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Identifying the Median Path of a Stochastic Processes

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Abstract

This paper presents a method to characterize the typical path of a stochastic process, which I refer to as the Median Path. The paper describes how to estimate the Median Path in simulation and compares it to a different estimate of the typical path that is defined as the median value at each time step, like an ensemble average. The Median Path is an actual path from the stochastic process, whereas the path of median values is not. Therefore, the two paths may have very different properties. The Median Path is a single path from a set of simulated paths and is identified using a Ranking Algorithm that calculates the rank of each path at each time and then averages the ranks over time, similar to a time average. The Median Path is potentially useful in simulation applications where it is important to characterize the actual behaviour of a path generated by the stochastic process rather than the behaviour of statistics of the process over time.

Keywords: Simulation, Ensemble Average, Time Average.

JEL Classification: C14, C15, C63.

Identifying the Median Path of a Stochastic Processes

1. Introduction

The paper introduces the Median Path, which is an estimate of the typical behaviour of a stochastic process over time. The Median Path is a particular path in a set of simulated paths generated by a stochastic process. The Median Path is identified by a Ranking Algorithm that compares the values of all paths at each time step, ranks each path at each time step, and then averages the ranks over time. In this way, the Ranking Algorithm can identify a path that is consistently in the midst of the set of simulated paths over time. In contrast, the typical path could be described by the median values for the stochastic process over time, which I refer to as the Benchmark Path. The Median Path is more representative of the typical behaviour of the stochastic process because it is an actual path generated by the stochastic process, rather than a path defined by statistics at each time step.

$$(1) \quad X(t,n)=X(t-1,n)+e.$$

The paper uses a stationary time series generated by Equation (1), where $X(t,n)$ denotes the value of n -th path at time t . I assume that the total number of paths is large, $N=10,000$ to ensure that the statistics calculated across paths are close to theoretical values. Note that the median rank is $N/2=5000$. I assume that there are a relatively small of number time steps, $T=100$, to provide enough length of path to show phenomenon of interest. I also assume that $X(0)=100$ and $e \sim U(-0.01,0.01)$ throughout for simplicity.

2. The Median Path

2.1 *Ranking Algorithm*

To identify the Median Path, I use a Ranking Algorithm that compares all of the simulated paths at each time step. I describe the algorithm below.

1. For each time t , rank the value of the n -th path against the value of all other paths at that time. Denote this ranking as $R(n,t)$, which takes values from 1 to N .
2. Calculate the average of the ranks over time, $R'(n)=\text{average}(R(n,t))$, and then rank the paths again based on the value of $R'(n)$. The new ranking $R''(n)$ takes values from 1 to N .
3. Identify the path n^* that produces the median value amongst the re-ranked scores, $R''(n^*)=N/2$. This path is the Median Path.

The Ranking Algorithm is designed to capture information about the ranking of a single path in comparison to others over time. If a particular path is consistently in the middle of the set of simulated paths, then that path will receive a medium ranking over time $R'(n)$ approximately $N/2$ and the re-ranked score will also be near the median $R''(n)$ approximately $N/2$. In general, these rankings can be used to identify paths that are consistently in the middle of the pack for all simulated paths. However, a particular path may have high ranking in some times and low rankings in others that offset each other and cause the average ranking to be approximately $N/2$, which raises a concern that I address with a Revised Ranking Algorithm below.

For comparison, I define the Benchmark Path as the median across all paths at each time $M(t)=\text{median}(X(n,t))$. The Benchmark Path provides a simple way to characterize the typical level of a stochastic process but has small variation over time in large samples because it is defined as an estimate of the median value at each time step.

2.2 *Estimates of the Median Path*

Figure 1 shows two estimates of the Median Path and one estimate of the Benchmark Path. The Median Paths are very different from the Benchmark Path, as expected. The Median Paths moves between 96 and 104, whereas the Benchmark Path has much smaller variation. This occurs because the Median Path has more natural variation that accurately represents the typical behaviour of this time series, whereas the Benchmark Path has less variation because it is constructed as a statistic with little variance in large samples.

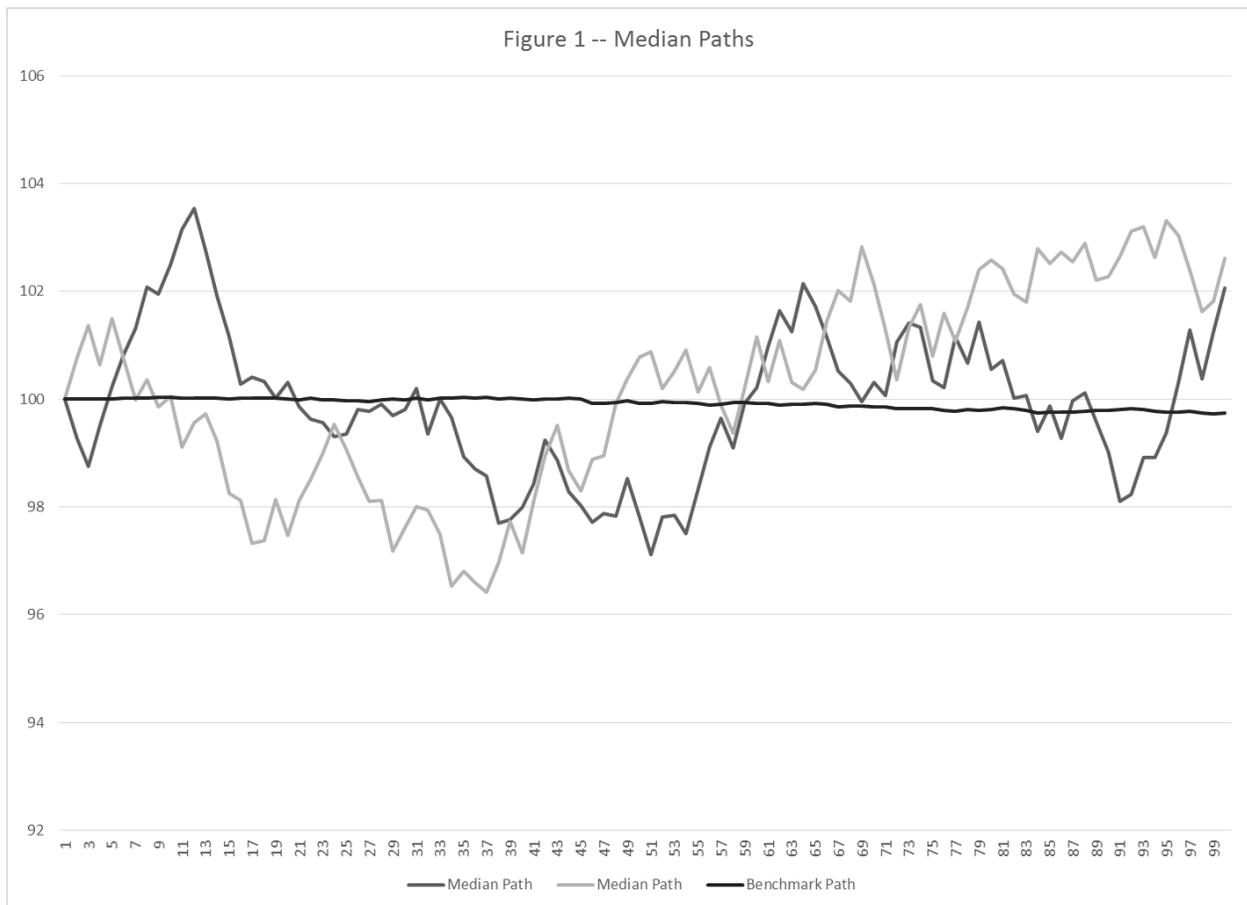


Figure 2 shows that the two estimates of the Median Path have very different behaviour over time, which reflects just how much variability is typical for the underlying stochastic process. In contrast, the Benchmark Path may accurately represent the level of the typical path but not the variability. It is possible to use confidence intervals around the Benchmark Path to characterize

the dispersion of a path around the typical level, but this will not give a sense for what a particular path generated by the stochastic process will actually look like in the same manner as an estimate of the Median Path.

2.3 *Revised Ranking Algorithm*

To address the concern that a particular path could offset high rankings in some periods with low rankings in other periods, I introduce a weighting scheme to penalize paths that have rankings far from the median value at any time. I use the sum squares of differences to emphasize that the ranking should be close to the median value over time. The Revised Ranking Algorithm is given below.

1. For each time t , rank the value of the n -th path against all other simulated values at that time. Denote this ranking as $R(n,t)$, which takes values from 1 to N .
2. Calculate the average over time of the sum of squared differences between the rank and median rank, $R'(n)=\text{average}[(R(n,t)-N/2)^2]$.
3. Identify the path that minimizes the re-ranked scores, $n^*:\min(R'(n))$. This path is the Median Path.

The Revised Ranking Algorithm identifies the path that minimizes the sum of squared differences between its rank and the median rank over time. Using the sum of squared differences in this ways penalizes paths that deviate far from the median at any time point and mitigates the risk that a path can offset a high ranking in one period with a low ranking in other period. The path that minimizes the sum of squared differences is close to the median of all paths in a stronger sense than the initial Ranking Algorithm. I provide two estimates of the Median Path using the Revised Ranking Algorithm below.

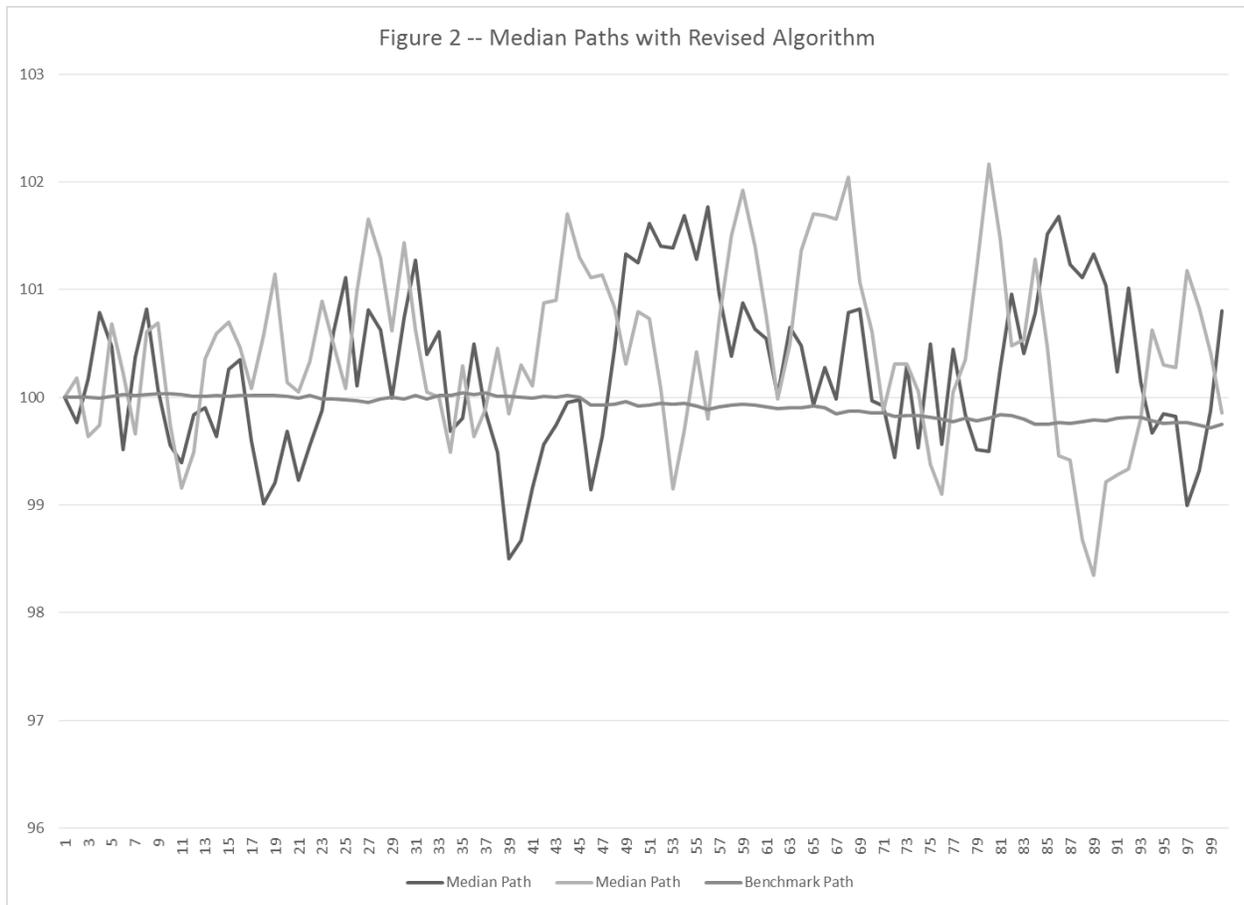


Figure 3 shows that the two estimates of the Median Path are still different from each other, but are both more closely concentrated around the Benchmark Path over time. The Median Paths here do not take values far above or below the median value over time and this suggests that Revised Ranking Algorithm more effectively identifies paths that are consistently the in midst of all the simulated paths.

3. Discussion

The Median Path is a numerical method that is useful for characterizing the typical behaviour of a stochastic process because it identifies an actual path generated by the process, rather than a path defined by a statistic at each time step. Therefore, the Median Path more accurately reflects the natural variation in data generated by the process. Understanding how a system responds to

such variation can give important insight that goes beyond what can be learned from the system response to the Benchmark Path. Furthermore, it is possible to extend the Median Path method to characterize other percentile-paths, such as the Upper-Quartile Path or 99-Percentile Path. These methods can follow similar a construction as described here and help characterize the extreme behaviour of a stochastic process.