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Abstract We study the efficiency of the uniform auction as an allocation mechanism for emission permits among polluting firms. In our model, firms have private information about their abatement costs, which differ across firms and across units, and bidders' demands are linear. We show that there is a continuum of interior Bayesian-Nash equilibria, and only one is efficient, minimizing abatement costs. We find that the existence of many bidders is not a sufficient condition to guarantee an efficient equilibrium in the uniform auction. Additionally, bidders' types have to be uncorrelated.

JEL classification D44(Auctions)

Keywords Emission permits · Uniform auction · Efficiency · Incomplete information simultaneous games

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1 Introduction

Emissions trading, or cap and trade, is an increasingly popular environmental policy instrument used to encourage firms to reduce pollution. A permit represents the right to emit an specific amount of a pollutant; firms are required to hold a number of permits equivalent to their emissions, and are legally forced to reduce them, from their business as usual level, i.e., their emissions in the absence of regulation, to the number of permits they hold, and bear any associated abatement costs. The main objective of this policy approach is to achieve the emission target reduction at the lowest economic cost, i.e., to achieve an efficient allocation of permits among polluting firms.

Under an emission trading system, the environmental authority sets a binding emission target that is allocated to firms in the form of emissions permits, and the permits can be traded in a secondary market. In this secondary market, buyers pay for the right to increase their emissions, while sellers gain a reward for decreasing them. To make the initial allocation among firms, until recently, the most widespread method was grandfathering, i.e., the distribution of permits for free based on historical emissions, but recently, auctions are increasingly being used. It is well established in the literature that if the secondary market is perfectly competitive, those firms who can reduce emissions most cheaply will do so, which ensures that the equilibrium allocation will be efficient.[1] If this is the case, the initial allocation of permits only matters for the distribution of the gains from trade but is irrelevant to achieve efficiency: both grandfathering and auctions attain efficiency.

On the other hand, as first noted by Hahn (1984), if the secondary market for emission permits is not perfectly competitive, the initial allocation of permits matters for efficiency. One of the main reasons for using auctions as an allocation mechanism is the usual belief that auctions attain allocation efficiency.[2] However, while efficiency is a classical result for single-unit auctions, the existing theory reveals that efficiency does not hold for multi-unit auctions when bidders demand multiple units, as it is the case of emission permit auctions.

We analyze the uniform auction format, the one used in some of the most important emission permit markets worldwide, with the EU ETS being the leading example. In a uniform auction, bidders submit bids for different quantities at different prices, and the auctioneer determines the market clearing or stop-out price, and accepts all bids above it. Bidders pay the stop-out price for all units won. We ignore the secondary market of emission permits in order to analyze the efficiency of the uniform auction format within a fairly simple theoretical model.

Two elements regarding the auction's efficiency need to be stressed. First, in single-unit auctions, efficiency can be simply achieved by assigning the good to the bidder with the highest valuation. Then, assuming ex-ante symmetric bidders, any single-unit auction is efficient whenever equilibrium bids are increasing in valuations. Efficiency is more complex in multi-unit multi-bid auctions, even assuming ex-ante symmetric bidders, since a bid is a demand schedule instead of an scalar. As Ausubel et al (2014)

[1] See, for example, Montgomery (1972).

[2] For example, the European Commission is strengthening the introduction of auctions in the European Emission Trading System (EU ETS) based on the belief that this method "best ensures the efficiency, transparency and simplicity of the system, creates the greatest incentives for investment in a low-carbon economy and eliminates windfall profits". See http://ec.europa.eu/clima/policies/ets/cap/auctioning/faq en.htm, Section "Why are allowances being auctioned?".

point out, bidders have an incentive to shade bids differently across units, since a bid for one unit may determine the payment for other units won, and therefore efficiency is not guaranteed. Second, within multi-unit multi-bid auctions, theoretical results state that there are multiple equilibria under the uniform auction format.[3] If there are multiple equilibria, conditions under which the efficient equilibrium is reached should be analyzed.

In this paper we follow Wilson (1979) *share auctions* approach, in which the good auctioned is assumed to be perfectly divisible, and bidders -polluting firms- bid a continuous demand schedule. Bidders' valuations of the permits are determined by their marginal abatement cost, that we assume to be decreasing, a reasonable assumption if the cost of reducing emissions is increasing as more emissions are reduced. We further assume that they decrease linearly. Different polluting firms might -and generally will- have different abatement technologies. We assume that at the time of bidding, bidders have some private information about their marginal abatement costs. Specifically, those marginal costs have a slope common to all bidders and an intercept, the bidder's *type*, that varies across bidders and that is private information. Therefore, in our model the marginal abatement costs are heterogeneous not only across bidders, but also for each bidder for different units.

We restrict the analysis to *interior* equilibria, in which all bidders buy permits at the auction and have positive abatement cost. We prove that interior equilibria are reached when the differences in the abatement cost functions across bidders are not too large. We think that this is an interesting scenario to analyze the auction's efficiency.[4]

At the auction, bidders' strategies map the privately observed types into demand functions, the bids. Accordingly to the assumption of linear marginal abatement costs, we consider equilibria conformed by strategies that are additively separable and linear in price and bidder's type, while we allow for non-linear responses. We prove that, as marginal abatement costs are linear in types, to restrict to equilibria conformed by linear strategies is not essential, since the efficient allocation rule is also linear in types. Our model results in multiplicity of equilibria under the uniform format. While some functional and distributional assumptions are made for the sake of tractability, we believe that our model does consider some of the relevant characteristics of the emission permits auctions.

Our results can be summarized as follows. We first characterize equilibria, and show that the bidding behavior under any equilibrium strategy can be interpreted in terms of marginal valuation and price. As the number of bidders tends to infinity, the highest marginal abatement cost bidder i has to pay, conditional on his type, under his expected allocation in the auction, i.e., his valuation for the last unit that he wins at the auction, is equal to the expected auction price. As the number of bidders increases, the strategic aspect of bidding disappears. In contrast, with only two bidders, a bidder's valuation for the last unit that he wins at the auction is larger than the expected auction price. Secondly, we show that multiplicity of equilibria is, somehow, limited. A linear strategy is characterized by three parameters: a

[3] See, for example, Wang and Zender (2002).

[4] For example, participation in the EU ETS is mandatory for companies in some sectors, but "only plants above a certain size are included, certain small installations can be excluded if governments put in place fiscal or other measures that will cut their emissions by an equivalent amount, and in the aviation sector, until 2016 the EU ETS applies only to flights between airports located in the European Economic Area (EEA)". See http://ec.europa.eu/clima/policies/ets/index en.htm

constant, the type's coefficient and the price's coefficient. We prove that there exist two differentiable functions that define two of the parameters in terms of the third one, so that any interior equilibrium can be characterized in terms of the type's coefficient. Moreover, the set of values of the type's coefficient defining equilibrium strategies is an open interval. This is important for two reasons. First, regarding the positive properties of equilibria, existence and uniqueness can be easily analyzed within the real line. Second, there is a normative side, since we next give necessary and sufficient conditions for efficiency in terms of that parameter: it has to be equal to the inverse of the coefficient of the bidder emission level on the marginal abatement cost function.

Next, we analyze the efficiency of equilibria. We use a concept of ex-post efficiency, as Ausubel et al (2014). We consider two cases. First, we analyze a pure private value case, in which the bidders' types are privately observed and are independent across bidders. Next, we consider a mineral right model, in with bidders' types are correlated. In both cases, we find that the unique efficient allocation of permits can be attained as an interior equilibrium for any number of bidders equal or greater than two. In the case of independent types, for a small number of bidders (we prove our result for two bidders), there are other equilibria beside the efficient equilibrium. We also provide a categorization of the inefficiency of equilibria. At the efficient allocation, marginal abatement cost are equalized across bidders. Then, a necessary, but not sufficient condition for efficiency, is that a firm with larger marginal abatement cost for all units (higher type), is awarded more permits at the auction than any other firm with lower marginal abatement cost. This is always the case, given that the equilibrium strategies have the same slope and different intercepts. However, some equilibrium are inefficient because firms with high types get too few permits with respect to the efficient allocation, while some other equilibria are inefficient because firms with high types get too many. We label those inefficiencies as *under* or *over* assignment, respectively, of the environmentally inefficient firms. The main difference between independent and correlated types is that when the number of bidders tends to infinity, while the number of permits per bidder remains constant, when types are independent the only equilibrium is the efficient equilibrium, while when types are correlated there are other equilibria besides the efficient equilibrium. This is one of the main results of our analysis: in our model, the existence of many bidders is a sufficient condition to guarantee the efficiency in the uniform auction if, additionally, the bidders' types are independent. In contrast, for a case of correlated types, we show that a large number of bidders is not sufficient for efficiency.

Our paper relates to Ausubel et al (2014), that using the share auction approach establish that every equilibrium of the uniform-price auction is ex post *inefficient* when bidders have flat demands, i.e., when they have constant marginal utility. The reason is differential bid shading. In their paper, when bidders have linearly decreasing marginal utility, as in our paper, they consider, as we do, linear equilibrium, in order to address the multiplicity of equilibria. The difference with our paper is that they assume that bidders' marginal utility functions are identical, and therefore efficiency is not an issue. In contrast, we allow for bidders' specific values. In their paper, supply is uncertain, while in our paper the number of permits auctioned is known. Ausubel and Cramton (2002) analyze efficiency of a uniform price auction in a pure private value where bidders' marginal values are decreasing in quantity, as we do, and conclude that there does not exist an ex post efficient equilibrium. The main difference with our model is that they impose bidding strategies such that bidders bid their valuation for the first unit, which implies that the bid curves lie strictly below the marginal-value curve at all positive quantities. We obtain such an equilibria in our model, that, as in their case, is not efficient because there is differential bid shading. However, allowing more general linear equilibria, we show that there are efficient equilibria in the uniform auction.

There is a branch of literature that analyze the efficiency of the cap-and-trade system when there is market power in the permit market. Hahn (1984) stated that if a dominant firm exists, the efficiency loss due to market power depends on the initial allocation of permits and the permit price is an increasing function of the dominant firm's allocation. Hagem and Westskog (1998) extended the Hahn setting in a dynamic framework. An overview of this literature can be found in Montero (2009).

We aim at filling a gap in the theoretical literature regarding auctions of emission permits. Some authors have addressed permit auctioning from a descriptive point of view (see. e.g., Hepburn et al (2006) or Cramton and Kerr (2002)) or by means of experimental studies (see, e.g., Ledyard and Szakaly-Moore (1994), Godby (1999), Godby (2000), Muller et al (2002) or Goeree et al (2010)). Also some theoretical approximations have been made. Antelo and Bru (2009) compare auctioning and grandfathering in a permit market with a dominant firm when the government is concerned both about cost-effectiveness and public revenue. Alvarez and André (2015b) and Alvarez and André (2015a) also compare auctioning and grandfathering when there is a secondary market with market power and firms have private information on their own abatement technologies. Kline and Menezes (1999) examine a stylized version of EPA double auctions between buyers and sellers. Nevertheless, none of this study addresses multi-unit, multi-bid auctions of permits as we do in a standard auction theory approach. Antelo and Bru (2009) assume perfect information, and hence omit one of the main ingredients of auctions. Alvarez and André (2015b) consider that the bidders act non-strategically and in Alvarez and André (2015a) only one firm (the dominant one) bids strategically. Kline and Menezes (1999) do not address the use of auctions to make the initial allocation of permits by the environmental authority (as it is done in the UE ETS), but only to exchange permits among firms. Moreover, they use a single-unit approach and restrict themselves to perfect information except for two specific examples under complete information. So, apart from addressing the specific question on efficiency, our aim is to contribute to the theoretical literature by providing a sound model for permit auctioning and bidding.

This paper is organized as follows. In Section 2 we present the model and in Section 3 we characterize the equilibria in Proposition 1, show that the set of interior equilibria conformed by linear strategies can be indexed by a parameter in Proposition 2, and establish the existence and multiplicity of equilibria in Proposition 3. In Section 4 we analyze the efficiency of the equilibria, when bidders' types are independent in Proposition 5 and when types are correlated in Proposition 6. Finally, Section 5 concludes the paper. The Appendix contains all proofs.

2 The model

Assume that Q perfectly divisible permits are inelastically supplied in a uniform auction with I bidders, the polluting firms, indexed by $i \in \{1, \ldots, I\}$, with $I \geq 2$.

Bidder i's marginal abatement cost is linear, defined by

$$\phi(e; \tilde{\alpha}_i) := \tilde{\alpha}_i - \beta e \tag{1}$$

where $\tilde{\alpha}_i$ is a bidder-specific random variable, his *type*, e is the bidder's emission level, and β is some positive constant.[5] Types are drawn from a joint continuous distribution, which is common knowledge. The realization of $\tilde{\alpha}_i$ is privately observed by bidder i before the auction, i.e., each firms knows his marginal abatement cost function before the auction, but not his rivals'. We assume that the marginal probability distribution of types is identical across types, i.e., bidders are *ex ante* symmetric, and has support $\Omega \equiv [\underline{\alpha}, \overline{\alpha}]$.

Note that equation (1) implies that bidders have marginal values for additional permits that are decreasing in the quantity received. As Ausubel et al (2014) point out, this is an aspect of multi-unit demands not present in auctions of unit demands.[6]

Let $C(q_i; \alpha_i)$ denote bidder i's total cost of complying with the emission cap when his type is α_i and he wins q_i permits at the auction. The q_i permits won at the auction give the bidder the right to pollute q_i units, and he has to incur in the cost of reducing pollution from his *business as usual* emission level, that is, bidder i's emission level in the absence of environmental regulation, to q_i. Let $e^*(\alpha_i)$ denote bidder's i business as usual emission level when his type is α_i. Since ϕ is strictly decreasing in e, it is defined as the value of e that solves $\phi(e; \alpha_i) = 0$; from (1), it is $e^*(\alpha_i) = \alpha_i/\beta$. Bidder i's total cost is the sum of the auction payment and his abatement cost. Under the uniform format bidders pay the same price for all permits, the auction stop-out price, p, defined as the maximum price at which all permits are sold, so that bidder i's total cost is defined by

$$C(q_i; \alpha_i) := pq_i + \int_{q_i}^{e^*(\alpha_i)} \phi(e; \alpha_i) de \tag{2}$$

The first term in (2) is bidder i's auction payment under the uniform format given stop-out price p, and the second term is his abatement cost.

We follow Wilson (1979) *share auctions* approach: we assume that the Q permits are perfectly divisible, and bidders' strategies are a continuous demand schedule, that we define next.

Definition 1 Strategy. Bidder i's strategy at the auction is a demand function, $\gamma_i(\alpha_i, p)$, that for each realizations of his type, $\alpha_i \in \Omega$, specifies the quantity demanded at different price levels, p:

$$\gamma_i : \Omega \times \mathcal{R}_+ \to \mathcal{R}_+$$

Denote by Γ the strategy space, which we restrict to the class of functions that are continuous and non-increasing in p.

A profile of strategies is a vector $\boldsymbol{\gamma} := (\gamma_1, \ldots, \gamma_I)$, which specifies a strategy for each bidder. We alternatively write $\boldsymbol{\gamma} = (\gamma_i, \boldsymbol{\gamma}_{-i})$, where $\boldsymbol{\gamma}_{-i}$ is the vector of strategies played by all bidders except i.

Bidder i's best response to $\boldsymbol{\gamma}_{-i}$ is the strategy γ_i that minimizes his expected cost:

$$E\{C(q_i; \alpha_i) \mid \gamma_i, \boldsymbol{\gamma}_{-i}\}$$

[5] A tilde denotes random variable. The same letter without tilde denotes an arbitrary realization.

[6] They also consider diminishing marginal values that are linear, as we do, but they assume that $\alpha_i = \alpha_j$ with probability one for all i and j.

where the expectation is taken with respect to $\boldsymbol{\alpha}_{-i}$, his rivals' types.

The game is a simultaneous game of incomplete information, for which the standard equilibrium concept is Bayesian-Nash equilibrium, that we define next.

Definition 2 Equilibrium. A profile of strategies $(\gamma_1^*, \ldots, \gamma_I^*)$ is an equilibrium if and only if, for all $i \in \{1, \ldots, I\}$, $\gamma_i \in \Gamma$ and $\alpha_i \in \Omega$, it is

$$E\{C(q_i; \alpha_i) \mid \gamma_i^*, \boldsymbol{\gamma}_{-i}^*\} \leq E\{C(q_i; \alpha_i) \mid \gamma_i, \boldsymbol{\gamma}_{-i}^*\}$$

that is, γ_i^* is a best response to $\boldsymbol{\gamma}_{-i}^*$: bidder i cannot lower his expected cost by deviating from γ_i^* when all other bidders are playing γ_{-i}^*.

We focus on symmetric equilibria of the form $(\gamma^*, \ldots, \gamma^*)$; i.e., any two bidders with the same type submit the same demand function. In the sequel, we refer to a Bayesian Nash symmetric equilibrium simply as an equilibrium. When $(\gamma^*, \ldots, \gamma^*)$ is an equilibrium, we say that γ^* is the equilibrium strategy. Moreover, p^* and q_i^* refer hereafter to the auction's stop-out price and to bidder i's allocation under an equilibrium strategy; these values depend on the vector of bidder's types.

3 Characterization of equilibria

In this Section we characterize the auction equilibria. We consider demands that are additively separable in bidder's type and price, and that are linear in price.[7] Specifically, we consider equilibria conformed by strategies of the form

$$\gamma(\alpha_i, p) = \tau(\alpha_i) - \delta p \tag{3}$$

where τ is an arbitrary function of the bidder's observed type and δ is some positive constant. Both $\tau(\cdot)$ and δ are to be determined at the equilibrium. We must remark that the strategy space is not restricted to this class of strategies, i.e. arbitrary responses within Γ are allowed.

Furthermore, we restrict ourselves to *interior* equilibria. We say that an allocation of permits is interior if each bidder receives a positive amount of permits and has strictly positive abatement cost. An equilibrium is interior if the stop-out price is non-negative and generates an interior allocation with probability one, that is, for any vector of types' realizations.

Our approach to characterize equilibria rests on the fact that, under the uniform format, a bidder's auction payment depends only on the stop-out price and the quantity demanded at that price. We proceed as follows. Select an arbitrary bidder, i. First, we characterize the stop-out price that minimizes bidder i's expected total cost, given that all of his rivals are playing some (and the same) arbitrary strategy as in (3). Then, we characterize the expected stop-out price if all bidders, including i, follow

[7] Wang and Zender (2002) and Ausubel et al (2014) also consider separable strategies when analyzing equilibria with asymmetric bidder information, as we do in this paper. As Ausubel et al (2014) mention, obtaining predictive results in multi-unit auctions in settings with decreasing marginal utilities, as we have in our model, requires strong assumptions.

that strategy. That strategy is the equilibrium strategy if and only if, for each $\alpha_i \in \Omega$, the expected stop-out price when all bidders play the strategy is equal to the stop-out price that minimizes bidder i's expected total cost, both expectations being conditional on α_i. The next Proposition characterizes the equilibrium strategy. All proofs are left to the Appendix.

Proposition 1 *Consider an strategy γ, as in (3). At any interior equilibrium, the following equality holds for all $i \in \{1, \ldots, I\}$ with probability 1*

$$\alpha_i - \beta \left(\frac{I-1}{I}\right) \left(\tau(\alpha_i) - \delta E\{p^*_{-i} \mid \gamma,\ \alpha_i\}\right) = \frac{2}{I} \times \frac{\tau(\alpha_i)}{\delta} + \left(\frac{I-2}{I}\right) \times E\{p^*_{-i} \mid \gamma,\ \alpha_i\} \tag{4}$$

where $E\{p^*_{-i} \mid \gamma,\ \alpha_i\}$ is the expected stop-out price if all bidders but i follow γ and bidder i bids zero.

To interpret (4), consider first the case in which there is a large number of bidders, $I \to \infty$. With many bidders, each bidder's contribution to aggregate demand is small, and thus $E\{p^*_{-i} \mid \gamma,\ \alpha_i\}$ tends to the expected auction price, $E\{p^*_{-i} \mid \gamma,\ \alpha_i\} \to E\{p^* \mid \gamma,\ \alpha_i\}$ as $I \to \infty$. Then equation (4) is

$$\alpha_i - \beta \left(\tau(\alpha_i) - \delta E\{p^* \mid \gamma,\ \alpha_i\}\right) = E\{p^* \mid \gamma,\ \alpha_i\} \tag{5}$$

The left hand side of (5) is the marginal abatement cost bidder i has to pay under his expected allocation in the auction,[8] $\tau(\alpha_i) - \delta E\{p^* \mid \gamma,\ \alpha_i\}$. At the equilibrium, that expected marginal cost, his saving on abatement cost from the last unit, equals the expected auction price, the right hand size of (5). The intuition behind this result is that the presence of many bidders eliminates strategical aspects from the bidders' strategies, and bidders bid to equalize their expected marginal abatement cost of the last unit won at the auction to the expected price.

Next, consider a finite number of bidders, $I \geq 2$, with bidders acting strategically. The left hand side of (4) is still an estimation of the highest marginal abatement cost bidder i has to pay under his expected equilibrium allocation. To see this, assume that all bidders but i follow some strategy γ as in (3). Let p^*_{-i} and p^* denote the auction's stop-out price considering the demand of all bidders but the i and all bidders, respectively. Thus, p^*_{-i} satisfies $\sum_{j \neq i} \tau(\alpha_j) - (I-1)\delta p^*_{-i} = Q$ whereas p^* satisfies $\sum_j \tau(\alpha_j) - I\delta p^* = Q$. From the linearity in price of the previous equations, it follows $\frac{I-1}{I}\left(\tau(\alpha_i) - \delta p^*_{-i}\right) = \tau(\alpha_i) - \delta p^*$. The term multiplying β on the left hand side of (4) is therefore bidder i's expected allocation, so that the left hand side is bidder i's highest expected marginal abatement cost.

To interpret the right hand side of (4), assume that bidder i expects to win some permits in the auction by following γ when all other bidders follow γ as well. In that case, he expects the auction price to be strictly above $E\{p^*_{-i} \mid \gamma,\ \alpha_i\}$ and strictly below $\tau(\alpha_i)/\delta$, this latter being the highest price -under γ- at which he demands a positive quantity. The right hand side of (4) is a convex combination of those lower and upper bounds, with the upper bound having less weight as I increases. This is illustrated in Figure 1. In the Figure, $D_{-i}(p)$ is the demand of all bidders but the i, and p^*_{-i} the corresponding stop-out price, determined by the intersection of $D_{-i}(p)$ and the vertical line representing the inelastic supply of Q permits. Bidder i's demand is $\tau(\alpha_i) - \delta p$. The aggregate demand is $D(p)$, and the corresponding stop out price is p^*. The Figure shows that bidder i wins permits at the auction if and only if p^* satisfies

[8] More precisely, we refer to the marginal abatement cost of the last unit.

$p^*_{-i} < p^* < \frac{\tau(\alpha_i)}{\delta}$. The quantity of permits awarded to bidder i at the equilibrium is q_i^*, which satisfies $\tau(\alpha_i) - \delta p^* = q_i^* = \frac{I-1}{I}\left(\tau(\alpha_i) - \delta p^*_{-i}\right)$.[9] Therefore,the right hand side of (4) is bidder i's estimate of the stop-out price at an interior equilibrium.

Note that with two bidders, $I = 2$, the right hand side of (4) is $\tau(\alpha_i)/\delta$, which is the upper bound for the expected auction price whenever bidder i wins permits at the auction. In contrast with the many bidders case, when there are only two bidders, the expected abatement cost of the last unit the bidder wins at the auction (left side of (4)) is larger than the expected auction's price. Therefore, with I finite there is an strategic component in bidding, that disappears when $I \to \infty$.

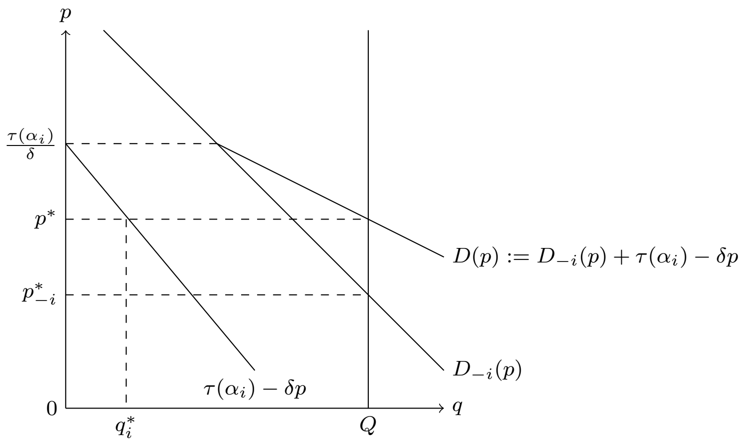


Fig. 1: The supply of permits is the vertical line at Q, and $D_{-i}(p)$ is the demand of all bidders but the i; the intersection determines the corresponding stop-out price, i p^*_{-i}. Bidder i's demand is $\tau(\alpha_i) - \delta p$. The aggregated demand is $D(p)$, and p^* the corresponding stop out price. Bidder i wins permits at the auction if and only if p^* satisfies $p^*_{-i} < p^* < \frac{\tau(\alpha_i)}{\delta}$. The quantity awarded to bidder i at the equilibrium is q_i^*.

Next, we focus on equilibria that are linear both in bidder's type and price, that is, such that the strategy conforming an equilibrium is linear in both arguments, α_i and p. Apart from its simplicity, linear strategies are natural candidates to conform an equilibrium as the marginal abatement cost function -which defines the valuation of the permits auctioned- is assumed to be linear. Moreover, our analysis will illustrate that if the marginal abatement cost and the strategies conforming the equilibria have similar shapes, there is a straightforward way to analyze efficiency.

The next Lemma characterizes necessary and sufficient conditions for a linear strategy (not necessarily conforming an equilibrium) to generate an interior allocation.

[9] Without loss of generality, in the Figure we have represented a case for which $\alpha_i < max_{j \neq i}(\alpha_j)$, but the graph is valid for all cases around p^*.

Lemma 1 *Consider an arbitrary linear strategy*

$$\gamma(\alpha_i, p) = \kappa_0 + \kappa_1 \alpha_i - \delta p$$

where κ_0, κ_1 and δ are constants. Assume that all bidders play γ. Under γ, each bidder demands a positive quantity at the stop-out price and the stop-out price is positive (with probability one) if and only if

$$I(\kappa_0 + \kappa_1 \underline{\alpha}) > Q > (I-1)\kappa_1 (\overline{\alpha} - \underline{\alpha}) \tag{6}$$

Furthermore, under the auction allocation, each bidder's marginal abatement cost is non-negative (with probability one) if and only if

$$\frac{\beta Q}{I} \leq \begin{cases} \underline{\alpha} & \text{if } \beta\kappa_1 \leq \frac{I}{I-1} \\ \overline{\alpha} - \beta\kappa_1 \frac{I-1}{I} (\overline{\alpha} - \underline{\alpha}) & \text{otherwise} \end{cases} \tag{7}$$

The first inequality in (6) gives the condition under which the stop-out price is strictly positive with probability one: if all bidders get the lowest possible signal, $\underline{\alpha}$, the aggregate quantity demanded at price 0, $I(\kappa_0 + \kappa_1 \underline{\alpha})$, has to be greater than Q. The second inequality gives the condition under which each bidder demands a positive quantity of permits at the stop-out price with probability one: the worst scenario is that all bidders but one get the highest signal, $\overline{\alpha}$, and one of them gets the lowest one, $\underline{\alpha}$; in that case, the last kink in the aggregate demand is at a price equal to the vertical intercept of the demand of the bidder with the lowest signal, $\frac{\kappa_0}{\delta} + \frac{\kappa_1}{\delta}\underline{\alpha}$, with a quantity demanded equal to $(I-1)\kappa_1 (\overline{\alpha} - \underline{\alpha})$. The second inequality in (6) states that Q has to be greater than that quantity. Note that it imposes that the dispersion of types, $\overline{\alpha} - \underline{\alpha}$ has to be *low enough.* Finally, condition (7) states that each bidder has a positive abatement cost if the number of permits auctioned is *low enough.*

Next, we characterize the set of linear and interior equilibria. We assume hereafter that the expected value of rivals' types conditional on the own observed type is linear. Therefore, for any two bidders, i and j, it is

$$E\{\tilde{\alpha}_j \mid \alpha_i\} = (1-\lambda)\, E\{\tilde{\alpha}_i\} + \lambda \alpha_i \tag{8}$$

where λ denotes the linear correlation between $\tilde{\alpha}_i$ and $\tilde{\alpha}_j$, which we assume to be common for any pair of types. In the sequel we assume $\lambda \in [0,1]$. Recall that, additionally, we assume equal marginal distribution of types.

The next Proposition shows that the set of interior equilibria conformed by linear strategies can be indexed by κ_1, the type's coefficient on the strategy's horizontal intercept: there is a one-to-one mapping from the values of κ_1 to the set of interior equilibria conformed by linear strategies. This is important for two reasons. First, regarding the positive properties of the equilibrium, its existence and uniqueness can be easily studied within the real line (as κ_1 is real-valued) instead of the more complex space of linear strategies. Second, there is a normative side: we will show that the efficiency of any equilibrium strategy depends only on κ_1.

Proposition 2 *There exists a unique pair of two real valued differentiable functions, (g_0, g_δ), such that a linear strategy with coefficients $(\kappa_0, \kappa_1, \delta)$, conforms an interior equilibrium if and only if $\kappa_0 = g_0(\kappa_1)$ and $\delta = g_\delta(\kappa_1)$. Moreover, g_δ is strictly increasing, with $g_\delta(0) = 0$.*

From Proposition 2, it is easy to prove that $\delta > 0$ for all $\lambda \in [0,1]$, i.e., the equilibrium strategy is downward sloping. Given the properties of g_δ, this implies that $\kappa_1 > 0$, i.e., the equilibrium strategy's horizontal intercept is increasing in α_i. Therefore, for any two firms i and j such that $\alpha_i > \alpha_j$, the demand of the less efficient firm, i, lies to the right of the demand of the more efficient firm, i.e., the former bids a higher price for each given amount of permits than his rival.

Note that the equilibrium strategies considered by Ausubel and Cramton (2002) and Ausubel et al (2014) correspond to $\kappa_1 = \delta$ and $\kappa_0 = 0$. Imposing $\kappa_1 = \delta$, $g_0(\kappa_1)$ from Proposition 2 implies $\kappa_0 = 0$, i.e., our equilibria includes theirs.[10]

The next Proposition analyzes the existence of equilibria conformed by linear strategies.

Proposition 3 *Assume that*

$$\overline{\alpha} - \underline{\alpha} < \frac{\beta Q}{I} < \underline{\alpha}. \tag{9}$$

Then there exists a non-empty interval, say (κ_1^l, κ_1^u), *with* $(\kappa_1^l, \kappa_1^u) \subset (0,\infty)$ *and a subinterval* $[0, \lambda^u) \subset [0,1)$, *such that, if* $\lambda \in [0, \lambda^u)$, *for any* $\kappa_1' \in (\kappa_1^l, \kappa_1^u)$ *the linear strategy with coefficients* $(\kappa_0, \kappa_1, \delta) = (g_0(\kappa_1'), \kappa_1', g_\delta(\kappa_1'))$ *constitutes an interior equilibrium, where* g_0 *and* g_δ *are defined in Proposition 2.*

Proposition 3 states a condition for the existence of interior equilibria: the range of types, $\overline{\alpha} - \underline{\alpha}$, has to be low in comparison with the infimum of Ω, $\underline{\alpha}$. Intuitively, if we allow for too different types across bidders, under a symmetric equilibrium it might occur that the bidder with the lowest type gets no permits. There is a simple conceptual reason to focus on interior equilibria, which is to limit the ex-ante heterogeneity across bidders. Equivalently, any such mechanism would (or at least should) produce rather obvious assignments. In the analysis that follows we will make use of (9) or stronger though qualitatively similar conditions.

Additionally, Proposition 3 states that even with linear strategies there is a continuum of equilibria, that is, there are multiple equilibria for the uniform auction. The values of κ_1 defining an equilibrium belong to an open interval in the real line, while the functions that map each value of κ_1 into the associated values of (κ_0, δ) are continuous, as stated in Proposition 2. It is important to notice that, in some sense, the equilibria are *close* to one another, since we are considering interior equilibria.

4 Efficiency

We use an *ex-post* efficiency concept, as in Ausubel et al (2014). Given a vector of types' realizations and a total amount of permits, Q, the efficient allocation of permits minimizes the total abatement cost among the I firms. The formal definition is presented next.

[10] In fact, imposing $\kappa_1 = \delta$ we obtain the unique equilibrium strategy in equation (9) of Ausubel et al (2014), allowing for different types.

Definition 3 Efficient allocation. Given a total amount of permits, Q, and a vector of types realizations, $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_I)$, an assignment of the Q permits among bidders, $\mathbf{q^o} = (q_1^o, \dots, q_I^o)$, is *ex post efficient* if it minimizes the total abatement cost:[11]

$$\mathbf{q^o} \equiv argmin_{\{q_1,\dots,q_I\}} \left\{ \sum_{i=1}^{I} \int_{q_i}^{e_i^*(\alpha_i)} \phi_i(e;\alpha_i)de \mid \sum_{i=1}^{I} q_i \leq Q \right\}$$

The definition of efficiency is contingent on Q, the quantity of permits auctioned: we do not define the efficient quantity of permits to be auctioned but take that quantity as exogenous and focus on its efficient distribution. This concept of efficiency is usually termed as *cost-effective* in environmental economics. Since in our model the marginal abatement cost for each firm and each type realization is strictly decreasing, that cost-minimizing or efficient allocation is unique.

We restrict the analysis to parameter values for which the cost-minimizing allocation is interior with probability one. Under an interior allocation, each firm buys permits at the auction and has positive abatement costs. If the efficient allocation is interior, marginal abatement costs are equalized across bidders, that is, $\phi(q_i^o;\alpha_i) = \phi(q_j^o;\alpha_j)$ for any i and j in $\{1,\dots,I\}$. The next Lemma characterizes the subspace of parameter values for which the efficient allocation is interior with probability one. We use the following terminology. An strategy is efficient if the permits allocation when all bidders play it is the efficient allocation with probability one. The Lemma also gives necessary and sufficient conditions for a strategy as in (3) to be efficient.

Lemma 2

1. *The efficient allocation is interior with probability one iff*

$$(I-1)(\overline{\alpha} - \underline{\alpha}) \leq \beta Q \leq I\underline{\alpha}$$

2. *Assume that the efficient allocation is interior and consider an strategy γ as in (3). Then γ is efficient if and only if τ is linear, $\tau(\alpha) = \kappa_0 + \kappa_1\alpha$, with $\beta\kappa_1 = 1$.*

Figure 2 shows the geometry of part 1 of Lemma 2 when there are two bidders, in the $\overline{q_1 q_2}$ plane. The feasible allocations of permits are delimited by the triangle $\{(0,0),(0,Q),(Q,0)\}$. The red lines are type-dependent. The dashed line $\alpha_1 - \beta q_1 = \alpha_2 - \beta q_2$, which represents allocations for which marginal abatement cost are equal across bidders, changes its intercept as the difference between types, $\alpha_2 - \alpha_1$, changes. If the types are *too* different, i.e., if $|\alpha_2 - \alpha_1|$ is *too big*, that line shifts *too much* either upwards or downwards so that the optimal allocation is to assign all permits to the highest type firm. Furthermore, if *too many* permits are available, that is, if Q is *too large*, the point A, which represents the *business as usual* emission level, lies inside the feasible triangle, so that the efficient allocation is A, and abatement cost are zero. Finally, if $|\alpha_2 - \alpha_1|$ and Q are both small enough, the efficient allocation, E, is an interior allocation: both bidders buy permits at the auction and have positive abatement costs. We arbitrarily have selected a realization of types such that $\alpha_2 - \alpha_1 < 0$, so that at the efficient allocation bidder 2 buys less permits than bidder 1.

[11] Since ex-post efficient is type-dependent, we should write $\mathbf{q^o}(\boldsymbol{\alpha})$ instead of $\mathbf{q^o}$, but we drop the argument for ease of exposition. For the same reason, we omit a non-negativity constraint which applies component-wise in $\mathbf{q^o}$.

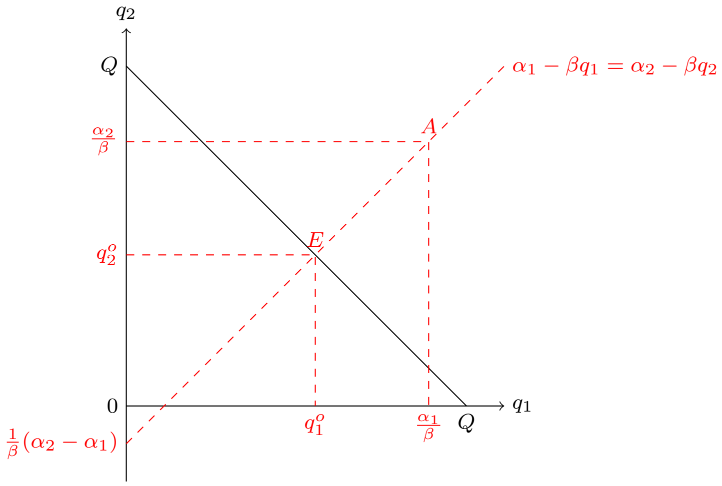
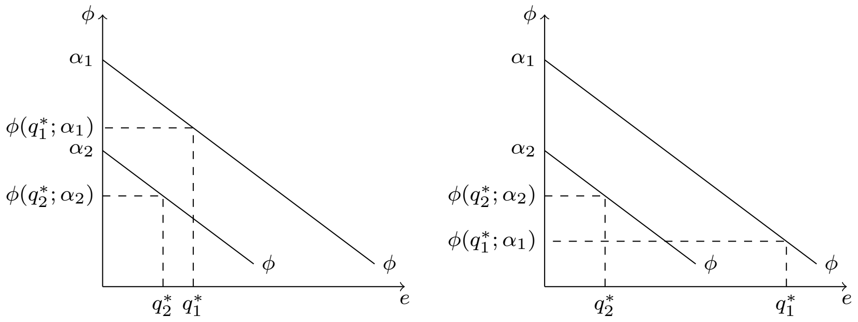


Fig. 2: Geometry of the efficient allocation for $I = 2$ in the $\overline{q_1 q_2}$ plane. The feasible allocations of permits are delimited by the triangle $\{(0,0),(0,Q),(Q,0)\}$. The red lines are type-dependent. The dashed line $\alpha_1 - \beta q_1 = \alpha_2 - \beta q_2$ changes its intercept as the difference $\alpha_2 - \alpha_1$ changes. If $|\alpha_2 - \alpha_1|$ is *big*, that line shifts *too much* either upwards or downwards so that the optimal allocation is to assign all permits to the highest type firm. Furthermore, if *too many* permits are available, that is, Q is too large, the point A, which corresponds to the *business as usual* emission level, lies inside the feasible triangle, so that the efficient allocation is A, under which the marginal abatement costs are zero. Finally, if $|\alpha_2 - \alpha_1|$ and Q are both small enough, the efficient allocation, E, is interior. We arbitrarily have selected a realization of types such that $\alpha_2 - \alpha_1 < 0$.

Part 2 of Lemma 2 states that ex-post efficient strategies are easily characterized in our model: κ_1, the type's coefficient in the strategy played by all bidders, has to be equal to $\frac{1}{\beta}$, the inverse of the slope of the marginal abatement cost function. To understand this condition, assume that all firms play an strategy γ as in Lemma 1, and that the corresponding equilibrium allocation is $(q_1^*, ..., q_I^*)$. Consider any two firms, i and j, such that $\alpha_i > \alpha_j$, i.e., such that firm i has higher marginal abatement cost for any emission level than his rival. If $\kappa_1 > 0$, the less efficient firm, firm i, bids more aggressively and thus gets more permits at the auction than his rival. Still, the less efficient firm might not obtain the efficient amount of permits. Given γ, the difference in marginal abatement costs among firms is

$$\phi(q_i^*; \alpha_i) - \phi(q_j^*; \alpha_j) = (1 - \beta\kappa_1)(\alpha_i - \alpha_j)$$

Clearly, equality of the marginal abatement costs is *ex-post* guaranteed only if κ_1 satisfies $1 - \beta\kappa_1 = 0$. Contrarily, if $\kappa_1 > 0$ and $1 - \beta\kappa_1 > 0$, the highest type firm has higher marginal abatement cost than his rival at the equilibrium. In other words, the highest type firm gets more permits that his rival, but fails to get enough permits as to equalize marginal abatement costs. We say that the less efficient firm is *under-assigned* in the auction with respect to the efficient allocation. Analogously, the case $\kappa_1 > 0$ and $1 - \beta\kappa_1 < 0$ leads to an *over-assignment* of the less efficient firm. These departures from the efficient strategy are depicted in Figure 3, for the case of two bidders. In the Figure, we represent in both panels the marginal abatement cost function, ϕ, for two bidders, 1 and 2, such that $\alpha_1 > \alpha_2$ and $q_1^* > q_2^*$, which necessarily rests on strategies (which are not plotted) with $\kappa_1 > 0$. However, the marginal abatement cost at the equilibrium are not equal across firms, as efficiency requires. Panel (a) represents a permit allocation and the corresponding marginal abatement costs when $1 - \beta\kappa_1 > 0$, and thus the less efficient firm (firm with type α_1) is *under-assigned*, so that $\phi(q_1^*; \alpha_1) > \phi(q_2^*; \alpha_2)$. Panel (b) represents a case in which the less efficient firm is *over-assigned*.



(a) $1 - \beta\kappa_1 > 0$. (b) $1 - \beta\kappa_1 < 0$.

Fig. 3: For $I = 2$, efficiency requires $\phi\left(q_1^*; \alpha_1\right) = \phi\left(q_2^*; \alpha_2\right)$. If both bidders play a strategy as in Lemma 1 (not plotted), efficiency requires $1 - \beta\kappa_1 = 0$. Panel (a) represents a permit allocation $\left(q_1^*, q_2^*\right)$ and the corresponding marginal abatement costs when $1 - \beta\kappa_1 > 0$, and thus the less efficient firm (firm with type α_1) is *under-assigned*, so that $\phi(q_1^*; \alpha_1) > \phi(q_2^*; , \alpha_2)$. Panel (b) represents a case in which the less efficient firm is *over-assigned*.

Next, we analyze equilibrium strategies. The two essential questions analyzed in this section are as follows. First, does the efficient strategy belong to the set of equilibria?[12] And second, if there other equilibria besides the efficient equilibrium, what sort of inefficiency characterize those equilibria?

As we use an ex-post concept of efficiency, the distribution of types is irrelevant for the efficient allocation. However, the distributional assumptions matter for the equilibria. If all bidders have the same type with probability one, with corresponds to $\lambda = 1$ in (8), under any symmetric equilibrium, all bidders submit the same demand function, permits are equally shared among bidders and, since types are identical, the allocation is efficient. In short, with $\lambda = 1$ any symmetric equilibrium is efficient. This is the case analyzed by (Ausubel et al, 2014). But things are not that simple if types vary across bidders. Subsection 4.1 analyzes the case of independent types, that is, $\lambda = 0$, and Subsection 4.2 considers an special case of positively correlated types, $\lambda = 1/2$.

4.1 Independent types

In this Subsection we consider the case of independent types, i.e, $\lambda = 0$ in (8). Using the standard terminology in auction theory, this is a pure *private value* case, as the valuations (marginal abatement costs in our model) are privately observed and independent across bidders. The basic idea to analyze efficiency follows from the previous Section. From Proposition 2, we know that the set of interior equilibria is indexed by κ_1, while from Lemma 2 we know that any linear strategy is characterized by $\beta\kappa_1 = 1$. Both results combined make the analysis of efficiency tractable, as it simply requires to find out the equilibrium values of a real valued parameter, κ_1. The next Proposition characterized interior equilibria.

Proposition 4 *Assume (9) and $\lambda = 0$. Then, there exists some κ_1^* satisfying $\kappa_1^* < \frac{1}{\beta}\frac{I}{I-1}$, such that a linear strategy constitutes an interior equilibrium if and only if $\kappa_1 \in \left(\kappa_1^*, \frac{1}{\beta}\frac{I}{I-1}\right)$. Furthermore, there*

[12] As mentioned above, we consider *ex-post* efficiency, that is, the efficient allocation depends on the realization of types. The analogous *ex-ante* concept leads to a trivial conclusion as any symmetric equilibrium is *ex-ante* efficient.

exists some finite value of I, say I^0, such that for any $I > I^0$: (i) κ_1^* increases as $E\{\tilde{\alpha}_i\}$, $E\{\tilde{\alpha}_i\} - \underline{\alpha}$ or Q increase (considering each of those variations separately), and (ii) the efficient equilibrium, characterized by $\beta\kappa_1 = 1$, is an interior equilibrium.

The basic message from Proposition 4 is that the set of (interior) equilibria is convex in the dimension of κ_1. In other words, the set of values of κ_1 defining an equilibrium is an open interval, which depends on parameters related to the types' distribution and the number of bidders, I. The next Proposition analyzes this latter dependence and defines efficient equilibria.

Proposition 5 *Assume $\lambda = 0$ and*

$$\frac{I}{I-1}\left(\overline{\alpha} - \underline{\alpha}\right) \leq \frac{\beta Q}{I-1} \leq \underline{\alpha} \tag{10}$$

Then:

1. *For all $I \geq 2$ the efficient strategy is an interior equilibrium strategy, given by*
$$\gamma_0^*(\alpha_i, p) = \frac{1}{\beta}\left(E\{\tilde{\alpha}_i\} - \frac{1}{I-1}\beta Q + \alpha_i - 2p\right)$$
For any bidder i and any $\alpha_i \in \Omega$, it satisfies
$$E\{p^* \mid \alpha_i\} \leq E\{\phi(q_i^*, \alpha_i) \mid \alpha_i\} \tag{11}$$
where p^* is the auction's stop-out price and q_i^* is bidder i's allocation of permits.
2. *Consider $I \to \infty$ and $Q \to \infty$ while $\frac{Q}{I}$ remains constant. Then the unique interior equilibrium is the efficient strategy.*
3. *For the case $I = 2$, there exists some $b^0 \in (0, 1)$ such that any κ_1 with $\beta\kappa_1 \in (b^0, 2)$ defines an equilibrium strategy. Additionally, in all equilibria, for any bidder i and any $\alpha_i \in \Omega$, (11) is satisfied.*

Part 1 of Proposition 5 presents the efficient strategy, that is an equilibrium strategy for all $I \geq 2$. From Proposition 2, given the value for κ_1 that characterizes the efficient strategy, $\beta\kappa_1 = 1$, there exists unique values for (κ_0, δ) that define the corresponding equilibrium strategy. The comparative statics properties of the efficient equilibrium strategy are straightforward. Under (10), for each possible type's realization, the relative position of the efficient equilibrium strategy, γ, and the marginal abatement cost function, ϕ, in the usual price-quantity axis, is as depicted in Figure 4: the efficient strategy has a lower vertical intercept and a higher horizontal intercept than the marginal abatement cost function. I.e., given that bidders are restricted to bid a linear strategy, the efficient equilibrium strategy is such that bid shading is decreasing as q increases, and is negative (bidders bid more that their valuation for those units) for q large enough.

Even if bidders bid higher than their valuation for some units, for all $I \geq 2$, at the efficient equilibrium, bidders expect to have a positive surplus: the expected auction's stop-out price is not greater than the marginal abatement cost of the last permit bought at the auction, as stated on condition (11). This is illustrated on Figure 4, for the realized stop-out price, p^*, and marginal abatement cost of the last unit won, $\phi(q_i^*; \alpha_i)$. In the Figure, given p^*, γ determines the assignment at the auction, q_i^*, which in turn determines the marginal saving in the abatement cost of the last unit won at the auction, $\phi(q_i^*; \alpha_i)$.

According to (2), the bidder's total cost is the payment in the auction (blue area) plus the abatement cost, the area below ϕ from his assignment of permits, q_i^*, up to $e^*(\alpha_i)$ (orange). If the firm had no permits, his total cost would be the whole area below ϕ from zero to $e^*(\alpha_i)$. Thus, his realized surplus from participating in the auction is the green area. Notice that $p^* \leq \phi(q_i^*, \alpha_i)$ implies a positive surplus: under the uniform auction format, a sufficient condition to have a positive surplus is that the auction's stop-out price, p^*, is not larger than the marginal saving in the abatement cost of the last unit won at the auction, $\phi(q_i^*, \alpha_i)$.

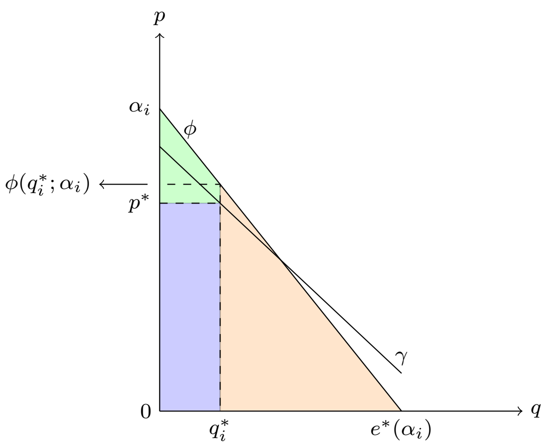


Fig. 4: An arbitrary linear strategy, γ, and the marginal abatement cost function, ϕ, are represented given the firm's type, α_i. The auction's stop-out price, p^*, and γ determine the assignment in the auction, q_i^*, which in turn determines the marginal saving in the abatement cost of the last unit won at the auction, $\phi(q_i^*;\alpha_i)$. The bidder total cost is the payment in the auction (blue area) plus the abatement cost, the area below ϕ from his assignment of permits up to $e^*(\alpha_i)$ (orange). If the firm had no permits, his total cost would be the whole area below ϕ from zero to $e^*(\alpha_i)$. Thus, his realized surplus from the auction is the green area. Notice that $p^* \leq \phi(q_i^*, \alpha_i)$ implies a positive surplus.

Part 2 of Proposition 5 states an interesting property of the set of interior equilibria. As $I \to \infty$ keeping the ratio Q/I constant, the *only* interior equilibrium is the efficient equilibrium: in a private value model, an increase in the number of bidders drives the auction toward the efficient equilibrium. This is not the case for I finite, as we illustrate on part 3 of Proposition 5, considering the case $I = 2$. With only two bidders, there are inefficient equilibria. In terms of the notation previously introduced, under some of those inefficient equilibria the firm with highest type is under-assigned while under some other equilibria it is over-assigned. In all of those equilibria condition (11) holds, i.e., bidders expect to have a positive surplus when participating in the auction.

4.2 Correlated types: mineral right model

In this Subsection we consider a specific probabilistic structure that implies private values with positively correlated types. The basic idea is to split the bidders' type into a common term plus a bidder specific term, so that the correlation among different types arises from the common term. This corresponds to the *mineral rights model*, using auction theory terminology.[13] In addition, we specify a probability distributions for both terms so that the marginal distribution is identical across types, has finite support and the conditional expectation of types is linear, as in (8).

Specifically, we assume that for all i

$$\tilde{\alpha}_i = \theta + \tilde{a} + \tilde{u}_i \tag{12}$$

where θ is a parameter, and $\{\tilde{a}, \tilde{u}_1, \dots, \tilde{u}_I\}$ is a set of globally independent and identically distributed zero-mean random variables. With this specification, $\tilde{a}$ is the common term for all types while the u's are bidder specific. Moreover, we assume that all the random variables are uniformly distributed in some finite interval $[-\sigma, \sigma]$, with σ fixed, which implies that the marginal distribution is identical across types, with the support of $\tilde{\alpha}_i$ being $[\theta - 2\sigma, \theta + 2\sigma]$. Straightforward calculation shows that the unconditional expectation is given by $E\{\tilde{\alpha}_i\} = \theta$, and, additionally,

$$E\{\tilde{\alpha}_j \mid \alpha_i\} = \frac{1}{2}(\theta + \alpha_i) \tag{13}$$

i.e, the expectation of the rivals' type conditional on the own type is linear, as required in our model. A direct comparison between (8) and (13) shows that the latter conveys $\lambda = \frac{1}{2}$. For our purposes, any probabilistic structure leading to the same correlation value among types is essentially equivalent to the one presented here.

Next, we analyze efficiency and equilibria when $\lambda = 1/2$. We focus in the comparison to the independent case presented on the previous Subsection. The next Proposition summarizes our main results.

Proposition 6 *Assume (10),* $\lambda = 1/2$ *and*

$$\frac{2}{I-1}\beta Q \leq E\{\tilde{\alpha}_i\} \qquad 2(\overline{\alpha} - \underline{\alpha}) \leq \frac{\beta Q}{I} \leq 2\underline{\alpha} - \overline{\alpha} \tag{14}$$

Then:

1. *For all* $I \geq 2$ *the efficient strategy is an interior equilibrium strategy, given by*
$$\gamma^*_{1/2}(\alpha_i, p) = \frac{1}{\beta}\left(\frac{1}{I+1}E\{\tilde{\alpha}_i\} - \frac{2}{(I+1)(I-1)}\beta Q + \alpha_i - \frac{I+2}{I+1}p\right)$$
For any bidder i *and any* $\alpha_i \in \Omega$*, it satisfies (11).*
2. *Consider* $I \to \infty$ *and* $Q \to \infty$ *while* $\frac{Q}{I}$ *remains constant. Then there exists some* $b^1 \in (0,1)$ *such that* κ_1 *defines an equilibrium strategy if and only if* $\beta\kappa_1 \in (b^1, 2)$*. Additionally, in all equilibria, for any* i *and* $\alpha_i \in \Omega$*, (11) is satisfied.*

Similar to part 1 of Proposition 5, part 1 of Proposition 6 presents the efficient strategy, that is an equilibrium strategy for all $I \geq 2$. As it is the case for $\lambda = 0$, in the efficient equilibria condition (11) holds, and bidders expect to have a positive surplus from participating in the auction.

[13] Usually, a mineral right model structure is used to correlate signals among bidders in a common value model, see Krishna (2009). While our model is not a common value model, an analogous structure is used to correlate types.

Part 2 of Proposition 6 states that, in contrast to the case of private values, there are many equilibria besides the efficient equilibrium when types are correlated and I and $Q \to \infty$, so that the ratio Q/I stays constant. In all of them condition (11) holds, and bidders expect to have a positive surplus from participating in the auction. This is one of the main results of our analysis: the existence of many bidders is not a sufficient condition to guarantee an efficient equilibrium in the uniform auction. Additionally, bidders' types have to be uncorrelated. The next Corollary presents an example of an equilibrium that is not efficient when $I \to \infty$ and $\lambda = \frac{1}{2}$.

Corollary 1 *Assume $I \to \infty$ and $Q \to \infty$ while $\frac{Q}{I}$ remains constant, and $\lambda = \frac{1}{2}$. An interior equilibrium is characterized by $\beta\kappa_1 = \frac{3}{2}$. The equilibrium strategy is*

$$\gamma_{1/2}(\alpha_i, p) = \frac{3}{\beta}\left(\frac{E\{\alpha_i\} + \alpha_i}{2} - p\right) \tag{15}$$

The stop-out price and the equilibrium allocation for bidder i are, respectively, $p^* = E\{\tilde{\alpha}_i\} - \frac{\beta Q}{3I}$ and $q^* = \frac{3}{2\beta}(\alpha_i - E\{\tilde{\alpha}_i\}) + \frac{Q}{I}$.

In the equilibrium presented in Corollary 1, all bidders with types α_i such that $\alpha_i < E\{\tilde{\alpha}_i\}$ bid more than their valuations for all units, and even if they expect to have a positive surplus participating in the auction, their realized surplus is negative. Only bidders with high types end up having a positive surplus, even if they are over-assigned, given that $1 - \beta\kappa_1 < 0$.

Finally, to conclude the analysis, the next Corollary compares efficient equilibrium strategies as I and $Q \to \infty$ so that the ratio Q/I stays constant for $\lambda = 0$ and $\lambda = 1/2$.

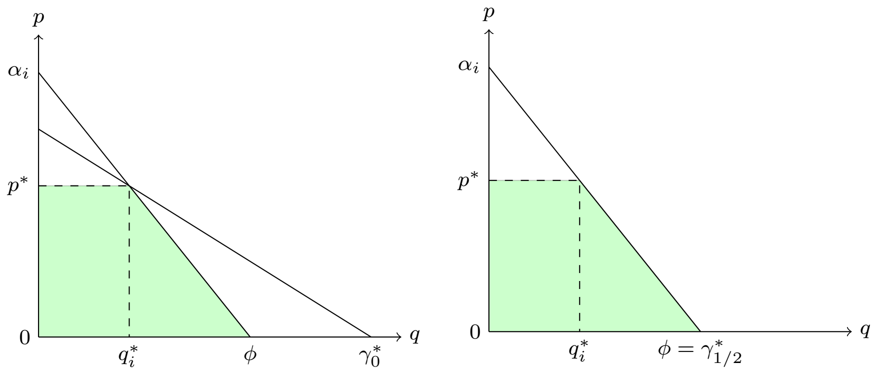
Corollary 2 *Assume $I \to \infty$ and $Q \to \infty$ while $\frac{Q}{I}$ remains constant. Denote by γ^*_λ for $\lambda \in \{0, \frac{1}{2}\}$ the efficient equilibrium strategy at the limiting value of I when the correlation between types is λ. It is*

$$\gamma^*_0(\alpha_i, p) = \frac{1}{\beta}\left(E\{\alpha_i\} - \frac{\beta Q}{I} + \alpha_i - 2p\right) \qquad \gamma^*_{1/2}(\alpha_i, p) = \frac{1}{\beta}(\alpha_i - p) \tag{16}$$

For $\lambda \in \{0, \frac{1}{2}\}$ the stop-out price and the equilibrium allocation for bidder i are, respectively, $p^* = E\{\tilde{\alpha}_i\} - \frac{\beta Q}{I}$ and $q^* = \frac{1}{\beta}(\alpha_i - E\{\tilde{\alpha}_i\}) + \frac{Q}{I}$.

Figure 5 shows the relative position of the efficient equilibrium strategy, γ_λ, for $\lambda = 0$ on panel (a), and for $\lambda = \frac{1}{2}$ on panel (b), and the marginal abatement cost function, $\phi(q; \alpha_i)$, common to both panels, as $I \to \infty$. Also common to both panels are the stop-out price, p^* and the equilibrium allocation for bidder i, q^*_i. When types are uncorrelated, $\lambda = 0$, the unique interior equilibrium is such that bidders bid lower than their marginal abatement cost for that particular unit for all quantities lower than q^*_i, and bid higher than their marginal abatement cost for that particular unit for all quantities greater than q^*_i. In contrast, when types are correlated, $\lambda = \frac{1}{2}$, in the efficient equilibrium bidders bid their marginal abatement cost for all units, and the equilibrium strategy is the inverse of the marginal abatement cost function. For both cases, uncorrelated and correlated types, bidder i's total cost is the same, the green area in the Figure, equal to the auction's payment, $p^* q^*_i$, plus the abatement cost.

From the expression for the efficient allocation in Corollary 2, note that bidders whose type, α_i, is greater than the *a priori* expected value of types, $E\{\tilde{\alpha}_i\}$, get more permits at the auction than they would get if permits were equally shared among bidders, $\frac{Q}{I}$, while bidder whose type is lower than the *a priori* expected value of types, get less.



(a) Uncorrelated types, $\lambda = 0$ (b) Correlated types, $\lambda = 1/2$

Fig. 5: The Figure shows the relative position of the efficient equilibrium strategy, γ_λ^*, for $\lambda = 0$ on panel (a), and for $\lambda = \frac{1}{2}$ on panel (b), and the marginal abatement cost function, $\phi(q; \alpha_i)$, common to both panels, as $I \to \infty$. The auction stop-out price is p^* and the equilibrium allocation for bidder i is q_i^*. The bidder's total cost is the green area, equal to the auction's payment, $p^* q_i^*$, plus the abatement cost.

5 Concluding remarks

Two final comments are in order. First, the concept of efficiency that we have used is taken from Ausubel et al (2014), and it is standard in the environmental economics literature: it follows from minimizing the sum of firms' abatement costs. However, we might have been more general by considering Pareto efficiency. An allocation of permits is Pareto efficient if it is not Pareto dominated, in the sense that it is impossible to reduce one firm's abatement cost without increasing at least other firm's. A Pareto efficient allocation minimizes the weighted sum of abatement costs for some given vector of weights. In our model, it is straightforward to show that an allocation is not Pareto dominated if, under it, the marginal abatement cost for each firm is strictly positive. Thus, any equilibrium strategy that satisfies condition (7) in Lemma 1, that guarantees that each bidder's marginal abatement cost is non-negative, leads to an allocation of permits that is Pareto efficient, although, of course, it is not necessarily *the* efficient allocation that we have considered in our analysis.

Second, once that we accept the concept of efficiency used in this paper, our analysis points to an *ideal* situation for the uniform auction to be efficient: many bidders with independent types. However, this is not necessarily a positive assessment for the uniform auction. Perhaps, the question is not whether (or when) the uniform auction does *good* but whether it does *better*. For instance, it remains to analyze how other auction-based mechanisms are expected to perform in a market fundamentally characterized by a large number of relatively similar participants with independent types and a rigid initial supply of permits. We claim that further research in this area would help to isolate the differential contribution of the uniform auction as an allocation mechanism of emission permits.

6 Proofs

The following notation is convenient for several of the proofs, while it is omitted from the main text to ease the overall exposition.

$$\mu := (1-\lambda)E\{\tilde{\alpha}_i\} \tag{17}$$

And (8) can be rewritten as

$$E\{\tilde{\alpha}_j \mid \alpha_i\} = \mu + \lambda\alpha_i$$

Proof **of Proposition 1.** Assume that all bidders but i follow some -and the same- linear strategy, as in (3). The residual supply for bidder i at price p is

$$S_{-i}(p) = Q - \sum_{j\neq i}\tau(\alpha_j) + (I-1)\delta p \tag{18}$$

where $\sum_{j\neq i}\tau(\alpha_j)$, depends on the types of all bidders but the i. For an arbitrary stop-out price p, bidder i's realized cost is $C(S_{-i}(p),\alpha_i)$. After some algebra from (1), (2) and (18), we have

$$\begin{aligned} E\{C(S_{-i}(p),\alpha_i) \mid \boldsymbol{\gamma}_{-i}\} &= \left(-\alpha_i(I-1)\delta + (Q-\hat{\rho}(\alpha_i))(1+\beta(I-1)\delta)\right)p \\ &\quad + \left(\frac{1}{2}\beta(I-1)\delta + 1\right)(I-1)\delta p^2 + \theta \end{aligned} \tag{19}$$

where θ contains terms that do not depend on p and $\hat{\rho}(\alpha_i) := E\{\sum_{j\neq i}\tau(\alpha_j) \mid \alpha_i\}$. It is

$$\theta = \alpha_i + (Q-\hat{\rho}(\alpha_i)) - \frac{\beta}{2}\left(e^* + (Q-\hat{\rho}(\alpha_i))^2\right) \tag{20}$$

Under the uniform format, bidder i's cost depends only on the stop-out price and on the quantity demanded at that price and, given a residual supply, there is a one-to-one mapping between that price and that quantity. Thus, given a residual supply and α_i, choosing the stop-out price is equivalent to choosing the quantity demanded at that price. Given α_i and the rivals' strategy, the stop-out price that minimizes bidder i's expected cost is

$$\min_p E\{C(S_{-i}(p),\alpha_i) \mid \boldsymbol{\gamma}_{-i}\}$$

Denote by $p^*(\alpha_i,\boldsymbol{\gamma}_{-i})$ the solution to that problem. From (19), first order conditions of bidder's i minimization problem imply that

$$p^*(\alpha_i,\boldsymbol{\gamma}_{-i}) = \frac{\alpha_i(I-1)\delta - (Q-\hat{\rho}(\alpha_i))(1+\beta(I-1)\delta)}{(2+\beta(I-1)\delta)(I-1)\delta} \tag{21}$$

Assuming that all bidders (including i) play γ defined in (3), bidder i has no incentives to deviate if and only if, for each $\alpha_i \in \Omega$, the expected stop out price when all bidders are playing γ, conditional on α_i, is precisely $p^*(\alpha_i,\boldsymbol{\gamma}_{-i})$.

On the other hand, if all bidders follow γ defined in (3), the stop-out price can be characterized as the solution in p to

$$S_{-i}(p) = \tau(\alpha_i) - \delta p$$

Solving this latter equation in p, denoting the solution by p_o, and taking expectations conditional on α_i, we get

$$E\{p_o \mid \alpha_i, \gamma\} = \frac{1}{\delta I}(\hat{\rho}(\alpha_i) + \tau(\alpha_i) - Q) \tag{22}$$

Thus, γ is an equilibrium if and only if (τ, δ) satisfy

$$p^*(\alpha_i, \boldsymbol{\gamma}_{-i}) = E\{p_o \mid \alpha_i, \gamma\} \qquad \forall \alpha_i \in \Omega \tag{23}$$

Using (21) and (22), we can re-write (23) as

$$\alpha_i = \frac{\xi}{I} \times \frac{\tau(\alpha_i)}{\delta} + \left(1 - \frac{\xi}{I}\right) \times \frac{1}{I-1}\frac{1}{\delta}((\hat{\rho}(\alpha_i) - Q) \tag{24}$$

where $\xi := 2 + \beta(I-1)\delta$. Consider the equation $S_{-i}(p) = 0$, where $S_{-i}(p)$ is given by (18), and denote its solution by p_{-i}. It is

$$E\{p_{-i} \mid \gamma,\ \alpha_i\} = \frac{1}{I-1}\frac{1}{\delta}((\hat{\rho}(\alpha_i) - Q)$$

which substituted in (24) leads to

$$\alpha_i = \frac{\xi}{I} \times \frac{\tau(\alpha_i)}{\delta} + \left(1 - \frac{\xi}{I}\right) \times E\{p_{-i} \mid \gamma,\ \alpha_i\}$$

This latter equality can be easily rewritten as in the statement in the Proposition. □

Proof **of Lemma 1.** Assuming that all players demand some positive quantity at the stop-out price, the market clearing condition is

$$\sum_i \gamma(\alpha_i, p) = Q \iff p = \frac{1}{\delta}\left(\kappa_0 + \kappa_1 \frac{1}{I}\sum_i \alpha_i - \frac{Q}{I}\right)$$

where the last equality characterizes the auction's stop-out price. Recall that the support of $\tilde{\alpha}_i$ is $[\underline{\alpha}, \overline{\alpha}]$. The auction's stop-out price is positive with probability 1 if and only if[14]

$$\kappa_0 + \kappa_1 \underline{\alpha} - \frac{Q}{I} > 0 \tag{25}$$

The quantity demanded by bidder i at the stop-out price is

$$\begin{aligned}\gamma_i(\alpha_i, p) &= \kappa_0 + \kappa_1 \alpha_i - \left(\kappa_0 + \kappa_1 \frac{1}{I}\sum_i \alpha_i - \frac{Q}{I}\right) \\ &= \kappa_1\left(\alpha_i - \frac{1}{I}\sum_i \alpha_i\right) + \frac{Q}{I} \\ &= \frac{I-1}{I}\kappa_1\left(\alpha_i - \frac{1}{I-1}\sum_{j\neq i}\alpha_j\right) + \frac{Q}{I}\end{aligned}$$

Thus

$$\gamma_i > 0 \iff (I-1)\kappa_1\left(\alpha_i - \frac{1}{I-1}\sum_{j\neq i}\alpha_j\right) + Q > 0$$

[14] Note that the sop-out price is, a priori, a random variable that depends on the realization of bidders' types. Positive with probability 1 means that it is positive for all the possible realizations of the bidders' types.

The latter equality holds with probability 1 iff

$$(I-1)\kappa_1(\underline{\alpha}-\overline{\alpha})+Q>0 \tag{26}$$

Combining (25) and (26) we obtain (6).

Next, we give conditions for a non-negative marginal abatement cost. It is

$$\phi(e;\alpha_i)\geq 0 \iff \alpha_i-\beta(\kappa_0+\kappa_1\alpha_i-\delta p)\geq 0$$

Substituting p from the market clearing condition for interior solution and collecting terms in α's, the latter inequality becomes

$$\left(1-\beta\kappa_1\left(1-\frac{1}{I}\right)\right)\alpha_i+\beta\kappa_1\frac{1}{I}\sum_{j\neq i}\alpha_j\geq\frac{\beta Q}{I}$$

The most adverse case on the second term on the left is $\alpha_j=\underline{\alpha}$ for all $j\neq i$, so that the inequality becomes

$$\left(1-\beta\kappa_1\left(1-\frac{1}{I}\right)\right)\alpha_i+\beta\kappa_1\frac{I-1}{I}\underline{\alpha}\geq\frac{\beta Q}{I} \tag{27}$$

The coefficient of α_i in (27) is non-negative if and only if

$$\beta\kappa_1\leq\frac{I}{I-1} \tag{28}$$

Thus, if (28) holds, the most adverse case of α_i for inequality (27) is $\alpha_i=\underline{\alpha}$. Substituting this value of α_i, the inequality (27) becomes

$$\underline{\alpha}\geq\frac{\beta Q}{I}$$

On the other hand, if the inequality in (28) does not hold, the most adverse case of α_i for inequality (27) is $\alpha_i=\overline{\alpha}$. Substituting this value of α_i, the inequality (27) becomes

$$\overline{\alpha}-\beta\kappa_1\frac{I-1}{I}(\overline{\alpha}-\underline{\alpha})\geq\frac{\beta Q}{I}$$

Inequality (7) summarizes these latter two inequalities. □

Proof **of Proposition 2.** Assume that τ is linear, i.e., we restrict to equilibria in which γ is linear in both arguments, α_i and p:

$$\tau(\alpha)=\kappa_0+\kappa_1\alpha \tag{29}$$

Using (29), (8) and (17), we can write

$$\hat{\rho}(\alpha_i)=(I-1)(\kappa_0+\kappa_1\mu)+(I-1)\kappa_1\lambda\alpha_i$$

In turn, with the above expression for $\hat{\rho}(\alpha_i)$, both sides of (23) are linear on α_i. Specifically, substituting in (21) we have

$$\begin{aligned}p^*(\alpha_i,\boldsymbol{\gamma}_{-i})=&\frac{1}{2+\beta(I-1)\delta}\left(1+(1+\beta(I-1)\delta)\frac{\lambda\kappa_1}{\delta}\right)\alpha_i\\&+\frac{1+\beta(I-1)\delta}{(2+\beta(I-1)\delta)(I-1)\delta}\left((I-1)(\kappa_0+\kappa_1\mu)-Q\right)\end{aligned}$$

and from (22)

$$E\{p_o \mid \alpha_i, \gamma\} = \frac{1}{I\delta}(\lambda(I-1)+1)\kappa_1\alpha_i + \frac{1}{I\delta}((I-1)(\kappa_0+\kappa_1\mu)+\kappa_0-Q)$$

Using these expressions, from (23), the coefficients of α_i and the intercept on both expressions have to be equal. Equalizing the coefficients of α_i we obtain the following equation

$$\frac{1}{\beta(I-1)} - \left(\frac{2}{\beta(I-1)} + \lambda\kappa_1\right)\frac{1}{\xi} = \frac{1}{I}(1-\lambda)\kappa_1 \tag{30}$$

where $\xi := 2+\beta(I-1)\delta$, as in the proof of Proposition 1. Notice that there is a one to one mapping between ξ (or ξ^{-1}) and δ. Equalizing the intercepts (the terms that do not depend on α_i), we obtain the following equation

$$\left(\frac{1}{I-1}Q - (\kappa_0+\kappa_1\mu)\right)\frac{1}{\xi} + \frac{1}{I}\kappa_1\mu = \frac{1}{I(I-1)}Q \tag{31}$$

Equations (30) and (31) characterize the equilibrium parameters for any linear strategy γ. The unknowns are κ_0, κ_1 and ξ^{-1}.

From (30), we have that

$$\frac{1}{\xi} = m(\kappa_1) \tag{32}$$

where we have denoted

$$m(\kappa_1) := \frac{I-\beta(I-1)(1-\lambda)\kappa_1}{2I+\beta(I-1)I\lambda\kappa_1} \tag{33}$$

It follows that

$$\delta = g_\delta(\kappa_1) = \left(\frac{1}{m(\kappa_1)} - 2\right)\frac{1}{\beta(I-1)} \tag{34}$$

Clearly, m is differentiable. Straightforward computations show that $m'(\kappa_1) < 0$ if $\lambda \geq 0$ and $m(0) = \frac{1}{2}$. The properties of g_δ follow from the properties of m.

Substituting (32) into (31) and solving for κ_0, we get

$$\kappa_0 = g_0(\kappa_1) = \left(1 - \frac{1}{Im(\kappa_1)}\right)\left(\frac{1}{I-1}Q - \kappa_1\mu\right) \tag{35}$$

The right hand side of equation (35) defines g_0. □

Proof **of Proposition 3.** Consider the first inequality in (6), substitute the expression for κ_0 given in (35) and re-order terms to obtain

$$\left(1-\frac{1}{m(\kappa_1)}\right)\frac{1}{I-1}Q + \left(\underline{\alpha} + \mu\left(\frac{1}{Im(\kappa_1)} - 1\right)\right)I\kappa_1 \geq 0 \tag{36}$$

Step 1. We show that (36) neither holds for $\kappa_1 = 0$ nor for $\kappa_1 \to \infty$. If $\kappa_1 = 0$, using that $m(0) = 1/2$, (36) collapses to

$$-\frac{1}{I-1}Q \geq 0$$

which does not hold. Furthermore, as $\kappa_1 \to \infty$, we have $m(\kappa_1) \to -\frac{1-\lambda}{I\lambda}$. For $\lambda \neq 1$, the sign of the left hand side in (36) as $\kappa_1 \to \infty$ is given by the coefficient of κ_1 with $m(\kappa_1)$ at its limiting value:

$$\underline{\alpha} - \mu \frac{1}{1-\lambda} = \underline{\alpha} - E\{\tilde{\alpha}\} \tag{37}$$

where we have used (17) . The expression on the right hand side of (37) is clearly negative.

Step 2. Consider κ_1^p, defined by $\beta\kappa_1(I-1) = I$. It is

$$m\left(\kappa_1^p\right) = \frac{\lambda}{2+\lambda I}$$

For $\kappa_1 = \kappa_1^p$, taking limits in (36) as $\lambda \to 0$ we have

$$-\frac{1}{I-1}Q + \mu\kappa_1^p \geq 0$$

where $\mu \to E\{\tilde{\alpha}_i\}$ as $\lambda \to 0$. Using the expression for κ_1^p, the latter inequality becomes

$$\frac{\beta Q}{I} \leq E\{\tilde{\alpha}_i\} \tag{38}$$

Now consider the second inequality in (6) for $\kappa_1 = \kappa_1^p$. It is

$$(I-1)\kappa_1^p(\overline{\alpha} - \underline{\alpha}) \leq Q \iff \overline{\alpha} - \underline{\alpha} \leq \frac{\beta Q}{I} \tag{39}$$

Finally, condition (7) for $\kappa_1 = \kappa_1^p$ becomes

$$\frac{\beta Q}{I} \leq \underline{\alpha} \tag{40}$$

which is more restrictive than (38). Combining (39) and (40) we obtain (9). Using a continuity argument, it follows the existence of an interval containing κ_1^p and some non-empty interval in $[0, \lambda^u)$. □

Proof **of Lemma 2.**

1. We solve the problem that characterizes the efficient allocation for a given vector of type realizations, say $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_I)$. We define the lagrangian

$$\mathcal{L} = \sum_{i=1}^{I} \int_{q_i}^{e_i^*(\alpha_i)} \phi(e; \alpha_i) de + \theta \left(\sum_{i=1}^{I} q_i - Q \right)$$

where non-negativity constraints are omitted for sake of simplicity and θ denotes the multiplier. The Kuhn-Tucker conditions are

$$\phi(q_i; \alpha_i) = \theta \qquad i \in \{1, \ldots, I\} \tag{41}$$

$$\theta \left(\sum_{i=1}^{I} q_i - Q \right) = 0 \qquad \theta \geq 0$$

An interior allocation occurs when $\phi(q_i; \alpha_i) > 0$ holds, which implies $\theta > 0$ and thus

$$\sum_{i=1}^{I} q_i - Q = 0 \tag{42}$$

Next, we solve in q's the set of equations given by (41) and (42). These equations conform a system of linear equations which can be solved using standard linear algebra. Alternatively, consider any $i \neq 1$ and use (41) to write

$$q_1 - q_i = \frac{1}{\beta}(\alpha_1 - \alpha_i) \tag{43}$$

Substituting in (42) for any $i \neq 1$ and then solving (42) for q_1 we have

$$q_1 = \frac{Q}{I} - \frac{1}{\beta I}\left(\sum_{i \neq 1} \alpha_i - (I-1)\alpha_1\right)$$

Substituting back in (43) and solving for q_i we obtain

$$q_i = \frac{Q}{I} - \frac{1}{\beta I}\left(\sum_{j \neq i} \alpha_j - (I-1)\alpha_i\right) \tag{44}$$

for any $i \in \{1, \ldots, I\}$. Using (44), the non-negativity requirement can be writen

$$q_i \geq 0 \iff Q \geq \frac{1}{\beta}\left(\sum_{j \neq i} \alpha_j - (I-1)\alpha_i\right)$$

Considering the most adverse realizations, the latter inequality becomes

$$Q \geq \frac{1}{\beta}(I-1)\left(\overline{\alpha} - \underline{\alpha}\right) \tag{45}$$

Analogously, using (44), the non-negativity of the marginal abatement cost (equivalently, the condition for $\theta > 0$) can be writen

$$\phi(q_i; \alpha_i) \geq 0 \iff Q \leq \frac{1}{\beta}\sum_i \alpha_i$$

Considering the most adverse realizations, the latter inequality becomes

$$Q \leq \frac{1}{\beta} I \underline{\alpha} \tag{46}$$

This part of the Lemma follows trivially from the combination of (45) and (46).

2. If the efficient allocation is interior, from (41), for any $i, j \in \{1, \ldots, I\}$, it is

$$\alpha_i - \alpha_j = \beta(q_i - q_j)$$

Now assume that q's in this latter equality come from an equilibrium in the auction in which all bidders play a strategy as in (3) and that leads to an interior allocation, that is, for any $h \in \{1, \ldots, I\}$ it is

$$q_h = \tau(\alpha_h) - \delta p^*$$

where p^* is the stop-out price in the auction. Combining the previous two equalities, we have

$$\frac{1}{\beta} = \frac{\tau(\alpha_i) - \tau(\alpha_j)}{\alpha_i - \alpha_j}$$

This latter equality holds for any arbitrary pair of realizations α_i and α_j in $[\underline{\alpha}, \overline{\alpha}]$ if and only if it is $\tau(\alpha) = \kappa_0 + \beta^{-1}\alpha$, where κ_0 is an arbitrary parameter. □

Proof **of Proposition 4.** Write the inequalities that characterize interior equilibria for $\lambda = 0$. First, take $\lambda = 0$ in (33) and substitute into (36) to obtain

$$\left(1 - \frac{2}{1-x}\right)\frac{\beta Q}{I} + \left(I\underline{\alpha} + \mu\left(\frac{2}{1-x} - I\right)\right)x \geq 0 \tag{47}$$

where we have denoted $x := \beta\frac{I-1}{I}\kappa_1$. From Proposition 2, interior equilibria are indexed by κ_1, so they are equivalently indexed by x. Recall that this latter inequality is the first inequality in (6). Using the definition of x, the second inequality in (6) becomes:

$$\frac{\beta Q}{I} \geq x(\overline{\alpha} - \underline{\alpha}) \tag{48}$$

Finally, the condition (7) can be written

$$\frac{\beta Q}{I} \leq \begin{cases} \underline{\alpha} & \text{if } x \leq 1 \\ \overline{\alpha} - x\left(\overline{\alpha} - \underline{\alpha}\right) & \text{otherwise} \end{cases} \tag{49}$$

Thus, for $\lambda = 0$, x defines an interior equilibrium if and only if it satisfies (47) to (49).

Rewrite (47) as

$$\frac{\beta Q}{I} + I(\underline{\alpha} - \mu)x + \left(\mu x - \frac{\beta Q}{I}\right)\frac{2}{1-x} \geq 0 \tag{50}$$

where we must recall that $x := \beta\frac{I-1}{I}\kappa_1$. Notice that for $x = 1$ this latter inequality collapses to

$$\mu \geq \frac{\beta Q}{I}$$

which is implied by (9). In addition, for $x = 1$ (48) and (49) collapse to (9). Thus, under (9) there is an interior equilibrium for $x = 1$ or, equivalently

$$\kappa_1 = \frac{1}{\beta}\frac{I}{I-1}$$

Now consider $1 - x > 0$. Write (50) as

$$I(\mu - \underline{\alpha})x^2 + \left(2\mu - I(\mu - \underline{\alpha}) - \frac{\beta Q}{I}\right)x - \frac{\beta Q}{I} \geq 0 \tag{51}$$

Let us denote by h a function of x such that (51) is $h(x) \geq 0$. Clearly, h is quadratic, convex since $\mu - \underline{\alpha} > 0$ holds and it satisfies $h(0) < 0$ and $h(1) = 2\left(\mu - \frac{\beta Q}{I}\right) > 0$. Thus, there is a unique value of x in $(0, 1)$, say x^*, such that (51) holds if and only if $x \in [x^*, 1]$. More concretely, $h(x) = 0$ has necessarily two real roots and x^* is the largest (and the only positive) root. Notice also that (48) and (49) are implied by (9) for any $x < 1$.

If $1 - x < 0$, then from (50) we obtain $h(x) \leq 0$, where h is still the left hand side of (51). Thus, this latter inequality cannot hold for $x > 1$. Therefore, the set of interior equilibria are characterized by $x \in [x^*, 1]$. Denoting κ_1^* such that $x^* = \beta\frac{I-1}{I}\kappa_1^*$, the interior equilibria are characterized equivalently by $\kappa_1 \in \left(\kappa_1^*, \frac{1}{\beta}\frac{I}{I-1}\right)$.

The sensitivity analysis of κ_1^* is equivalent to x^*. Consider first a variation in μ keeping all other parameters constant. Taking total differential in the latter equality, we have

$$(2 + I(x^* - 1))\, x^* d\mu + \left(2\mu + I(\mu - \underline{\alpha})(2x^* - 1) - \frac{\beta Q}{I}\right) dx^* = 0 \tag{52}$$

Since $x^* < 1$ holds, the coefficient of $d\mu$ is negative if I is large enough. To analyze the sign of the coefficient of dx we must notice first that (9) implies $2\mu - \frac{\beta Q}{I} > 0$, so a sufficient condition for that coefficient to be positive is $2x^* - 1 > 0$ or, equivalently, $x^* > \frac{1}{2}$. It is

$$h\left(\frac{1}{2}\right) = \mu - \frac{1}{4} I(\mu - \underline{\alpha}) - \frac{3}{2}\frac{\beta Q}{I}$$

Thus, $h\left(\frac{1}{2}\right) < 0$ holds if I is large enough, which in turn implies that $x^* > \frac{1}{2}$. Therefore, for I large enough, the coefficient of $d\mu$ is negative whereas the coefficient of dx is positive, which implies $d\mu$ and dx^* must have the same sign.

Let $s := \mu - \underline{\alpha}$. Consider a variation in s keeping all other parameters (in particular μ) constant. Taking total differential in $h(x^*) = 0$ we have

$$I x^*(x^* - 1) ds + \left(2\mu + Is(2x^* - 1) - \frac{\beta Q}{I}\right) dx^* = 0$$

The coefficient of ds in the previous expression is negative since $x^* < 1$, whereas, following an argument as above, the coefficient of dx^* is positive if I is large enough. Thus, for I large enough ds and dx^* must have the same sign.

Consider a variation in Q keeping all other parameters constant. Taking total differential in $h(x^*) = 0$ we have

$$-\frac{1}{I}(x^* + 1)\beta dQ + \left(2\mu - \frac{\beta Q}{I} + I(\mu - \underline{\alpha})(2x^* - 1)\right) dx^* = 0$$

Using an analogous reasoning, when I is large enough we obtain that dQ and dx^* must have the same sign.

It rests to prove that the efficient equilibrium, $\beta\kappa_1 = 1$, belongs to the set of interior equilibrium for I large enough. Notice that $\beta\kappa_1 = 1$ implies $x = \frac{I-1}{I}$. It conforms an interior equilibrium if and only if it satisfies (47) to (49). Substituting in (47), we have

$$\frac{I - 1}{2I - 1}(\underline{\alpha} + \mu) \geq \frac{\beta Q}{I} \tag{53}$$

whereas substituting in (48) and (49) we have

$$\underline{\alpha} \geq \frac{\beta Q}{I} \geq \frac{I - 1}{I}(\overline{\alpha} - \underline{\alpha}) \tag{54}$$

For any I finite, (54) is implied by (9). In addition, (53) is also implied by (9) if

$$\frac{I - 1}{2I - 1}(\underline{\alpha} + \mu) \geq \underline{\alpha}$$

The coefficient of $\underline{\alpha} + \mu$ in this latter inequality is strictly increasing and continuous in I. It converges to $\frac{1}{2}$ as $I \to \infty$. The inequality clearly holds at the limiting value of that coefficient, thus it must hold for any I larger than some finite threshold. □

Proof **of Proposition 5.**

1. The necessary and sufficient conditions for κ_1 to constitute an interior equilibrium are (47) to (49), in Proposition 4, where $x = \beta\frac{I-1}{I}\kappa_1$. Taking $\beta\kappa_1 = 1$, (47) becomes

$$\underline{\alpha} + \mu \geq \left(1 + \frac{I}{I-1}\right)\frac{\beta Q}{I} \tag{55}$$

Also (48) and (49) become

$$\overline{\alpha} - \underline{\alpha} \leq \frac{\beta Q}{I-1} \leq \frac{I}{I-1}\underline{\alpha} \tag{56}$$

The inequalities (55) and (56) are implied by (10) just notting that $\lambda = 0$ implies $\mu = E\{\tilde{\alpha}\} > \underline{\alpha}$. The efficient equilibrium strategy follows from taking $\kappa_1 = \beta^{-1}$ and $\lambda = 0$ in (32) to (35), in the proof of Proposition 2. To show that the efficient strategy satisfies (11) is left to the part 4 of this proof.

2. Take again the characterization of equilibrium as (47) to (49). We write $I \to \infty$ to represent: $I \to \infty$ and $Q \to \infty$ while $\frac{Q}{I}$ remains constant. If $I \to \infty$ for $\beta\kappa_1 \neq 1$, then (47) converges to

$$(\underline{\alpha} - \mu)\, x \geq 0$$

which cannot hold: $\underline{\alpha} - \mu < 0$ holds for any non-degenerated distribution of types, while $x > 0 \iff \kappa_1 > 0$ and, from Proposition 2, this latter equality must hold at any equilibrium under which firms submit downward sloping demand functions. In addition, for $\beta\kappa_1 = 1$, the conditions for an interior equilibrium are (55) and (56), which hold in the limiting case $I \to \infty$ under (10).

3. Take $I = 2$ and let $\beta\kappa_1$ be arbitrary. Using (32) to (35), in the proof of Proposition 2, we have

$$m(\kappa_1) = \frac{1}{4}(2 - \beta\kappa_1);\ \kappa_0 = \frac{\beta\kappa_1}{2 - \beta\kappa_1}(\kappa_1\mu - Q);\ \delta = \frac{2\kappa_1}{2 - \beta\kappa_1} \tag{57}$$

From the latter equality, we have a downward sloping demand if and only if $\beta\kappa_1 \in (0, 2)$. The conditions for an interior equilibrium in Lemma 1, for $I = 2$, can be written as follows. (6) is

$$\beta\kappa_0 + \beta\kappa_1\underline{\alpha} > \frac{\beta Q}{2} > \frac{1}{2}\beta\kappa_1(\overline{\alpha} - \underline{\alpha}) \tag{58}$$

In addition, since $\beta\kappa_1 \leq 2$, the condition (7) is

$$\frac{\beta Q}{2} \leq \underline{\alpha} \tag{59}$$

The second equality in (58) and (59) are implied by (10). Use the expression for κ_0 above to write the first inequality in (58) as

$$\beta\kappa_1\mu + (2 - \beta\kappa_1)\underline{\alpha} \geq \left(\frac{2}{\beta\kappa_1} + 1\right)\frac{\beta Q}{2}$$

The left hand side of the previous inequality is continuous and strictly increasing in $\beta\kappa_1$ as $\mu > \underline{\alpha}$ holds. The right hand side is continuous and strictly decreasing. Clearly, the inequality fails to hold as $\beta\kappa_1 \to 0$, whereas it is implied by (10) at $\beta\kappa_1 = 1$. Thus, under (10), there must exist a unique $b^0 \in (0, 1)$ such that for any $\beta\kappa_1 \in (0, 2)$ the inequality holds iff $\beta\kappa_1 \in (b^0, 2)$.

4. We prove (11) for $I = 2$ and $\beta\kappa_1 \in (0, 2)$. For any linear strategy, the stop-out price in the auction is defined by the market clearing condition

$$2\kappa_0 + \kappa_1 \sum_i \alpha_i - 2\delta p^* = Q$$

Using the expressions for κ_0 and δ in (57), we have

$$p^*(\hat{\alpha}) = \frac{1}{\beta\delta}\left(\beta\kappa_0 - \frac{1}{2}\beta Q + \beta\kappa_1\hat{\alpha}\right) \tag{60}$$

where $\hat{\alpha}$ denotes the sample mean of types and the notation emphasizes that the stop out price depends on it. Denoting by q_i^* the auction assignment for bidder i, his marginal saving on abatement cost is

$$\phi_i(q_i^*, \alpha_i) = \alpha_i - \beta q_i^* = \alpha_i - \beta\left(\kappa_0 + \kappa_1\alpha_i - \delta p^*(\hat{\alpha})\right) \tag{61}$$

Taking expectations in (61), substituting in (11) and re-arranging terms, (11) is equivalent to

$$(1 - \beta\delta)E\{p^*(\hat{\alpha}) \mid \alpha_i\} \leq (1 - \beta\kappa_1)\alpha_i - \beta\kappa_0 \tag{62}$$

Taking expectations in (60) and substituting into (62), we can rewrite (62) as

$$\left(\frac{1}{\beta\delta} - 1\right)\left(\beta\kappa_0 - \frac{1}{2}\beta Q + \frac{1}{2}\beta\kappa_1\left(\alpha_i + E\{\tilde{\alpha}_i\}\right)\right) \leq (1 - \beta\kappa_1)\alpha_i - \beta\kappa_0$$

where we have used that, since bidder i only observes his own type and types are independent, it is

$$E\{\hat{\alpha} \mid \alpha_i\} = \frac{1}{2}\left(\alpha_i + E\{\tilde{\alpha}_i\}\right)$$

The latter inequality is equivalent to

$$\frac{1}{\beta\delta}\beta\kappa_0 - \frac{1}{2}\left(\frac{1}{\beta\delta} - 1\right)\beta Q + \left(\left(\frac{1}{\beta\delta} - 1\right)\frac{1}{2}\beta\kappa_1 - 1 + \beta\kappa_1\right)\alpha_i + \frac{1}{2}\left(\frac{1}{\beta\delta} - 1\right)\beta\kappa_1 E\{\tilde{\alpha}_i\} \leq 0$$

Using the expression for δ in this latter inequality yields

$$\frac{2 - \beta\kappa_1}{2\beta\kappa_1}\beta\kappa_0 - \frac{2 - 3\beta\kappa_1}{4\beta\kappa_1}\beta Q - \frac{1}{4}(2 - \beta\kappa_1)\alpha_i + \frac{1}{4}\left(2 - 3\beta\kappa_1\right)E\{\tilde{\alpha}_i\} \leq 0$$

Finally, using the expression for κ_0 in this latter inequality yields

$$-\frac{2 - \beta\kappa_1}{4}\left(\frac{\beta Q}{\beta\kappa_1} + \alpha_i - E\{\tilde{\alpha}_i\}\right) \leq 0$$

since $\beta\kappa_1 \in (0, 2)$, the latter inequality is equivalent to

$$\beta Q \geq \beta\kappa_1\left(E\{\tilde{\alpha}_i\} - \alpha_i\right) \tag{63}$$

But (63) is implied by the first inequality in (10) since $\beta\kappa_1 \leq 2$ and $\overline{\alpha} > E\{\tilde{\alpha}_i\}$.

Finally, we prove (11) for $I > 2$ and $\beta\kappa_1 = 1$. Notice that in this case (11) can still be written as (62). The efficient strategy satisfies $\beta\kappa_1 = 1$. In addition, using the value for κ_0 and δ for the efficient strategy, in the part 1 of the Proposition, we can rewrite (62) as

$$E\{p^*(\hat{\alpha}) \mid \alpha_i\} \geq E\{\tilde{\alpha}_i\} - \frac{I}{I-1}\frac{\beta Q}{I} \tag{64}$$

To obtain the stop out price, we consider the market clearing condition

$$\sum_i \gamma(\alpha_i, p) = Q$$

Use the expression for γ in the part 1 of the Proposition, solve for p, to obtain

$$p^*(\hat{\alpha}) = \frac{1}{2}\left(E\{\tilde{\alpha}_i\} + \hat{\alpha}\right) - \frac{1}{2}\left(\frac{I}{I-1} + 1\right)\frac{\beta Q}{I}$$

where, as before, $\hat{\alpha}$ denotes the sample mean of types. Analogously to the two-bidder case, notice that

$$E\{\hat{\alpha} \mid \alpha_i\} = \frac{I-1}{I} E\{\tilde{\alpha}_i\} + \frac{1}{I}\alpha_i$$

Taking conditional expectations on the previous expression for $p^*(\hat{\alpha})$, substituting in (64) and rearranging terms, (64) is equivalent to

$$\frac{\beta Q}{I-1} > E\{\tilde{\alpha}_i\} - \alpha_i$$

This latter inequality is clearly implied by the first inequality in (10) just noting that it is

$$E\{\tilde{\alpha}_i\} - \alpha_i \leq \overline{\alpha} - \underline{\alpha}$$

□

Proof **of Proposition 6.**

1. We write the inequalities that characterize interior equilibria for $\lambda = 1/2$. Note that $\lambda = 1/2$ implies $\mu = \frac{E\{\tilde{\alpha}_i\}}{2}$. First, (33) is

$$m(\kappa_1) = \frac{1-x}{2+Ix}$$

where $x := \beta \frac{I-1}{2I}\kappa_1$. From Proposition 2, interior equilibria are indexed by κ_1, so they are equivalently indexed by x. Substitute the expressions for $m(\kappa_1)$ and μ into (36) to obtain

$$\left(1 - \frac{2+Ix}{1-x}\right)\frac{\beta Q}{2I} + \left(I\underline{\alpha} + \frac{E\{\,\tilde{\alpha}_i\}}{2}\left(\frac{2+Ix}{1-x} - I\right)\right)x \geq 0 \tag{65}$$

Recall that this latter inequality is the first inequality in (6). Using the definition of x, the second inequality in (6) becomes:

$$\frac{\beta Q}{I} \geq 2x(\overline{\alpha} - \underline{\alpha}) \tag{66}$$

Finally, condition (7) can be written

$$\frac{\beta Q}{I} \leq \begin{cases} \underline{\alpha} & \text{if } x \leq 1/2 \\ \overline{\alpha} - 2x\left(\overline{\alpha} - \underline{\alpha}\right) & \text{otherwise} \end{cases} \tag{67}$$

Thus, for $\lambda = 1/2$, x defines an interior equilibrium if and only if it satisfies (65) to (67).

Take $\beta\kappa_1 = 1$. Then

$$\frac{2+Ix}{1-x} = \frac{I(I+3)}{I+1}$$

Thus, (65) can be written

$$\underline{\alpha} + \frac{1}{2}E\{\tilde{\alpha}_i\}\left(\frac{I+3}{I+1} - 1\right) \geq \left(\frac{I+3}{I+1} - \frac{1}{I}\right)\frac{\beta Q}{I-1}$$

or, equivalently,

$$\underline{\alpha} + \frac{1}{I+1}E\{\tilde{\alpha}_i\} \geq \frac{\beta Q}{I} + \frac{2}{I+1}\frac{\beta Q}{I-1}$$

Using (10), it suffices to add the first inequality in (14) for this latter inequality to hold. In addition, (66) and (67), for $\beta\kappa_1 = 1$, collapse to

$$\frac{I-1}{I}\left(\overline{\alpha} - \underline{\alpha}\right) \leq \frac{\beta Q}{I} \leq \underline{\alpha}$$

These latter two inequalities are implied by (10). The efficient equilibrium strategy follows from taking $\kappa_1 = \beta^{-1}$ and $\lambda = \frac{1}{2}$ in (32) to (35), in the proof of Proposition 2.

As in the proof of Proposition 5, we can write (11) as (62). For the case $\lambda = \frac{1}{2}$, the efficient equilibrium strategy is presented in the part 1 of this Proposition, in particular

$$\kappa_0 = \frac{1}{\beta}\frac{1}{I+1}\left(E\{\tilde{\alpha}_i\} - 2\frac{I}{I-1}\frac{\beta Q}{I}\right); \qquad \kappa_1 = \frac{1}{\beta}: \qquad \delta = \frac{1}{\beta}\frac{I+2}{I+1}$$

Substituting in (62) and re-arranging terms, it is

$$E\{p^*(\hat{\alpha}) \mid \alpha_i\} \geq E\{\tilde{\alpha}_i\} - 2\frac{I}{I-1}\frac{\beta Q}{I} \tag{68}$$

In the other hand, the stop-out price follows from the usual market-clearing condition, as in the proof of Proposition 5, leading to

$$p^*(\hat{\alpha}) = \frac{1}{\beta\delta}\left(\beta\kappa_0 + \beta\kappa_1\hat{\alpha} - \frac{\beta Q}{I}\right) \tag{69}$$

where $\hat{\alpha}$ denotes the sample mean of the types. Under $\lambda = \frac{1}{2}$, it is

$$E\{\hat{\alpha} \mid \alpha_i\} = \frac{I-1}{I}E\{\tilde{\alpha}_j \mid \alpha_i\} + \frac{1}{I}\alpha_i = \frac{I-1}{2I}E\{\tilde{\alpha}_i\} + \frac{I+1}{2I}\alpha_i \tag{70}$$

where $j \neq i$. Taking the conditional expectation on the stop-out price, using the parameter values of the equilibrium strategy and substituting into (68), we can rewrite it as

$$\left(\frac{2I}{I-1} - \frac{I+1}{I+2}\left(\frac{1}{I+1}\frac{2I}{I-1} + 1\right)\right)\frac{\beta Q}{I} \geq \left(1 - \frac{I+1}{I+2}\left(\frac{1}{I+1} + \frac{I-1}{2I}\right)\right)E\{\tilde{\alpha}_i\} - \frac{I+1}{I+2}\frac{I+1}{2I}\alpha_i$$

or, equivalently

$$\frac{\beta Q}{I-1} \geq \frac{1}{2}\left(E\{\tilde{\alpha}_i\} - \alpha_i\right)$$

which is implied for all $\alpha_i \in \Omega$ by the first inequality in (10).

2. In the remainder of the proof we write $I \to \infty$ to represent: $I \to \infty$ and $Q \to \infty$ while $\frac{\beta Q}{I}$ remains constant. Taking $I \to \infty$, $x \to \frac{\beta\kappa_1}{2}$. Additionally, considering $x \neq 0$, equation (65) converges to

$$\underline{\alpha} + \frac{1}{2}v(x) \geq 0 \tag{71}$$

where we have denoted

$$v(x) := \frac{1}{1-x}\left(E\{\tilde{\alpha}_i\}(2x-1) - \frac{\beta Q}{I}\right)$$

Note that v is unbounded at $x = 1$. Under (9), it is

$$E\{\tilde{\alpha}_i\} - \frac{\beta Q}{I} > \underline{\alpha} - \frac{\beta Q}{I} > 0$$

Thus, as $x \to 1$ from the left and from the right, it is $\lim_{x\to1^-} v(x) = \infty$ and $\lim_{x\to1^+} v(x) = -\infty$, respectively. Furthermore

$$v'(x) = \frac{1}{(1-x)^2}\left(E\{\tilde{\alpha}_i\} - \frac{\beta Q}{I}\right)$$

which is positive from (9). In turn, (9) is implied by (10). In fact, in the remainder of this part of the proof it will suffice to use (9) instead of (10). Note that $\lim_{x\to\infty} v(x) = -2E\{\tilde{\alpha}_i\}$. Using this latter limit in (71) and taking into account that v is strictly increasing, (71) cannot hold for any $x > 1$. Equivalently, there cannot be interior equilibrium for any κ_1 such that $\beta\kappa_1 > 2$ holds as $I \to \infty$.

Note that $v(1/2) = -2\frac{\beta Q}{I}$. Thus, (71) holds with strict inequality under (9) for $x = 1/2$. Taking into account that v is strictly increasing, (71) holds for any $x \in [1/2, 1]$. Notice that (66) and (67) are unaffected by taking $I \to \infty$. For $x = 1/2$, (66) and (67) are implied by (9), so that $x = 1/2$, or equivalently $\beta\kappa_1 = 1$, that is, the efficient allocation, constitutes an interior equilibrium. Within $x \in (1/2, 1]$, the most restrictive case for (66) and (67) to hold is at $x = 1$. For that value of x, (66) and (67) are equivalent to the second and third inequalities in (14). Now consider $x \in \left(0, \frac{1}{2}\right)$. (66) and (67) are implied by (9). Under (9), (71) holds with strict inequality for $x = 1/2$. Since v is continuous and increasing at any $x \leq 1/2$, (71) must also hold if and only if $x \in (x^*, 1/2)$, for some $x^* \in (0, 1/2)$. The relation between x^* and b^1 follows from $x \to \frac{\beta\kappa_1}{2}$ as $I \to \infty$.

As in the proof of Proposition 5, we can write (11) as (62). Combining with the expression of the stop-out price, in (69), we can write (11) as

$$\frac{1}{\beta\delta}\beta\kappa_0 + \left(\frac{1}{\beta\delta} - 1\right)\left(\beta\kappa_1 E\{\hat{\alpha} \mid \alpha_i\} - \frac{\beta Q}{I}\right) \leq (1 - \beta\kappa_1)\alpha_i$$

Taking limits as $I \to \infty$ in (70) and substituting in the later inequality, we can rewrite it as

$$\frac{1}{\beta\delta}\beta\kappa_0 + \left(\frac{1}{\beta\delta} - 1\right)\left(\frac{1}{2}\beta\kappa_1 E\{\tilde{\alpha}_i\} - \frac{\beta Q}{I}\right) \leq \left(1 - \beta\kappa_1 - \left(\frac{1}{\beta\delta} - 1\right)\frac{1}{2}\beta\kappa_1\right)\alpha_i \tag{72}$$

Next, we use the expressions in (32) to (35), in the proof of Proposition 2. Taking $\lambda = \frac{1}{2}$ and $I \to \infty$, it is

$$\lim_{I\to\infty} \frac{I}{\xi} = \lim_{I\to\infty} Im(\kappa_1) \iff \frac{1}{\beta\delta} = \frac{2 - \beta\kappa_1}{\beta\kappa_1};$$

and

$$\lim_{I\to\infty} \beta\kappa_0 = \left(1 - \frac{\beta\kappa_1}{2 - \beta\kappa_1}\right)\left(\frac{\beta Q}{I} - \frac{1}{2}E\{\tilde{\alpha}_i\}\beta\kappa_1\right)$$

Using these limit values into (72) we have $0 \leq 0$ for any $\beta\kappa_1 \neq 0$. □

Proof **of Corollary 1.** With $\beta\kappa_1 = \frac{3}{2}$, x, as defined in the proof of Proposition (6) is $x = \frac{3(I-1)}{4I}$, so that $m(\kappa_1) = \frac{I+3}{I(3I+5)}$. From the definition of ξ in the proof of Proposition 2, substituting in equations (32)

$$\delta = \frac{I(3I+5) - 2(I+3)}{\beta(I+3)(I-1)} \tag{73}$$

that tends to $\frac{3}{\beta}$ as $I \to \infty$. From (35)

$$\kappa_0 = \left(1 - \frac{3I+5}{I+3}\right)\left(\frac{1}{I-1}Q - \frac{3}{2\beta}\frac{E(\alpha_i}{2}\right) \tag{74}$$

that tends to $\frac{3}{2\beta}E(\alpha_i)$ as $I \to \infty$. □

Proof **of Corollary 2.** It follows from the definition of ξ in the proof of the Proposition 2, substitution of the corresponding value of λ and $\kappa_1 = 1/\beta$ in equations (32) to (35), and then taking limits as $I \to \infty$, $Q \to \infty$ and $\frac{Q}{I}$ stays constant. □

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