) = \int_{\mathbb{H}} \sum_{i \in \mathbb{I}} (u \circ \mathcal{W}_i)\, dL$.

VARIABLE POPULATION. *Suppose the social preorder $\succsim^*$ is generated by $\succsim_{\mathbb{P}^*}$.*

(iii) Then $\succsim_{\mathbb{P}^*}$ satisfies EUT or Vector EUT if and only if $\succsim^*$ does too.
(iv) In either case, if $\succsim_{\mathbb{P}^*}$ is represented by $U(p) = \int_{\mathbb{W}^*} u\, dp$, then $\succsim^*$ is represented by $V(L) = \int_{\mathbb{H}^*} \sum_{i \in \mathbb{I}^*} (u \circ \mathcal{W}_i^* - u(\Omega))\, dL$.

The $u(\Omega)$ appearing in Proposition 3.3.1(iv) is needed to ensure that the sum has finitely many non-zero terms. Since, given u, $\succsim_{\mathbb{P}^*}$ also satisfies EUT with respect to $u - u(\Omega)$, this can be seen as merely normalising the utility function to have value zero at Ω. If there is a $c \in \mathbb{W}$ such that $1_c \sim_{\mathbb{P}^*} 1_\Omega$ (there need not be), then c is a critical level, and $u(c) = u(\Omega)$. Informally, however, we will speak of $u(\Omega)$ itself as a critical level.

[23] This integral is the weak (or Pettis) integral corresponding to the (locally convex) topological vector space whose topology is induced by $\mathcal{A}$ (Rudin [94, 3.10]). We will therefore refer to it as the weak integral. If $\mathbb{V} = \mathbb{R}^N$, with N an arbitrary set, and $\mathcal{A}$ is the set of projections $x \mapsto x_i$, $i \in N$, then this is the coordinatewise integral.

A popular approach to handling failures of (O) is to look for a set $\mathcal{U}$ of $\mathcal{P}(Y)$-integrable utility functions mapping Y to a completely preordered vector space $(\mathbb{V}, \succsim_{\mathbb{V}})$ such that $p \succsim_X p' \iff (\forall u \in \mathcal{U})(\int_Y u\, dp \succsim_{\mathbb{V}} \int_Y u\, dp')$ in the appropriate sense of integration.[24] Any such so-called 'multi-[vector] EUT' representation of the individual preorder will be inherited by the social preorder; just apply Proposition 3.3.1 to one $u \in \mathcal{U}$ at a time.

However, although multi-representations may be technically and heuristically useful, they do not really provide further generality as they convert to special cases of vector EUT representations. For write $\mathcal{U} = \{u_i \,|\, i \in \mathcal{I}\}$ for some index set $\mathcal{I}$. Let $(\mathbb{V}^+, \succsim_{\mathbb{V}^+})$ be the preordered vector space where $\mathbb{V}^+ := \mathbb{V}^{\mathcal{I}}$ and $x \succsim_{\mathbb{V}^+} y \iff (\forall i \in \mathcal{I})(x_i \succsim_{\mathbb{V}} y_i)$. Let $\mathcal{A}$ be the set of projections $x \mapsto x_i$ on $\mathbb{V}^+$. Define weakly $\mathcal{P}(Y)$-integrable (with respect to $\mathcal{A}$) $u^+ \colon Y \to \mathbb{V}^+$ by setting $(u^+(y))_i = u_i(y)$. Then $U^+(p) := \int_Y u^+ dp$ is a vector EUT representation of $\succsim_X$.

In the ordinary EUT version, the claim that the social preorder is represented by $V(L) = \int_{\mathbb{H}} \sum_{i \in \mathbb{I}} (u \circ \mathcal{W}_i)\, dp$ is the conclusion of Harsanyi's anonymous theorem. The considerably more general vector EUT version allows for failures of continuity and completeness, but maintains the 'total utility' form. Total utility representations are also preserved in the $V(L) = \int_{\mathbb{H}^*} \sum_{i \in \mathbb{I}^*} (u \circ \mathcal{W}_i^* - u(\Omega))\, dL$ representation in the variable population case, in both the ordinary and vector EUT cases. Proposition 3.3.1 shows that these representations arise from the assumptions of our aggregation theorems combined with a simple further assumption about the individual preorder. But it will turn out that even more general 'Harsanyi-like' total utility representations can be obtained from our aggregation theorems.

3.4. **Mixture-preserving representations.** Without going into details, the existence of integral expected-utility-style representations typically depends upon the individual preorder satisfying normatively natural axioms, like those in section 3.1, together with technical or structural conditions. If the latter fail, there may be no such representation.[25] By contrast, our aggregation theorems do not rely upon the individual or social preorders having any such representation. For example, Theorem 2.3.1 explicitly characterizes the social preorder in terms of the individual preorder, and Proposition 3.1.1 adds that if the individual preorder satisfies the normatively central expected utility axioms (O), (Ar) and (I_3), then so does the social preorder. This is also shown by Proposition 3.3.1, but only with further technical assumptions.

Nevertheless, we can bridge these two styles of results if we generalize the notion of an expected-utility-style representation. In particular, we allow utility functions to be vector-valued, require them to be mixture-preserving, but do not insist on an integral representation. Thus, a *mixture-preserving* (MP) representation of $\succsim_X$ is a mixture-preserving map $U \colon X \to \mathbb{V}$ to a preordered vector space $(\mathbb{V}, \succsim_{\mathbb{V}})$ such that $x \succsim_X y \iff U(x) \succsim_{\mathbb{V}} U(y)$.

Theorem 3.4.1. *$\succsim_X$ satisfies (I_3) if and only if it has an MP representation.*

(Compare this with Theorem 2.4.2. (I_3) corresponds exactly to mixture-preservation.)

The theorem applies in particular to $X = \mathbb{P}^*$. (For brevity we focus on the variable population case, the constant population case being exactly parallel.) Moreover:

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[24]The case where $(\mathbb{V}, \succsim_{\mathbb{V}}) = (\mathbb{R}, \geq)$ is discussed in Baucells and Shapley [8], Dubra, Maccheroni and Ok [47]; and Evren [49].

[25]See e.g. Fishburn [50] and Dillenberger and Krishna [43] for discussion.

Proposition 3.4.2. *Suppose $\succsim_{\mathbb{P}^*}$ generates $\succsim^*$. Then $\succsim_{\mathbb{P}^*}$ has an MP representation (i.e. satisfies* (I$_3$)*) if and only if $\succsim^*$ does. More precisely, if U is an MP representation of $\succsim_{\mathbb{P}^*}$, then $\sum_{i\in\mathbb{I}^*}(U\circ\mathcal{P}_i^* - U(1_\Omega))$ is an MP representation of $\succsim^*$.*

In many ways, this is the best result of section 3, and we could finish it here. First, it shows that we obtain an extremely general Harsanyi-like 'total utility' mixture-preserving representation of the social preorder based merely on the assumption that the individual preorder satisfies (I$_3$). It is worth remarking that the conceptual import of additive representations has often been taken to be separability (see e.g. Blackorby et al. [14]). In our framework, Proposition 3.2.1 shows that separability is all but equivalent to (I$_3$). Second, Propostion 3.4.2 opens up the use of theorems concerning preordered vector spaces to further analyze the social preorder.

In particular, we can apply structure theorems for preordered vector spaces to reinterpret an MP representation as a family of more familiar real-valued representations. Informally, elements of the value space $\mathbb{V}$ can be represented by matrices of real numbers. The space of row-vectors is lexicographically ordered, and one matrix ranks higher than another if and only if it ranks higher in each row.

Formally, a *lexicographic mixture-preserving* (LMP) representation of $\succsim_X$ consists of a family $\{U_{ab}\}$ of mixture-preserving functions $X \to \mathbb{R}$, indexed by a in a set $\mathcal{R}$ (for 'rows') and b in a completely ordered set $\mathcal{C}$ (for 'columns'). It must satisfy the following two conditions for all $p, q \in X$:

LMP1. For each $a \in \mathcal{R}$, either $U_{ab}(p) = U_{ab}(q)$ for all $b \in \mathcal{C}$, or there is a largest $b \in \mathcal{C}$ such that $U_{ab}(p) \neq U_{ab}(q)$.

LMP2. We have $p \succsim_X q$ if and only if, for each $a \in \mathcal{R}$, either $U_{ab}(p) = U_{ab}(q)$ for all $b \in \mathcal{C}$, or $U_{ab}(p) > U_{ab}(q)$ for the largest $b \in \mathcal{C}$ such that $U_{ab}(p) \neq U_{ab}(q)$.

Theorem 3.4.3. *$\succsim_X$ satisfies* (I$_3$) *if and only if it has an LMP representation. It satisfies* (I$_3$) *and* (O) *if and only if it has an LMP representation with $\#\mathcal{R} = 1$. It satisfies* (I$_3$) *and* (MC) *if and only if it has an LMP representation with $\#\mathcal{C} = 1$. And it satisfies* (I$_3$)*,* (O)*, and* (MC) *if and only if it has an LMP representation with $\#\mathcal{R} = \#\mathcal{C} = 1$.*

This shows, in particular, that the three main axioms of EUT are jointly equivalent to having a representation by a single mixture-preserving function $X \to \mathbb{R}$. Note also the special role of (MC) rather than (Ar).

The theorem applies in particular for $X = \mathbb{P}^*$. Moreover, in parallel to Proposition 3.4.2, we have:

Proposition 3.4.4. *Suppose $\succsim^*$ is generated by $\succsim_{\mathbb{P}^*}$. Then $\succsim_{\mathbb{P}^*}$ has an LMP representation (i.e. it satisfies* (I$_3$)*) if and only if $\succsim^*$ does. More precisely, if $\{U_{ab}\}$ is an LMP representation of $\succsim_{\mathbb{P}^*}$, then $\{\sum_{i\in\mathbb{I}^*}(U_{ab}\circ\mathcal{W}_i^* - U_{ab}(1_\Omega))\}$ is an LMP representation of $\succsim^*$.*

The proof of Theorem 3.4.3 is found in the appendix. It depends on the following fundamental structure theorem for preordered vector spaces $\mathbb{V}$.

Suppose first that $\succsim_{\mathbb{V}}^{\text{com}}$ is a complete preorder on $\mathbb{V}$. Say that a *lexicographic filtration* of $(\mathbb{V}, \succsim_{\mathbb{V}}^{\text{com}})$ is a pair $(\mathcal{O}, \mathcal{G})$ consisting of a set $\mathcal{O}$ of subspaces of $\mathbb{V}$ that are completely ordered by inclusion, and a family $\mathcal{G} = \{g_W\}_{W\in\mathcal{O}}$ of linear maps $g_W : W \to \mathbb{R}$, satisfying the following conditions:

(a) For each $v \in \mathbb{V}$ there is a smallest $W_v \in \mathcal{O}$ containing v, and every $W \in \mathcal{O}$ equals W_v for some v.

(b) For each $v \in W \in \mathcal{O}$, $g_W(v) = 0 \iff W_v \subsetneq W$ or $W = W_0$.

(c) For $v, w \in \mathbb{V}$ with $v, w \succsim_{\mathbb{V}}^{\text{com}} 0$,

$$v \succsim_{\mathbb{V}}^{\text{com}} w \iff W_v \supseteq W_w \text{ and } g_{W_v}(v) \geq g_{W_v}(w).$$

When $\succsim_{\mathbb{V}}$ is not complete, we consider complete extensions of it; here a preorder $(Y, \succsim_2)$ *extends* a preorder $(Y, \succsim_1)$ if for all $x, y \in Y$, $x \sim_1 y \implies x \sim_2 y$, and $x \succ_1 y \implies x \succ_2 y$.

Theorem 3.4.5. *(1) If $(\mathbb{V}, \succsim_{\mathbb{V}})$ is a preordered vector space, then $v \succsim_{\mathbb{V}} w$ if and only if $v \succsim_{\mathbb{V}}^{\mathrm{com}} w$ for all complete vector preorders $\succsim_{\mathbb{V}}^{\mathrm{com}}$ extending $\succsim_{\mathbb{V}}$.*

(2) If $(\mathbb{V}, \succsim_{\mathbb{V}}^{\mathrm{com}})$ is a completely preordered vector space, then it admits a lexicographic filtration $(\mathcal{O}, \mathcal{G})$. Moreover, the set $\mathcal{O}$ is uniquely determined, and each function $g_W \in \mathcal{G}$ is uniquely determined up to positive scale.

(3) If $\succsim_{\mathbb{V}}$ satisfies (MC)*, then in (1) we can consider only complete extensions $\succsim_{\mathbb{V}}^{\mathrm{com}}$ that satisfy* (MC)*. If $\succsim_{\mathbb{V}}^{\mathrm{com}}$ satisfies* (MC)*, then in (2) we have $\mathcal{O} = \{\mathbb{V}, \{v : v \sim_{\mathbb{V}}^{\mathrm{com}} 0\}\}$, and thus a single function $g_{\mathbb{V}}$ such that $v \succsim_{\mathbb{V}}^{\mathrm{com}} w \iff g_{\mathbb{V}}(v) \geq g_{\mathbb{V}}(w)$.*

Several similar theorems are known in the literature, going back to Hahn for ordered abelian groups; see Conrad [37] for references and results in even greater generality. The best-known result for ordered vector spaces in particular is due to Hausner and Wendel [69].[26] They show that every completely ordered vector space can be embeded in a lexicographically ordered function space, but their construction depends on transfinite induction and is extremely non-unique. In contrast, the lexicographic filtration that we emphasise is essentially unique and constructively definable, while amply illustrating the lexicographic nature of vector preorders.

3.5. **Utilitarian preorders.** We propose to define as *utilitarian preorders* those social preorders which are generated by some individual preorder which satisfies (I_3). There are two main reasons for this. First, Proposition 3.4.2 shows that these are precisely the social preorders which admit a Harsanyi-like mixture-preserving total utility representation of a very general form. Second, the premises of Theorem 2.3.1 seem to be mandatory for a utilitarian to accept, and when we further add (I_3) to the individual preorder, we immediately obtain further conditions which have a natural utilitarian flavor, namely (P_3) and (S_3). In fact, as already noted, those conditions could be used in place of (I_3). We further develop this argument in section 5.1, where we show that any social preorder which satisfies the conditions of Theorem 2.3.1 has features that are arguably essential to utilitarianism, namely indifference to both *ex ante* and *ex post* equality.

4. Non-expected utility

In the constant population case, our aggregation theorems are compatible with any non-expected utility theory at the individual level, provided the theory can be understood in terms of a preorder on probability measures. In the variable case, the individual preorder must satisfy Omega Independence, but we saw in Proposition 2.4.1 that this imposes very little restriction.

There are two reasons to further explore non-expected utility theory. The first is normative. Although independence remains very popular at the normative level, it continues to have its critics; see, for example, Buchak [28]. Parallel to the expected utility case, it is therefore natural to ask what non-expected utility conditions on the individual preorder imply about the social preorder.

The second is positive. Even if one sides with independence at the normative level, it is hard to ignore its widespread violation at the empirical level. Of course, some subjects may violate independence because they accept, for example, social judgments concerning fairness between individuals. The most famous of these is Diamond's example, to be discussed in section 5.1. But the empirical literature

[26] For a survey of more recent developments in lexicographic approaches to expected utility, see Pivato [90, §7.1]

has mostly focused on subjects who violate independence when only self-interest is at play, as in Allais's example. Such subjects may on occasion put themselves in the position of the social observer to make judgments about social distribution. But they are unlikely to suspend their views about risk in the process, and it is natural to ask whether their views about risk are reflected in their views about distribution, even in risk-free cases. Answering this first requires having models of what independence-violating judgments about risk imply about social distribution.

In what follows, we discuss separately and then in combination two standard approaches to non-expected utility, what we call axiomatic and functional approaches. This will lead to a further class of total utility representations, adding to the case for a utilitarian interpretation of our aggregation theorems.

4.1. **Axioms.** One strand of non-expected utility theory has been to articulate axioms which weaken independence in natural ways. Some non-expected utility axioms are straightforwardly inherited by the social preorder in both the constant and variable population cases. These include Betweenness, Quasiconcavity, Quasiconvexity, Very Weak Substitution, and Mixture Symmetry. In addition, Weak Substitution and Ratio Substitution are inherited in at least the constant population case.[27] These results follow easily from linearity of the map $L \mapsto p_L$.

These conditions are typically combined with (O) and (C) in the non-expected utility literature, but there is work aimed at allowing for failures of each of those conditions. Just to give one example, Karni and Zhou [71] propose an axiom they call Weak Substitution for Noncomparable Lotteries, a condition which relaxes Weak Substitution to accommodate incompleteness.[28] At least in the constant population case, this is also inherited by the social preorder.

Inheritance of other non-expected utility axioms is less straightforward, as they are designed for the case where the set of outcomes is an interval I of real numbers. But if $\mathbb{W} = I$, for example, then typically $\mathbb{H} = I^n$, and such axioms would not then apply to the social preorder.

Those axioms aside, the ease with which inheritance can be shown for the axioms so far discussed might lead one to guess that inheritance is quite ubiquitous. Nevertheless, some non-expected utility axioms are not inherited.

Say that a preorder $\succsim_X$ on $\mathcal{P}(Y)$ is *upper-measurable* if $U_x := \{y \in X\colon 1_y \succsim_X 1_x\}$ is measurable for every $x \in X$. Suppose $\succsim$ is upper-measurable. Define a preorder $\succsim_X^{\mathrm{SD}}$ on $\mathcal{P}(Y)$ by $p \succsim_X^{\mathrm{SD}} q \iff p(U_y) \geq q(U_y)$ for all $y \in Y$. We say that p *stochastically dominates* q when $p \succsim_X^{\mathrm{SD}} q$. Consider the following axiom, which requires consistency with stochastic dominance.

Monotonicity (M) For an upper-measurable preorder $\succsim_X$, (i) $p \sim_X^{\mathrm{SD}} q \implies p \sim_Y q$; and (ii) $p \succ_X^{\mathrm{SD}} q \implies p \succ_X q$.

This axiom is very widely assumed in non-expected utility theory. As a constraint on the individual preorder, it provides non-expected utility theorists with an answer to an earlier worry.

To explain, whereas Pareto relates superiority for each individual to social superiority, the following condition on the social preorder weakens it by relating stochastic dominance for each individual to social superiority.

SD-Pareto Suppose $\succsim^*$ is upper-measurable. Then
(i) $\mathcal{P}_i(L) \sim_{\mathbb{P}^*}^{\mathrm{SD}} \mathcal{P}_i(L')$ for all $i \in \mathbb{I}^* \implies L \sim^* L'$; and

[27] See Chew, Epstein, and Segal [33] for definitions of Quasiconcavity, Quasiconvexity, and Mixture Symmetry, and Chew [32] for the other axioms. Schmidt [97] provides a survey.

[28] For discussion of failures of continuity, see Schmidt [96].

(ii) $\mathcal{P}_i(L) \succsim^{\mathrm{SD}}_{\mathbb{P}^*} \mathcal{P}_i(L')$ for all $i \in \mathbb{I}^*$ and $\mathcal{P}_j(L) \succ^{\mathrm{SD}}_{\mathbb{P}^*} \mathcal{P}_j(L')$ for some $j \in \mathbb{I}^*$ $\implies L \succ^* L'$.

We noted in §3.2 that when the individual preorder violates an independence axiom, the dual Pareto axiom is violated. But when the individual preorder satisfies (M), there is a limit to the severity of Pareto violations.

Proposition 4.1.1. *Suppose $\succsim^*$ is generated by Omega Independent and upper-measurable $\succsim_{\mathbb{P}^*}$. Then $\succsim_{\mathbb{P}^*}$ satisfies (M) $\implies$ $\succsim^*$ satisfies SD-Pareto.*

In response to criticism of their forced rejection of Pareto axioms, non-expected utility theorists may therefore reply that they accept (M), and thus endorse SD-Pareto, and claim that the transition from SD-Pareto to full-blown Pareto involves stronger assumptions about risk than have been acknowledged.

However, the status of (M) in non-expected utility theory leads to some problems. When $\succsim_X$ satisfies EUT, it also satisfies (M). Thus when the individual preorder satisfies EUT, it automatically satisfies (M), and by Proposition 3.3.1, so does the social preorder. However, the following example shows that the social preorder does not in general inherit (M), even in the constant population case.

Example 4.1.2. Make the assumptions of Example 1.5.4, again with the concrete assumption that $r(x) = x^2$, and equip $\mathbb{W}$ and $\mathbb{H}$ with the Borel sigma algebras. Assume a population of two people. Then $\succsim$ ranks a history h with welfare levels $w_1 \leq w_2$ according to the aggregate score $V(h) = \frac{3}{4}w_1 + \frac{1}{4}w_2$. Both $\succsim_{\mathbb{P}}$ and $\succsim$ are upper measurable, and $\succsim_{\mathbb{P}}$ satisfies (M). Let $h_1 = [0,0]$, $h_2 = [-1,3]$ and $h_2 = [-2,6]$. Then $h_1 \sim h_2 \sim h_3$, so that $h_1 \sim^{\mathrm{SD}} L := \frac{1}{2}h_2 + \frac{1}{2}h_3$. But $U(p_{h_1}) = 0$ and $U(p_L) = -\frac{1}{4}$, hence $h_1 \succ L$, violating (M)(i). For a violation of (M)(ii), let $h_4 = [-\frac{1}{8}, -\frac{1}{8}]$. Then $L \succ^{\mathrm{SD}} h_4$ but $h_4 \succ L$.

This example reveals tension in a common line of thought. For in some variant, (M) has been seen as '[t]he most widely acknowledged principle of rational behavior under risk' (Schmidt [97, p. 19]). But it has also often been said that rationality requires applying at the social level whatever conditions one imposes at the individual level (e.g. Harsanyi [64, p. 637]).

One response would be insist that (M) does apply at the social level, and say so much the worse for non-expected utility theories which are forced to reject it at that level. But the following result suggests that this view would lead to a significant restriction on non-expected utility theories, which universally reject (I_3).

Proposition 4.1.3. *Make the assumptions of Theorem 2.3.1. Suppose that $\mathbb{H}^*$ consists of every logically possible history whose population is finite; $\mathbb{P}^*$ and $\mathbb{L}^*$ consist of all finitely supported prospects and lotteries respectively; and $\succsim_{\mathbb{P}^*}$ is complete and continuous.*[29] *Then $\succsim^*$ satisfies (M) $\implies$ $\succsim_{\mathbb{P}^*}$ satisfies* (I_3).

Thus it appears that the relationship between individual and social rationality is more complex, and that endorsing a non-expected utility theory which takes (M) as a premise at the individual level while rejecting (M) at the social level remains a live option. But this leads to a technical difficulty, as the representation theorem for that theory will not be directly applicable to the social preorder. We therefore now examine other ways of obtaining non-expected utility representations of the social preorder.

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[29] Say that a sequence (P_n) in $\mathbb{P}^*$ *converges strongly* to $P \in \mathbb{P}^*$ (written $P_n \xrightarrow{s} P$) whenever $P_n(U) \to P(U)$ for all measurable U in $\mathbb{W}^*$. The continuity condition we adopt is that whenever $P_n \xrightarrow{s} P$, (i) $P_n \succsim_{\mathbb{P}^*} Q$ for all $n \implies P \succsim_{\mathbb{P}^*} Q$; and (ii) $Q \succsim_{\mathbb{P}^*} P_n$ for all $n \implies Q \succsim_{\mathbb{P}^*} P$.

4.2. **The functional approach.** The axiomatic approach to non-expected utility theory tries to respect the normative plausibility of independence by focusing on axioms which only mildly weaken it. By contrast, what we will call the functional approach, pioneered by Machina [74], abandons independence entirely while imposing technical conditions on preorders which are just strong enough to allow one to apply expected utility techniques. Machina imposed representability by a Fréchet differentiable function, but this turned out to exclude a number of popular non-expected utility models. However, Chew et al. [34] and Chew and Nishimura [36] showed that Machina's main results could be obtained under the weaker assumption of representability by a Gâteaux differentiable function. Other differentiability concepts have since been discussed, but Gâteaux remains the most popular and has yielded many important applications; see, for example, Cerreia-Vioglio et al [30] for entry into the literature. For brevity, however, we only consider Gâteaux differentiability. We first define this and relate it to expected utility. We then give results concerning the inheritance of Gâteaux differentiability, and elaborate on the significance of these results for our aggregation theorems. Roughly speaking, we generalise Proposition 3.3.1(ii, iv), showing how 'local' expected utility theory for the individual preorder determines a 'locally' utilitarian social preorder.

Let P be a convex subset of a vector space. We say that $V\colon P \to \mathbb{R}$ is *Gâteaux differentiable at* $p \in P$ if the limit

$$V'_p(q-p) \coloneqq \lim_{t \to 0+} \frac{V(p + t(q-p)) - V(p)}{t}$$

exists for all $q \in P$.[30] Thus $V'_p(q-p)$ is a directional derivative of V at p in the direction $q-p$. We use obvious terminology, saying, for example, that V is Gâteaux differentiable when it is Gâteaux differentiable at all $p \in P$.

Suppose further that P is a convex set of finite signed measures on a measurable space Y. We say that V is *integrally Gâteaux differentiable at* $p \in P$ when it is Gâteaux differentiable at p and there exists a P-integrable $v_p\colon Y \to \mathbb{R}$ such that $V'_p(q-p) = \int_Y v_p \, \mathrm{d}(q-p)$ for all $q \in P$.[31] Let ∇V_p be the set of such v_p; thus $\nabla V_p \neq \varnothing$ if and only if V is integrally Gâteaux differentiable at p.

While Gâteaux differentiability has a simple interpretation, the integral version can be made intuitive by connecting it with expected utility theory, as we now explain. Suppose henceforth that P is a convex set of probability measures. A function $V\colon P \to \mathbb{R}$ is an *expected utility function* if there is a P-integrable function v such that, for any $q \in P$, $V(q) = \int_Y v \, dq$. Note that for any basepoint $p \in P$, we can rewrite this as $V(p + t(q-p)) = \int_Y v \, \mathrm{d}(p + t(q-p))$ for all $q \in P$ and $t \in [0,1]$. Following Machina [74], it is natural to say that V is a *local expected utility* (LEU) function at p if there is a measurable v_p satisfying this equation up to first order in t. To be precise, for each $q \in P$, v_p satisfies

$$V(p + t(q-p)) = \int_Y v_p \, \mathrm{d}(p + t(q-p)) + o(t) \quad \text{for } t \in [0,1]. \tag{3}$$

We say that V is an LEU function when there is such a v_p for every $p \in P$.

Suppose an ordering $\succsim_P$ on P has an LEU representation, i.e. is represented by an LEU function V. When $\int_Y v_p \, dq > \int_Y v_p \, dp$, (3) implies that for all sufficiently small $t > 0$, $p + t(q-p) \succ_P p$. Thus small linear perturbations of a point $p \in P$ are governed by classical expected utility theory with respect to the local utility

[30] Our notion of Gâteaux differentiability is very weak, as it only requires a one-sided limit, only considers $q \in P$, and does not make any topological assumptions.

[31] Similar definitions, but with more restrictions on v_p or Y, are found in Chew et al. [34], Chew and Mao [35], and Cerreia-Vioglio et al [30]. For example, the latter, from whom we borrow notation, assume v_p is continuous and bounded, but we make no such assumption.

function v_p. Machina's insight was that important global properties of orderings could be inferred from such local expected utility behavior. However, the following is immediate from definitions.

Lemma 4.2.1. *Suppose P is a convex set of probability measures. Then $V : P \to \mathbb{R}$ is an LEU function if and only if it is integrally Gâteaux differentiable.*

Proposition 4.2.2 (Gâteaux: constant population inheritance)**.** *Suppose $\succsim_{\mathbb{P}}$ generates $\succsim$. Suppose $\succsim_{\mathbb{P}}$ can be represented by a function $U : \mathbb{P} \to \mathbb{R}$. Fix $L \in \mathbb{L}$. Then*

(i) $\succsim$ can be represented by $V : \mathbb{L} \to \mathbb{R}$ defined by $V(L) := \#\mathbb{I}U(p_L)$.
(ii) If U is Gâteaux differentiable at p_L, then so is V is at L.
(iii) If U is integrally Gâteaux differentiable at p_L, then so is V at L; moreover, for any $u_L \in \nabla U_{p_L}$, we have $\sum_{i\in\mathbb{I}} u_L \circ \mathcal{W}_i \in \nabla V_L$.

This result has two significant implications. Let us say a preorder is *Gâteaux representable* if it is representable by a (integrally or otherwise) Gâteaux differentiable function. First, then, the social preorders generated by the Gâteaux representable individual preorders inherit Gâteaux representability. Thus the wide range of tools and results associated with the Gâteaux concepts can be accessed at the social level merely by imposing Gâteaux representability at the individual level.

Second, specialising now to integral Gâteaux representations, the resulting social preorders have a natural interpretation. In classical (Harsanyi-style) utilitarianism, the constant population social welfare function V is given by expected total utility. Adapting our discussion of local expected utility, it is natural to say that V is *locally utilitarian* at a lottery $L \in \mathbb{L}$ if this is true to first order for linear perturbations of L, i.e. (more precisely) if there exists a utility function $u_L : \mathbb{W} \to \mathbb{R}$ such that, for each fixed $M \in \mathbb{L}$,

$$(4)\quad V(L + t(M - L)) = \int_{\mathbb{H}} \left(\sum_{i\in\mathbb{I}} u_L \circ \mathcal{W}_i \right) \mathrm{d}(L + t(M - L)) + o(t) \quad \text{for } t \in [0, 1].$$

Proposition 4.2.2(iii) shows that, under the conditions of our constant population aggregation theorem, if the individual preorder is integrally Gâteaux representable, then the social preorder has a locally utilitarian representation. This is a 'local' version of Proposition 3.3.1(ii) and bolsters the utilitarian interpretation of the aggregation theorem. In fact, the locally utilitarian representations constructed in Proposition 4.2.2(iii) are 'more utilitarian' than suggested merely by (4), because the local utility function u_L is characterised in terms of the individual preorder, as it is in ordinary utilitarianism.[32]

Let us extend this result to the variable population setting. We will show that if the individual preorder has an integral Gâteaux representation U^*, satisfying one extra condition, then the social preorder has a local utilitarian representation, now including a critical level; this is a local version of Proposition 3.3.1(iv).

For any $L \in \mathbb{L}^*_{\mathbb{I}}$, write

$$(5)\qquad V^*(L) := \#\mathbb{I}U^*(p^{\mathbb{I}}_L) - \#\mathbb{I}U^*(1_\Omega).$$

It is easy to check that this is a well-defined function of $L \in \mathbb{L}^*$, independent of $\mathbb{I}$, if U^* is 'Omega-linear' in the sense that, for all $P \in \mathbb{P}^*$ and $\alpha \in [0, 1]$,

$$(6)\qquad U^*(\alpha P + (1 - \alpha)1_\Omega) = \alpha U^*(P) + (1 - \alpha)U^*(1_\Omega).$$

[32]A slightly different notion of local utilitarianism was discussed by Machina [74, §5.2]. His notion applies to social welfare functions on histories, or rather on 'wealth distributions', which he idealises as probability measures on $\mathbb{W}$.

Note that Omega-linearity is a natural correlate of Omega Independence, which it immediately implies. Of course, there may be ways to define an appropriate V^* when U^* is not Omega-linear, but this particular version gains interest from the following two propositions.

Proposition 4.2.3 (Gâteaux: variable population inheritance)**.** *Suppose $\succsim_{\mathbb{P}^*}$ generates $\succsim^*$. Assume that the sigma algebra on $\mathbb{H}^*$ is coherent.*[33] *Suppose $\succsim_{\mathbb{P}^*}$ can be represented by an Omega-linear function $U^*\colon \mathbb{P}^* \to \mathbb{R}$. Fix $L \in \mathbb{L}^*_{\mathbb{I}}$. Then*

(i) $\succsim^*$ can be represented by $V^*\colon \mathbb{L}^* \to \mathbb{R}$ as defined by (5)*;*
(ii) If U^* is Gâteaux differentiable at $p^{\mathbb{I}}_L$, then so is V^* at L;
(iii) If U^* is integrally Gâteaux differentiable at $p^{\mathbb{I}}_L$, then so is V^* at L; moreover, for any $u_L \in \nabla U^*_{p^{\mathbb{I}}_L}$, we have $\sum_{i\in\mathbb{I}^*}(u_L \circ \mathcal{W}^*_i - u_L(\Omega)) \in \nabla V^*_L$.

The constant and variable population cases are related in the following result.

Proposition 4.2.4 (Gâteaux: extension)**.** *Let $\mathbb{P}^*$ include $\mathbb{P}$.*[34] *Suppose $\succsim_{\mathbb{P}}$ is representable by a Gâteaux differentiable function $U\colon \mathbb{P} \to \mathbb{R}$. Fix any critical level $c \in \mathbb{R}$. Then U can be extended to an Omega-linear Gâteaux differentiable function $U^*\colon \mathbb{P}^* \to \mathbb{R}$ with $U^*(1_\Omega) = c$, and the preorder $\succsim_{\mathbb{P}^*}$ represented by U^* is Omega-independent and includes $\succsim_{\mathbb{P}}$. If U is integrally Gâteaux differentiable, then we can choose U^* to be integrally Gâteaux differentiable on $\mathbb{P}^* \setminus \{1_\Omega\}$.*

One could extend these results to the Vector LEU case defined in parallel to Vector EUT, but we will not pursue this.

In summary, let us recapitulate for the integral case. If the individual preorder is integrally Gâteaux representable in the constant population case, then it can be embedded in many Omega Independent individual preorders in the variable population case, with free choice of the critical level, which are also integrally Gâteaux representable (except perhaps at 1_Ω), and the constant and variable social preorders these generate are locally utilitarian.

4.3. **In combination.** To pursue both the normative and positive projects outlined in the introduction to this section, one wishes to impose natural conditions on the individual preorder, then derive conclusions about the social preorder. But the axiomatic and functional approaches face complementary difficulties. The axiomatic approach articulates natural conditions on the individual preorder, but it may not be easy to work out what they imply about the social preorder. The conditions may not be inherited by the social preorder, and even if they are, the social domain conditions may be too complicated for the relevant representation theorems to be directly applicable. Conversely, it is far from obvious what normative or behavioral significance technical conditions such as Gâteaux representability have, so even when they are inherited by the social preorder, it may be unclear why this matters. In combination, however, each approach may remedy the other's defects. Much of the interest in Gâteaux representability arises because preorders which satisfy natural non-expected utility axioms turn out to be Gâteaux representable. Thus by using the results of the previous section, we obtain Gâteaux representations of the social preorder merely by imposing natural conditions on the individual preorder.

[33]We defined 'coherent' just before Proposition 2.3.3. This assumption is only necessary for the results concerning integral Gâteaux differentiability.

[34]This was defined in section 2.4.

Having defined utilitarian preorders as those social preorders generated by individual preorders which satisfy (I_3), it is natural to say that social preorders generated by any individual preorder are *generalized utilitarian*.[35] This proposal is supported by the fact that even without (I_3), many natural conditions on the individual preorder are at least compatible with, and sometimes guarantee,[36] integral Gâteaux representations of the individual preorder, and in both the constant and variable population settings, those generate locally utilitarian social preorders. In addition, Proposition 3.4.2 makes it natural to label *non-expected utilitarian* those generalized utilitarian preorders generated by individual preorders which violate (I_3).

5. Comparisons

We now relate our aggregation theorems to several standard topics: egalitarianism; the *ex ante* versus *ex post* distinction; and interpersonal comparisons. We end with a comparison to Harsanyi's approach to social aggregation.

5.1. **Generalized utilitarianism and egalitarianism.**

We know of no discussion of generalized utilitarian preorders. Our goal in this section is to contrast them with egalitarian social preorders. But first we show that they form a rich class.

To simplify the discussion, let us assume that $\mathbb{H}^*$ is the set of all possible histories with finite populations, and that $\mathbb{H} \subset \mathbb{H}^*$ is a set of all possible histories with some constant population. Let $\mathbb{P}^*$ and $\mathbb{P}$ be the sets of prospects with finite support on $\mathbb{W}^*$ and $\mathbb{W}$ respectively. Let $\mathbb{L}^*$ and $\mathbb{L}$ be the sets of lotteries with finite support on $\mathbb{H}^*$ and $\mathbb{H}$ respectively. In particular, for each $h \in \mathbb{H}^*$ there is a lottery $1_h \in \mathbb{L}^*$.

Say that a preorder $\succsim_0^*$ on $\mathbb{H}^*$ is consistent with generalized utilitarianism if there exists some generalized utilitarian preorder $\succsim^*$ on $\mathbb{L}^*$ such that for all h, $h' \in \mathbb{H}^*$, $h \succsim_0^* h' \iff 1_h \succsim^* 1_{h'}$. We can similarly ask whether a preorder $\succsim_0$ on $\mathbb{H}$ is consistent with generalized utilitarianism for the given finite population: whether $h \succsim_0 h' \iff 1_h \succsim 1_{h'}$ for all $h, h' \in \mathbb{H}$. Discussions of the ethics of distribution often focus solely on risk-free cases. So, in the first instance, it is natural to ask which preorders on histories are consistent with generalized utilitarianism.

We answer this question in terms of the following two conditions.

> **Weak Anonymity** Given $h \in \mathbb{H}$ and $\sigma \in \Sigma$, we have $h \sim_0 \sigma h$.

Next, say that $k \in \mathbb{H}^*$ is an *m-scaling* of $h \in \mathbb{H}^*$ if it consists of 'm copies' of h – that is, there is an m-to-1 map s of $\mathbb{I}^*$ onto itself such that $\mathcal{W}_i(k) = \mathcal{W}_{s(i)}(h)$ for every individual i. For example, $[x, x, y, y, \Omega, \Omega, \dots]$ is a 2-scaling of $[x, y, \Omega, \dots]$.

> **Scale Invariance** If, for some $m > 0$, $k, k' \in \mathbb{H}^*$ are m-scalings of $h, h' \in \mathbb{H}^*$ respectively, then $k \succsim_0^* k' \iff h \succsim_0^* h'$.

Weak Anonymity is obviously a very weak and uncontroversial constraint, while Scale Invariance is not that much stronger. But these are the only constraints imposed by consistency with generalized utilitarianism.

Proposition 5.1.1. *(i) A preorder on $\mathbb{H}$ is consistent with constant population generalized utilitarianism if and only if it satisfies Weak Anonymity.*
(ii) A preorder on $\mathbb{H}^*$ is consistent with generalized utilitarianism if and only if it satisfies Scale Invariance.

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[35] The term 'generalized utilitarianism' has been used differently in the literature; in the case of risk it has been used to refer to social preorders which are represented the sum of transformed individual expected utilities. See e.g. Grant et al [56]. We give reasons for preferring our usage in the next section.

[36] See Chew and Mao [35] for a summary for the case where $\mathbb{W}$ is a real interval and $\mathbb{P}$ is the set of Borel probability measures.

This result shows that all the seemingly reasonable egalitarian (and other) preorders of histories are included in the generalized utilitarian preorders of histories. This raises questions about the significance of generalized utilitarian preorders as a class. In particular, why do they merit the 'utilitarian' name, if they include preorders with apparently egalitarian properties?

In fact, despite this worry, the axioms of the aggregation theorems precisely rule out certain features of the social preorder that may be considered essential to standard egalitarian concerns. Thus, even if some generalized utilitarian preorders are egalitarian in *some* useful sense, this class still excludes the main lines of egalitarianism. To see this, suppose given welfare levels x and z with $x \succ_{\mathbb{P}} z$, and a population $\mathbb{I} = \{a, b\}$. Consider the following lotteries, with columns corresponding to histories.

L_E	$\frac{1}{2}$	$\frac{1}{2}$
a	x	z
b	x	z

L_F	$\frac{1}{2}$	$\frac{1}{2}$
a	x	z
b	z	x

L_U	$\frac{1}{2}$	$\frac{1}{2}$
a	x	x
b	z	z

It is arguable that L_E is socially better than L_F on the grounds that while the two individuals face identical prospects, L_E ensures *ex post* equality (Myerson [85]). It is also arguable that L_F is better than L_U on the grounds that while there is nothing to chose between their outcomes, under L_F there is *ex ante* equality, so L_F is arguably fairer (Diamond [42]).

In our view, suitable generalizations of '$L_E \succ L_F$' and '$L_F \succ L_U$' are essential to *ex post* and *ex ante* egalitarianism respectively.[37] If so, there is a principled distinction between generalized utilitarianism and each of *ex post* and *ex ante* egalitarianism, even if they take generalized forms.[38] In particular, generalized utilitarianism is inconsistent with *ex post* egalitarianism because it accepts Anteriority, and inconsistent with *ex ante* egalitarianism because it accepts Two-Stage Anonymity. Thus despite the egalitarian appearance of some generalized utilitarian preorders of histories, there is a sharp distinction between generalized utilitarianism and standard forms of egalitarianism, adding to the case for our definition of the former.[39]

Steps towards further understanding egalitarianism and generalized utilitarianism could be taken as follows. Any condition on the social preorder of histories is equivalent to some condition on the individual preorder, and vice versa. It would be worthwhile to make such equivalences explicit for standard egalitarian-seeming conditions at the social level, and also for standard conditions from non-expected utility theory at the individual level. It could turn out that standard conditions at the two levels are nicely paired; alternatively, standard conditions at one level may turn out to generate novel conditions at the other level. Such results would have both normative and positive applications.

Example 5.1.2. We saw that RDSs are naturally paired with RDIs in Example 1.5.4. At the normative level, downward increasing RDSs have been seen as plausible forms of egalitarianism. This might be taken as normative support for the risk-avoidant RDIs which generate them. On the other hand, the empirically supported RDIs have S-shaped risk functions.[40] Provided the population is large enough, such RDIs lead to RDSs which are apparently inegalitarian at the high end, favoring

[37]For similar views, see Broome [23, 24], Ben-Porath, Gilboa, and Schmeidler [10], Fleurbaey [53], Saito [95] and McCarthy [75] among others.

[38]McCarthy and Thomas [78] is an attempt to model generalized *ex ante* and generalized *ex post* egalitarianism.

[39]The alternative usage of 'generalized utilitarianism' mentioned in note 35 typically violates Two Stage Anonymity. In the popular case where the transform is a strictly increasing, strictly concave function, the corresponding social preorder ranks $L_F \succ L_U$, and it seems to us much more natural to see this as a form of *ex ante* egalitarianism, not a version of utilitarianism.

[40]See Schmidt [97, §4.2.2] for references.

unit transfers from the relatively well-off (but perhaps absolutely badly off) to the relatively better off. Given the lack of enthusiasm for inegalitarian ideas, this might call into question actual attitudes to risk.

5.2. ***Ex ante* and *ex post*.** We now explain why there is a natural sense in which generalized utilitarian preorders are those social preorders which are weakly *ex ante* and anonymously *ex post*. We focus on the constant population case, the variable case being parallel. The Pareto conditions are therefore understood to be relative to a fixed population.

5.2.1. *Ex ante.* Let (RP) and (Ant) stand for Reduction to Prospects and Anteriority. The following irreversible implications are obtained by noting that (RP) is equivalent to the restriction of (P_3) to lotteries in $\mathcal{L}(\mathbb{P})$.

$$(\mathrm{RP}) \Leftarrow (\mathrm{P}_3) \Rightarrow (\mathrm{P}_2) \Rightarrow (\mathrm{P}_1) \Rightarrow (\mathrm{Ant})$$

Social preorders are said to be *ex ante* if they respect unanimous 'before the event' judgments of individual welfare. Each of the above principles expresses some such notion of respect, which helps explain why '*ex ante*' is used quite flexibly. But the most popular interpretation sees social preorders as *ex ante* if they satisfy (P_2) (Mongin and d'Aspremont [83, §5.4]). This corresponds to a relatively strong notion of unanimity: respect the unanimous judgments of non-indifferent individuals. But this notion of unanimity is more fully captured by the stronger (P_3). We therefore suggest that it is social preorders which satisfy (P_3) which should be seen as *ex ante*.

Requiring a social preorder to be *ex ante* in this sense carries an implicit commitment: given modest assumptions, it means that the individual preorder has to satisfy Strong Independence (see Proposition 3.2.1). But that rules out a wide range of possibilities for individual welfare comparisons, so it is natural to ask which principle is as *ex ante* as possible while remaining neutral on whether the individual preorder satisfies any independence axiom. Given Proposition 3.2.1 again, and the principles displayed above, that principle is the conjunction of (RP) and (Ant). We will therefore say that social preorders satisfying that conjunct are *weakly ex ante*.

Similarly, in the variable population case, we will say that a social preorder satisfying (P_3) is *ex ante*, and one satisfying Anteriority and Reduction to Prospects (Variable) is weakly *ex ante*.

5.2.2. *Ex post.* Social preorders are often said to be *ex post* when they satisfy expected utility (Mongin and d'Aspremont [83, §5.4]). But this seems distant from the ordinary meaning of the term, which suggests that lotteries should be socially evaluated from some sort of 'after the event' perspective in which all risk has resolved. In particular, if two lotteries are in some natural sense equivalent from that perspective, then they should be ranked as equals.

To approach the matter more directly, let us temporarily suppose that for all h in $\mathbb{H}$, $\{h\}$ is measurable and 1_h is in $\mathbb{L}$. Say that a 'level of social welfare' is an equivalence class of histories under the social indifference relation $\sim_0$. Two lotteries are naturally said to be equivalent from an 'after the event' perspective whenever they define the same probability distribution over levels of social welfare. Formally, say that a subset U of $\mathbb{H}$ is 'closed under indifference' if $h \in U$ and $h \sim_0 h'$ entail $h' \in U$. Then we will say that a social preorder is *ex post* if it satisfies the following.[41]

[41]Note that, when the social preorder is upper-measurable, so that the stochastic dominance relation is defined, Posteriority is implied by the first part of Monotonicity (M)(i), and these conditions often coincide in practice. Even then, though, Posteriority is logically weaker, and gets more directly at the *ex post* idea.

Posteriority. Given $L, L' \in \mathbb{L}$, suppose that $L(U) = L'(U)$ whenever U is a measurable subset of $\mathbb{H}$ that is closed under indifference. Then $L \sim L'$.

Continuing with the temporary domain assumptions, if U is a measurable subset of $\mathbb{H}$ that is closed under indifference, Anonymity implies that U is permutation-invariant. Hence given Anonymity, Posterior Anonymity emerges as a much weaker, special case of Posteriority. It is therefore natural to call social preorders satisfying Posterior Anonymity *anonymously ex post.* The same applies in the variable population case.

An appealing feature of this terminology is that anonymously *ex post* social preorders rule out *ex ante* egalitarianism, and weakly *ex ante* social preorders rule out *ex post* egalitarianism.

It should be noted that this derivation of Posterior Anonymity does not always make sense in our very general framework. For example, the definition of Posteriority presupposes that 1_h is in $\mathbb{L}$ for all h in $\mathbb{H}$. Nevertheless, Posterior Anonymity has self-standing appeal, and is always well-defined in our framework.

5.2.3. *The aggregation theorems redux.* Two-Stage Anonymity is entailed by Posterior Anonymity, and although Posterior Anonymity is our conceptually favored principle, it was simpler to work with Two-Stage Anonymity. Nevertheless, granted some modest measurability assumptions in the variable population case, Propositions 1.3.2 and 2.3.3 show that this makes no difference. Thus we can recapitulate our aggregation theorems as follows: given an individual preorder (satisfying Omega Independence in the variable case), the social preorder it generates is the unique social preorder which is weakly *ex ante* and anonymously *ex post.*

5.3. **Interpersonal comparisons and incompleteness.** We develop our discussion of interpersonal comparisons by starting with what we will call Harsanyi's original theorem. This theorem shows that if intrapersonal comparisons for each individual satisfy EUT, the social preorder satisfies EUT, and strong Pareto holds, then the social preorder is represented by the sum of strictly positively weighted individual expected utilities.

Harsanyi's result, however, is not much use without supplementation. Since the utility weights are undetermined beyond being strictly positive, unless two lotteries are already ranked by strong Pareto, Harsanyi's conclusion does not determine their social ranking. Of course, as Harsanyi [63] immediately notes, this problem can be solved by tacking on interpersonal comparisons and anonymity to the end of his conclusion, thereby arriving at the conclusion of Harsanyi's anonymous theorem.

But there is a more serious problem connected with interpersonal comparisons. Harsanyi's original theorem assumes that the social preorder is complete. But without making any assumptions about interpersonal comparisons, it is difficult to see what justifies the assumption that the social preorder is complete.[42] Adding on interpersonal comparisons at the end obviously does nothing to help as completeness has already been assumed to get to that point.

This points to a peculiarity in Harsanyi's method. If one is happy to add on interpersonal comparisons at the end, it is hard to see the cost of including them at the beginning. This suggests that Harsanyi may have been underexploiting interpersonal comparisons. While interpersonal comparisons can be used to fix the

[42]For example, if social completeness is dropped from Harsanyi's premises, the social preorder could be the highly incomplete social preorder induced by strong Pareto (Danan *et al* [40]). Unease on this topic this is reflected in the large debate on whether Harsanyi was implicitly assuming interpersonal comparisons. See e.g. Harsanyi [67, p. 294], [66, p. 227], [65, pp. 81–2]; Broome [24, p. 219]; Mongin [80, pp. 348–50]; and Mongin and d'Aspremont [83, p. 432].

weights *given* social completeness, using them from the outset opens up the possibility of using them to *justify* social completeness. More generally, if one follows Harsanyi by encoding both inter and intrapersonal comparisons within a single preorder, and uses this preorder at the start, one has the possibility of explicitly showing how completeness of the social preorder depends on completeness of the individual preorder; or more generally, showing precisely how gaps in the individual preorder determine gaps in the social preorder.

The main result of Pivato [89] makes progress in this direction. Consider the following independence and Pareto axioms, which are respectively intermediate in strength between (I_1) and (I_2), and (P_1) and (P_2).

($I_{1.5}$) For all $\alpha \in (0,1)$, $p \succsim_X p' \implies \alpha p + (1-\alpha)q \succsim_X \alpha p' + (1-\alpha)q$.

($P_{1.5}$) (i) $\mathcal{P}_i(L) \succsim_{\mathbb{P}} \mathcal{P}_i(L')$ for all $i \in \mathbb{I} \implies L \succsim L'$; and (ii) $\mathcal{P}_i(L) \succ_{\mathbb{P}} \mathcal{P}_i(L')$ for all $i \in \mathbb{I} \implies L \succ L'$.

Assuming that the individual and social preorders satisfy ($I_{1.5}$), ($P_{1.5}$) and Anonymity, Pivato shows that the social preorder must extend the one generated by the individual preorder. This gives an upper limit on how incomplete the social preorder can be, but forces the individual preorder to satisfy ($I_{1.5}$), precluding most non-expected utility theories. In contrast, the conclusion of Theorem 1.3.1 uniquely determines the social preorder without imposing any restrictions on the individual preorder. In particular, it shows precisely which gaps in the social preorder are generated by gaps in the individual preorder.

This raises the question of how our conclusion can be stronger than Pivato's, for in most ways our premises are much weaker: Anteriority is strictly weaker than ($P_{1.5}$) (or even P_1); and Two-Stage Anonymity is strictly weaker than Anonymity conjoined with ($I_{1.5}$) (or even I_1). The answer has to do with Reduction to Prospects.

Part of the answer is that we use a stronger domain condition, namely $\mathcal{L}(\mathbb{P}) \subset \mathbb{L}$, in order to apply Reduction to Prospects; defense was given in Remark 1.1.2. But the main answer is that Reduction to Prospects is not implied by Pivato's assumptions, or any standard Pareto condition. However, Reduction to Prospects is equivalent to the restriction of ($P_{1.5}$) to lotteries in $\mathcal{L}(\mathbb{P})$ conjoined with the following principle: for any P, P' in $\mathbb{P}$, $P \curlywedge_{\mathbb{P}} P' \implies \mathcal{L}(P) \curlywedge \mathcal{L}(P')$. But we suggest that this principle is very plausible.

We can now give another perspective on Theorem 1.3.1. On its own, Reduction to Prospects provides the desired dependence of the social preorder on the individual preorder in the special case of comparisons between lotteries guaranteeing perfect equality. In that situation, it tells us precisely which gaps in the social preorder are generated by gaps in the individual preorder. When combined with Anteriority and Two-Stage Anonymity, these conclusions extend to the general case of comparisons between arbitrary lotteries.

5.4. **Harsanyi.** Theorem 1.3.1 is an improvement on Harsanyi's anonymous theorem. In common with Harsanyi's anonymous theorem, it shows how the social preorder is uniquely determined by the individual preorder. But Theorem 1.3.1 achieves this with no constraints on the individual preorder whatsoever, thus permitting a very wide range of specializations. One specialization results in a very general Harsanyi-like representation of the social preorder merely by supposing that the individual preorder satisfies (I_3) (Proposition 3.4.2). Another specialization results in the conclusion of Harsanyi's anonymous theorem merely by assuming that the individual preorder satisfies EUT (Proposition 3.3.1). That specialization uses much weaker assumptions than Harsanyi's version: it derives, rather than assumes, both Strong Pareto and, at the social level, all of the expected utility axioms. Total utility representations which are locally Harsanyi-like can be obtained even when the individual preorder violates (I_3) (Proposition 4.2.2(iii)).

However, despite resting on much weaker premises, Theorem 1.3.1 shares fundamental distributive assumptions with Harsanyi's theorem. In particular, both results accept premises which express indifference to the two ways of valuing equality discussed in §5.1, namely those expressed by *ex ante* and *ex post* egalitarianism. More generally, Theorem 1.3.1 weakens but preserves the *ex ante* and *ex post* flavor of Harsanyi's result. It uses Harsanyi's device of encoding interpersonal and intrapersonal comparisons within a single preorder, but makes much weaker assumptions about it. For these reasons, we suggest that Theorem 1.3.1 should be seen as articulating the core of Harsanyi's anonymous theorem.

In one way, this acts as a corrective to some of Harsanyi's remarks. Harsanyi [64, p. 627] describes the extensive use of expected utility as a "very crucial ingredient" for his theory, and it is easy to get the impression that Harsanyi's anonymous theorem is in some sense all about expected utility theory. But we are suggesting that the core of Harsanyi's approach turns out to have nothing to do with expected utility theory.

In more important ways, it amplifies Harsanyi's insights. First, by dropping all of the expected utility axioms, it shows that the core of his approach can accommodate a much wider range of views about welfare comparisons than permitted by his anonymous theorem. In particular, in dropping completeness, Theorem 1.3.1 allows for any amount of incomparability between welfare levels; in dropping continuity, it allows some welfare levels to be infinitely more valuable than others; and in dropping independence, it allows for all sorts of views about risk in ranking prospects over welfare levels.

Second, our constant population Theorem 1.3.1 generalizes very easily to our variable population Theorem 2.3.1, requiring no significantly new ethical idea or further element of justification. This contrasts with other approaches to the extension. In particular, along with the full expected utility framework, Hammond [58] and Blackorby, Bossert and Donaldson [15, 16, 18] introduce novel ethical ideas to effect that extension. Hammond uses a strong separability principle which says that the welfare levels of the long-dead can be ignored. Blackorby *et al* assume that at least some histories have a critical level. Under one of several important interpretations, Pivato [90, Thm. 1] shows, roughly, that for variable but finite populations, there is a Harsanyi-like total utility representation into a linearly ordered abelian group if and only if the social preorder is complete, anonymous, and satisfies a separability condition. Under this interpretation, the separability condition implies both strong independence and strong separability across individuals. Thus the main advance in terms of ethical assumptions is to have dispensed with continuity. By contrast, our Theorem 2.3.1 neither assumes nor implies completeness, continuity, strong independence, strong separability across people, or the existence of a critical level for some history. When we further assume that the individual preorder is strongly independent, we obtain a mixture-preserving total utility representation into a preordered vector space. As Pivato [90] notes, linearly ordered abelian groups can always be embedded in lexicographically ordered vector spaces by the Hahn embedding theorem, so subject to one qualification, our Propositions 3.4.2 and 3.4.4 cover this kind of representation as a special case. The qualification is that Pivato's framework is designed to allow for infinitesimal probabilities, whereas we have assumed standard real-valued probabilities. But this assumption plays no real role in either of our aggregation theorems. In brief, all we require is the possibility of forming rational mixtures of lotteries.

Harsanyi himself does not seem to have thought that his anonymous theorem extends to the variable population case. His public discussion of the variable case seems to be limited to an exchange reported in Ng [86]. Instead of applying his anonymous theorem, Harsanyi recommends using the veil of ignorance described in [62], along with the extension of the individual preorder to the variable case discussed in Example 2.5.2, leading to a form of average utilitarianism.

This leads to a third way in which our work amplifies Harsanyi's insights. We think that appeals to a veil of ignorance require justification, especially in the variable population case, where it is quite unclear how best to formulate the principle. But a version of the veil turns out to be vindicated by Theorem 2.3.1. For any finite population $\mathbb{I} \subset \mathbb{I}^*$ and lottery $L \in \mathbb{L}_{\mathbb{I}}$, the lottery $p_L^{\mathbb{I}}$ defined in Theorem 2.3.1 can be interpreted as the prospect faced by an individual behind a veil of ignorance, in the sense that he has an equal chance of being any member of $\mathbb{I}$ under L. The theorem then gives necessary and sufficient conditions for the social preorder to be governed by the individual preorder for individuals behind the veil. The flexibility of the individual preorder allows for a wide range of views about individual rationality to be represented.

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Appendix A. Proofs

Section 1.

Proof of Proposition 1.3.2. Suppose that $L(U) = L'(U)$ for every measurable and permutation-invariant $U \subset \mathbb{H}$. We want to show $L \sim L'$. Suppose given measurable $V \subset \mathbb{W}$. We can write

$$\begin{aligned} \#\,\mathbb{I}\cdot p_L(V) &= \sum_{i\in\mathbb{I}} \mathcal{P}_i(L)(V) \\ &= \sum_{i\in\mathbb{I}} L(\mathcal{W}_i^{-1}(V)) = \sum_{n=1}^{\#\,\mathbb{I}} L\left(\bigcup_{\substack{I\subset\mathbb{I}\\ \#I=n}} \bigcap_{i\in I} \mathcal{W}_i^{-1}(V)\right). \end{aligned} \tag{7}$$

(For the last equation, note that if a history contributes its probability to exactly k summands of the left-hand sum, then it also contributes to exactly k summands of the right-hand sum, namely those with $n = 1, 2, \ldots, k$.) On the right hand side, all arguments of L are measurable and permutation-invariant. We therefore find that

$$\#\,\mathbb{I}\cdot p_L(V) = \#\,\mathbb{I}\cdot p_{L'}(V)$$

for arbitrary measurable V. Hence $p_L = p_{L'}$. According to (1), we therefore have $L \sim L'$, as required for Posterior Anonymity. □

Section 2.

Proof of Lemma 2.1.3. For (i), suppose given $L \in \mathbb{L}^*_{\mathbb{I}}$ and $i \in \mathbb{I}^* \setminus \mathbb{I}$. Let U be measurable in $\mathbb{W}^*$ with $\Omega \in U$. Then $\mathbb{H}^*_{\mathbb{I}} \subset (\mathcal{W}^*_i)^{-1}(U)$, hence $\mathcal{P}^*_i(L)(U) = L((\mathcal{W}^*_i)^{-1}(U)) = 1$. Since this is true for every such U, we must have $\mathcal{P}^*_i(L) = 1_\Omega$.

For (ii), for any finite $\mathbb{I}$, we have $h_\Omega = \mathcal{H}_{\mathbb{I}}(\Omega) \in \mathbb{H}^*$.

For (iii), we have $\mathcal{L}_{\mathbb{I}}(1_\Omega) \in \mathbb{L}^*$, and we claim that $\mathcal{L}_{\mathbb{I}}(1_\Omega) = 1_{h_\Omega}$. Indeed, for any measurable $U \subset \mathbb{H}^*$ with $h_\Omega \in U$, we have $\Omega \in \mathcal{H}_{\mathbb{I}}^{-1}(U)$. Therefore $\mathcal{L}_{\mathbb{I}}(1_\Omega)(U) = 1_\Omega(\mathcal{H}_{\mathbb{I}}^{-1}(U)) = 1$, as desired. Moreover, for any measurable $U \subset \mathbb{W}^*$ with $\Omega \in U$, we have, for any $i \in \mathbb{I}^*$, $h_\Omega \in \mathcal{W}^{*\,-1}_i(U)$. Therefore $\mathcal{P}^*_i(1_{h_\Omega})(U) = 1_{h_\Omega}(\mathcal{W}^{*\,-1}_i(U)) = 1$. Therefore $\mathcal{P}^*_i(1_{h_\Omega}) = 1_\Omega$.

For (iv), suppose we have $L \in \mathbb{L}^*_{\mathbb{I}}$, and $\mathcal{P}^*_i(L) = 1_\Omega$ for all $i \in \mathbb{I}^*$. Then $L(\mathcal{W}^{*\,-1}_i(\mathbb{W})) = \mathcal{P}^*_i(L)(\mathbb{W}) = 0$ for all i. Defining $V := \bigcup_{i\in\mathbb{I}} \mathcal{W}^{*\,-1}_i(\mathbb{W})$, we must have $L(V) = 0$. Suppose given measurable $U \subset \mathbb{H}^*$ with $h_\Omega \notin U$. We have $U \cap \mathbb{H}^*_{\mathbb{I}} \subset V \cap \mathbb{H}^*_{\mathbb{I}}$, so $L(U) \le L(V)$, so $L(U) = 0$. Therefore $L = 1_{h_\Omega}$. □

Proof of Proposition 2.3.3. Suppose that $L(U) = L'(U)$ for every measurable and Σ^*-invariant $U \subset \mathbb{H}^*$. We want to show $L \sim^* L'$, and it suffices to show that $p^{\mathbb{I}}_L = p^{\mathbb{I}}_{L'}$ if $L, L' \in \mathbb{L}^*_{\mathbb{I}}$. Now, for any measurable $W \subset \mathbb{W}^*$, $p^{\mathbb{I}}_L(W) = 1 - p^{\mathbb{I}}_L(\mathbb{W}^* - W)$. Since either $\Omega \notin W$ or $\Omega \notin \mathbb{W}^* - W$, it suffices to show that $p^{\mathbb{I}}_L(V) = p^{\mathbb{I}}_{L'}(V)$ for every measurable $V \subset \mathbb{W}^*$ such that $\Omega \notin V$.

For each number n, $1 \le n \le \#\mathbb{I}$, let U_n be the set of histories in which at least n individuals have welfare levels in V; and, for any finite population $\mathbb{J}$, let $U^{\mathbb{J}}_n$ be the set of histories in which at least n individuals in $\mathbb{J}$ have welfare levels in V. That is:

$$U_n := \bigcup_{\substack{I\subset\mathbb{I}^*\\ \#I=n}} \bigcap_{i\in I} \mathcal{W}^{*\,-1}_i(V) \qquad U^{\mathbb{J}}_n := \bigcup_{\substack{I\subset\mathbb{J}\\ \#I=n}} \bigcap_{i\in I} \mathcal{W}^{*\,-1}_i(V).$$

On the assumption that $\Omega \notin V$, we have $U_n \cap \mathbb{H}^*_{\mathbb{J}} = U^{\mathbb{J}}_n \cap \mathbb{H}^*_{\mathbb{J}}$. The sets $U^{\mathbb{J}}_n$ are measurable, and therefore $U_n \cap \mathbb{H}^*_{\mathbb{J}}$ is measurable in $\mathbb{H}^*_{\mathbb{J}}$. Since we assume that the sigma algebra on $\mathbb{H}^*$ is coherent, this shows that U_n itself is measurable in $\mathbb{H}^*$.

Following the proof of Proposition 1.3.2, and especially formula (7), we find

$$\#\,\mathbb{I}\cdot p^{\mathbb{I}}_L(V) = \sum_{n=1}^{\#\mathbb{I}} L(U^{\mathbb{I}}_n). = \sum_{n=1}^{\#\mathbb{I}} L(U_n).$$

Since the sets U_n are Σ^*-invariant, $L(U_n) = L'(U_n)$, so $p^{\mathbb{I}}_L(V) = p^{\mathbb{I}}_{L'}(V)$, as desired. □

The following proof was suggested by a result due to Milgram [79].

Proof of Theorem 2.4.2. Let $(X, \succsim_X)$ be a preordered set. Define $U_x := \{y \in X \mid y \succsim_X x\}$, and let $\mathcal{U} := \{U_x \mid x \in X\}$.

Let λ be the least ordinal whose cardinality is equal to that of $\mathcal{U}$. Let $\ge_L$ be the lexicographic order, a complete vector preorder, on the vector space $\mathbb{V}_0 := \mathbb{R}^\lambda$. That is, $\ge_L$ is an ordering, and $(v_\beta) >_L (w_\beta)$ if and only if the least $\beta < \lambda$ such that $v_\beta \neq w_\beta$ is one such that $v_\beta > w_\beta$. Let $(U_\beta)_{\beta<\lambda}$ be a well-ordering of the members of $\mathcal{U}$ indexed by the ordinals less than λ.

Define a function $f\colon X \to \mathbb{V}_0$ by $(f(x))_\beta = 1$ if $x \in U_\beta$, 0 otherwise. It is easy to see that if $\succsim_X$ is complete, then f represents $\succsim_X$. But in general, let Σ be the group of permutations on λ. Define $\sigma f\colon X \to \mathbb{V}_0$ by $((\sigma f)(x))_\beta = (f(x))_{\sigma^{-1}\beta}$. Define $\mathbb{V}$ to be the product space $(\mathbb{V}_0)^\Sigma$, and equip $\mathbb{V}$ with the vector preorder $\succsim_\mathbb{V}$ given by $v \succsim_\mathbb{V} w \iff v_\sigma \geq_L w_\sigma$ for all $\sigma \in \Sigma$. Finally, define $F\colon X \to \mathbb{V}$ by $F(x)_\sigma = (\sigma f)(x)$. We claim that F represents $\succsim_X$. Equivalently,

$$(*) \qquad x \succsim_X y \iff (\sigma f)(x) \geq_L (\sigma f)(y) \quad \forall \sigma \in \Sigma.$$

To see this, note that for all $\sigma \in \Sigma$, x, $y \in X$, $x \succ_X y \implies (\sigma f)(x) >_L (\sigma f)(y)$ and $x \sim y \implies (\sigma f)(x) = (\sigma f)(y)$. Suppose $x \curlywedge_X y$. Then there is a least ordinal γ such that $(f(x))_\gamma = 1$ and $(f(y))_\gamma = 0$, and a least ordinal δ such that $(f(x))_\delta = 0$ and $(f(y))_\delta = 1$. Let $\sigma' \in \Sigma$ be the permutation $(\gamma\delta)$. If $\gamma < \delta$, then $f(x) >_L f(y)$ but $(\sigma' f)(y) >_L (\sigma' f)(x)$. Similarly, if $\delta < \gamma$, then $f(y) >_L f(x)$ but $(\sigma' f)(x) >_L (\sigma' f)(y)$. These observations establish $(*)$. □

Proof of Proposition 2.4.1. (i) Applying Theorem 2.4.2 to $X = \mathbb{P}_\Omega$, we have a representation $U\colon \mathbb{P}_\Omega \to \mathbb{V}$ of $\succsim_{\mathbb{P}_\Omega}$, for some preordered vector space $(\mathbb{V}, \succsim_\mathbb{V})$. Since $\mathbb{P}^*$ includes $\mathbb{P}$, each member of $\mathbb{P}^*$ can be written in the form $P_\alpha := \alpha P + (1-\alpha)1_\Omega$ for some $P \in \mathbb{P}$, $\alpha \in [0,1]$. This presentation is unique except when $\alpha = 0$. Define a function $\tilde{U}\colon \mathbb{P}^* \to \mathbb{V}$ by the rule

$$\tilde{U}(P_\alpha) = \alpha U(P) + (1-\alpha)U(1_\Omega).$$

Let $\succsim_{\mathbb{P}^*}$ be the preorder on $\mathbb{P}^*$ represented by $\tilde{U}$. We claim that $\succsim_{\mathbb{P}^*}$ is Omega Independent and includes $\succsim_{\mathbb{P}_\Omega}$.

For all $P \in \mathbb{P}_\Omega$, $\tilde{U}(P) = U(P)$, so $\succsim_{\mathbb{P}^*}$ includes $\succsim_{\mathbb{P}_\Omega}$. To show that $\succsim_{\mathbb{P}^*}$ satisfies Omega Independence, suppose P, $P' \in \mathbb{P}^*$, and let $\alpha \in (0,1) \cap \mathbb{Q}$. We wish to show that $P \succsim_{\mathbb{P}^*} P' \iff P_\alpha \succsim_{\mathbb{P}^*} P'_\alpha$. We have $P = Q_\beta$ and $P' = Q'_\gamma$ for some Q, $Q' \in \mathbb{P}$, β, $\gamma \in [0,1]$. Then $P_\alpha = Q_{\alpha\beta}$ and $P'_\alpha = Q'_{\alpha\gamma}$. Thus:

$$\begin{aligned} P \succsim_{\mathbb{P}^*} P' &\iff \tilde{U}(P) \succsim_\mathbb{V} \tilde{U}(P') \\ &\iff \beta U(Q) + (1-\beta)U(1_\Omega) \succsim_\mathbb{V} \gamma U(Q') + (1-\gamma)U(1_\Omega) \\ &\iff \alpha\beta U(Q) + (1-\alpha\beta)U(1_\Omega) \succsim_\mathbb{V} \alpha\gamma U(Q') + (1-\alpha\gamma)U(1_\Omega) \\ &\iff \tilde{U}(P_\alpha) \succsim_\mathbb{V} \tilde{U}(P'_\alpha) \\ &\iff P_\alpha \succsim_{\mathbb{P}^*} P'_\alpha \end{aligned}$$

Here the third line uses the fact that $v \mapsto \alpha v + (1-\alpha)U(1_\Omega)$ is an order-preserving transformation of $\mathbb{V}$. This establishes that $\succsim_{\mathbb{P}^*}$ is Omega Independent and includes $\succsim_{\mathbb{P}_\Omega}$.

(ii) Suppose first that for some $Q \neq Q' \in \mathbb{P}_\Omega$, $Q \succsim_{\mathbb{P}_\Omega} Q'$. For P, $P' \in \mathbb{P}^*$, define $P \succsim_{\mathbb{P}^*} P' \iff (P = P') \vee (P \succsim_{\mathbb{P}_\Omega} P')$. This is a preorder on $\mathbb{P}^*$ that includes $\succsim_{\mathbb{P}_\Omega}$. But since $Q \succsim_{\mathbb{P}^*} Q'$ and $\frac{1}{2}Q + \frac{1}{2}1_\Omega \curlywedge_{\mathbb{P}^*} \frac{1}{2}Q' + \frac{1}{2}1_\Omega$, $\succsim_{\mathbb{P}^*}$ violates Omega Independence.

Suppose instead that for all $Q \neq Q' \in \mathbb{P}_\Omega$, $Q \curlywedge_{\mathbb{P}_\Omega} Q'$. Let $P_0 \in \mathbb{P}$. For P, $P' \in \mathbb{P}^*$, define $P \succsim_{\mathbb{P}^*} P' \iff (P = P') \vee ((P = \frac{1}{2}P_0 + \frac{1}{2}1_\Omega) \wedge (P' = 1_\Omega))$. This is also a preorder on $\mathbb{P}^*$ that includes $\succsim_{\mathbb{P}_\Omega}$. But since $\frac{1}{2}P_0 + \frac{1}{2}1_\Omega \succ_{\mathbb{P}^*} 1_\Omega$ and $P_0 \curlywedge_{\mathbb{P}^*} 1_\Omega$, $\succsim_{\mathbb{P}^*}$ violates Omega Independence.

(iii) The fact that $\mathbb{M}_\mathbb{I}$ is a constant population model is easy to verify from the construction in the text; then (1) is immediate from (2).

(iv) Suppose $\mathbb{M} = \langle \mathbb{I}, \mathbb{W}, \mathbb{P}, \succsim_\mathbb{P}, \mathbb{H}, \mathbb{L}, \succsim \rangle$ is a constant population model satisfying (1). $\mathbb{P}^*$ includes $\mathbb{P}$, so that we are given $\mathbb{W}^*$, and we are also given an Omega Independent $\succsim_{\mathbb{P}^*}$ that includes $\succsim_\mathbb{P}$.

We wish to describe a variable population model $\mathbb{M}^* = \langle \mathbb{I}^*, \mathbb{W}^*, \mathbb{P}^*, \succsim_{\mathbb{P}^*}, \mathbb{H}^*, \mathbb{L}^*, \succsim^* \rangle$ satisfying (2). Let $\mathbb{I}^*$ be any infinite population with $\mathbb{I} \subset \mathbb{I}^*$ and associated group of permutations Σ^*; we need to define $\mathbb{H}^*$, $\mathbb{L}^*$ and $\succsim^*$.

We first define $\mathbb{H}^*$. For $h \in \mathbb{H}$, let $\iota(h)$ be the history in $(\mathbb{W}^*)^{\mathbb{I}^*}$ such that $(\iota(h))(i) = h(i)$ if $i \in \mathbb{I}$, Ω otherwise. For $w \in \mathbb{W}^*$ and finite $\mathbb{J} \subset \mathbb{I}^*$, let $\mathcal{H}^*_{\mathbb{J}}(w)$ denote the history in $(\mathbb{W}^*)^{\mathbb{I}^*}$ given by $\mathcal{H}^*_{\mathbb{J}}(w)(i) = w$ if $i \in \mathbb{J}$, Ω otherwise.

Define

$$\mathbb{H}^* := \Sigma^*(\iota(\mathbb{H})) \cup \{\mathcal{H}^*_{\mathbb{J}}(w) \colon \mathbb{J} \subset \mathbb{I}^* \text{ is finite}, w \in \mathbb{W}^*\}$$

For any $i \in \mathbb{I}^*$ and finite $\mathbb{J} \subset \mathbb{I}^*$, the functions $\iota \colon \mathbb{H} \to \mathbb{H}^*$, $\mathcal{W}^*_i \colon \mathbb{H}^* \to \mathbb{W}^*$ and $\mathcal{H}^*_{\mathbb{J}} \colon \mathbb{W}^* \to \mathbb{H}^*$ are then well-defined, and $\mathbb{H}^*$ is Σ^*-invariant.

Let $\mathcal{F}$ be the sigma algebra of $\mathbb{H}$. Equip $\mathbb{H}^*$ with the sigma algebra $\mathcal{F}^*$ defined as the smallest sigma algebra containing $\iota(\mathcal{F})$ which makes the action of Σ^* on $\mathbb{H}^*$ and the functions $\mathcal{W}^*_i$ measurable. Thus $\mathcal{F}$ is generated by

$$\mathcal{G} := \Sigma^*(\iota(\mathcal{F})) \cup \{(\mathcal{W}^*_i)^{-1}(U) \colon i \in \mathbb{I}^*, U \text{ is measurable in } \mathbb{W}^*\}$$

One can now show each $\mathcal{H}^*_{\mathbb{J}}$ is measurable; we omit verification.

We define the preorder $\succsim^*$ on $\mathbb{L}^*$ via (2). Since $\succsim_{\mathbb{P}^*}$ is Omega Independent, this is well-defined.

One can now show that $\mathbb{M}^* = \langle \mathbb{I}^*, \mathbb{W}^*, \mathbb{P}^*, \succsim_{\mathbb{P}^*}, \mathbb{H}^*, \mathbb{L}^*, \succsim^* \rangle$ is a variable population model which satisfies (2); we again omit detailed verification. Let $\mathbb{M}_{\mathbb{I}} = \langle \mathbb{I}, \mathbb{W}, \mathbb{P}, \succsim_{\mathbb{P}}, \mathbb{H}_{\mathbb{I}}, \mathbb{L}_{\mathbb{I}}, \succsim_{\mathbb{I}} \rangle$ be the restriction of $\mathbb{M}^*$ to $\mathbb{I}$. It is straightforward to show that $\mathbb{H} = \mathbb{H}_{\mathbb{I}}$, $\mathbb{L} = \mathbb{L}_{\mathbb{I}}$, and $\succsim = \succsim_{\mathbb{I}}$. This shows that $\mathbb{M}_{\mathbb{I}} = \mathbb{M}$, establishing that $\mathbb{M}^*$ includes $\mathbb{M}$, contains $\succsim_{\mathbb{P}^*}$, and satisfies (2) as needed. $\square$

Section 3.

Proof of Proposition 3.1.1. The proofs of (i) and (iii) are exactly parallel, so of these we will present only (iii). Moreover, the claim about (O) is obvious; the proofs for the other cases use the fact that for any L, $M \in \mathbb{L}^*_{\mathbb{I}}$, $\alpha \in (0,1)$, $p^{\mathbb{I}}_{\alpha L + (1-\alpha)M} = \alpha p^{\mathbb{I}}_L + (1-\alpha) p^{\mathbb{I}}_M$. They are very similar, so we only present the proof for $(\mathrm{I_i})$.

To show that $\succsim_{\mathbb{P}^*}$ satisfies $(\mathrm{I_i}) \iff \succsim^*$ satisfies $(\mathrm{I_i})$ for $i = 1, 2, 3$, it is sufficient to establish it for $i = a, b, c$. Let the symbol $\diamond$ stand for $\sim$, $\succ$, or $\curlywedge$, corresponding $i = a, b, c$.

Suppose given L, L', $M \in \mathbb{L}^*$, $\alpha \in (0,1)$. For some finite $\mathbb{I} \subset \mathbb{I}^*$, L, L', $M \in \mathbb{L}^*_{\mathbb{I}}$. Then for $i = a, b, c$:

$$\begin{aligned} L \diamond^* L' &\implies p^{\mathbb{I}}_L \diamond_{\mathbb{P}^*} p^{\mathbb{I}}_{L'} && (\succsim_{\mathbb{P}^*} \text{ generates } \succsim^*) \\ &\implies \alpha p^{\mathbb{I}}_L + (1-\alpha) p^{\mathbb{I}}_M \diamond_{\mathbb{P}^*} \alpha p^{\mathbb{I}}_{L'} + (1-\alpha) p^{\mathbb{I}}_M && ((\mathrm{I_i}) \text{ for } \succsim_{\mathbb{P}^*}) \\ &\implies p^{\mathbb{I}}_{\alpha L + (1-\alpha)M} \diamond_{\mathbb{P}^*} p^{\mathbb{I}}_{\alpha L' + (1-\alpha)M} \\ &\implies \alpha L + (1-\alpha) M \diamond^* \alpha L' + (1-\alpha) M && (\succsim_{\mathbb{P}^*} \text{ generates } \succsim^*) \end{aligned}$$

This shows that $(\mathrm{I_i})$ for $\succsim_{\mathbb{P}^*}$ implies $\implies$ $(\mathrm{I_i})$ for $\succsim^*$. Conversely, suppose given $P, Q, R \in \mathbb{P}^*$. Then

$$\begin{aligned} P \diamond_{\mathbb{P}^*} Q &\implies \mathcal{L}_{\mathbb{I}}(P) \diamond^* \mathcal{L}_{\mathbb{I}}(Q) && (\succsim_{\mathbb{P}^*} \text{ generates } \succsim^*) \\ &\implies \alpha \mathcal{L}_{\mathbb{I}}(P) + (1-\alpha) \mathcal{L}_{\mathbb{I}}(R) \diamond^* \alpha \mathcal{L}_{\mathbb{I}}(Q) + (1-\alpha) \mathcal{L}_{\mathbb{I}}(R) && ((\mathrm{I_i}) \text{ for } \succsim^*) \\ &\implies \alpha P + (1-\alpha) R \diamond_{\mathbb{P}^*} \alpha P + (1-\alpha) R && (\succsim_{\mathbb{P}^*} \text{ generates } \succsim^*). \end{aligned}$$

So $(\mathrm{I_i})$ for $\succsim^*$ implise $(\mathrm{I_i})$ for $\succsim_{\mathbb{P}^*}$.

Now let us turn to (ii). First a general observation. Suppose given topological spaces X, Y with preorders $\succsim_X, \succsim_Y$, and a function $f \colon X \to Y$. Assume (A) that

f is continuous, and (B) that for all $a, b \in X$, $a \succsim_X b \iff f(a) \succsim_Y f(b)$. Then, we claim, if $\succsim_Y$ is continuous, so is $\succsim_X$. Indeed, for any $q \in X$, we find

$$\{p \in X : p \succsim_X q\} = \{p \in X : f(p) \succsim_Y f(q)\} = f^{-1}\{y \in Y : y \succsim_Y f(q)\}.$$

The right-hand side is the inverse image of a closed set under a continuous function, so it is closed. A similar calculation shows that $\{p \in X : q \succsim_X p\}$ is closed; hence $\succsim_X$ is continuous.

Taking $f = \mathcal{L}\colon \mathbb{P} \to \mathbb{L}$, assumption (A) is part of (Top), and assumption (B) follows from Reduction to Prospects. We conclude that, if $\succsim$ is continuous, so is $\succsim_{\mathbb{P}}$. Conversely, define $f\colon \mathbb{L} \to \mathbb{P}$ by $f(L) = p_L$. Assumption (A) follows from the continuity of mixing and of every $\mathcal{P}_i$, whereas (B) is part of what it means for $\succsim$ to be generated by $\succsim_{\mathbb{P}}$. We conclude that, if $\succsim_{\mathbb{P}}$ is continuous, so is $\succsim$.

As for (iv), the same logic just used shows that $\succsim_{\mathbb{P}^*}$ is continuous if and only if the restriction of $\succsim^*$ to each and every $\mathbb{L}^*_{\mathbb{I}}$ is continuous. Now, if $\succsim^*$ is continous on $\mathbb{L}^*$, then its restriction to every $\mathbb{L}^*_{\mathbb{I}}$ is continuous; given topological coherence from (Top*), we may conclude the converse. Indeed, assume that the restriction of $\succsim^*$ to every $\mathbb{L}^*_{\mathbb{I}}$ is continuous. It suffices to show that, for any $L_0 \in \mathbb{L}^*$, the set $X = \{L \in \mathbb{L}^* : L \succsim^* L_0\}$ is closed in $\mathbb{L}^*$ (and similarly that $\{L \in \mathbb{L}^* : L_0 \succsim^* L\}$ is closed). By topological coherence, it suffices to show that $X \cap \mathbb{L}^*_{\mathbb{I}}$ is closed in $\mathbb{L}^*_{\mathbb{I}}$, for every $\mathbb{I}$. Suppose that L_0 is in $\mathbb{L}^*_{\mathbb{J}}$, and let $\mathbb{K} = \mathbb{I} \cup \mathbb{J}$, so L_0 is also in $\mathbb{L}^*_{\mathbb{K}}$. Then $X \cap \mathbb{L}^*_{\mathbb{K}}$ is closed in $\mathbb{L}^*_{\mathbb{K}}$, by the continuity of $\succsim^*$ on $\mathbb{L}^*_{\mathbb{K}}$. That means there is some closed $V \subset \mathbb{L}^*$ such that $V \cap \mathbb{L}^*_{\mathbb{K}} = X \cap \mathbb{L}^*_{\mathbb{K}}$. But then $X \cap \mathbb{L}^*_{\mathbb{I}} = V \cap \mathbb{L}^*_{\mathbb{I}}$ is closed in $\mathbb{L}^*_{\mathbb{I}}$, as desired. □

Proof of Proposition 3.2.1. We have $\succsim_{\mathbb{I}}$ satisfies $(\mathrm{I}^{\mathbb{Q}}_{\mathrm{i}}) \iff \succsim_{\mathbb{P}^g}$ satisfies $(\mathrm{I}^{\mathbb{Q}}_{\mathrm{i}})$ from Proposition 3.1.1, so it will be enough to argue that $(\mathrm{S_i}) \iff (\mathrm{I}^{\mathbb{Q}}_{\mathrm{i}}) \iff (\mathrm{P_i})$ for $i = 1, 2, 3$, where $(\mathrm{I}^{\mathbb{Q}}_{\mathrm{i}})$ is the condition on $\succsim_{\mathbb{P}^g}$.

We first argue that $(\mathrm{S_i}) \impliedby (\mathrm{I}^{\mathbb{Q}}_{\mathrm{i}}) \implies (\mathrm{P_i})$ for $i = 1, 2, 3$. It will be sufficient to show that $(\mathrm{S_i}) \impliedby [(\mathrm{I}^{\mathbb{Q}}_{\mathrm{a}})\&(\mathrm{I}^{\mathbb{Q}}_{\mathrm{i}})] \implies (\mathrm{P_i})$ for $i = a, b, c$. So suppose we have $(\mathrm{I}^{\mathbb{Q}}_{\mathrm{a}})$ and $(\mathrm{I}^{\mathbb{Q}}_{\mathrm{i}})$. Let the symbol $\diamond$ stand for $\sim$, $\succ$, or $\curlywedge$, corresponding $i = a, b, c$. We claim

(D1) For $i = a, b, c$, the antecedent of each of $(\mathrm{S_i})$ and $(\mathrm{P_i})$ implies $p^{\mathbb{K}}_L \sim_{\mathbb{P}^g} p^{\mathbb{K}}_{L'}$.
(D2) For $i = a, b, c$, the antecedent of each of $(\mathrm{S_i})$ and $(\mathrm{P_i})$ implies $p^{\mathbb{J}}_L \diamond_{\mathbb{P}^g} p^{\mathbb{J}}_{L'}$.

Granted (D1) and (D2), we can deduce $p^{\mathbb{I}}_L \diamond_{\mathbb{P}^*} p^{\mathbb{I}}_{L'}$ by assuming the antecedent of either $(\mathrm{S_i})$ or $(\mathrm{P_i})$:

$$\begin{aligned} p^{\mathbb{I}}_L &= \tfrac{\#\mathbb{J}}{\#\mathbb{I}} p^{\mathbb{J}}_L + \tfrac{\#\mathbb{K}}{\#\mathbb{I}} p^{\mathbb{K}}_L \\ &\sim_{\mathbb{P}^g} \tfrac{\#\mathbb{J}}{\#\mathbb{I}} p^{\mathbb{J}}_L + \tfrac{\#\mathbb{K}}{\#\mathbb{I}} p^{\mathbb{K}}_{L'} \qquad (\mathrm{I}^{\mathbb{Q}}_{\mathrm{a}}) \text{ and (D1)} \\ &\diamond_{\mathbb{P}^g} \tfrac{\#\mathbb{J}}{\#\mathbb{I}} p^{\mathbb{J}}_{L'} + \tfrac{\#\mathbb{K}}{\#\mathbb{I}} p^{\mathbb{K}}_{L'} \qquad (\mathrm{I}^{\mathbb{Q}}_{\mathrm{i}}) \text{ and (D2)} \\ &= p^{\mathbb{I}}_{L'} \end{aligned}$$

Since $\succsim_{\mathbb{P}^*}$ generates $\succsim^*$, we find $L \diamond_{\mathbb{I}} L'$, validating both $(\mathrm{S_i})$ and $(\mathrm{P_i})$. It remains to prove (D1) and (D2).

Suppose the antecedent of $(\mathrm{S_i})$ is satisfied, so that $L|_{\mathbb{K}} \sim_{\mathbb{K}} L'|_{\mathbb{K}}$. Then $p^{\mathbb{K}}_L = p^{\mathbb{K}}_{L|_{\mathbb{K}}} \sim_{\mathbb{P}^g} p^{\mathbb{K}}_{L'|_{\mathbb{K}}} = p^{\mathbb{K}}_{L'}$, as claimed by (D1). Similar reasoning shows $p^{\mathbb{J}}_L = p^{\mathbb{J}}_{L|_{\mathbb{J}}} \diamond_{\mathbb{P}^g} p^{\mathbb{J}}_{L'|\mathbb{J}} = p^{\mathbb{J}}_{L'}$, as claimed by (D2).

Suppose instead that the antecedent of $(\mathrm{P_i})$ is satisfied, so that $L \approx^{\mathbb{K}}_{\mathbb{P}^g} L'$. This means that $\mathcal{P}_k(L) \sim_{\mathbb{P}^g} \mathcal{P}_k(L')$ for all $k \in \mathbb{K}$. We obtain $p^{\mathbb{K}}_L \sim_{\mathbb{P}^g} p^{\mathbb{K}}_{L'}$, as claimed by (D1), by repeatedly applying $(\mathrm{I}^{\mathbb{Q}}_{\mathrm{a}})$. If $i = a$ or $i = b$, then $p^{\mathbb{J}}_L \diamond_{\mathbb{P}^g} p^{\mathbb{J}}_{L'}$ follows by a similar method. The case $i = c$ is slightly more complicated. Choose any $j \in \mathbb{J}$. Since $\mathcal{P}_k(L) \sim_{\mathbb{P}^g} \mathcal{P}_j(L)$ for any other $k \in \mathbb{J}$, we can deduce $p^{\mathbb{J}}_L \sim_{\mathbb{P}^g} \mathcal{P}_j(L)$

by repeatedly applying $(\mathrm{I}^{\mathbb{Q}}_{\mathrm{a}})$. Similarly, $p^{\mathbb{J}}_{L'} \sim_{\mathbb{P}^g} \mathcal{P}_j(L')$. Since $\mathcal{P}_j(L) \curlywedge_{\mathbb{P}^g} \mathcal{P}_j(L')$, we obtain $p^{\mathbb{J}}_{L} \diamond_{\mathbb{P}^g} p^{\mathbb{J}}_{L'}$, completing the proof of (D2).

We now argue that $(\mathrm{S_i}) \implies (\mathrm{I}^{\mathbb{Q}}_{\mathrm{i}}) \impliedby (\mathrm{P_i})$ for $i = 1, 2, 3$, and indeed for each of $i = a, b, c$. Suppose given p, p', $q \in \mathbb{P}^g$ and $\alpha \in (0, 1) \cap \mathbb{Q}$. Let $P := \alpha p + (1 - \alpha)q$, $P' := \alpha p' + (1 - \alpha)q$. It suffices to show that, given $i \in \{a, b, c\}$, each of $(\mathrm{S_i})$ and $(\mathrm{P_i})$ implies $p \diamond_{\mathbb{P}^g} p' \implies P \diamond_{\mathbb{P}^g} P'$.

Let $\mathbb{J}$, $\mathbb{K} \subset \mathbb{I}^*$ be finite with $\mathbb{J} \cap \mathbb{K} = \varnothing$ such that $\frac{\#\mathbb{J}}{\#\mathbb{K}} = \frac{\alpha}{1-\alpha}$. Let $\mathbb{I} := \mathbb{J} \cup \mathbb{K}$. If $\mathcal{F}$ is compositional, we can find $L, L' \in \mathbb{L}_{\mathbb{I}}$ such that $\mathcal{P}_j(L) = p$ and $\mathcal{P}_j(L') = p'$ for all $j \in \mathbb{J}$, and $\mathcal{P}_k(L) = \mathcal{P}_k(L') = q$ for all $k \in \mathbb{K}$. Then $p^{\mathbb{I}}_{\mathbb{L}} = P$ and $p^{\mathbb{I}}_{\mathbb{L}'} = P'$. Alternatively, if $\mathcal{F}$ is the restriction of a variable population model, then set $L := \frac{1}{2}\mathcal{L}_{\mathbb{J}}(p) + \frac{1}{2}\mathcal{L}_{\mathbb{K}}(q)$ and $L' := \frac{1}{2}\mathcal{L}_{\mathbb{J}}(p') + \frac{1}{2}\mathcal{L}_{\mathbb{K}}(q)$. Then $p^{\mathbb{I}}_{\mathbb{L}} = \frac{1}{2}P + \frac{1}{2}1_\Omega$ and $p^{\mathbb{I}}_{\mathbb{L}'} = \frac{1}{2}P' + \frac{1}{2}1_\Omega$. Under either assumption, $L \diamond_{\mathbb{I}} L' \iff p^{\mathbb{I}}_{\mathbb{L}} \diamond_{\mathbb{P}^g} p^{\mathbb{I}}_{\mathbb{L}'} \iff P \diamond_{\mathbb{P}^g} P'$ (using Omega Independence in the second case). In addition, the antecedent of each of $(\mathrm{S_i})$ and $(\mathrm{P_i})$ holds if and only if $p \diamond_{\mathbb{P}^g} p'$. Tracing backwards, we see that if $p \diamond_{\mathbb{P}^g} p'$, then $P \diamond_{\mathbb{P}^g} P'$, as claimed by $(\mathrm{I}^{\mathbb{Q}}_{\mathrm{i}})$.

□

Lemma A.0.1 (Bogachev [20, Thm 3.6.1])**.** *Let X and Y be measurable spaces. Let $f : X \to Y$ and $g : Y \to \mathbb{R}$ be measurable functions, and let μ be a nonnegative measure on X. Then $\mu \circ f^{-1}$ is a measure on Y and g is integrable with respect to $\mu \circ f^{-1}$ on Y precisely when $g \circ f$ is integrable with respect to μ. In addition, one has*

$$\int_Y g \, d(\mu \circ f^{-1}) = \int_X g \circ f \, d\mu.$$

Proof of Proposition 3.3.1. We will present the proofs in the variable population case, since those in the constant population case are exactly parallel.

Suppose $\succsim_{\mathbb{P}^*}$ satisfies Vector EUT with respect to some $(\mathbb{V}, \succsim_{\mathbb{V}}, \mathcal{A})$; the standard EUT case is covered by $(\mathbb{R}, \geq, \{\mathrm{id}\})$. For any $L \in \mathbb{L}^*$ and $\Lambda \in \mathcal{A}$, Lemma A.0.1 above yields

$$\int_{\mathbb{W}^*} \Lambda \circ u \, d\mathcal{P}_i(L) = \int_{\mathbb{W}^*} \Lambda \circ u \, d(L \circ (\mathcal{W}^*_i)^{-1}) = \int_{\mathbb{H}^*} \Lambda \circ u \circ \mathcal{W}^*_i \, dL.$$

Using this we compute that for any finite $\mathbb{I} \subset \mathbb{I}^*$,

$$\begin{aligned} \int_{\mathbb{W}^*} \Lambda \circ u \, dp^{\mathbb{I}}_L &= \int_{\mathbb{W}^*} \Lambda \circ u \, d\left(\frac{1}{\#\,\mathbb{I}} \sum_{i \in \mathbb{I}} \mathcal{P}_i(L) \right) = \frac{1}{\#\,\mathbb{I}} \sum_{i \in \mathbb{I}} \int_{\mathbb{W}^*} \Lambda \circ u \, d\mathcal{P}_i(L) \\ &= \int_{\mathbb{H}^*} \frac{1}{\#\,\mathbb{I}} \sum_{i \in \mathbb{I}} \Lambda \circ u \circ \mathcal{W}^*_i \, dL = \int_{\mathbb{H}^*} \Lambda \circ \frac{1}{\#\,\mathbb{I}} \sum_{i \in \mathbb{I}} u \circ \mathcal{W}^*_i \, dL. \end{aligned}$$

Therefore, by definition of the $\mathbb{V}$-valued integral,

$$\int_{\mathbb{W}^*} u \, dp^{\mathbb{I}}_L = \int_{\mathbb{H}^*} \frac{1}{\#\,\mathbb{I}} \sum_{i \in \mathbb{I}} u \circ \mathcal{W}^*_i \, dL.$$

Let L, $L' \in \mathbb{L}^*_{\mathbb{I}}$. Since $\succsim^*$ is generated by $\succsim_{\mathbb{P}^*}$,

$$\begin{aligned} L \succsim^* L' &\iff p^{\mathbb{I}}_L \succsim_{\mathbb{P}^*} p^{\mathbb{I}}_{L'} \\ &\iff \int_{\mathbb{H}^*} \frac{1}{\#\,\mathbb{I}} \sum_{i \in \mathbb{I}} u \circ \mathcal{W}^*_i \, dL \succsim_{\mathbb{V}} \int_{\mathbb{H}^*} \frac{1}{\#\,\mathbb{I}} \sum_{i \in \mathbb{I}} u \circ \mathcal{W}^*_i \, dL' \\ &\iff \int_{\mathbb{H}^*} \sum_{i \in \mathbb{I}^*} (u \circ \mathcal{W}^*_i - u(\Omega)) \, dL \succsim_{\mathbb{V}} \int_{\mathbb{H}^*} \sum_{i \in \mathbb{I}^*} (u \circ \mathcal{W}^*_i - u(\Omega)) \, dL'. \end{aligned}$$

The last step uses the fact that, in a preordered vector space, the transformation $v \mapsto \#\mathbb{I}v - \#\mathbb{I}u(\Omega)$ is order-preserving. This shows that $\succsim^*$ satisfies Vector EUT for the same triple $(\mathbb{V}, \succsim_{\mathbb{V}}, \mathcal{A})$, and is in particular represented by the stated function V.

A similar calculation shows that if v is a utility function (integrand) for $\succsim^*$, then $v \circ \mathcal{H}_{\mathbb{I}}^*$ is a utility function for $\succsim_{\mathbb{P}^*}$, for any finite nonempty $\mathbb{I}$. This shows that if $\succsim^*$ satisfies Vector EUT, then so does $\succsim_{\mathbb{P}^*}$. □

Proof of Theorem 3.4.1. It is clear that $\succsim_X$ satisfies (I_3) if it has an MP representation. Indeed, for $p, p', q \in X$ and $\alpha \in (0,1)$, $p \succsim_X p' \iff U(p) \succsim_{\mathbb{V}} U(p') \iff \alpha U(p) + (1-\alpha)U(q) \succsim_{\mathbb{V}} \alpha U(p') + (1-\alpha)U(q) \iff \alpha p + (1-\alpha)q \succsim_X \alpha p' + (1-\alpha)q$. (The second biconditional follows from the definition of 'preordered vector space'.)

Conversely, suppose that $\succsim_X$ satisfies (I_3). Let $\mathbb{V} = \mathrm{span}(X)$. (By definition, a convex set like X is a subset of a vector space. When $X = \mathbb{P}^*$, the vector space is the space of signed measures on $\mathbb{W}^*$.) Define $C \subset \mathbb{V}$ by $C := \{\lambda(q - q') \colon \lambda > 0, q \succsim_X q'\}$. Define a binary relation $\succsim_{\mathbb{V}}$ on $\mathbb{V}$ by $r \succsim_{\mathbb{V}} s \iff r - s \in C$. It is well known, and easy to check, that this construction makes $\mathbb{V}$ into a preordered vector space.

Let U be the inclusion of X into $\mathbb{V}$. It is obviously mixture preserving, and $p \succsim_X p' \implies U(p) \succsim_{\mathbb{V}} U(p')$. It remains to show that $U(p) \succsim_{\mathbb{V}} U(p') \implies p \succsim_X p'$.

If $U(p) \succsim_{\mathbb{V}} U(p')$, then there must be $\lambda > 0$ and $q, q' \in X$ with $q \succsim_X q'$ and $p - p' = \lambda(q - q')$. Let $\alpha := \frac{1}{1+\lambda}$. Then $\alpha p + (1-\alpha)q' = \alpha p' + (1-\alpha)q$. This, together with the fact that $\succsim_X$ satisfies (I_3), yields $q \succsim_X q' \implies \alpha p' + (1-\alpha)q \succsim_X \alpha p' + (1-\alpha)q' \implies \alpha p + (1-\alpha)q' \succsim_X \alpha p' + (1-\alpha)q' \implies p \succsim_X p'$, as desired. □

Proof of Proposition 3.4.2. The first statement follows from Theorem 3.4.1 and Proposition 3.1.1. Suppose now given an MP representation $U \colon \mathbb{P}^* \to \mathbb{V}$. For $L, L' \in \mathbb{L}_{\mathbb{I}}^*$, we have

$$\begin{aligned} L \succsim^* L' &\iff p_L^{\mathbb{I}} \succsim_{\mathbb{V}} p_{L'}^{\mathbb{I}} && (\succsim_{\mathbb{P}^*} \text{ generates } \succsim^*) \\ &\iff U(p_L^{\mathbb{I}}) \succsim_{\mathbb{V}} U(p_{L'}^{\mathbb{I}}) && (U \text{ represents } \succsim_{\mathbb{P}^*}) \\ &\iff \textstyle\sum_{i\in\mathbb{I}} U(\mathcal{P}_i^*(L)) \succsim_{\mathbb{V}} \sum_{i\in\mathbb{I}} U(\mathcal{P}_i^*(L')) && (U \text{ mixture preserving}). \end{aligned}$$

This last inequality is in turn equivalent to

$$\sum_{i\in\mathbb{I}^*} [U(\mathcal{P}_i^*(L)) - U(1_\Omega)] \succsim_{\mathbb{V}} \sum_{i\in\mathbb{I}^*} [U(\mathcal{P}_i^*(L')) - U(1_\Omega)].$$

This shows that $\sum_{i\in\mathbb{I}^*} (U \circ \mathcal{P}_i - U(1_\Omega))$ represents $\succsim^*$. It is easy to check that this function is mixture preserving, so it is an MP representation of $\succsim^*$. □

Proof of Theorem 3.4.3. Given a set $\mathcal{R}$ and an ordered set $\mathcal{C}$, we can make $\mathbb{V} := \mathbb{R}^{\mathcal{R}\times\mathcal{C}}$ into a preordered vector space, by the following rule: $(x_{ab}) \succsim_{\mathbb{V}} (y_{ab})$ if and only if, for each $a \in \mathcal{R}$, either $x_{ab} = y_{ab}$ for all $b \in \mathcal{C}$, or there is a largest $b \in \mathcal{C}$ such that $x_{ab} \neq y_{ab}$, and, for that b, $x_{ab} > y_{ab}$. An LMP representation is an MP representation with values in a preordered vector space of this form. In particular, by Theorem 3.4.1, to have an LMP representation, $\succsim_X$ must satisfy (I_3).

Conversely, by Theorem 3.4.1, if $\succsim_X$ satisfies (I_3) then it has an MP representation U with values in some preordered vector space $\mathbb{V}$. To prove the first statement of the theorem, it remains to find an LMP representation $\{\tilde{U}_{ab}\}$ of $\succsim_{\mathbb{V}}$, for then $\{\tilde{U}_{ab} \circ U\}$ is an LMP representation of $\succsim_X$.

To do this, apply Theorem 3.4.5. Let $\mathcal{R}$ be the set of complete vector preorders extending $\succsim_{\mathbb{V}}$. For $a \in \mathcal{R}$, let $(\mathcal{O}_a, \mathcal{G}_a)$ be a corresponding lexicographic filtration, guaranteed to exist by Theorem 3.4.5. Let $\mathcal{C}$ be the disjoint union of the sets $\mathcal{O}_a$; choose any ordering on it that restricts to the given ordering on each $\mathcal{O}_a$. For each

$b \in \mathcal{O}_a \subset \mathcal{C}$, we have a function $g_b \in \mathcal{G}_a$; extend g_b arbitrarily to a linear map $\mathbb{V} \to \mathbb{R}$. Now, for each $(a, b) \in \mathcal{R} \times \mathcal{C}$ and $v \in \mathbb{V}$, define

$$\tilde{U}_{ab}(v) = \begin{cases} g_b(v) & \text{if } b \text{ is in } \mathcal{O}_a \subset \mathcal{C} \\ 0 & \text{if not.} \end{cases}$$

Let us verify that this is an LMP representation of $\succsim_{\mathbb{V}}$. Fix $v, w \in \mathbb{V}$. For LMP1, fix $a \in \mathcal{R}$, and suppose that $U_{ab}(v) \neq U_{ab}(w)$ for some $b \in \mathcal{C}$. Then $\tilde{U}_{ab}(v) = \tilde{U}_{ab}(w) = 0$ unless $b \in \mathcal{O}_a$, so it suffices to show that there is a largest $b \in \mathcal{O}_a$ with $g_b(v) \neq g_b(w)$. Since $(\mathcal{O}_a, \mathcal{G}_a)$ is a lexicographic filtration, we can can take b to be the largest element of $\mathcal{O}_a$ containing $v - w$. Moreover, for this b, $g_b(v) > g_b(w)$ if and only if $v \succsim_{\mathbb{V}}^{\text{com}} w$ for the complete extension $\succsim_{\mathbb{V}}^{\text{com}}$ corresponding to $(\mathcal{O}_a, \mathcal{G}_a)$. LMP2 therefore follows from the first statement of Theorem 3.4.5.

For the third statement, concerning (MC), having an LMP representation with $\mathcal{C} = 1$ means there is a set $\mathcal{R}$ indexing a family of mixture-preserving functions $\{U_a : X \to \mathbb{R}\}_{a \in \mathcal{R}}$ such that $p \succsim_X q$ iff $U_a(p) \succsim U_a(q)$ for all a. It is easy to see that, in this case, $\succsim_X$ must satisfy (MC); for example, given $p, q, r \in X$, the set $\{\alpha \in [0, 1] : \alpha p + (1 - \alpha) r \succsim_X q\}$ is the intersection of the closed sets $\{\alpha \in [0, 1] : \alpha U_a(p) + (1 - \alpha) U_a(r) \geq U_a(q)\}$, and is therefore closed. (It is worth noting that $\succsim_X$ may not satisfy (Ar).)

Conversely, suppose that $\succsim_X$ satisfies (MC) as well as (I_3). As before, we obtain an MP representation of $\succsim_X$ from Theorem 3.4.1. We claim that the preordered vector space $(\mathbb{V}, \succsim_{\mathbb{V}})$ constructed in the proof of that theorem satisfies (MC). Indeed, suppose given $u, v, w \in \mathbb{V}$, and let $\mathbb{A} = \{\lambda(x - x') : \lambda > 0, x, x' \in X\}$ be the affine hull of X in $\mathbb{V}$. Note that the positive cone C is contained in $\mathbb{A}$. If $u - v \notin \mathbb{A}$ or $w - v \notin \mathbb{A}$, there is therefore at most one $\alpha \in [0, 1]$ such that $\alpha u + (1 - \alpha) w \succsim_{\mathbb{V}} v$, so the set of such α is closed. Otherwise, we have $u - v = \lambda_1(x - x')$ and $w - v = \lambda_2(y - y')$ for some $x, x', y, y' \in X$ and $\lambda_1, \lambda_2 > 0$. Set $\beta_i = \lambda_i/(\lambda_1 + \lambda_2)$, $p = (\beta_1 x + (1 - \beta_1) x' + y')/2$, $q = (x' + y')/2$, $r = (x' + \beta_2 y + (1 - \beta_2) y')/2$. Then we have $p, q, r \in X$ and a direct calculation shows that, for any $\alpha \in [0, 1]$,

$$\alpha p + (1 - \alpha) r - q = \frac{1}{\lambda_1 + \lambda_2}(\alpha u + (1 - \alpha) w - v).$$

Therefore $\alpha u + (1 - \alpha) w \succsim_{\mathbb{V}} v \iff \alpha p + (1 - \alpha) r \succsim_{\mathbb{V}} q$. Since the restriction of $\succsim_{\mathbb{V}}$ to X equals $\succsim_X$, and $\succsim_X$ satisfies (MC), $\succsim_{\mathbb{V}}$ must also satisfy (MC).

It suffices, then, to show that a preordered vector space that satisfies (MC) has an LMP representation with $\#\mathcal{C} = 1$. Let $\mathcal{R}$ be the set of complete, mixture-continuous vector preorders extending $\succsim_{\mathbb{V}}$; by the last part of Theorem 3.4.5, each $a \in \mathcal{R}$ gives rise to a representing function $g_{\mathbb{V}}^a : \mathbb{V} \to \mathbb{R}$. Moreover, we get the desired LMP representation $\{\tilde{U}_a\}$ by defining $\tilde{U}_a = g_{\mathbb{V}}^a$.

Finally, consider the second and fourth statements to be proved, concerning (O). It is easy to see the any LMP representation with $\#\mathcal{R} = 1$ ensures that $\succsim_X$ is complete. Conversely, suppose that $\succsim_X$ satisfies (I_3) and (O), and again let $(\mathbb{V}, \succsim_{\mathbb{V}})$ be the preordered vector space constructed in the proof of Theorem 3.4.1. As constructed, $\succsim_{\mathbb{V}}$ may not be complete. However, choose any $x_0 \in X$, and set $\mathbb{V}_0 = \text{Span}(\{x - x_0 : u \in X\}) = \{\lambda(p - q) : \lambda > 0, p, q \in X\}$, and define $U_0(x) = x - x_0$; then the restriction $\succsim_{\mathbb{V}_0}$ of $\succsim_{\mathbb{V}}$ to $\mathbb{V}_0$ is complete, and U_0 is an MP representation of $\succsim_X$ in $\mathbb{V}_0$. As before, $\succsim_{\mathbb{V}_0}$ also satisfies (MC) if and only if $\succsim_X$ does.

Now, whether or not $\succsim_{\mathbb{V}_0}$ satisfies (MC), the above constructions of LMP representations of preordered vector spaces produce LMP representations of $\succsim_{\mathbb{V}_0}$ with $\#\mathcal{R} = 1$. Composing with U_0, these in determine LMP representations of $\succsim_X$, proving the second and fourth statements of the theorem. □

Proof of Proposition 3.4.4. The first statement follows from Theorem 3.4.3 and Proposition 3.1.1. As for the second, we explained in the proof of Theorem 3.4.4 that an LMP representation $\{U_{ab}\}$ amounts to an MP representation U with values in $\mathbb{V} = \mathbb{R}^{\mathcal{R}\times\mathcal{C}}$. Proposition 3.4.2 applies to this MP representation to give the desired result. □

Proof of Theorem 3.4.5. Suppose that $(\mathbb{V}, \succsim_{\mathbb{V}})$ is a preordered vector space, as in the first part of the theorem.

Lemma A.0.2. *Suppose given $v_0 \in \mathbb{V}$ such that $v_0 \not\succsim_{\mathbb{V}} 0$. Then there exists a complete vector preorder $\succsim_{\mathbb{V}}^{\text{com}}$ extending $\succsim_{\mathbb{V}}$ such that $v_0 \prec_{\mathbb{V}}^{\text{com}} 0$. If $\succsim_{\mathbb{V}}$ satisfies* (MC)*, then we can choose $\succsim_{\mathbb{V}}^{\text{com}}$ also to satisfy* (MC)*.*

Proof. Let us show that there exists a vector preorder $\succsim'_{\mathbb{V}}$, not necessarily complete, extending $\succsim_{\mathbb{V}}$ and such that $v_0 \prec'_{\mathbb{V}} 0$. In fact, we can define $\succsim'_{\mathbb{V}}$ by the rule:

$$w \succsim'_{\mathbb{V}} 0 \iff \exists \lambda \geq 0 : w + \lambda v_0 \succsim_{\mathbb{V}} 0.$$

Then, by Zorn's Lemma, a maximal such extension $\succsim_{\mathbb{V}}^{\text{com}}$ exists. This extension must be complete, since otherwise we could find a further extension using the same trick.

For the claim about (MC), we just have to show that $\succsim'_{\mathbb{V}}$ satisfies (MC) if $\succsim_{\mathbb{V}}$ does. Given $u, v, w \in \mathbb{V}$, let $A(u, v, w) = \{\alpha \in [0, 1] : \alpha u + (1 - \alpha)w - v \succsim_{\mathbb{V}} 0\}$ and $A'(u, v, w) = \{\alpha \in [0, 1] : \alpha u + (1 - \alpha)w - v \succsim'_{\mathbb{V}} 0\}$. (MC) for $\succsim_{\mathbb{V}}$ implies that $A(u, v, w)$ is always closed, and we have to check that $A'(u, v, w)$ is closed.

Let $\mathbb{V}_0 = \text{Span}\{u - v, w - v, v_0\}$, and $C_0 = \{v \in \mathbb{V}_0 : v \succsim_{\mathbb{V}} 0\}$. We use the fact that a convex subset C of a finite-dimensional vector space like $\mathbb{V}_0$ is closed if and only if, for all $x, y \in \mathbb{V}_0$, the set $B(x, y, C) = \{\alpha \in [0, 1] : \alpha x + (1 - \alpha)y \in C\}$ is closed. (The left-to-right direction is obvious. Right-to-left, suppose x is in the relative boundary of C. Choose y in the relative interior; by Rockafellar [93, Theorem 6.1], $B(x, y, C)$ contains $[0, 1)$, so must equal $[0, 1]$; hence $x \in C$.) Since $B(x, y, C_0) = A(x, 0, y)$ is closed for all $x, y \in \mathbb{V}_0$, C_0 is closed. It follows from Rockafellar [93, Corollary 9.1.3] that $C'_0 := C_0 + \{\lambda v_0 : \lambda \leq 0\}$ is also closed in $\mathbb{V}_0$. Since $A'(u, v, w) = B(u - v, w - v, C'_0)$, it is closed, as desired. □

To prove the first part of the theorem, suppose that $v \succsim_{\mathbb{V}} w$. Then $v \succsim_{\mathbb{V}}^{\text{com}} w$, for all complete vector preorders $\succsim_{\mathbb{V}}^{\text{com}}$ extending $\succsim_{\mathbb{V}}$, by definition of 'extending'. Conversely, suppose that $v \succsim_{\mathbb{V}}^{\text{com}} w$, or equivalently $v_0 := v - w \succsim_{\mathbb{V}}^{\text{com}} 0$, for all such $\succsim_{\mathbb{V}}^{\text{com}}$. We cannot have $v_0 \prec_{\mathbb{V}} 0$, for that would require $v_0 \prec_{\mathbb{V}}^{\text{com}} 0$. Nor can we have $v_0 \not\succsim_{\mathbb{V}} 0$: by Lemma A.0.2, we would then have some $\succsim_{\mathbb{V}}^{\text{com}}$ with $v_0 \prec_{\mathbb{V}}^{\text{com}} 0$. Therefore we must have $v_0 \succsim_{\mathbb{V}} 0$, hence $v \succsim_{\mathbb{V}} w$, as desired.

Now for the second part of the theorem. For each $v \in \mathbb{V}$, write $|v| = v$ if $v \geq 0$, and $|v| = -v$ if $v < 0$. Define

$$W_v^+ = \{w \in \mathbb{V} : \lambda|v| \succsim_{\mathbb{V}}^{\text{com}} |w| \text{ for } \textit{some} \text{ real number } \lambda > 0\}$$

$$W_v^- = \{w \in \mathbb{V} : \lambda|v| \succ_{\mathbb{V}}^{\text{com}} |w| \text{ for } \textit{all} \text{ real numbers } \lambda > 0\}.$$

Let $\mathcal{O} = \{W_v^+ : v \in \mathbb{V}\}$. Clearly $W_v^- \subset W_v^+$. It is also easy to check

(a) $W_w^+ \subseteq W_v^+ \iff w \in W_v^+ \iff v \notin W_w^- \iff W_v^+ \not\subseteq W_w^-$.

It follows that $\mathcal{O}$ is completely ordered by inclusion: either $W_w^+ \subseteq W_v^+$, or else $W_v^+ \subseteq W_w^- \subseteq W_w^+$. Moreover, W_w^+ is the smallest element of $\mathcal{O}$ containing w: for if $w \in W_v^+$ then $W_w^+ \subseteq W_v^+$.

To get a lexicographic filtration, it suffices to produce for each $W \in \mathcal{O}$ a linear map $g_W : W \to \mathbb{R}$ satisfying the following two conditions:

(b) For $w \in W$, $g_W(w) = 0 \iff W_w^+ \subsetneq W$ or $W = W_0^+$.

(c) For $v, w \in \mathbb{V}$ with $v, w \succsim_{\mathbb{V}}^{\mathrm{com}} 0$,

$$v \succsim_{\mathbb{V}}^{\mathrm{com}} w \iff W_v^+ \supseteq W_w^+ \text{ and } g_{W_v^+}(v) \geq g_{W_v^+}(w).$$

To do it, choose v_0 such that $W = W_{v_0}^+$. If $v_0 \sim_{\mathbb{V}} 0$, so that $W = W_0^+$, define $g_W = 0$; otherwise, define $g_W(w) = \inf\{\lambda \in \mathbb{R} : \lambda|v_0| \succsim_{\mathbb{V}}^{\mathrm{com}} w\}$. (The infimum exists. If $w = |w|$, then this set of λ is non-empty by definition of $W_{v_0}^+$, and bounded below by 0; if $w = -|w|$, then the infimum equals $-g_W(|w|)$.)

First let us check that g_W is linear. This is trivial when $W = W_0^+$, so suppose $W = W_{v_0}^+$ with $v_0 \not\sim_{\mathbb{V}} 0$. In fact we claim that $W_v^+ = W_v^- \oplus \mathbb{R}|v_0|$, and that $w \mapsto g_W(w)|v_0|$ is the projection onto the second factor. Therefore g_W is linear. To prove the claim, it suffices to show that $w - g_W(w)|v_0|$ lies in $W_{v_0}^-$. By definition of g_W, we know that

$$(g_W(w) + \mu_1)|v_0| \succsim_{\mathbb{V}}^{\mathrm{com}} w \succsim_{\mathbb{V}}^{\mathrm{com}} (g_W(w) - \mu_2)|v_0|$$

for any real $\mu_1, \mu_2 > 0$. Rearranging,

$$\mu_1|v_0| \succsim_{\mathbb{V}}^{\mathrm{com}} w - g_W(w)|v_0| \succsim_{\mathbb{V}}^{\mathrm{com}} -\mu_2|v_0|.$$

If $w - g_W(w)|v_0| \succsim_{\mathbb{V}}^{\mathrm{com}} 0$, then the inequality on the left shows that $w - g_W(w)|v_0|$ is in $W_{v_0}^-$; on the other hand, if $w - g_W(w)|v_0| \prec_{\mathbb{V}} 0$, then the inequality on the right does the same.

Now let us check that the g_W we have defined satisfy condition (b). Assuming that $W \neq W_0$, we have from the definitions that $g_{W_v^+}(w) = 0 \iff w \in W_v^-$. We also have from (a) that $w \in W_v^- \iff W_w^+ \subsetneq W_v^+$. That proves (b).

As for (c), the left-to-right implication follows easily from the definition of the subspaces W_v^+ and the functions $g_{W_v^+}$. Conversely, suppose that $W_v^+ \subseteq W_w^+$ and $g_{W_v^+}(v) \geq g_{W_v^+}(w)$. Then $v - w \in W_v^+$, and $g_{W_v^+}(v - w) = g_{W_v^+}(v) - g_{W_v^+}(w) \geq 0$. From the definition of $g_{W_v^+}$ we must have $v \succsim_{\mathbb{V}}^{\mathrm{com}} w$.

Finally, let us show that the lexicographic filtration is unique in the stated sense. First, given a lexicographic filtration $(\mathcal{O}, \mathcal{G})$, it is easy to check that W_v satisfies the definition of W_v^+. This shows that $\mathcal{O}$ is uniquely determined. Second, $g_{W_0} = 0$, so is completely determined. For $W_v \neq W_0$, the kernel of g_{W_v} is $\{w : W_w \subsetneq W_v\}$. So the kernel is completely determined, and g_{W_v} assigns $|v|$ a positive value. This determines g_{W_v} up to positive scale.

As for part (3) of the theorem, if $\succsim_{\mathbb{V}}$ satisfies (MC), then the construction of Lemma A.0.2 yields complete extensions that also satisfy (MC), and the subsequent argument goes through to prove the addendum to part (1). As for the addendum to part (2), suppose that the lexicographic filtration for $\succsim_{\mathbb{V}}^{\mathrm{com}}$ contains $W_v \supsetneq W_w \supsetneq W_0$. Then the set of $\alpha \in [0, 1]$ such that $\alpha|v| + (1 - \alpha)0 \succsim_{\mathbb{V}}^{\mathrm{com}} |w|$ equals $(0, 1]$, contradicting (MC). So, given (MC), $\mathcal{O} = \{\mathbb{V}, W_0\}$ with $g_{\mathbb{V}} \neq 0$, or $\mathcal{O} = \{W_0\}$, in which case $\mathbb{V} = W_0$ and $g_{\mathbb{V}} = 0$. □

Section 4.

Proof of Proposition 4.1.1. Suppose upper-measurable $\succsim_{\mathbb{P}^*}$ satisfies (M). For (ii), let $L, L' \in \mathbb{L}^*$. Suppose that $\mathcal{P}_i(L) \succsim_{\mathbb{P}^*}^{\mathrm{SD}} \mathcal{P}_i(L')$ for all $i \in \mathbb{I}^*$ and $\mathcal{P}_j(L) \succ_{\mathbb{P}^*}^{\mathrm{SD}} \mathcal{P}_j(L')$ for some $j \in \mathbb{I}^*$. Then $L, L' \in \mathbb{L}_{\mathbb{I}}$ for some finite $\mathbb{I} \subset \mathbb{I}^*$ with $j \in \mathbb{I}$, and for all upper $A \subset \mathbb{W}^*$, $\mathcal{P}_i(L)(A) \geq \mathcal{P}_i(L')(A)$ for all $i \in \mathbb{I}$ and $\mathcal{P}_j(L)(A) \geq \mathcal{P}_j(L')(A)$, hence $p_L^{\mathbb{I}}(A) > p_{L'}^{\mathbb{I}}(A)$. This implies $p_L \succ_{\mathbb{P}^*}^{\mathrm{SD}} p_{L'}$ by (M), hence $L \succ^* L'$ by Theorem 2.3.1. The proof of (i) is similar. □

Proof of Proposition 4.1.3. Suppose given $P, Q, R \in \mathbb{P}^*$ and $\alpha \in (0, 1)$. Write $[P, R]$ for the mixture $\alpha P + (1 - \alpha)R$.

Suppose first of all that P and Q have rational probability values. It follows that, for some common denominator N, any population $\mathbb{I}$ of size N, and some $P_i, Q_i \in \mathbb{W}^*$, we can write

$$P = \frac{1}{N}\sum_{i\in\mathbb{I}} 1_{P_i} \quad Q = \frac{1}{N}\sum_{i\in\mathbb{I}} 1_{Q_i}.$$

It follows that there is some $H_P \in \mathbb{H}^*_{\mathbb{I}}$ with $\mathcal{W}^*_i(H_P) = P_i$ for all $i \in \mathbb{I}$; we therefore have a lottery $L_P := 1_{H_P}$ with $p^{\mathbb{I}}_{L_P} = P$. Similarly for Q.

Since $\succsim_{\mathbb{P}^*}$ is complete, either $P \succsim_{\mathbb{P}^*} Q$ or $Q \succsim_{\mathbb{P}^*} P$; hence, by Theorem 2.3.1, either $L_P \succsim^* L_Q$ or $L_Q \succsim^* L_P$. Since L_P and L_Q are delta-measures, $[L_P, \mathcal{L}_{\mathbb{I}}(R)]$ stochastically dominates $[L_Q, \mathcal{L}_{\mathbb{I}}(R)]$ if $L_P \succsim^* L_Q$, and vice versa if $L_Q \succsim^* L_P$. Applying (M), we find

$$[L_P, \mathcal{L}_{\mathbb{I}}(R)] \succsim^* [L_Q, \mathcal{L}_{\mathbb{I}}(R)] \iff L_P \succsim^* L_Q.$$

Now, $p^{\mathbb{I}}_{[L_P,\mathcal{L}_{\mathbb{I}}(R)]} = [P, R]$ and $p^{\mathbb{I}}_{[L_Q,\mathcal{L}_{\mathbb{I}}(R)]} = [Q, R]$. So the aggregation theorem gives

$$[P, R] \succsim_{\mathbb{P}^*} [Q, R] \iff [L_P, \mathcal{L}_{\mathbb{I}}(R)] \succsim^* [L_Q, \mathcal{L}_{\mathbb{I}}(R)] \iff L_P \succsim^* L_Q \iff P \succsim_{\mathbb{P}^*} Q.$$

This establishes (I_3) for $\succsim_{\mathbb{P}^*}$ under the assumption that P, Q have rational values.

Suppose now that P, Q are general. Then we can find a sequence (A_i) in $\mathbb{P}^*$ strongly converging to P such that each A_i has rational values, and each A_i stochastically dominates P. Similarly choose a sequence (B_i) converging to Q such that each B_i has rational values and Q stochastically dominates each B_i. Note that (M) for $\succsim^*$ and Reduction to Prospects imply that $\succsim_{\mathbb{P}^*}$ satisfies (M). Thus $A_i \succsim_{\mathbb{P}^*} P$ and $Q \succsim_{\mathbb{P}^*} B_i$. Using this, the result for rational-probability prospects, and continuity, we have

$$P \succsim_{\mathbb{P}^*} Q \implies \forall ij. A_i \succsim_{\mathbb{P}^*} B_j \iff \forall ij.[A_i, R] \succsim_{\mathbb{P}^*} [B_j, R] \implies [P, R] \succsim_{\mathbb{P}^*} [Q, R].$$

Moreover, the first and last implications are reversible using continuity and (M) respectively. □

Proof of Lemma 4.2.1. Suppose $V : P \to \mathbb{R}$ is integrally Gâteaux differentiable at p. There exists $v_p \in \nabla V_p$ such that $V(p) = \int_Y v_p\, dp$: for any $u_p \in \nabla V_p$, set $v_p := u_p + V(p) - \int_Y u_p\, dp$. By definition of ∇V_p, for any $q \in P$ and $t \in [0, 1]$,

$$V(p + t(q - p)) = V(p) + t\int_Y v_p\, d(q - p) + o(t).$$

This rearranges to (3). Conversely, if V satisfies (3) for some function v_p, it follows that $V'_p(q - p) = \int_Y v_p\, \mathrm{d}(q - p)$, hence V is integrally Gâteaux differentiable. □

Proof of Proposition 4.2.2. Part (i) is immediate from the constant population aggregation theorem and the fact that U, hence $\#\mathbb{I}U$, represents $\succsim_{\mathbb{P}}$.

For part (ii), suppose that U is Gâteaux differentiable at p_L. Then $V(L+t(M-L)) = \#\mathbb{I}U(p_L + t(p_M - p_L))$, so $V'_L(M - L) = \#\mathbb{I}U'_{p_L}(p_M - p_L)$.

For part (iii), suppose that U is integrally Gâteaux differentiable at p_L. Fix $u_L \in \nabla U_{p_L}$. For any $M \in \mathbb{P}$ we have $\#\mathbb{I}\int_{\mathbb{W}} u_L\, dp_M = \int_{\mathbb{W}} \sum_{i\in\mathbb{I}} u_L\, d(\mathcal{P}_i(M)) = \int_{\mathbb{W}} \sum_{i\in\mathbb{I}} u_L\, d(M \circ \mathcal{W}_i^{-1}) = \int_{\mathbb{H}} \sum_{i\in\mathbb{I}} u_L \circ \mathcal{W}_i\, dM$, using Lemma A.0.1. This shows that $\sum_{i\in\mathbb{I}} u_L \circ \mathcal{W}_i$ is $\mathbb{L}$-integrable. Moreover, $V'_L(M - L) = \#\mathbb{I}U'_{p_L}(p_M - p_L) = \#\mathbb{I}\int_{\mathbb{W}} u_L\, \mathrm{d}(p_M - p_L) = \int_{\mathbb{H}} \sum_{i\in\mathbb{I}} u_L \circ \mathcal{W}_i\, \mathrm{d}(M - L)$. This establishes part (iii) of the proposition. □

Proof of Proposition 4.2.3. Part (i) is immediate from the variable population aggregation theorem and the fact that U^*, hence $\#\mathbb{I}U^* - \#\mathbb{I}U^*(\Omega)$, represents $\succsim_{\mathbb{P}^*}$.

For (ii) and (iii) we use the following Lemma, proved below.

Lemma A.0.3. *Suppose U^* is an Omega-linear function on $\mathbb{P}^*$. For any $P \in \mathbb{P}^*$ and $\alpha \in [0,1]$, define $P_\alpha = \alpha P + (1-\alpha)1_\Omega$. Then:*

(a) If U^* is Gâteaux differentiable at $P \in \mathbb{P}^*$, then it is Gâteaux differentiable at P_α, for any $\alpha \in [0,1]$.

(b) If U^* is integrally Gâteaux differentiable at $P \in \mathbb{P}^*$, then it is integrally Gâteaux differentiable at P_α, for any $\alpha \in (0,1]$; moreover, $\nabla U^*_P \subseteq \nabla U^*_{P_\alpha}$.

For (ii), suppose that U^* is Gâteaux-differentiable at $p^{\mathbb{I}}_L$. Note that, for $\mathbb{J} \supset \mathbb{I}$, $p^{\mathbb{J}}_L$ is a mixture of $p^{\mathbb{I}}_L$ and 1_Ω; thus, by the lemma, U^* is Gâteaux-differentiable at $p^{\mathbb{J}}_L$ for all $\mathbb{J} \supset \mathbb{I}$. Given $M \in \mathbb{L}^*$, we can therefore enlarge $\mathbb{I}$ to ensure $M \in \mathbb{L}^*_{\mathbb{I}}$. We then have $V^*(L + t(M-L)) = \#\mathbb{I}U^*(p^{\mathbb{I}}_L + t(p^{\mathbb{I}}_M - p^{\mathbb{I}}_L)) - \#\mathbb{I}U^*(1_\Omega)$, so $(V^*)'_L(M-L) = \#\mathbb{I}(U^*)'_{p^{\mathbb{I}}_L}(p^{\mathbb{I}}_M - p^{\mathbb{I}}_L)$.

For (iii), suppose that U^* is integrally Gâteaux differentiable at $p^{\mathbb{I}}_{\mathbb{L}}$. By the lemma, we can find some u_L such that in fact $u_L \in \nabla U_{p^{\mathbb{J}}_L}$ for every $\mathbb{J} \supset \mathbb{I}$. So we can again enlarge $\mathbb{I}$ to ensure that any given M is in $\mathbb{L}^*_{\mathbb{I}}$.

Now define $f_{\mathbb{I}} := \sum_{i\in\mathbb{I}}(u_L \circ \mathcal{W}^*_i - u_L(\Omega))$. Then $f := \sum_{i\in\mathbb{I}^*}(u_L \circ \mathcal{W}^*_i - u_L(\Omega))$ is a well-defined function on $\mathbb{H}^*$, and $f(h) = f_{\mathbb{I}}(h)$ for all $h \in \mathbb{H}^*_{\mathbb{I}}$. Since each $f_{\mathbb{I}}$ is measurable, we find that f is measurable on each $\mathbb{H}^*_{\mathbb{I}}$, so, by coherence, measurable on $\mathbb{H}^*$. Moreover, a calculation similar to that in the proof of Proposition 4.2.2 shows that $f_{\mathbb{I}}$ is $\mathbb{L}^*_{\mathbb{I}}$-integrable; therefore f is $\mathbb{L}^*$-integrable. In particular the calculation gives $\int_{\mathbb{H}^*} f\, dM = \int_{\mathbb{H}^*_{\mathbb{I}}} f_{\mathbb{I}}\, dM = \#\mathbb{I}\int_{\mathbb{W}^*}(u_L - u_L(\Omega))\, dp^{\mathbb{I}}_M$ for $M \in \mathbb{L}^*_{\mathbb{I}}$. Now, we automatically have $u_L - u_L(\Omega) \in \nabla U_{p^{\mathbb{I}}_L}$. Therefore, for $M, L \in \mathbb{L}^*_{\mathbb{I}}$, $(V^*)'_L(M-L) = \#\mathbb{I}(U^*)'_{p^{\mathbb{I}}_L}(p^{\mathbb{I}}_M - p^{\mathbb{I}}_L) = \#\mathbb{I}\int_{\mathbb{W}}(u_L - u_L(\Omega))\, \mathrm{d}(p^{\mathbb{I}}_M - p^{\mathbb{I}}_L) = \int_{\mathbb{H}^*} f\, \mathrm{d}(M-L)$. This establishes part (iii) of the proposition. □

Proof of Lemma A.0.3. For any $Q \in \mathbb{P}^*$ and $\beta \in [0,1]$, we claim

(8)
$$(U^*)'_{P_\alpha}(Q_\beta - P_\alpha) = \begin{cases} \beta(U^*(Q) - U^*(1_\Omega)), & \text{if } \alpha = 0; \\ \beta(U^*)'_P(Q-P) + (\beta-\alpha)(U^*(P) - U^*(1_\Omega)), & \text{if } \alpha \in (0,1]. \end{cases}$$

To see this, set $f(t) := U^*(P_\alpha + t(Q_\beta - P_\alpha))$. First consider $\alpha = 0$. Then $P_\alpha = 1_\Omega$. We have $f(t) = U^*(Q_{t\beta}) = t\beta U^*(Q) + (1-t\beta)U^*(1_\Omega)$. Hence $(U^*)'_{P_\alpha}(Q_\beta - P_\alpha) = \partial_+ f(t)|_{t=0} = \beta(U^*(Q) - U^*(1_\Omega))$. (Here ∂_+ is the right-sided derivative with respect to t.)

Now consider $\alpha \neq 0$. Set $x(t) = \frac{\beta t}{\alpha + t(\beta-\alpha)}$ and $R(t) := P + x(t)(Q-P)$. This is in $\mathbb{P}^*$ for all t small enough. Moreover, a straightforward calculation shows $P_\alpha + t(Q_\beta - P_\alpha) = R(t)_{\alpha+t(\beta-\alpha)}$. Therefore, by Omega-linearity (6),

$$f(t) = U^*(R(t)_{\alpha+t(\beta-\alpha)}) = (\alpha + t(\beta-\alpha))U^*(R(t)) + (1 - (\alpha + t(\beta-\alpha)))U^*(1_\Omega).$$

Then $(U^*)'_{P_\alpha}(Q_\beta - P_\alpha)$ is the partial derivative

$$\partial_+ f(t)|_{t=0} = \alpha\partial_+ U^*(R(t))|_{t=0} + (\beta-\alpha)U^*(R(0)) - (\beta-\alpha)U^*(1_\Omega).$$

Note that $\partial_+ x(t)|_{t=0} = \frac{\beta}{\alpha}$, so $\partial_+ U^*(R(t))|_{t=0} = \frac{\beta}{\alpha}\partial_+ U^*(P + t(Q-P))|_{t=0} = \frac{\beta}{\alpha}(U^*)'_P(Q-P)$. Therefore

$$(U^*)'_{P_\alpha}(Q_\beta - P_\alpha) = \beta(U^*)'_P(Q-P) + (\beta-\alpha)(U^*(P) - U^*(1_\Omega))$$

as claimed. In particular, U^* is Gâteaux differentiable at P_α.

Next, suppose that U^* is integrally Gâteaux differentiable at $P \in \mathbb{P}^*$; fix $u \in \nabla U^*_P$. We claim $u \in \nabla U^*_{P_\alpha}$ for any $\alpha \in (0,1]$. First note that, by (8), we have $\int_{\mathbb{W}^*} u\, \mathrm{d}(P - 1_\Omega) = -(U^*)'_P(1_\Omega - P) = U^*(P) - U^*(1_\Omega)$. Then calculate: $\int_{\mathbb{W}^*} u\, d(Q_\beta - P_\alpha) = \int_{\mathbb{W}^*} u\, d(Q_\beta - P_\beta) + \int_{\mathbb{W}^*} u\, d(P_\beta - P_\alpha) = \beta\int_{\mathbb{W}^*} u\, d(Q-P) + (\beta-\alpha)\int_{\mathbb{W}^*} u\, d(P - 1_\Omega) = \beta(U^*)'_P(Q-P) + (\beta-\alpha)(U^*(P) - U^*(1_\Omega)) = (U^*)'_{P_\alpha}(Q_\beta - P_\alpha)$. □

Proof of Proposition 4.2.4. Since $\mathbb{P}^*$ includes $\mathbb{P}$, every element of $\mathbb{P}^*$ is of the form $P_\alpha := \alpha P + (1-\alpha)1_\Omega$ for some $P \in \mathbb{P}$ and $\alpha \in [0,1]$. Define $U^*(P_\alpha) = \alpha U(P) + (1-\alpha)c$. Then U^* is Omega-linear. Let $\succsim_{\mathbb{P}^*}$ be the preorder represented by U^*. Omega Independence follows immediately from Omega-linearity. As for Gâteaux differentiability, it suffices to prove the following minor variant on Lemma A.0.3:

(a) If U is Gâteaux differentiable at $P \in \mathbb{P}$, then U^* is Gâteaux differentiable at P_α, for all $\alpha \in [0,1]$.
(b) If U is integrally Gâteaux differentiable at $P \in \mathbb{P}$, then U^* is Gâteaux differentiable at P_α, for all $\alpha \in (0,1]$.

For (a), we use (8) to calculate $(U^*)'_{P_\alpha}(Q_\beta - P_\alpha)$ for any $Q \in \mathbb{P}$, $\beta \in [0,1]$; note that, in (8), $(U^*)'_P(P-Q) = U'_P(P-Q)$. For (b), given $u \in \nabla U_P$, extend it to $\mathbb{W}^*$ by $u(\Omega) = c + \int_{\mathbb{W}} u\,dP - U(P)$. This ensures that $\int_{\mathbb{W}^*} u\,\mathrm{d}(P - 1_\Omega) = U^*(P) - U^*(\Omega)$. As in the proof of Lemma 8, direct calculation then shows that $u \in \nabla U^*_{P_\alpha}$ for any $\alpha \in (0,1]$. □

Section 5.

Proof of Proposition 5.1.1. The proof of (i) is an easy version of the proof of (ii), so we present only the latter.

Suppose that $\succsim_0^*$ is consistent with generalised utilitarianism, and specifically corresponds to an individual preorder $\succsim_{\mathbb{P}^*}$. For any finite, non-empty $\mathbb{I} \subset \mathbb{I}^*$ and $h \in \mathbb{H}^*_{\mathbb{I}}$, define $p_h^{\mathbb{I}} := \frac{1}{\#\mathbb{I}} \sum_{i \in \mathbb{I}} 1_{\mathcal{W}_i^*(h)}$. Thus for $h, h' \in \mathbb{H}^*_{\mathbb{I}}$, we have $h \succsim_0^* h'$ iff $p_h^{\mathbb{I}} \succsim_{\mathbb{P}^*} p_{h'}^{\mathbb{I}}$. Suppose that $k \in \mathbb{H}^*$ is an m-scaling of $h \in \mathbb{H}^*_{\mathbb{I}}$, and that s is a corresponding m-to-1 map. Then it is easy to see that $p_h^{\mathbb{I}} = p_k^{s^{-1}(\mathbb{I})}$. Now, given $h, h' \in \mathbb{H}^*_{\mathbb{I}}$, their m-scalings k, k', and corresponding m-to-1 maps s, s', we can, by applying a permutatation to k, ensure that $s^{-1}(\mathbb{I}) = (s')^{-1}(\mathbb{I}) =: \mathbb{J}$. Since then $k, k' \in \mathbb{H}^*_{\mathbb{J}}$, we have

$$k \succsim_0^* k' \iff p_k^{\mathbb{J}} \succsim_{\mathbb{P}^*} p_{k'}^{\mathbb{J}} \iff p_h^{\mathbb{I}} \succsim_{\mathbb{P}^*} p_{h'}^{\mathbb{I}} \iff h \succsim_0^* h'.$$

Therefore $\succsim_0^*$ satisfies Scale Invariance.

Conversely, suppose that $\succsim_0^*$ satisfies Scale Invariance; we need to define a corresponding individual preorder. Recall that, by stipulation, we are only dealing with finitely supported prospects. Let $\mathbb{P}_0^*$ be the set of prospects on $\mathbb{W}^*$ with finite support and rational probabilities. We will first define a preorder on $\mathbb{P}_0^*$ and then embed it in a preorder on $\mathbb{P}^*$.

Choose a sequence of populations $\mathbb{I}_1 \subset \mathbb{I}_2 \subset \dots$ such that $\#\mathbb{I}_n = n$. For any $p \in \mathbb{P}_0^*$, there is some $n > 0$ and $h \in \mathbb{I}_n$ such that $p = p_h^{\mathbb{I}_n}$. In this case say that h is a realization of p at n. Note that if h is a realization of p at n, and k is a realization of p at mn, then k must be an m-scaling of h.

For any $p \in \mathbb{P}_0^*$, let $I(p)$ be the set of natural numbers n such that p has a realization at n. It is easy to see that, given $m, n \in I$, we also have $m + n \in I$, and if furthermore $m > n$, then also $m - n \in I$. It follows from the Euclidean algorithm that $I(p)$ contains the greatest common divisor $N(p)$ of its elements, so that $I(p)$ is the set of multiples of $N(p)$.

Similarly, for any pair $p, p' \in \mathbb{P}_0^*$, let $I(p, p')$ be the set of natural numbers n such that p, p' both have realizations at n. It follows from what we just said that $I(p, p')$ consists of all multiples of the least common multiple $N(p, p')$ of $N(p)$ and $N(p')$. The scale-invariance of $\succsim_0^*$ yields the following observation. If h, h' are realizations of p, p' at $m \in I(p, p')$, and k, k' are realizations of p, p' at $n \in I(p, p')$, then $h \succsim_0^* h'$ if and only if $k \succsim_0^* k'$.

This allows us to define $\succsim_{\mathbb{P}_0^*}$ on $\mathbb{P}_0^*$ as follows.

$$p \succsim_{\mathbb{P}_0^*} p' \iff \begin{array}{l}\text{for some (therefore any) } n \in I(p,p'), \text{ there are}\\ \text{realizations } h,h' \text{ of } p,p' \text{ at } n \text{ with } h \succsim_0^* h'.\end{array}$$

This is a preorder. In particular it is transitive, since, given $p, p', p'' \in \mathbb{P}_0^*$, we can consider realizations h, h', h'' of p, p', p'' at some common n. If $p \succsim_{\mathbb{P}_0^*} p' \succsim_{\mathbb{P}_0^*} p''$ then we must have $h \succsim_0^* h' \succsim_0^* h''$. Since $\succsim_0$ is transitive, $h \succsim_0 h''$, and therefore $p \succsim_{\mathbb{P}_0^*} p''$.

Let us also check that $\succsim_{\mathbb{P}_0^*}$ satisfies Omega Independence. Suppose given $p, p' \in \mathbb{P}_0^*$, and $m/n =: \alpha \in (0,1) \cap \mathbb{Q}$. Then realizations of p, p' at $N(p,p')m$ are realizations of $\alpha p + (1-\alpha)1_\Omega$ and $\alpha p' + (1-\alpha)1_\Omega$ at $N(p,p')n$. It follows that $p \succsim_{\mathbb{P}_0^*} p'$ if and only if $\alpha p + (1-\alpha)1_\Omega \succsim_{\mathbb{P}_0^*} \alpha p' + (1-\alpha)1_\Omega$, as desired.

We now embed $\succsim_{\mathbb{P}_0^*}$ in a preorder $\succsim_{\mathbb{P}^*}$ on $\mathbb{P}^*$, the convex hull of $\mathbb{P}_0^*$:

$$p \succsim_{\mathbb{P}^*} p' \iff \begin{cases} p, p' \in \mathbb{P}_0^* \text{ and } p \succsim_{\mathbb{P}_0^*} p', \text{ or} \\ p = p'. \end{cases}$$

Then $\succsim_{\mathbb{P}^*}$ is a preorder on $\mathbb{P}^*$ which satisfies Omega Independence. Let $\succsim^*$ be the social preorder on $\mathbb{L}^*$ it generates. Then, for any finite non-empty set $\mathbb{I} \subset \mathbb{I}^*$ such that h and h' are in $\mathbb{H}_{\mathbb{I}}^*$, $h \succsim_0^* h' \iff p_h^{\mathbb{I}} \succsim_{\mathbb{P}^*} p_{h'}^{\mathbb{I}} \iff p_{1_h}^{\mathbb{I}} \succsim_{\mathbb{P}^*} p_{1_{h'}}^{\mathbb{I}} \iff 1_h \succsim^* 1_{h'}$. This shows that $\succsim_0^*$ is consistent with the generalized utilitarian preorder $\succsim^*$. □

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