Testing Gollier and Weitzman's Solution of the "Weitzman-Gollier Puzzle"

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TESTING GOLLIER AND WEITZMAN'S SOLUTION OF THE "WEITZMAN-GOLLIER PUZZLE"

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Abstract: Despite the fact that the "Weitzman-Gollier Puzzle" arose in the context of risk neutrality, Gollier and Weitzman (2009) claimed to have solved the puzzle by showing that, in case of risk aversion, discounting and compounding approaches yield the same result, and that these can be expressed in ways that are morphologically similar to the conflicting formulations of the original risk neutral model. This paper replicates their analysis with a simple numerical example and shows that the equality of results obtained is due to discount and compound factors being each other's reciprocals in the risk averse model, while the inequality of the puzzle is due to this condition not being met in the risk neutral case. Their claim to have solved the puzzle is not sustained. It is shown that the source of the puzzle is Weitzman's incorrect specification of the present value factor and that, correcting for this, the right conclusion under his assumptions is that certainty equivalent discount rates are growing functions of time. Gollier and Weitzman (2009) also claimed that "the 'effective' discount rate must decline over time toward its lowest possible value." This paper finds that when long term market yields are a growing function of time, it makes no sense to invest in projects of similar risk but lesser yield, irrespective of one's degree of risk aversion.

JEL Codes: D61, H43

1. Introduction

In their paper entitled "How Should the Distant Future Be Discounted When Discount Rates Are Uncertain?" Christian Gollier and Martin L. Weitzman (2009) claim to have solved the "Weizman-Gollier puzzle," which had spawned a substantial literature. They also confirm Weitzman's original recommendation that the discount rate applicable to the distant future should be a declining function of time: "The bottom-line message that we wish for readers to take away from this paper is the following. When future discount rates are uncertain but have a permanent component, then the 'effective' discount rate must decline over time toward its lowest possible value."

In "Why the Far-Distant Future Should Be Discounted at Its Lowest Possible Rate" Martin L. Weiztman (1998) concluded that "Uncertainty about future discount rates provides a strong generic rationale for using certainty-equivalent social discount rates that decline over time." Assuming risk neutrality, Weitzman defined the certainty equivalent discount factor A as the probability $\{p_i\}$ weighted average of the discount factors corresponding to the possible interest rate scenarios $\{r_i\}$ of a safe future payoff, where p_i is the probability of r_i occurring.

$$A = \sum p_i e^{-r_i t} \tag{1}$$

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from which the following certainty equivalent discount rate can be derived[2]:

$$R_W = -\left(1/t\right)\ln\left(\sum_i p_i e^{-r_i t}\right) \tag{2}$$

This – the same as expression (2) in Gollier and Weitzman (2009:4) – is a declining function of time and tends to the lowest possible r_i, because present values of distant benefits computed at high rates of interest are much smaller than those computed at low rates of interest. At the limit only the lowest rate counts in determining the certainty equivalent. This is also observed in expression (4) of Gollier and Weitzman (2009:4).

The "Weizman-Gollier puzzle" arose with the publication of Christian Gollier's (2004) "Maximizing the Expected Net Future Value as an Alternative Strategy to Gamma Discounting," in which, under similar assumptions, Gollier derived a certainty equivalent rate (CER) from the expected compound factor *F,* obtained by probability weighting possible future values of a safe initial investment.

$$F = \sum p_i e^{r_i t} \tag{3}$$

from which the following certainty equivalent rate can be derived:

$$R_G = \left(1/t\right)\ln\left(\sum_i p_i e^{r_i t}\right) \tag{4}$$

This – the same as expression (7) in Gollier and Weitzman (2009:5) – is an increasing function of time and tends to the highest possible r_i, because future values of a given investment computed at high rates of interest are much higher than those computed at low rates of interest. At the limit only the highest rate counts in determining the certainty equivalent. This is also observed in expression (8) of Gollier and Weitzman (2009:5)

In the Weitzman formulation the future value is certain while the present value is stochastic, whereas in the Gollier formulation the present value is certain while the future value is stochastic. This observation led Gollier (2004) to state that "Taking the expected net future value is equivalent to assuming that all risks will be borne by the future generation." *Even though risk is irrelevant in the context of risk neutrality, assumed by both fundamental papers of the puzzle*, most of the literature trying to reconcile the two approaches appeals to the notion of risk aversion.

This is the approach taken by Gollier and Weitzman (2009) as well, and because it is not immediately obvious how a contradiction that exists in the context of risk neutrality can be resolved in the context of risk aversion, the claims made by the authors are worth testing. This is what the present paper aims to do.

"How might a person resolve this distressing paradox by choosing between two such seemingly symmetric formulations, with each one having diametrically opposed implications for distant-future discounting? The answer can only come from a 'careful rigorous analysis' " state Gollier and Weitzman (2009), who suggest a specific model from

[2] Subscripts W attribute to Weitzman, G to Gollier.

which they derive their conclusions. This paper summarizes their arguments, and creates a numerical example of their model to replicate their "rigorous analysis," and attempts to verify their stated conclusions. The claims to be verified specifically are the following:

1. "The two rigorous formulations [of the certainty equivalent rate] give the *same* discount rate (as a function of time), thereby resolving the 'Weitzman-Gollier puzzle.' " The "two rigorous formulations" referred to in this quote are the ones found in Gollier and Weitzman (2009). Do the presented results derived assuming risk aversion transfer to the risk neutrality case and do they solve the puzzle thereby?
2. "The bottom-line message that we wish for readers to take away from this paper is the following. When future discount rates are uncertain but have a permanent component, then the 'effective' discount rate must decline over time toward its lowest possible value." What is the scope of this sweeping statement?
    a. Does it apply to any degree of risk aversion?
    b. Does it apply only to the certainty equivalents of project cash-flows, or does it apply directly to the expected monetary cash flows of projects as well?

This paper proceeds as follows. In Section 2, the line of reasoning of Gollier and Weitzman (2009) is summarized. In Section 3, a numerical example is built of their analysis, and the calculations performed within its framework are used to evaluate their claims. It is found that while it is true that the same expected discount rate can be obtained from either discount or compound factors in the risk averse model formulated, this fails to solve the puzzle, for in the original risk neutral context the discrepancy remains. In Section 4, it is explained that the reason why the same discount rates are obtained through both discounting and compounding in the Gollier and Weitzman (2009) model is that their expected discount and compound factors are each other's reciprocals, while in the original puzzle this is not the case, and therefore the corresponding CERs (certainty equivalent rates) are different. The real solution to the puzzle lies in the recognition that Weitzman's (1998) original formulation of the expected discount factor (1) is not correct. Using the correct formulation resolves the puzzle. Section 5 is devoted to checking Gollier and Weitzman's (2009) second claim. Under the assumption of perfectly auto-correlated stochastic interest rates, risk-neutral certainty equivalent discount rates are a positive function of time. While the discount rates applicable to risk-averse certainty equivalents of future payoffs are declining functions of time in the example presented, the monetary yield from which certainty equivalent payoffs are computed must increase with time on a par with market yields, if their risks are similar, as the calculation of certainty equivalents treats market and project yields similarly. Section 6 presents conclusions.

The supporting calculations performed with the data of the numerical example are presented in Appendix A, references to which, including the relevant section number, are provided in parentheses where needed.

2. Summary of the Gollier and Weitzman argument

The model proposed in Gollier and Weitzman (2009:3) is as follows:

"In the highly stylized model of this paper, time $t = 0, 1, 2, ...,$ is measured in discrete periods of unit length. To state loosely the issue at hand, a decision must be taken now, just before time zero (call it time 0−), whether or not to invest a marginal cost δ that will yield a marginal benefit ε at future time t. Right now, at time 0−, it is unknown what will be the appropriate future rate of return on capital in the economy. There are n possible future states of the economy, indexed by $i = 1, 2, ..., n$. As of now (time 0^{-}), future state i is viewed as having marginal product of capital r_i with probability $p_i > 0$, where $\Sigma_i p_i = 1$. A decision must be made now (at time 0−, just before the "true" state of the world is revealed at time t = 0) about whether or not to invest δ now in order to gain payoff ε at future time t. To pose the problem sharply, it is assumed that immediately after the investment decision is made, at time 0, the true state of the world i is revealed and the marginal product of capital will thenceforth be r_i, from time $t = 0$ to time $t = \infty$."

Gollier and Weitzman (2009) postulate that if a decision maker optimizes his consumption path by reference to a linear budget constraint represented by interest rate r_i of state of the world i, then the optimal consumption trajectory for each scenario i must satisfy the following first order condition

$$\mathrm{V'}_i\,(C_0) = \mathrm{V'}_i\,(C_t)\; exp(r_i\, t) \tag{5}$$

where $\mathrm{V'}_i\,(C_t)$ is the marginal utility of consumption of the period indicated by the subscript of C, in the state of the world identified by subscript i of V'.

At time t = 0^{-} a safe investment opportunity arises that expends marginal cost of δ in time period 0 to yield a marginal benefit of ε in time period t. The investment project will increase the expected utility of the decision maker if and only if

$$\varepsilon \sum p_i\; \mathrm{V'}_i\,(C_t) \geq \delta \sum p_i\; \mathrm{V'}_i\,(C_0) \tag{6}$$

Using optimality condition (5) this can be rewritten in two ways. The one called the "Weitzman approach," which first defines an expected discount factor, eliminates $\mathrm{V'}_i(C_t)$ from (6) yielding:

$$\varepsilon \sum q^W{}_i\; exp(-r_i\, t) \geq \delta \tag{7}$$

where $q^W{}_i = p_i\; \mathrm{V'}_i\,(C_0)\, /\, \sum p_i\; \mathrm{V'}_i\,(C_0)$.

According to Gollier and Weitzman (2009) this is equivalent to discounting ε at the following rate[3]:

$$R_d = -\left(1/t\right) \ln\left(\sum_i q_i^W e^{-r_i t}\right) \tag{8}$$

Alternatively, the "Gollier approach," which first defines an expected compound factor, consist of eliminating $\mathrm{V'}_i(C_0)$ from (6) yielding:

$$\varepsilon \geq \delta \sum q^G{}_i\; exp(r_i\, t) \tag{9}$$

[3] The subscript d of R refers to the discounting approach and the subscript c to the compounding approach.

where $q^G{}_i = p_i \text{ V'}_i(C_t) / \Sigma\, p_i \text{ V'}_i(C_t)$

According to Gollier and Weitzman (2009) this is equivalent to discounting ε at the following rate:

$$R_c = \left(1/t\right)\ln\left(\sum_i q_i^G e^{r_i t}\right) \tag{10}$$

Since both (8) and (10) were derived from (6), it must be true that $R_d = R_c$. The authors go on to state that "This means that the adjustment of the valuation for risk resolves the 'Weitzman-Gollier puzzle,' " denote this the "risk adjusted discount rate" R_*, and go on to state that "qualitatively the properties of the efficient discount rate $R_*(t)$ resemble closely those of $R_W(t)$ recommended by Weitzman, with the only quantitative difference being the substitution of "Weitzman-adjusted probabilities" $\{q^W{}_i\}$ for the unadjusted probabilities $\{p_i\}$."

3. A numerical example of the Gollier-Weitzman model

The utility function proposed by the authors is:

$$V(C) = \sum_t^{\infty} e^{-\rho t} U(C_t) \tag{11}$$

where $\rho>0$ is the pure rate of time preference and $U(C_t)$ is a utility function that the authors did not specify, but which will be taken here to be of the constant-inter temporal-elasticity-of-substitution (CIES) type:

$$U(C) = \frac{C^{1-\sigma} - 1}{1-\sigma} \tag{12}$$

where consumption $C > 0$, and the elasticity of marginal utility with respect to consumption $\sigma > 0$ but not equal to 1. This is also the measure of the decision maker's constant proportional risk aversion.

The above utility function will be maximized subject to a budget constraint given by an original endowment of consumptions C_t for time periods 0 and t, and a constant interest rate r_i for each scenario i. Under these circumstances the welfare impact of investing at time 0 for a yield of $\exp(r\times t)$ at time t can be measured by the changes in the value of the utility function (11) used.

The following parameters are assumed in the simple two-scenario numerical example proposed:

Table 1

Parameters of the numerical example

Scenario 1 interest rate, r_1	1%
Scenario 2 interest rate, r_2	5%
Probability of scenario 1, p_1	0.5
Probability of scenario 2, p_2	0.5
Consumption at time = 0	$2,000
Annual growth of consumption	0.75%
Constant proportional risk aversion, σ	0.8
Pure rate of time preference, ρ	0.5%
Time t in years	200

The data of this simple example will be used to verify the claims made in Gollier and Weitzman (2009).

As the arguments of Gollier and Weitzman are predicated on the decision maker being on his optimal consumption trajectory, the first thing to do is to calculate it for each scenario. This means that the investor pondering whether to invest in the safe project of the Gollier Weitzman model at time 0^- must also prepare to take action to bring himself into optimality once the interest rate to prevail in an instant and forever thence is revealed. To this end, he must solve for the optimal capital market action by maximizing, subject to his budget constraint, the following expression for each interest rate scenario, where x is the amount to invest (or borrow if negative):

$$V(C) = \frac{(C_0 - x)^{1-\sigma} - 1}{1-\sigma} + \frac{(C_t + e^{rt}x)^{1-\sigma} - 1}{e^{\rho t}(1-\sigma)} \qquad (13)$$

The calculation of the optimal investment or borrowing is shown in Appendix A, Section 1 (referred to as A.1 henceforth). With the data of our simple example, the investor would borrow $182.24 in the low interest rate scenario, and would invest $1,554.51 in the high interest rate scenario. It will be on the basis of the resulting alternative optimal consumption paths (A.1) that he will ponder whether to invest in a safe project costing δ and yielding ε.

At this point we can compute R_d. To this end expression (6) will be adapted to our simple example, as follows, keeping the notation of the previous Section:

$$\varepsilon\,(\,p_1\, V'_1(C_t) + p_2\, V'_2(C_t)\,) \geq \delta\,(\,p_1\, V'_1(C_0) + p_2\, V'_2(C_0)\,) \qquad (14)$$

Employing the "Weitzman approach," which seeks to define a discount factor, this becomes:

$$\frac{p_1 V_1'(C_t) + p_2 V_2'(C_t)}{p_1 V_1'(C_0) + p_2 V_2'(C_0)} \geq \frac{\delta}{\varepsilon} \qquad (15)$$

If we convert the above relationship into a strict equality, and replace δ/ε by D, we can interpret D as the expected value of the Weitzman approach discount factor, from which we can compute:

$$R_d = -\,(1/t)\ln(\,D\,) \qquad (16)$$

To follow the "Gollier approach" of finding a compound factor instead, all we have to do is invert expression (15), make it a strict equality, and replace ε/δ by F, which can be interpreted as the expected value of the "Gollier approach" compound factor, from which we can compute:

$$R_c = (1/t)\ln(\,F\,) \qquad (17)$$

It is clear from this that $D = 1/F$, which is as it should be[4], as expected discount factors are the inverses of expected compound factors, and therefore $R_d = R_c$. This means that CERs can be computed from either expected discount or compound factors. Notice that the transformations defined by (7) and (9) are not needed for computational purposes. Gollier and Weitzman (2009) only performed those to show that expressions that look like (2) and (3) can be derived from (6).

With the data of our example we obtain (A.2) that $R_d = R_c = 1.76\%$. This result was calculated directly from (14), without going through the transformations that Gollier and Weitzman (2009) used to obtain expressions morphologically similar to the original Weitzman (1998) and Gollier (2004) formulations, or using "risk adjusted probabilities." The same result is obtained, however, if one works through those transformations (A.3).

Gollier and Weitzman's (2009) assertion that "The two rigorous formulations give the *same* discount rate (as a function of time), thereby resolving the 'Weitzman-Gollier puzzle.' " is incorrect, however, for the fact that the two formulations R_d and R_c, (8) and (10), derived above in the context of risk aversion, are the same, does not solve the puzzle, as they are unrelated to R_W and R_G, (2) and (3), which are defined in the context of risk neutrality. As the following table shows, these rates are not the same (A.4).

Table 2

Certainty equivalent rates for selected time horizons

Years till time t	50	100	200	300	400	500
Risk neutral R_W	2.13%	1.67%	1.35%	1.23%	1.17%	1.14%
Risk averse R_d or R_c	2.65%	2.18%	1.76%	1.62%	1.57%	1.54%
Risk neutral R_G	3.87%	4.33%	4.65%	4.77%	4.83%	4.86%

The pairwise morphological resemblance between the definitions of R_d and R_c, as defined in expressions (8) and (10) respectively, on the one hand, and those of R_W and R_G, as defined in expressions (2) and (4) respectively, on the other, does not make them all equal. The equality between R_d and R_c neither makes $R_W = R_G$ nor provides an explanation for $R_W \neq R_G$. Furthermore, there is no reason why R_d should equal R_W, as the former pertains to a

[4] If $D = \delta/\varepsilon$ and $F = \varepsilon/\delta$, then $D = 1/F$.

risk averse investor, whereas the latter applies to a risk neutral one. This is why Gollier and Weitzman (2009) fail to resolve the puzzle.

4. Solving the Puzzle

The puzzle resides in the fact that $R_W \neq R_G$. Solving it requires finding the conditions that would make $R_W = R_G$. Using the notation of the previous section in which D is the expected discount factor and F is the expected compound factor, we can say that the puzzle will be solved if the following expression, obtained by setting expressions (2) and (4) to be equal, is true:

$$R_W = -\left(1/t\right)\ln\left(D\right) = R_G = \left(1/t\right)\ln\left(F\right) \tag{18}$$

For this to be true, the following must hold:

$$-\ln\left(D\right) = \ln(F) \tag{19}$$

Which means that it must be true that

$$D = \frac{1}{F} \tag{20}$$

This condition is met in the Gollier-Weitzman's (2009) model, analyzed in the previous section. In both sections 2 and 3 of Appendix A we can see that the expected compound factor F is 33.69026 and the expected discount factor is D is 0.029682, which are each other's reciprocals.

Furthermore, D complies with the definition of present value[5], as demonstrated in A.5: investing the present value of δ in the stochastic market has the same expected utility as the safe amount ε. Gollier and Weitzman's (2009) statement that "the two rigorous formulations give the *same* discount rate (as a function of time)" is therefore perfectly justified, but it refers exclusively to the values calculated in their model, R_d and R_c (see A.2). There is no puzzle in Gollier and Weitzman's (2009) because the expected discount and compound factors are each other's reciprocals. However, Weitzman's original expected discount factor and Gollier's original expected compound factor, both of which pertain to risk neutral investors, are not each other's reciprocals, as can be seen by multiplying (1) and (3):

$$\sum p_i e^{-r_i t} \sum p_i e^{r_i t} \neq 1 \tag{21}$$

The puzzle can be solved by extending the "rigorous formulation" proposed by the authors to the case of risk neutrality. As the marginal utility of income of risk neutral investors is constant over their entire utility function, the expected compound and discount factors cannot be defined from expressions involving marginal utilities, as in (6), but only by considering what the safe yield ε of investing δ should be in order to increase the

[5] "The present value of an asset is obtained by calculating how much money invested today would be needed, at the going interest rate, to generate the asset's future stream of receipts." (Paul A. Samuelson and William D. Nordhaus, 1992:271.)

investor's utility. Given that the investor can invest at the stochastic market yield $\{r_i\}$, investing δ for a safe yield of ε is only worthwhile if the following holds:

$$\varepsilon \geq \delta \sum p_i e^{r_i t} \tag{22}$$

By making the relationship a strict equality and dividing through by δ we get the expected compound factor:

$$F = \frac{\varepsilon}{\delta} = \sum p_i e^{r_i t} \tag{23}$$

Alternatively, we can rearrange (22) to get the expected discount factor:

$$D = \frac{\delta}{\varepsilon} = \frac{1}{\sum p_i e^{r_i t}} \tag{24}$$

As D is the inverse of F in (23) and (24) above, there will be no paradox.

With the data of our simple example we have F = \$11,016.93 (A.6), and using expression (18) of the compounding approach, the CER is:

$$R_c = (1/t) \ln(F) = (1/200) \ln(11{,}016.93) = 4.65\% \tag{25}$$

This is the same as the value of R_G in Table 2.

Alternatively, D = 1 / 11,016.93 = 9.07694E-05, and using expression (16) of the discounting approach we have:

$$R_d = -\,(1/t) \ln(D) = -\,(1/200) \ln(9.07694\text{E-}05) = 4.65\% \tag{26}$$

which confirms the absence of the paradox in the risk neutral case as well, provided that the correct expected discount factor is used. Risk-neutral $R_c = R_G \neq R_W$ because the correct expected discount factor D is 9.07694E-05 (A.6), and not Weitzman's discount factor A, which is 0.067690342 (A.7). The paradox was due to the Weitzman's adoption of expression (1) as the expected discount factor[6], which results in a different CER value of 1.35% with our data (A.7).

It should be clear enough from (22) that the correct discount factor is D, expression (24), and not A, expression (1). None the less, it might be useful to point out two additional properties of D and A.

First, discount factor D complies with the definition of present value, as the expected present value of a safe \$1 in year 200 is \$ 9.07694E-05, which if invested in the market for 200 years will have and expected yield of \$1. Weitzman's discount factor A does not comply, as investing its expected present value of \$0.067690342 has an expected yield of \$745.74 (A.8).

[6] Weitzman discounting is equivalent to discounting conventionally with the negatives of the interest rates assumed, and then multiplying the resulting expected CER value by –1, which is tantamount to reversing the arrow of time. For an explanation of this, and an exploration of the properties of Weitzman discounting, see Szabolcs Szekeres (2015a) "Governments Should Not Use Declining Discount Rates in Project Analysis."

Second, making a small investment yielding $R_d = R_c$ (4.65% in our example) will leave the welfare of the risk neutral investor unchanged, because the opportunity cost of making the small investment is equal to its yield. Someone investing in a small project yielding R_W (1.35% in our example), however, will incur a loss, as the opportunity cost of the funds exceeds the small investment's yield. Therefore, using Weitzman's discount factor *A* to judge investment decisions will lead to welfare loss. Someone investing A = \$0.067690342 for a gain of \$1 in year 200, when he could instead invest just D =\$ 9.07694E-05 in the market for the same gain, will lose the difference between *A* and *D*. This is equivalent to wasting 99.9% of the investment with the data of our example (A.9). This is just an illustrative example of an assertion that will hold in all cases under the assumptions of the Weitzman model: an investor paying a Weitzman present value for any future yield will always suffer an opportunity loss, for the Weitzman present value, computed at declining discount rates, will always be higher than that computed at the growing discount rates of the market.

In arguing for the correctness of Weitzman discounting under risk aversion, Gollier and Weitzman (2009:9) make special mention of the case in which the utility function is logarithmic, and point out that "In this case, Weitzman's rule was qualitatively and quantitatively correct." This happens to be a very special case, however, in which the standard and the Weitzman CER calculation methods yield the same result, but display neither growing nor declining trends. This is due to the coincidence that the natural logarithm of the utility function is the inverse function of the exponentiation that continuous compounding involves. This case therefore neither proves that the Weitzman method is correct, nor supports declining or growing CERs. See A.10.

In conclusion, the rigorous analysis proposed by Gollier and Weitzman (2009) shows that CERs can be computed either from compound or discount factors, provided that these are each other's reciprocals, which their analysis shows to be the case. This is true irrespective of the degree of risk aversion assumed, and this paper has shown that risk neutrality, which means constant proportional risk aversion of zero, is no exception. The Weitzman method of CER calculation under risk neutrality is incorrect therefore, contrary to the authors' assertion, precisely because it fails to comply with this proviso.

5. Which discount rates should be declining?

"The bottom-line message that we wish for readers to take away from this paper is the following. When future discount rates are uncertain but have a permanent component, then the "effective" discount rate must decline over time toward its lowest possible value" (Gollier and Weitzman, 2009:9). The aspects of this statement to be addressed are:

- "When future discount rates are uncertain but have a permanent component…" That this condition holds, is an implicit assumption of the Weitzman model, because representing the long run with a single stochastic market rate implies that annual interest rates are perfectly correlated. Otherwise, this source of alleged declining rates would vanish, as according to Christian Gollier (2009:4) "the term structure [of interest rates] obtained in this framework depends heavily upon the assumption that shocks on the interest rate are permanent. If they are purely transitory, the term structure of discount rates should be flat."
- " 'effective' discount rate…" To address this comprehensively we will consider:

- The expected monetary return of an investment. This is the internal rate of return (IRR) between the expected value of the investment and the expected value of the payoff (or, more generally, the IRR of the expected project cash-flow).
- The CER of the investment, which is the IRR defined by the certainty equivalent of the investment and the certainty equivalent of its payoff(s). For risk neutral investors this is the same as the above, as the certainty equivalent of a payoff is its expected monetary value, but for risk averse investors it is different.

Numerical examples will be used to explore these questions. Calculations will be made of expected monetary returns and CERs corresponding to small investment projects that have the same expected annual return as the market, but (a) are riskier, or (b) are equally risky, but their risk is negatively correlated with the market's risk. Their monetary CERs and risk averse CERs will be computed for selected time horizons, in order to check whether they indeed "decline over time toward [their] lowest possible value."

The data of the previous section will be used again, but one assumption of the Gollier-Weitzman (2009) model will be dropped, namely the assumption of clairvoyance. This will make the model much more realistic, as investors don't normally know the future values of distant future stochastic interest rates in advance of investing their portfolios. This means that instead of optimizing their portfolio separately for each scenario, they will choose an investment amount that maximizes their expected utility, knowing only the probability distribution of the market rate. In such cases the optimal consumption path under uncertainty will not prove to have been optimal in any scenario, ex-post. Consequently, the computational formulas used in Gollier and Weitzman (2009), which are based on the assumption of scenario specific optimization, will no longer be useful. Utility maximizing actions and CERs can still be identified, however, through numerical methods.

The analysis proceeds as follows. For each of the selected time horizons, the following is done:

1. The optimal investment to be made in the market is calculated by maximizing expected utility. This fixes the consumption paths (C_0 will be the same in both scenarios, C_t will be scenario specific).
2. The market's expected monetary return is computed. This is also the CER of risk neutral investors.
3. The risk averse market CER is computed by finding the rate of return of an investment of $1 with a safe yield such that the investor is indifferent between investing in it or at the stochastic market rate.
4. A small project is defined with an investment cost of $1, an expected annual return of x, and high and low rates defined in such a way that the coefficient of variation of rate x is 1.5 times the coefficient of variation computed from the low and high market rates (1% and 5%, respectively). It is assumed that the market risk and project risk are perfectly correlated. For each time horizon the value of x that leaves the total utility of the risk averse decision maker unchanged is found, and from that the expected monetary CER of the small project so defined is computed.
5. Another small project costing $1 is defined, the coefficient of variation of its expected annual return x is the same as that of the market rates, but this time the

risk of this project is inversely correlated with the risk of the market. The corresponding monetary CER of the small project so defined is computed.

With these assumptions, the following results were obtained (see A.11 for details of the calculations):

Table 3

Certainty equivalent rates for selected time horizons

Years till time *t*	**50**	**100**	**200**	**300**	**400**	**500**
Risk neutral CER	3.87%	4.33%	4.65%	4.77%	4.83%	4.86%
Risk averse CER	3.13%	2.65%	2.13%	1.95%	1.88%	1.85%
Risky project's required monetary CER	3.99%	4.44%	4.70%	4.79%	4.83%	4.86%
Negatively correlated project's required monetary CER	2.19%	2.14%	1.80%	1.72%	1.71%	1.71%

What the above results show is that risk neutral CERs increase with time. Risk averse CERs decline, but these can only be applied to the certainty equivalent payoffs of investment projects. Investment projects with risk equal to that of the market must have (growing) expected monetary returns as high as those of the market in order to have sufficiently high (declining) risk averse CERs to be acceptable to risk averse investors. Projects that are riskier than the market must have (growing) expected monetary returns even higher than those of the market. If the risks are the same as those of the market, but negatively correlated with it, then the required expected monetary returns decline. These projects are better than safe projects, hence their required monetary CER is below the risk averse market CER[7].

Weitzman's original model was framed in the context of risk neutrality, and therefore the recommendation to use declining discount rates is equivalent to the prescription of discounting the expected monetary flows of projects at declining rates. As the results of Table 3 illustrates – but expressions (24) and (25) demonstrate – this recommendation is not justified for risk neutral investors, and neither is it justified for projects that are as risky or riskier than the market for risk averse investors. Risk averse CERs of market rates do decline[8], because risk averse investors are willing to trade yield for safety, but by the same reasoning risk averse CERs of project yields also decline. Therefore, for a project to be preferred to the market by a risk averse investor, its risk averse CER would have to be higher than that of the market, and for that to occur, its expected monetary returns should be higher as well, provided that its risks are not lower than those of the market, or are not negatively correlated with it.

Clearly the above results are just illustrative examples, but two generalizations are possible:

1. Risk neutral CERs are a growing function of time regardless of the probability distribution of interest rates assumed, as long as the Weitzman assumption of

[7] For completeness Table 3 has been calculated also with the clairvoyance assumptions of the Weitzman (1998) model (A.11). The values obtained are slightly different, but the substantive conclusions are the same.

[8] For a comprehensive review of how CERs behave as a function of risk aversion and time see Szabolcs Szekeres (2015b).

perfectly correlated interest rates holds, for the reasons given in the explanation of (4) above, which is identical to the expression for expected risk neutral discount rate given in (26) above.

2. The actual behavior of risk averse CERs could very well differ from that derived from the current example (see Szekeres 2015b) because effective risk averse behavior is both a function of the degree of risk aversion and of the probability distribution of the decision maker's endowment. What can be generalized from the example, however, is that the process of converting monetary risk to expected utility affects market and project yields equally, and for that reason projects with risks similar to those of the market (and correlated with them) will have to yield at least as much as the market for them to be acceptable to risk averse investors. Required yields may be lower for independent, lower, or negatively correlated risks.

Gollier and Weitzman's (2009:9) "bottom-line message that […] when future discount rates are uncertain but have a permanent component, then the 'effective' discount rate must decline over time toward its lowest possible value" is not sustained. CERs do decline for some degrees of risk aversion, but the implication that the monetary CERs of projects can be declining over time (meaning that the longer the life of the project, the lower its required monetary return) is not true, unless their risks are much lower than, or negatively correlated with, those of the market.

6. Conclusions

In their paper entitled "How Should the Distant Future Be Discounted When Discount Rates Are Uncertain?" Christian Gollier and Martin L. Weitzman (2009) claim to have solved the "Weitzman-Gollier puzzle." To do so they conducted a "careful rigorous analysis" through a model that replicates the assumptions of the original Weitzman (1998) paper, but in which the behavior of a risk averse investor is analyzed, rather than that of the risk neutral investor of Weitzman's model and the puzzle. This paper created a numerical example with which to verify the validity of their assertions by carrying out the calculation suggested in their paper, and arrived at the following conclusions:

1. Regarding the claim of having solved the "Weizman-Gollier puzzle:"
    a. CERs derived from either discounting or compounding are the same in the model proposed in Gollier and Weitzman (2009), just as claimed.
    b. Gollier and Weitzman (2009) show that it is possible to find expressions involving "risk adjusted probabilities" that are morphologically similar to the Weitzman (1998) and Gollier (2004) expressions for risk neutral CERs. However, their equality in the risk averse case does not translate into the equality of the analogous expressions of the risk neutral case, and therefore their claim to have solved the puzzle thereby is not sustained.
    c. CERs derived from either discounting or compounding in the Gollier-Weitzman (2009) model are the same because the corresponding expected compound and discount factors are each other's reciprocals.
    d. The "Weitzman-Gollier Puzzle" is due to the fact that Weitzman's (1998) expected discount factor is not the inverse of Gollier's (2004) expected compound factor. Using the reciprocal of the latter as a discount factor eliminates the puzzle.

e. Weitzman's risk neutral expected discount factor does not comply with the definition of present value, and will lead to significant welfare loss if relied upon for investment decisions.

2. The claim that "when future discount rates are uncertain but have a permanent component, then the 'effective' discount rate must decline over time toward its lowest possible value" is not unequivocally sustained.
   a. When interest rates are perfectly correlated, risk neutral discount rates are not a declining, but an increasing function of time, and tend not to the lowest, but to the highest possible rate.
   b. Risk averse CERs of market rates can decline, but so can risk averse CERs of project yields.
   c. For a project to be preferred to the market by a risk averse investor, its risk averse CER would have to be higher than that of the market, and for that to occur, its expected monetary returns should be higher than the market's as well, provided its risks is not lower than that of the market, or is not negatively correlated with it.
   d. Risk averse CERs of projects with the same risk as that of the market, but which is negatively correlated with it, are declining functions of time.

In summary, an attempt to verify the claims made in Gollier and Weitzman (2009) by replicating the "careful rigorous analysis" of their suggested model through a numerical example has yielded the conclusion that Gollier and Weitzman (2009) have not solved the "Weitzman Gollier Puzzle." The true solution of the puzzle lies in the recognition that Weitzman's (1998) expected discount factor is incorrect. Their "bottom-line message that [] when future discount rates are uncertain but have a permanent component, then the "effective" discount rate must decline over time toward its lowest possible value" has not been unequivocally sustained. Risk neutral discount rates increase instead to their highest possible value. Risk averse CERs can decline, but barring a more favorable risk profile, the monetary CERs of projects must match the growing monetary CERs of the market for them to be attractive to risk averse investors.

This is what corresponds to common sense. In the (probably unlikely) event of market interest rates being sufficiently auto-correlated to make CERs of long term market yields a growing function of time, it makes no sense to invest in projects of similar risk but lesser yield, irrespective of one's degree of risk aversion.

References

Gollier, Christian (2004), "Maximizing the Expected Net Future Value as an Alternative Strategy to Gamma Discounting," *Finance Research Letters* 1(2), 85–89

Gollier, Christian (2009), "Expected Net Present Value, Expected Net Future Value, and the Ramsey Rule." CESifo Working Paper No. 2643, May 2009.

Gollier, Christian and Martin L. Weitzman. 2009. "How Should the Distant Future Be Discounted When Discount Rates are Uncertain?" TSE working paper series 09-107.

Samuelson, Paul A. and William D. Nordhaus (1992), *Economics*, 14th Edition, New York: McGraw-Hill, Inc.

Szekeres, Szabolcs (2015a), "Governments Should Not Use Declining Discount Rates in Project Analysis." Working Paper. MPRA Paper No. 63438. http://mpra.ub.uni-muenchen.de/63438/

Szekeres, Szabolcs (2015b) “The Mechanics of the Weitzman-Gollier Puzzles.” Working Paper. MPRA Paper No. 64286, http://mpra.ub.uni-muenchen.de/64286/
Weitzman, Martin L. 1998, “Why the far distant future should be discounted at its lowest possible rate,” *Journal of Environmental Management*, Volume 36.

Appendix A

1. Optimal investment or borrowing in each scenario

Maximize

$$V(C) = \frac{(C_0 - x)^{1-\sigma} - 1}{1-\sigma} + \frac{(C_t + e^{rt}x)^{1-\sigma} - 1}{e^{\rho t}(1-\sigma)}$$

Differentiating[9] this expression with respect to *x,* and setting the result equal to 0, gives the first order condition of the optimization:

$$\frac{e^{rt-\rho t}}{(e^{rt}x + C_t)^{\sigma}} + \frac{1}{(C_0 - x)^{\sigma}} = 0$$

The optimal market action to be taken can be obtained by solving for x in the above expression[10]:

$$x = \frac{e^{\frac{rt}{\sigma}}C_0 - e^{\frac{\rho t}{\sigma}}C_t}{e^{\frac{\rho t}{\sigma}+rt} + e^{\frac{rt}{\sigma}}}$$

With the data of our simple example, the investor would borrow \$182.24 in the low interest rate scenario and would invest \$1,554.51 in the high interest rate scenario. This defines the following scenario dependent optimal consumption paths:

Consumption	Scenario 1	Scenario 2
Time 0	2,000 + 182.24 = 2,182.24	2,000 – 1,554.51 =445.49
Time t	2,000 × exp(0.0075 × 200) – 182.24 × exp(0.01 × 200) = 7,616.78	2,000 × exp(0.0075 × 200) + 1554.51 × exp(0.05 × 200) = 34,249,304.84

2. Calculation of the certainty equivalent discount factor

Differentiating[11] (11) with respect to C_0 and C_t we obtain the marginal utilities:

$$V_i^{'}(C_0) = \frac{1}{C_0^{\sigma}}$$

[9] Result courtesy of http://www.derivative-calculator.net
[10] Solution courtesy of http://www.wolframalpha.com
[11] Results courtesy of http://www.derivative-calculator.net

$$V_i^{'}(C_t) = \frac{1}{e^{\rho t} C_t^{\sigma}}$$

With the data of the quantitative example, these assume the following values:

Marginal utilities	Scenario 1	Scenario 2
Time 0	$\frac{1}{2{,}182.24^{0.8}} = 0.002132444$	$\frac{1}{445.49^{0.8}} = 0.007602019$
Time t	$\frac{1}{\exp(0.005 \times 200) \times 7616.8^{0.8}}$ = 0.000288595	$\frac{1}{\exp(0.005 \times 200) \times 3424304.8^{0.8}}$ = 3.45131E-07

Using the computed marginal utilities in expression (15) allows us to calculate the expected discount factor *D* as follows:

$$\frac{0.5 \times 0.000288595 \;\; + 0.5 \times 3.45131\,E-07}{0.5 \times 0.002132444 \;\; + 0.5 \times 0.007602019} = 0.029682$$

and from that R_d can be computed using (16):

R_d = – (1/200) ln(0.029682) = 1.76%

Inverting the expected discount factor we obtain the expected compound factor:

F = 1 / 0.029682 = 33.69026

from which R_c can be computed using (17) as follows:

R_c = (1/200) ln(33.6903) = 1.76%

3. Test of the Gollier and Weitzman (2009) calculation method

Gollier and Weitzman's (2009) calculation method can be tested as well. From (7) the expected discount factor *D* following the "Weitzman" approach is

$D = \Sigma\, q^W{}_i \, exp(-r_i\, t)$

For the "Weitzman approach" the "risk adjusted" probabilities are defined as

$q^W{}_i = p_i\, V'_i(C_0) / \Sigma\, p_i\, V'_i(C_0)$

which using our data become:

	Scenario 1	**Scenario 2**
q_W	$\frac{0.5\times 0.00213244}{0.5\times 0.002132444 + 0.5\times 0.007600203}$ = 0.21906126	$\frac{0.5\times 0.007602019}{0.5\times 0.002132444 + 0.5\times 0.007600203}$ = 0.78093874

The expected discount factor therefore can be calculated as follows:

$$D = 0.21906126 \times \exp(-0.01 \times 200) + 0.78093874 \times \exp(-0.05 \times 200) = 0.029682$$

which is the same result as was obtained in A.2, and from which, using (16), we get

$$R_d = -(1/200)\ln(0.029682) = 1.76\%$$

Alternatively, from (9), the expected compound factor F following the "Gollier" approach is

$$F = \Sigma\, q^G{}_i\, exp(r_i\, t)$$

For the "Gollier approach" the "risk adjusted" probabilities are defined as

$$q^G{}_i = p_i\, V'_i(C_t) / \Sigma\, p_i\, V'_i(C_t)$$

which using our data become:

	Scenario 1	**Scenario 2**
q^G	$\frac{0.5\times 0.00288595}{0.5\times 0.00288595 + 0.5\times 3.45131\text{E}-07}$ = 0.998805527	$\frac{0.5\times 3.45131\text{E}-07}{0.5\times 0.00288595 + 0.5\times 3.45131\text{E}-07}$ = 0.001194473

The expected compound factor therefore can be calculated as follows:

$$F = 0.998805527\times \exp(0.01 \times 200) + 0.001194473\times \exp(0.05 \times 200) = 33.69026$$

which is the same result as was obtained before, and from which, using (17), we get again

$$R_c = (1/200)\ln(33.69026) = 1.76\%$$

4. Certainty equivalent rates for selected time horizons

Certainty equivalent rates for selected time horizons

Years till time *t*	**50**	**100**	**200**	**300**	**400**	**500**
Risk neutral R_W	2.13%	1.67%	1.35%	1.23%	1.17%	1,14%
Risk averse R_d or R_c	2.65%	2.18%	1.76%	1.62%	1.57%	1.54%
Risk neutral R_G	3.87%	4.33%	4.65%	4.77%	4.83%	4.86%

The values of R_W and R_G (risk neutral case) were obtained by evaluating expressions (2) and (4) of the main text for the values of t shown in the column headings, with all other required data being the same as given in Table 1 for our numerical example. The values of R_d or R_c (risk averse case) were obtained by repeating the calculations shown in Sections 1 and 2 of this Appendix for the relevant time horizons.

5. Test of compliance with the definition of present value (risk averse)

The calculated expected discount factor will calculate a correct expected present value of a future safe sum if investing the expected present value in the capital market will yield a stochastic return such that its expected utility is the same as that of the future safe sum to which the discount factor was applied. In this section we will test if D = 0.029682 complies with this requirement.

Concept	**Calculation**	**Result**
Safe yield at time 200 (an arbitrary small sum).	ε	0.001
Utility of consumption at time 200 in scenario 1, including safe yield of 0.001.	$\frac{(7{,}616.775814+0.001)^{1-.8}-1}{1-.8}$	24.8761238
Utility of consumption at time 200 in scenario 2, including safe yield of 0.001.	$\frac{(34{,}249{,}304.84+0.001)^{1-.8}-1}{1-.8}$	155.6572603
Expected utility with safe yield of 0.001.	(24.8761238+155.6572603)/2	**90.26669203**
Expected present value of safe yield at time 200.	δ = D × ε =0.029682 × 0.001	2.96822E-05
Yield at time 200 of investing δ in scenario 1	2.96822E-05 × exp(200 × 0.01)	0.000219323

Yield at time *t* of investing δ in scenario 2	2.96822E-05 × exp(200 × 0.05)	0.653793353
Utility of consumption at time *t* in scenario 1, including yield of investing δ.	$\frac{(7{,}616.775814+0.000219323)^{1-.8}-1}{1-.8}$	24.87612319
Utility of consumption at time *t* in scenario 2, including yield of investing δ.	$\frac{(34{,}249{,}304.84+0.653793353)^{1-.8}-1}{1-.8}$	155.6572609
Expected utility having invested present value δ.	0.5×24.87612319+0.5×155.6572609	**90.26669203**

The above table shows that the expected utility of the safe yield is the same as the expected utility of investing the expected present value of that yield[12].

6. Standard risk neutral compound and discount factors

The risk neutral compound factor for the numerical example can be computed as follows, on the assumption that $1 is invested:

F = 0.5 × exp (0.01 × 200) + 0.5 × exp (0.05 × 200) = 11,016.93

And consequently the discount factor is:

D = 1 / 11,016.93 = 9.07694E-05

from which, using (16), the CER computes to:

R_d = (1/t) ln(D) = – (1/200) ln(9.07694E-05) = 4.65%

7. Weitzman risk neutral discount factor

Weitzman's risk neutral discount factor for the numerical example can be computed as follows, on the assumption that a safe $1 in year 200 is to be discounted:

A = 0.5 × exp(– 0.01 × 200) + 0.5 × exp(– 0.05 × 200) = 0.067690342

from which the CER computes to:

R_W = – (1/t) ln(A) = – (1/200) ln(0.067690342) = 1.35%

[12] The expected value of the yield is of course not equal to the safe yield [(0.000219323 + 0.653793353)/2 = 0.327006338].

8. Test of compliance with the definition of present value (risk neutral)

The standard discount factor D is compliant:

$D \times F = 1$

9.07694E-05 × 11,016.93 = 1

Weitzman's discount factor A is not compliant:

$A \times F \neq 1$

0.067690342 × 11,016.93 = 745.74

9. Loss from following the Weitzman present value rule

As the Weitzman discounting rule results in present values much higher than those of the conventional method, anyone investing in a project that has a positive present value according to the Weitzman rule would suffer an opportunity loss, for the same future benefits could be had much more cheaply by investing in the market instead. The loss can be quantified using the difference between Weitzman's and the correct discount factors. For each $1 of future benefits, this is as follows with the data of our example:

Absolute loss =$0.067690342 – $9.07694E-05 = $0.067599573

Fractional loss = $0.067599573 / $0.067690342 = 0.998659056

The fact that investing Weitzman's present value A in a project yielding a safe $1 leads to welfare loss can be shown for the formal Gollier-Weitzman (2009) model by assuming that the decision maker is risk neutral. In that case the utility function to be used in (11) should be linear. In its simplest form $U(C) = C$, and expression (11) becomes:

$$V(C) = \sum_{t}^{\infty} e^{-\rho t} C_t$$

The risk neutral investor would invest all he could in both the scenarios, because both the low and high interest rates exceed his pure rate of time preference, making the postponement of consumption the welfare maximizing action in both scenarios. Therefore, the welfare consequence of investing A = $ 0.067690342 (A.7) in the safe project yielding $1 is the opportunity cost at t = 200 less the safe yield obtained, which is (see A.8) 745.74 – 1 = 744.74. Discounting this at the pure rate of time preference of 0.5% yields a utility loss of 273.97. Had A been calculated using the correct discount rate, the welfare loss would have been zero, as the opportunity loss of investing in any project with an expected return equal to that of the market is zero.

10. Standard and Weitzman CERs when the utility function is logarithmic

For the standard CER we compute the certainty equivalent of future payoffs, noting that the utility function is logarithmic and that the utility function is the inverse function of exponentiation:

$$F = e^{\sum p_i \ln(e^{r_i t})} = e^{t \sum p_i r_i} = e^{t\bar{r}}$$

where $\bar{r}$ is the expected value of the annual interest rate. Consequently,

$$CER = \frac{1}{t}\ln(e^{t\bar{r}}) = \bar{r}$$

which means that the CER rate is constant as a function of time, and equal to the expected annual rate of interest.

The same happens with the Weitzman CER method:

$$A = e^{\sum p_i \ln(e^{-r_i t})} = e^{-t \sum p_i r_i} = e^{-t\bar{r}}$$

$$CER = -\frac{1}{t}\ln(e^{-t\bar{r}}) = \bar{r}$$

So indeed the two methods yield quantitatively identical results, but only in this case, which does not make the Weitzman method correct for other cases, nor does it support declining or growing discount rates.

A.11 CERs and required monetary returns

The analysis proceeds as described in the Section 5 of the main text. The risk neutral CERs are calculated as in A.6. The following table shows the calculation of the required monetary CER of a project that is riskier than the market. The small project has an investment cost of $1, the coefficient of variation of its annual rate is 1.5 times that of the market, the correlation between market and project risks is 1, and t = 200. The calculation of the risk averse CER would be identical to this, except that the small project's annual return probability would then be the same as that of the market.

Concept	Calculation	Result
Optimal market investment.	Total expected utility is maximized by changing the amount invested at time 0, taking into account, under each scenario, how much extra consumption would result from it at time 200.	1,255.321051
Consumption at time 0. This is the same in both scenarios.	Consumption endowment less optimal investment in the market. 2,000 – 1,255.321051	744.6789487
Consumption Scenario 1, time 200.	Consumption endowment plus market yield of the optimal investment 8,963.378141+1,255.321051 × EXP(200×0.01)	18,239.01581
Consumption Scenario 2, time 200.	Consumption endowment plus market yield of the optimal investment 8,963.378141+1,255.321051 × EXP(200×0.05)	27,659,249.58
Expected total utility	$0.5 \times ((744.6789487^{(1-0.8)} - 1) / (1 - 0.8))$ $+0.5 \times ((744.6789487^{(1-0.8)} - 1) / (1 - 0.8))$ $+(0.5 \times ((18{,}239.01581^{(1-0.8)} - 1) / (1 - 0.8))$ $+0.5 \times ((27{,}659{,}249.58^{(1-0.8)} - 1) / (1 - 0.8))) /$ $\mathrm{EXP}(200 \times 0.005)$	**46.78510666**
Expected annual yield x of the small project.	This is the solution of the iterative search for annual yield x. The search stops when expected total utilities are the same in the row above and in the last row of this table.	2.524429%
Coefficient of variation multiple.	Assumption.	1.5
Coefficient of variation of the project's rates.	The coefficient of variation of the market rates is $((0.01–0.03)^2+(0.05–0.03)^2)^{0.5}/0.03=0.9428$, which multiplied by 1.5 gives the coefficient of variation of the project.	1.414213562
Deviation d from annual mean of project high and low rates.	$d = x \times v / 2^{0.5}$ $= 0.02524429 \times 1.414213562 / 2^{0.5}$	2.524429%
Project low rate.	0.02524429 – 0.02524429	0.00%
Project high rate.	0.02524429 + 0.02524429	5.0489%

Project monetary CER	(1/200) × LN (0.5 × EXP(0 × 200) + 0.5 × EXP(0.050489 × 200))	**4.70%**
Correlation with market.	Assumption.	1
Consumption at time 0. This is the same in both scenarios.	Consumption endowment less investment in the market and in the project. 2,000 – 1,255.321051 – 1	743.6789487
Consumption Scenario 1, time 200.	Consumption endowment plus market and project yields. 8,963.378141+1,255.321051 × EXP(200×0.01) + EXP(200 × 0)	18,240.01581
Consumption Scenario 2, time 200.	Consumption endowment plus market and project yields. 8,963.378141+1,255.321051 × EXP(200×0.05) + EXP(200 × 0.050489)	27,683,537.09
Expected total utility	0.5 × ((743.6789487$^{(1-0.8)}$ – 1) / (1 – 0.8)) +0.5 × ((743.6789487$^{(1-0.8)}$ – 1) / (1 – 0.8)) +(0.5 × ((18,240.01581$^{(1-0.8)}$ – 1) / (1 – 0.8)) +0.5 × ((27,683,537.09$^{(1-0.8)}$ – 1) / (1 – 0.8))) / EXP(200 × 0.005)	**46.78510666**

Repeating these calculations for the selected values of *t* yields the following results:

No clairvoyance certainty equivalent rates for selected time horizons

Years till time *t*	**50**	**100**	**200**	**300**	**400**	**500**
Optimal investment	765.18	987.00	1,255.32	1,494.10	1,683.45	1,813.88
Risk neutral CER	3.87%	4.33%	4.65%	4.77%	4.83%	4.86%
Risk averse CER	3.13%	2.65%	2.13%	1.95%	1.88%	1.85%
Risky project's required monetary rate	3.99%	4.44%	4.70%	4.79%	4.83%	4.86%
Negative correlation project's required monetary CER.	2.19%	2.14%	1.80%	1.72%	1.71%	1.71%

For completeness, these values have been calculated with the clairvoyance assumption of the Weitzman (1998) model. Details of the calculations are not shown. In this case the optimal consumption path also differs in the values of C_0, which become scenario specific and are calculated as in A.1. The rest of the calculations follow the same pattern as given above for the no clairvoyance case.

Clairvoyant certainty equivalent rates for selected time horizons

Years till time *t*	**50**	**100**	**200**	**300**	**400**	**500**
Optimal investment in Scenario 1	–58.46	–108.47	–182.24	–223.02	–236.67	-230.89
Optimal investment in Scenario 2	1054.07	1292.76	1554.51	1734.07	1848.28	1915.82
Risk neutral CER	3.87%	4.33%	4.65%	4.77%	4.83%	4.86%
Risk averse CER	2.65%	2.18	1.76%	1.62%	1.57%	1.54%
Risky project's required monetary rate	4.13%	4.55	4.76%	4.82	4.85%	4.88%
Negative correlation project's required monetary CER.	2.10%	1.68%	1.44%	1.40%	1.40%	1.40%