Continuous time, continuous decision space prisoner's dilemma: A bridge between game theory and economic GCD-models

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A bridge between game theory and economic GCD-models

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Inhalt

1. Abstract

General Constrained Dynamic models (GCD – models) in economics are inspired by classical mechanics with constraints. Most macroeconomic models can be understood as special cases of GCD – models. Moreover, in this paper it will be shown that not only macroeconomic models but also game theoretic models are strongly related to GCD – models.

GCD models are characterized by a system of differential equations in continuous time while most game theoretical models are set up in discrete time. Therefore it is necessary to build a bridge from game theoretical models denominated in discrete time to game theoretical models using continuous time. This bridge is illustrated in the following using the example of a continuous time, continuous decision space prisoner's dilemma. Furthermore, it is shown that the differential equations which determine other continuous game theoretic models can be understood to a certain extent as special cases of the GCD – differential equations. Well known types of continuous game theoretic models include for instance "Evolutionary Game Theory" with the replicator equation, "Adaptive Dynamics" with the canonical equation, which is nothing else than a replicator – mutator equation, and the so called "Differential Games", which are strongly related to optimal control theory with two controls and two different objectives (goals).

Most of the GCD – models are characterised by 3 key feature:

- mutual influence,
- Power-factors
- Constraints

Nowak (2006b) and Taylor & Nowak (2007) show that there are five mechanisms which, under certain conditions, can lead to the evolution of cooperation in an iterated prisoner's dilemma. Inspired by this, we apply the 3 key features of GCD – models to the standard prisoner's dilemma in discrete time which yields 3 additional mechanisms which enable the evolution of cooperation.

The assumption or axiom of the free market economy is that an individual optimisation strategy will lead to an overall optimum by virtue of Adam Smith's invisible hand. Without additional conditions this assumption alone is fundamentally wrong. As in prisoner's dilemma also in economics cooperation is essential to get an overall optimum. The big question of political economy is to analyse which additional measures could guarantee that the individual optimisation strategy characterising a free market economy leads to cooperation as precondition to get an overall optimum.

From this point of view the different economic theories could be characterised in terms of which measures they assume to be sufficient to guarantee an overall optimum without abandoning the principle of individual optimisation.

2. Motivation

The history of the development of human societies is the history of the overcoming of prisoner's dilemma situations.

Egoistic behaviour is not only an important principle of evolution theory but also the basic principle of free market economy. More formally egoistic behaviour could be thought as individual optimisation strategy.

One of the big questions of evolution theory is: How can cooperation evolve? Under what conditions can it happen that individuals do not behave egoistically? To phrase it in economic terms: How can it happen that individuals do not behave according to what yields the highest utility for themselves, but set aside their interest in favour of a better overall result by cooperation? The principle answer to this question from evolution theory is, that the evolution of cooperation really is indeed very challenging and only can occur under very special conditions. In the context of Evolutionary Game Theory Nowak (2006b) and Taylor & Nowak (2007) elaborate necessary conditions in order for it to be possible that cooperation can evolve and characterised the following 5 mechanisms: direct reciprocity, indirect reciprocity, kin selection, group selection and graph selection (network reciprocity). They show that essentially all these mechanisms lead to a transformation of the payoff matrix of the prisoner's dilemma. To summarise: evolution theory provides the correct fundamental insight that the evolution of cooperation to achieve an overall optimum is bound to very special conditions being fulfilled.

In a small tribal group economy in principle it should be possible to recognise, which behaviour all the members need to display in order to achieve an overall optimum. In such a small community behaviour that is optimal for the group can be enforced by the chief. But the bigger and therefore the more complex an economy is, the more difficult it will be to recognise the overall optimising behaviour for all agents even by the use of supercomputers. In its core that is the reason why any larger centrally planned economy has failed. But if a planned overall optimisation is to fail, the only possibility to organise an economy is to organise it by individual optimisation strategies by their members. That is just the key organisation principle of free market economy. However, the assumption or axiom of the free market economy that this individual optimisation strategy will lead to an overall optimum by virtue of Adam Smith's invisible hand is fundamentally wrong. The big question of political economy is to analyse which additional measures could guarantee that the individual optimisation strategy characterising a free market economy leads to an overall optimum.

As the prisoner's dilemma is just the simplest model to describe situations where individual optimisation does not lead to an overall optimum, or in fact leads to the worst outcome for both of the players, it is a sound starting point to analyse all these questions. Especially the question of how the prisoner's dilemma can be overcome to achieve an overall optimum without abandoning the assumption of an individual optimisation strategy is important, not only for economic systems but also for evolution theory.

In E. Glötzl (2015) it was shown that most macroeconomic models can be understood as special cases of General Constrained Dynamic Models (GCD – models). In this paper we continue to show in which sense also game theoretical models can be understood as special cases of GCD – models. Exemplarily we illustrate the steps to go from the standard prisoner's dilemma to the prisoner's dilemma with continuous time and continuous decision space, which turns out to be a very simple GCD – model. Generally the dynamics of all continuous game theoretical models is characterised by a differential equation system. We will show that the “replicator equation” of Evolutionary Game Theory, the “canonical equation” of Adaptive Dynamics and the “Euler equations like“ characteristic equations of Differential Games are special cases of the basic equations of GCD – models.

The key features of GCD – models are.

- mutual influence, which means the following: In standard game theory usually a player A can only influence his own decisions. But in GCD – models it is quite natural that also a player B has an influence on the decisions of A and vice versa.
- Power-factors: The extent to which a player can assert his own interest and influence the decisions of the other player.
- Constraints: As in classical mechanics the dynamic in GCD – models can be restricted by constraints, which give rise to Lagrangian multiplicators.

As an application of the relationship between standard game theoretical models and GCD – models we show that these key features of GCD – models could be useful also in game theoretical models. Introducing mutual influence and power factors or introducing constraints in the standard prisoner’s dilemma leads to a transformation of the payoff matrix of the prisoner’s dilemma. Therefore such mechanisms can bring about cooperation in a similar vein as the 5 mechanisms proposed by Nowak.

3. From standard prisoner's dilemma to continuous prisoner's dilemma and other general continuous game theoretical models

The decision space of standard prisoner's dilemma is $\{cooperation, defection\}$. If one identifies $cooperation \equiv 0$ and $defection \equiv 1$ the decision space is $\{0,1\}$. The pay off matrix usually is written in the following way

A	cooperate $y^B = 0$	defect $y^B = 1$
cooperate $y^A = 0$	**R**	**S**
defect $y^A = 1$	**T**	**P**

We will wright this payoff matrix in the following changed way because writing it in this way the relationship to the vector field of GCD-models becomes more obviously. For the same reason we will call the payoff matrix also utility matrix $U^A(y^A, y^B)$ for player A depending on the decision y^A of A and the decision y^B of player B. Doing it in this way the utility matrices of player A and player B are:

defect $y^B = 1$	**S**	**P**
cooperate $y^B = 0$	**R**	**T**
A	cooperate $y^A = 0$	defect $y^A = 1$

defect $y^B = 1$	**T**	**P**
cooperate $y^B = 0$	**R**	**S**
B	cooperate $y^A = 0$	defect $y^A = 1$

This utility matrix leads to a prisoner's dilemma situation if

$T > R > P > S$

and avoids alternating decisions in the iterated prisoner's dilemma if

$$2R > S + T$$

For the sake of simplicity we use in the following

$S = 0 \qquad P = 1 \qquad R = 2 \qquad T = 3$

which fulfils the above conditions.

The decision of A depends on the direction in which his utility increases. The same is true for player B. The direction of the change of the decision for player A is given by the red dotted arrows, that for player B by the blue dotted arrows.

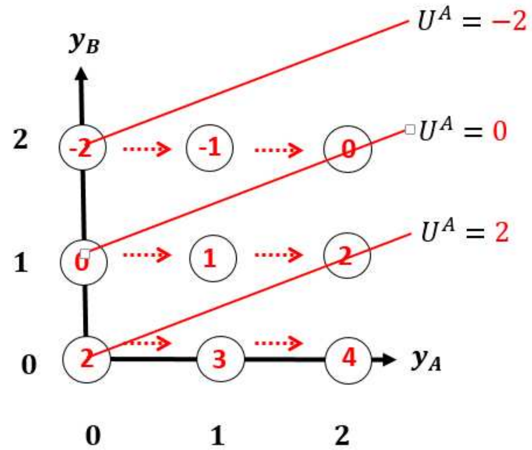
defect $y^B = 1$	**0**	**1**
cooperate $y^B = 0$	**2**	**3**
A	cooperate $y^A = 0$	defect $y^A = 1$

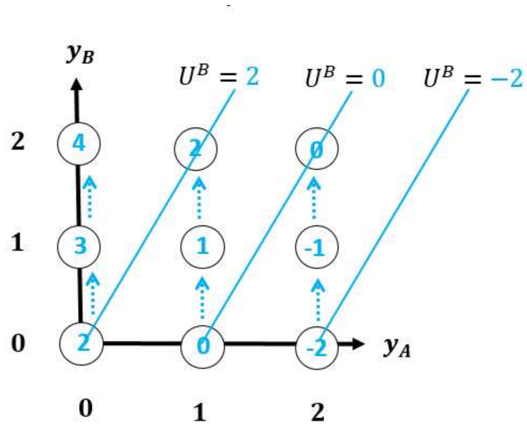
defect $y^B = 1$	**3**	**1**
cooperate $y^B = 0$	**2**	**0**
B	cooperate $y^A = 0$	defect $y^A = 1$

Obviously the utility matrices for decisions $\in \{0,1\}$ are given by the 2 utility functions

$$U^A(y^A, y^B) = y^A - 2y^B + 2$$
$$U^B(y^A, y^B) = -2y^A + y^B + 2$$

Drawing these utility functions in Cartesian coordinates together with the isolines of constant utility yields





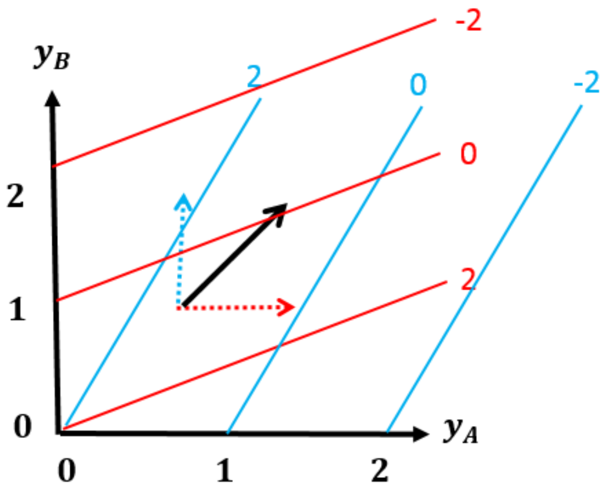
and means nothing else than to expand the decision space from $\{0,1\}$ to $\mathbb{R}_+$.

In discrete time and discrete decision space the players choose the direction in which they change in each round (or do not change) their decision to cooperate or to defect corresponding to the direction in which their individual utilities increase. In continuous time and continuous decision space each time the players change their decision during the infinitesimal time dt by the infinitesimal amount dy according to the increase of their individual utilities along the change of their decision. This gives rise to the following differential equation system

$$\frac{dy^A}{dt}=\frac{\partial U^A(y^A,y^B)}{\partial y^A}$$
$$\frac{dy^B}{dt}=\frac{\partial U^B(y^A,y^B)}{\partial y^B}$$

As this differential equation system is determined by the vector field $\left(\frac{\partial U^A(y^A,y^B)}{\partial y^A},\frac{\partial U^B(y^A,y^B)}{\partial y^B}\right)$, the relationship of this vector field with the payoff matrices becomes clear.

Next we combine the utility functions of both players together with the isolines in one graph. For each point in the decision space you can visualise the partial derivatives of the utilities with a red arrow for A and a blue arrow for B and the resulting move with a black arrow.



In the case if we choose S,R,P,T as above we get the following system of differential equations

$$\frac{dy^A}{dt}=\frac{\partial U^A(y^A,y^B)}{\partial y^A}=\frac{\partial(y^A-2y^B+2)}{\partial y^A}=1$$
$$\frac{dy^B}{dt}=\frac{\partial U^B(y^A,y^B)}{\partial y^B}=\frac{\partial(-2y^A+y^B+2)}{\partial y^B}=1$$

which describes the **continuous prisoner's dilemma**.

This system of differential equations has the trivial solution

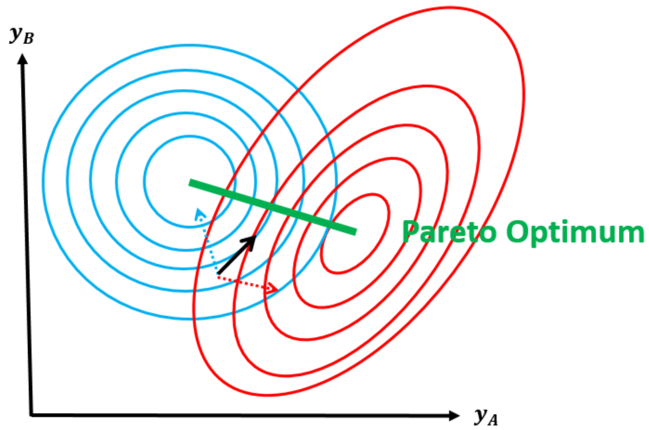
$$y^A(t) = t + y^A(0)$$
$$y^B(t) = t + y^B(0)$$

from which it follows that the utilities for both of the players in the continuous prisoner's dilemma decrease with time as in the discrete standard prisoner's dilemma:

$$U^A(y^A(t), y^B(t)) = -t + (y^A(0) - 2y^B(0) + 2)$$
$$U^B(y^A(t), y^B(t)) = -t + (-2y^A(0) + y^B(0) + 2)$$

Obviously this concept to describe the time derivative of the decisions of the players is not restricted to the special values we have chosen for S, P, R, T . For each case with distinct values for S, P, R, T we can find continuous utility functions U^A and U^B which coincide with the utility matrix of the discrete time case. So one can get not only continuous prisoner's dilemma situations but continuous generalisations of any 2 player game theoretic model. Obviously this concept can be generalised to any number of players.

In the typical example of the "Edgeworth box" in economics which describes the utilities and behaviour of two agents bargaining with two goods we get a graph similar to the following graph:



The individual optimisation strategy of each of the players can be described by the following behavioural equations:

$$y_A' == \frac{\partial U^A(y_A, y_B)}{\partial y_A} + \frac{\partial U^B(y_A, y_B)}{\partial y_A}$$
$$y_B' == \frac{\partial U^A(y_A, y_B)}{\partial y_B} + \frac{\partial U^B(y_A, y_B)}{\partial y_B}$$

The red arrow denotes

$$\begin{pmatrix} \dfrac{\partial U^A(y_A, y_B)}{\partial y_A} \\ \dfrac{\partial U^A(y_A, y_B)}{\partial y_B} \end{pmatrix}$$

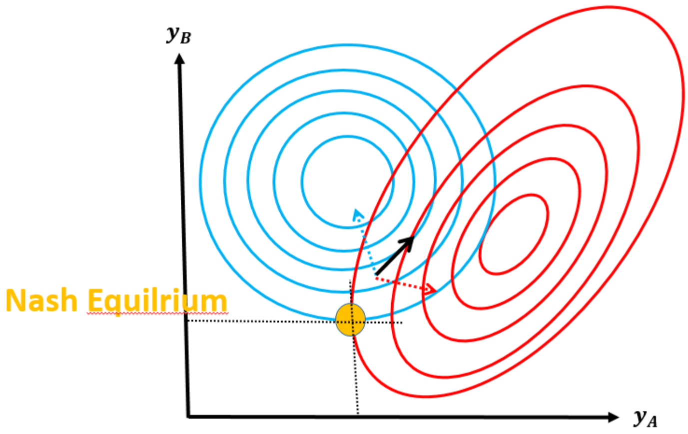
The blue arrow denotes

$$\begin{pmatrix} \dfrac{\partial U^B(y_A, y_B)}{\partial y_A} \\ \dfrac{\partial U^B(y_A, y_B)}{\partial y_B} \end{pmatrix}$$

And the black arrow denotes the sum of the red and the blue arrow and points in the direction of the resultant movement in the decision space.

This behavioural equation is obviously a special case of the behavioural equation of an "utility GCD – model" we will describe in the following chapter.

In the special case of this graph there exists even a Nash equilibrium:



The way back from a continuous game theoretical model to the discrete case can be done in the same way as the discretisation of a differential equation system. Of course continuous and discrete models do not necessarily show the same qualitative behaviour in every case, as not in every case the convergence of the discrete model to the continuous model can be guaranteed.

4. The basic behavioural equations of GCD-models

Inspired by constrained dynamics in classical mechanics GCD – models where introduced by Glötzl (2015) as a framework in which most of the economic models can be described. For the sake of simplicity in the following we will present all the formulas for the case of dimension 2 and 1 non holonomic constrained condition. The extension to higher dimensions and more constraints (and holonomic constraints) is obvious. In the case of time independent forces the dynamics in classical mechanics as well as in **"general GCD – models"** in economics is given by the following differential algebraic equation system

$$x_1' = y_1$$

$$x_2' = y_2$$

$$y_1' == f_1(x_1, x_2, y_1, y_2) + \lambda \frac{\partial CC(x_1, x_2, y_1, y_2)}{\partial y_1}$$

$$y_2' = f_2(x_1, x_2, y_1, y_2) + \lambda \frac{\partial CC(x_1, x_2, y_1, y_2)}{\partial y_2}$$

$$CC(x_1, x_2, y_1, y_2) = 0$$

In the case of there being a holonomic constrained condition CC, i.e. if $\left(\frac{\partial CC}{\partial y_1}, \frac{\partial CC}{\partial y_2} \right) \equiv (0,0)$, the partial derivatives of the constrained condition have to be substituted by the partial derivatives with respect to (x_1, x_2). In classical mechanics (x_1, x_2) denote the space coordinates and (y_1, y_2) the velocity coordinates. In economic models (x_1, x_2) in most cases denote stock variables and (y_1, y_2) in most cases denote flow variables. As in classical mechanics the forces usually can be split into an x – dependent and a y – dependent part

$$f(x_1, x_2, y_1, y_2) = f^x(x_1, x_2) + f^y(y_1, y_2)$$

In physics the most important cases are cases where all the forces are only x – dependent, in economics the most important cases are the cases where all the forces are y- dependent.

In physics the Helmholtz decomposition of a vector field in dimension 3 into a rotation free gradient field and a divergence free rotation field is well known and widely used. But in economics the applications are usually not restricted to dimension 3 but could be of any dimension. Therefore it is essential to realize that a Helmholtz decomposition is not only possible in dimension 3, but in any dimension n (Glötzl 2016).Without going into details of the definition of the operator ROT therefore in economics the forces have the form

$$f(x_1, x_2, y_1, y_2) = f^y(y_1, y_2) = grad\, U(y_1, y_2) + ROT\, V(y_1, y_2)$$

where U could be called (overall -) utility function. and V could be called (overall -) rotation potential.

Usually in economics the variables are related not to coordinates but to agents $A, B, C, ...$ If we assume for the sake of simplicity that there is only just 1 variable related to 1 agent we get the following Helmholtz Decomposition of the forces in economics

$$f(y^A, y^B) = grad\, U(y^A, y^B) + ROT\, V(y^A, y^B)$$

In economics two of the most influential schools are Keynesian school and the Neoclassical school. There is one big difference (of course beside many others) according to the relevance of utility functions. Neoclassic economists on the one hand assume that the behaviour of economy could be described alone with the gradient of 1 (overall -) utility function $grad\, U(y^A, y^B)$, which in this case should be called master utility function. Keynesians on the other hand refuse the significance of utility functions and describe the economy with general forces $f(y^A, y^B)$. With respect to Helmholtz Decomposition one could state this Keynesian refusal of utility functions in the following positive way: Keynesians assume that rotational forces play an important role in economics.

Beside the description of the above formulated general case of GCD – models, the most important case of GCD – models is the case of **"utility GCD – models"** where the forces could be described by individual utility functions U^A, U^B and power-factors $\mu_A^A, \mu_A^B, \mu_B^A, \mu_B^B$

$$x_A' = y_A$$
$$x_B' = y_B$$
$$y_A' == \mu_A^A \frac{\partial U^A(y_A, y_B)}{\partial y_A} + \mu_A^B \frac{\partial U^B(y_A, y_B)}{\partial y_A} + \lambda \frac{\partial CC(x_A, x_B, y_A, y_B)}{\partial y_A}$$
$$y_B' == \mu_B^A \frac{\partial U^A(y_A, y_B)}{\partial y_B} + \mu_B^B \frac{\partial U^B(y_A, y_B)}{\partial y_B} + \lambda \frac{\partial CC(x_A, x_B, y_A, y_B)}{\partial y_B}$$
$$CC(x_A, x_B, y_A, y_B) = 0$$

Utility GCD – models are insofar important as this system of differential equations is a formalisation of the **individual optimisation strategies** of agents A, B. Beside the constraint this formula could be interpreted in the following way:

Both agents A, B act with a force to change the variable y_A. A tries to change y_A to optimize his individual utility U^A along the gradient of his utility corrected by the power he has to enforce his decision. But also B has a so called mutual influence on y_A and tries to change y_A to optimize his individual utility U^B along the gradient of his utility corrected by the power he has to enforce his decision. In the same way both agents A, B act with a force to change the variable y_B.
Thus summarising utility GCD – models are characterised by three key features:

1. mutual influence

2. power-factors

3. constraints.

In chapter 6 we discuss how these features could also be applied in game theoretical models especially the prisoner's dilemma.

Remark:

The description of the forces in utility GCD – models by individual utilities U^A, U^B does in general not imply the existence of a master utility function U , which means that the forces cannot be described alone by the gradient of a master utility function but are in general (such as general forces) the sum of a gradient part and a rotation part according to the Helmholtz Decomposition. But of course there exist necessary conditions for the form of the individual utility functions and the power-factors which guarantee the existence of a master utility function (e.g. "quasi – linearity of utility functions or U^A depends only on y_A and not on y_B or all power factors concerning an agent are equal).

5. Comparison of the equations of continuous game theory with the equations of GCD – models.

Beside the trivial case of the continuous prisoner's dilemma we discussed in chapter 3 and the case of GCD – models we discussed in chapter 4 there are 3 types of models which could be seen as continuous time, continuous decision space game theoretical models:

1. Evolutionary Game Theory

An excellent introduction to Evolutionary Game Theory can be found in Nowak (2006a) or Sigmund (2011). The first ideas about Evolutionary Game Theory appeared in papers by Hamilton (1964), Trivers (1971), and Maynard Smith & Price (1973).
Evolutionary Game Theory is the study of frequency – dependent selection. Games can be formulated in terms of a payoff matrix, which specifies the payoff for one strategy when interacting with another. Evolutionary Game Theory interprets payoff as fitness: successful strategies reproduce faster. The so called "replicator equation" describes the deterministic evolutionary game dynamics. If M is some payoff matrix the time evolution of the frequency y_i of the phenotype i is given by

$$\dot{y}_i' = y_i\left((My)_i - y^T My\right)$$

2. Adaptive Dynamics

Adaptive Dynamics was introduced by Nowak & Sigmund (1990) to study the evolution of cooperation in the repeated prisoner's dilemma. A comprehensive list of papers about Adaptive Dynamics is continuously updated by E. Kisdi (2015).

Adaptive Dynamics extends Evolutionary Game Theory in a number of respects. The basic equation is called "canonical equation" which is also called replicator-mutator equation and describes the evolution of the frequency of mutating phenotypes along the gradient of the "invasion fitness function f"

$$y_i' = \frac{1}{2}\mu_i n_i \sigma_i^2 \frac{\partial f_i(y)}{\partial y_i}$$

3. Differential Games

The first to study Differential Games was Rufus Isaacs in 1951 (published Isaacs, R. (1965)).

Differential Games are related closely to optimal control problems. In an optimal control problem there is a single control $u(t)$ and a single criterion to be optimized. Differential game theory generalizes this to two controls $u^A(t), u^B(t)$ and 2 different objectives (goals)

$$J^A = \int \mathcal{L}^A\,(\mathrm{t}, \mathrm{y}, u^A, u^B)dt\,,\ J^B = \int \mathcal{L}^B\,(t, y, u^A, u^B)dt$$

- one for each player. Each player attempts to control the state of the system, as to achieve his goal the system responds to the inputs of all players.

The basic equations are Euler like equations. Without going into details they have the form

$$y_i' = f\left(t,\, y,\, u^A,\, u^B\right)$$
$$q1' = -\, q1\, \frac{\partial f\left(t,\, y,\, u^A,\, u^B\right)}{\partial y} + \frac{\partial L^A\left(t,\, y,\, u^A,\, u^B\right)}{\partial y}$$
$$q2' = -\, q2\, \frac{\partial f\left(t,\, y,\, u^A,\, u^B\right)}{\partial y} + \frac{\partial L^B\left(t,\, y,\, u^A,\, u^B\right)}{\partial y}$$

Comparing the basic equations of continuous game theoretical models one can state:

- Explicit constraints can only be found in GCD-models. The term $y^T My$ in the replicator equation of Evolutionary Game Theory is due to the constrained condition of constant population size.
- All of the continuous game theoretic models can be seen in some sense as special cases of the general GCD – model or the utility GCD – model.

For an overview we put all equations together in the next table. For the sake of simplicity we consider only GCD – models which do not depend on "stock variables x_i" and omit the equations $x_i' = y_i$ which hold by definition as well as the constraints (assumed to be non-holonomic).

general GCD (without utility)	$y' = f(y) + \lambda.grad^y CC(x,y) = grad\, U(y) + ROT\, V(y) + \lambda.grad^y CC(x,y)$
utility GCD	$y_i' = \sum_j \mu_i^j \frac{\partial U^j(y)}{\partial y_i} + \lambda.\frac{\partial CC(y)}{\partial y_i}$
continous prisoners dilemma	$y_i' = \frac{\partial U^i(y_A,\, y_B)}{\partial y_i} = 1$
evolutionary game theory : "replicator equation"	$y_i' = y_i\left(\left(My\right)_i - y^T My\right)$
adaptive dynamics : "canonical equation"	$y_i' = \frac{1}{2}\mu_i n_i \sigma_i^2 \frac{\partial f_i(y)}{\partial y_i}$
differential games	$y_i' = f\left(t,\, y,\, u^A,\, u^B\right)$ $q1' = -\, q1\, \frac{\partial f\left(t,\, y,\, u^A,\, u^B\right)}{\partial y} + \frac{\partial L_1\left(t,\, y,\, u^A,\, u^B\right)}{\partial y},$ $q2' = -\, q2\, \frac{\partial f\left(t,\, y,\, u^A,\, u^B\right)}{\partial y} + \frac{\partial L_2\left(t,\, y,\, u^A,\, u^B\right)}{\partial y},$

6. Applying the key features of GCD – models to the standard prisoner's dilemma, some more mechanisms for evolution of cooperation.

Nowak (2006b) and Taylor & Nowak (2007) present 5 mechanisms which make it possible that in the iterated prisoner's dilemma cooperation can evolve:

- direct reciprocity
- indirect reciprocity
- kin selection
- group selection
- graph selection

Each of these mechanisms leads to a transformation of the standard prisoner's dilemma payoff matrix

$$M = \begin{pmatrix} S & P \\ R & T \end{pmatrix}$$

to a new Matrix

$$\tilde{M} = \begin{pmatrix} b & d \\ a & c \end{pmatrix}$$

where a, b, c, d depend on R, S, T, P and the special parameters of the particular mechanism.

A necessary condition that in the iterated prisoner's dilemma cooperation can evolve is that cooperation is an ESS (evolutionary stable strategy) or defection is not an ESS which is equivalent to the **"cooperation condition"**

$$a > c \quad or \qquad b \geq d$$

For each mechanism Nowak (2006b) and Taylor & Nowak (2007) give conditions (which depend on R, S, T, P and special parameters of the particular mechanism) such that the cooperation condition holds.

Now the question is whether there are other reasonable mechanisms which lead to a transformation of the payoff matrix such that the cooperation condition hold. Let us call the 5 above mechanisms "stage 1 mechanisms". We propose 3 more possible mechanisms which could enable the evolution of cooperation where at least the last two stages are inspired by the key features of GCD – models. All of these mechanisms are of big importance especially for the cooperation between humans. Stage 2 and stage 3 are also possible mechanisms for animal cooperation. The stage 4 mechanism, which is based on the ability of humans to negotiate agreements, is only possible for humans and not for animals.

stage 2: external change of payoff matrix (utility function)

Cooperation cannot only be enabled by internal interactions of the members of the population but also by an "external" intervention, i.e. the punishment or taxation of non-cooperative behaviour. In animal societies this external intervention to enforce cooperative behaviour can come from the "top dog". In human societies the role of the top dog was first taken on by god as a virtual top dog, followed by priests, chieftains and finally by governments. Assume that the punishment resp. taxation of non-cooperative behaviour is 2 units, then the payoff matrices of A and B change in the following way:

$$M^A = \begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix} \quad \rightarrow \quad \tilde{M}^A = \begin{pmatrix} 0 & 1-2 \\ 2 & 3-2 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ \boxed{2} & 1 \end{pmatrix}$$

$$M^B = \begin{pmatrix} 3 & 1 \\ 2 & 0 \end{pmatrix} \quad \rightarrow \quad \tilde{M}^B = \begin{pmatrix} 3-2 & 1-2 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ \boxed{2} & 0 \end{pmatrix}$$

Therefore both A and B will cooperate and get the maximum payoff of 2 units.

stage 3: "mutual influence" and power factors

The external influence on the behaviour does not have to be based only on the intervention of 1 top dog on all the members of the population, but could come from all individual members. That means for 2 players that the behaviour of A is not only determined by individual optimisation of his individual payoff (utility), but is also determined by the payoff which B can get if B is able to influence the behaviour of A respectively the influence of B to change the payoff matrix of A. The power of A to assert his own behaviour and the power of A to alter the behaviour of B is determined by the particular power factors in full analogy to GCD – models.

In animal societies such a mechanism could be the basis for the evolution of pecking orders. In human societies the political power of different political groups can lead to the formation of labour unions and other democratic or feudal structures.

Denoting the payoff matrix of A depending on his own decisions M_A^A ,

denoting the payoff matrix of B depending on his own decisions M_B^B ,

and setting

$$M_A^A = \begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix} \qquad M_B^B = \begin{pmatrix} 3 & 1 \\ 2 & 0 \end{pmatrix}$$

and assuming that B can change the payoff matrix of A by a payoff matrix M_B^A

and assuming that A can change the payoff matrix of B by a payoff matrix M_A^B

and setting

$$M_B^A = \begin{pmatrix} 0 & -2 \\ 0 & -2 \end{pmatrix} \qquad M_A^B = \begin{pmatrix} -2 & -2 \\ 0 & 0 \end{pmatrix}$$

and denoting the particular power factors with μ_A^A, μ_B^A, μ_A^B, μ_B^B,

then the payoff matrices of A and B will transform in the following way:

$$M_A^A = \begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix} \qquad \rightarrow \qquad \tilde{M}_A^A = \mu_A^A M_A^A + \mu_B^A M_B^A = \mu_A^A \begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix} + \mu_B^A \begin{pmatrix} 0 & -2 \\ 0 & -2 \end{pmatrix}$$

$$M_B^B = \begin{pmatrix} 3 & 1 \\ 2 & 0 \end{pmatrix} \qquad \rightarrow \qquad \tilde{M}_B^B = \mu_B^B M_B^B + \mu_A^B M_A^B = \mu_B^B \begin{pmatrix} 3 & 1 \\ 2 & 0 \end{pmatrix} + \mu_A^B \begin{pmatrix} -2 & -2 \\ 0 & 0 \end{pmatrix}$$

Setting e.g. $\mu_A^A = \mu_B^B = 1$ yields, dependent on the particular power factors to influence the payoff matrix of the other player μ_B^A, μ_A^B, the following resulting payoffs and the following behaviour

$$for\ \mu_B^A = 0\ \ and\ \ \mu_A^B = 0 \qquad \tilde{M}_A^A = \begin{pmatrix} 0 & \boxed{1} \\ 2 & 3 \end{pmatrix} \qquad \tilde{M}_B^B = \begin{pmatrix} 3 & \boxed{1} \\ 2 & 0 \end{pmatrix} \qquad A\ defect,\ B\ defect$$

$$for\ \mu_B^A = 1\ \ and\ \ \mu_A^B = 0 \qquad \tilde{M}_A^A = \begin{pmatrix} \boxed{0} & -1 \\ 2 & 1 \end{pmatrix} \qquad \tilde{M}_B^B = \begin{pmatrix} \boxed{3} & 1 \\ 2 & 0 \end{pmatrix} \qquad A\ cooperate,\ B\ defect$$

$$for\ \mu_B^A = 0\ \ and\ \ \mu_A^B = 1 \qquad \tilde{M}_A^A = \begin{pmatrix} 0 & 1 \\ 2 & \boxed{3} \end{pmatrix} \qquad \tilde{M}_B^B = \begin{pmatrix} 1 & -1 \\ 2 & \boxed{0} \end{pmatrix} \qquad A\ defect,\ B\ cooperate$$

$$for\ \mu_B^A = 1\ \ and\ \ \mu_A^B = 1 \qquad \tilde{M}_A^A = \begin{pmatrix} 0 & -1 \\ \boxed{2} & 1 \end{pmatrix} \qquad \tilde{M}_B^B = \begin{pmatrix} 1 & -1 \\ \boxed{2} & 0 \end{pmatrix} \qquad A\ cooperate,\ B\ cooperate$$

stage 4: “agreements”, behavioural constrained conditions

Because only humans have such a highly developed brain to be able to think logically and speak with abstract language to each other, only humans are able to recognise whether they are in a prisoner’s dilemma situation. After this awareness they are able to negotiate a cooperation agreement. Thus the highly developed brain of humans is the reason why it is much easier for humans to develop cooperation than for all other animals. As cooperation is a fundamental advantage in evolution, mankind is able to dominate the whole biosphere. Finally, nearly all laws imposed by governments are agreements to overcome prisoner’s dilemma situations. Formally all such cooperation agreements between two agents or laws can be understood as constraints on behaviour, which guarantee that the individual optimisation strategies of the individuals lead to an overall optimum for the community. Formally these constraints lead to a smaller decision space.

E.g. the constraint

$$CC(y^A, y^B) = U^A(y^A, y^B) - U^B(y^A, y^B) = 0$$

would restrict the decision space to the diagonal in the following way:

$$M_A^A = \begin{pmatrix} \times & 1 \\ 2 & \times \end{pmatrix} \qquad M_B^B = \begin{pmatrix} \times & 1 \\ 2 & \times \end{pmatrix}$$

Some typical economic examples are agreements on a minimum wage or other restrictions with respect to inequality.

We discuss the concept of these additional 3 mechanisms with respect to political economy in the next chapter 7.

Of course not only behavioural constraints are important, but also constraints on resources are important in economics. But obviously such constraints on resources are important not only in economic but also in biological systems.

Important concluding remark:

Of course one could argue that these stage 2, 3 and 4 mechanisms are very simple compared with the 5 mechanisms for the evolution of cooperation of stage 1. But that is just the point. For the 5 stage 1 mechanisms there are at most only very weak complementary conditions necessary for the evolution of cooperation besides reproduction, mutation and selection. But the “higher” the evolution gets, the more skills individuals have and therefore the mechanisms which enable the evolution of cooperation become easier and easier.

7. Application to political economy

In a small tribal group economy in principle it should be possible to recognise, which behaviour all the members need to display in order to achieve an overall optimum. In such a small community behaviour that is optimal for the group can be enforced by the chief. But the bigger and therefore the more complex an economy is, the more difficult it will be to recognise the overall optimising behaviour for all agents even by the use of supercomputers. In its core that is the reason why any larger centrally planned economy has failed. But if a planned overall optimisation is to fail, the only possibility to organise an economy is to organise it by individual optimisation strategies by their members. That is just the key organisation principle of free market economy. However, the assumption or axiom of the free market economy that this individual optimisation strategy will lead to an overall optimum by virtue of Adam Smith's invisible hand is fundamentally wrong. The big question of political economy is to analyse which additional measures could guarantee that the individual optimisation strategy characterising a free market economy leads to an overall optimum. For all such measures it should be possible to formalise them as 1 of the 3 additional mechanisms.

From this point of view the different economic theories could be characterised in terms of which measures they assume to be sufficient to guarantee an overall optimum without abandoning the principle of individual optimisation:

Neoliberalism:

The fundamental axiom of Neoliberalism is the assumption that in general no strong measures are necessary to guarantee an overall optimum i.e. individual optimisation is sufficient. This assumption is fundamentally wrong if one recognises that prisoner's dilemma situations are not the exceptional case but the normal case when individuals live together.

Social Market Economy:

Social Market Economy in fact rejects the fundamental axiom of neoliberalism, but assumes that constraints on the distribution of welfare are enough to guarantee so that individual optimisation leads to an overall optimum.

Real – Socialism:

The aim of Real – Socialism is to achieve equality in society. But obviously what one can learn from prisoner's dilemma is that a constraint condition in the sense of equal utility for all individuals does not necessarily lead to an overall optimum as in the prisoner's dilemma even the worst solution fulfils the equality condition. The fundamental axiom of Real – Socialism is that it is impossible to reach an overall optimum by individual optimisation strategy no matter how strictly the constraints are set. The logical conclusion of Real-Socialism therefore is that only a planned economy can reach an overall optimum. As mentioned above this assumption underestimates the complexity of a large economy and therefore Real – Socialism has failed in the past.

Keynes:

In some sense the Keynesian economic theory may be the best compromise. On the one hand it accepts that because of the complexity of economics an individual optimisation strategy is unavoidable as a starting point. On the other hand Keynesians recognise that a steering of core parameters of the macro economy is possible and necessary and strong constraints are necessary to lead the economy to an overall optimum. They propose e.g.

- strong external influence by the government in the sense of stage 2 mechanisms, which means a broad range of strong regulations
- measures to equalise the economic power of the economic agents in the sense of stage 3 mechanisms
- strong constraints e.g. with respect to the distribution in the sense of stage 4 mechanisms.

8. Conclusion

GCD – models are inspired by the dynamics of classical mechanics with constraints. They can be thought of as a framework not only for most of the economic models but also for game theoretical models in continuous time. They are also strongly related to standard game theoretical models in discrete time. A simple bridge between discrete time game theoretical models and continuous time game theoretical models is the continuous time, continuous decision space prisoner's dilemma.

This relationship between all these models could be the basis for mutual fruitful inspiration. Applying the key features of GCD – models to the standard prisoner's dilemma leads to the formulation of 3 more mechanisms which can enable the evolution of cooperation especially in human societies. All these considerations lead to a more formal classification of important economic theories from the point of view of political economy.

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