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Nowcasting Nominal GDP with the Credit-Card Augmented Divisia Monetary Aggregates

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While credit cards provide transactions services, as do currency and demand deposits, credit cards have never been included in measures of the money supply. The reason is accounting conventions, which do not permit adding liabilities, such as credit card balances, to assets, such as money. However, economic aggregation theory and index number theory measure service flows and are based on microeconomic theory, not accounting. We derive theory needed to measure the joint services of credit cards and money. Carried forward rotating balances are not included in the current period weakly separable block, since they were used for transactions services in prior periods. The theory is developed for the representative consumer, who pays interest for the services of credit cards during the period used for transactions. This interest rate is reported by the Federal Reserve as the average over all credit card accounts, including those not paying interest. Based on our derived theory, we propose an empirical measurement of the joint services of credit cards and money. These new Divisia monetary aggregates are widely relevant to macroeconomic research.[1] We evaluate the ability of our money aggregate measures to nowcast nominal GDP. This is currently topical, given proposals for nominal GDP targeting, which require monthly measures of nominal GDP. The nowcasts are estimated using only real time information, as available for policy makers at the time predictions are made. We use a multivariate state space model that takes into account asynchronous information inflow, as proposed in Barnett, Chauvet, and Leiva-Leon (2016). The model considers real time information that arrives at different frequencies and asynchronously, in addition to mixed frequencies, missing data, and ragged edges. The results indicate that the proposed parsimonious model, containing information on real economic activity, inflation, and the new augmented Divisia monetary aggregates, produces the most accurate real time nowcasts of nominal GDP growth. In particular, we find that inclusion of the new aggregate in our nowcasting model yields substantially smaller mean squared errors than inclusion of the previous Divisia monetary aggregates.

Keywords: Credit Cards, Money, Credit, Aggregation Theory, Index Number Theory, Divisia Index, Risk, Asset Pricing, Nowcasting, Indicators.

JEL Classification: C43, C53, C58, E01, E3, E40, E41, E51, E52, E58, G17.

[1]The source of our data is the Center for Financial Stability, which is preparing to update and publicly release our new extended Divisia monetary aggregates monthly. The new aggregates will also be available monthly to Bloomberg terminal users. Our model is specific to credit cards, and not to store cards or charge cards.

1. Introduction

Most models of the monetary policy transmission mechanism operate through interest rates, and often involve a monetary or credit channel, but not both. See, e.g., Bernanke and Blinder (1988) and Mishkin (1996). In addition, there are multiple versions of each mechanism, usually implying different roles for interest rates during the economy's adjustment to central bank policy actions. However, there is a more fundamental reason for separating money from credit. While money is an asset, credit is a liability. In accounting conventions, assets and liabilities are not added together. But aggregation theory and economic index number theory are based on microeconomic theory, not accounting conventions. Economic aggregates measure service flows. To the degree that money and some forms of credit produce joint services, those services can be aggregated.

A particularly conspicuous example is credit card services, which are directly involved in transactions and contribute to the economy's liquidity in ways not dissimilar to those of money.[2] While this paper focuses on aggregation over monetary and credit card services, the basic principles could be relevant to some other forms of short term credit that contribute to the economy's liquidity services, such as checkable lines of credit.

While money is both an asset and part of wealth, credit cards are neither. Hence credit cards are not money. To the degree that monetary policy operates through a wealth effect (Pigou effect), as advocated by Milton Friedman, credit cards do not play a role. But to the degree that the flow of monetary services is relevant to the economy, as through the demand for monetary services or as an indicator measure, the omission of credit card services from "money" measures induces a loss of information. For example, Duca and Whitesell (1995) showed that a higher probability of credit card ownership was correlated with lower holdings of monetary transactions balances. Clearly credit card services are a substitute for the

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[2] We are indebted to Apostolos Serletis for his suggestion of this topic for research. His suggestion is contained in his presentation as discussant of Barnett's Presidential Address at the Inaugural Conference of the Society for Economic Measurement at the University of Chicago, August 18-20, 2014. The slides for Serletis's discussion can be found online at http://sem.society.cmu.edu/conference1.html.

services of monetary transactions balances, perhaps to a much higher degree than the services of many of the assets included in traditional monetary aggregates, such as the services of nonnegotiable certificates of deposit.

In this seminal paper, we use strongly simplifying assumptions. We assume credit cards are used to purchase consumer goods. All purchases are made at the beginning of periods, and payments for purchases are either by credit cards or money. Credit card purchases are repaid to the credit card company at the end of the current period or at the end of a future period, plus interest charged by the credit card company. Stated more formally, all discrete time periods are closed on the left and open on the right. After aggregation over consumers, the expected interest rate paid by the representative credit card holder can be very high, despite the fact that some consumers pay no interest on credit card balances. Future research is planned to disaggregate to heterogeneous agents, including consumers who repay soon enough to owe no interest. In the current model, such consumers affect the results only by decreasing the average interest rate paid by consumers on credit card balances aggregated over consumers.

To reflect the fact that money and credit cards provide services, such as liquidity and transactions services, money and credit are entered into a derived utility function, in accordance with Arrow and Hahn's (1971) proof.[3] The derived utility function absorbs constraints reflecting the explicit motives for using money and credit card services. Since this paper is about measurement, we need only assume the existence of such motives. In the context of this research, we have no need to work backwards to reveal the explicit motives. As has been shown repeatedly, any

[3] Our research in this paper is not dependent upon the simple decision problem we use for derivation and illustration. In the case of monetary aggregation, Barnett (1987) proved that the same aggregator functions and index numbers apply, regardless of whether the initial model has money in the utility function or production function, so long as there is intertemporal separability of structure and separability of components over which aggregation occurs. That result is equally as applicable to our current results with augmented aggregation over monetary asset and credit card services. While this paper uses economic index number theory, it should be observed that there also exists a statistical approach to index number theory. That approach produces the same results, with the Divisia index interpreted to be the Divisia mean using expenditure shares as probability. See Barnett and Serletis (1990).

of those motives, including the highly relevant transactions motive, are consistent with existence of a derived utility function absorbing the motive.[4]

Based on our derived theory, we propose an empirical measurement of the joint services of credit cards and money. These new Divisia monetary aggregates are widely relevant to macroeconomic research.[5] We evaluate the ability of our monetary services aggregate to nowcast nominal GDP. This objective is currently topical, given proposals for nominal GDP targeting, which requires monthly measures of nominal GDP. The nowcasts are estimated using only real time information as available for policy makers at the time predictions are made. We use a multivariate state space model that takes into account asynchronous information inflow and potential parameter instability: the DYMIBREAK model proposed in Barnett, Chauvet, and Leiva-Leon (2016). The model considers real time information arriving at different frequencies and asynchronously, and takes into account potential nonstationarity, in addition to mixed frequencies, missing data, and ragged edges. The results indicate that the proposed model, containing information on real economic activity, inflation, interest rates, and the new Divisia monetary aggregates, produces the most accurate real time nowcasts of nominal GDP growth. In particular, we find that the inclusion of the new aggregate measures in our nowcasting model yields substantially smaller mean squared errors than inclusion of the previous Divisia monetary aggregates, which in turn had performed substantially better than the official simple sum monetary aggregates.

2. Intertemporal Allocation

[4] The aggregator function is the derived function that always exists, if monetary and credit card services have positive value in equilibrium. See, e.g., Samuelson (1948), Arrow and Hahn (1971), Stanley Fischer (1974), Phlips and Spinnewyn (1982), Quirk and Saposnik (1968), and Poterba and Rotemberg (1987). Analogously, Feenstra (1986, p. 271) demonstrated “a functional equivalence between using real balances as an argument of the utility function and entering money into liquidity costs which appear in the budget constraints.” The converse mapping from money and credit in the utility function back to the explicit motive is not unique. But in this paper we are not seeking to identify the explicit motives for holding money or credit card balances.

[5] The source of our data is the Center for Financial Stability, which is preparing to update and publically release our new extended Divisia monetary aggregates monthly. The new aggregates will also be available monthly to Bloomberg terminal users. Our model is specific to credit cards, and not to store cards or charge cards.

We begin by defining the variables in the risk neutral case for the representative consumer:

$\mathbf{x}_s$ = vector of per capita (planned) consumptions of N goods and services (including those of durables) during period s.

$\mathbf{p}_s$ = vector of goods and services expected prices, and of durable goods expected rental prices during period s.

m_{is} = planned per capita real balances of monetary asset i during period s $(i = 1,2, \ldots, n)$.

c_{js} = planned per capita real expenditure with credit card type j for transactions during period s $(j = 1,2, \ldots, k)$. In the jargon of the credit card industry, those contemporaneous expenditures are called "volumes."

z_{js} = planned per capita rotating real balances in credit card type j during period s from transactions in previous periods $(j = 1,2, \ldots, k)$.

$y_{js} = c_{js} + z_{js}$= planned per capita total balances in credit type j during period s $(j = 1,2, \ldots, k)$.

r_{is} = expected nominal holding period yield (including capital gains and losses) on monetary asset i during period s $(i = 1,2, \ldots, n)$.

e_{js} = expected interest rate on c_{js}.

$\overline{e}_{js}$ = expected interest rate on z_{js}.

A_s = planned per capita real holdings of the benchmark asset during period s.

R_s = expected (one-period holding) yield on the benchmark asset during period s.

L_s = per capita labor supply during period s.

w_s = expected wage rate during period s.

The benchmark asset is defined to provide no services other than its expected yield, R_s, which motivates holding of the asset solely as a means of accumulating wealth. As a result, R_s is the maximum expected holding period yield available to consumers in the economy in period s from holding a secured asset. The benchmark asset is held to transfer wealth by consumers between multiperiod planning horizons, rather than to provide liquidity or other services. In contrast, $\overline{e}_{js}$ is not the interest rate on an asset and is not secured. It is the interest rate on an unsecured liability, subject to substantial default and fraud risk. Hence, $\overline{e}_{js}$ can be higher than the benchmark asset rate, and historically has always been much higher than the benchmark asset rate.[6]

It is important to recognize that the decision problem we model is not of a single economic agent, but rather of the "representative consumer," aggregated over all consumers. All quantities are therefore averaged over all consumers. Gorman's assumptions for the existence of a representative consumer are implicitly accepted, as is common in almost all modern macroeconomic theory having microeconomic foundations. This modeling assumption is particularly important in understand the credit card quantities and interest rates used in our research. About 20% of credit card holders in the United States do not pay explicit interest on credit card balances, since those credit card transactions are paid off by the end of the period. But the 80% who do pay interest pay very high interest rates.[7] The Federal Reserve provides two interest rate series for credit card debt. One, $\overline{e}_{js}$, includes interest only on accounts that do pay interest to the credit card issuing banks, while the other series, e_{js}, includes the approximately 20% that do not pay interest. The latter interest rate is thereby lower, since it is averaged over interest paid on both categories of accounts. Since we are modeling the representative consumer, aggregated over all consumers, e_{js} is always less than $\overline{e}_{js}$ for all j and s. The interest

[6] We follow the Center for Financial Stability (CFS) and the Bank of Israel in using the short term bank loan rate as a proxy for the benchmark rate. That interest rate has always exceeded the interest rate paid by banks on deposit accounts and on all other monetary assets used in the CFS Divisia monetary aggregates, and has always been lower than the Federal Reserve's reported average interest rate charged on credit card balances. For detailed information on CFS data sources, see Barnett, Liu, Mattson, and Noort (2013). For the additional data sources used by the CFS to extend to credit card services, see Barnett and Su (2016).

[7] The following statement is from www.motherjones.com/kevin-drum/2011/10/americans-are-clueless-about-their-credit-card-debt. "In the four working age categories, about 50% of households think they have outstanding credit card debt, but the credit card companies themselves think about 80% of households have outstanding balances." Since these percentages are of total households, including those having no credit cards, the percent of credit card holders paying interest might be even higher.

rate on rotating credit card balances, $\overline{e}_{js}$, is paid by all consumers who maintain rotating balances in credit cards. But e_{js} is averaged over those consumers who maintain such rotating balances and hence pay interest on contemporaneous credit card transactions (volumes) and those consumers who pay off such credit card transactions before the end of the period, and hence do not pay explicit interest on the credit card transactions. The Federal Reserve provides data on both $\overline{e}_{js}$ and e_{js}. Although e_{js} is less than $\overline{e}_{js}$, e_{js} also has always been higher than the benchmark rate. This observation is a reflection of the so-called credit card debt puzzle.[8]

We use the latter interest rate, e_{js}, in our augmented Divisia monetary aggregates formula, since the contemporaneous per capita transactions volumes in our model are averaged over both categories of credit card holders. We do not include rotating balances used for transactions in prior periods, since to do so would involve double counting of transactions services.

The expected interest rate, e_{js}, can be explicit or implicit, and applies to the aggregated representative consumer. For example, an implicit part of that interest rate could be in the form of an increased price of the goods purchased or in the form of a periodic service fee or membership fee. But we use only the Federal Reserve's average explicit interest rate series, which is lower than the one that would include implicit interest. Nevertheless, that downward biased explicit rate of return to credit card companies, e_{js}, aggregated over consumers, tends to be very high, far exceeding R_s, even after substantial losses from fraud.

It is also important to recognize that we are using the credit card industry's definition of "credit card," which excludes "store cards" and "charge cards." According to the trade's definition, "store cards" are issued by businesses providing credit only for their own goods, such as gasoline company credit cards or department store cards. To be a "credit card" by the trade's definition, the card must be accepted for all goods in the economy not constrained to cash-only sales.

[8]See, e.g., Telyukova and Wright (2008), who view the puzzle as a special case of the rate dominance puzzle in monetary economics. The "credit card debt puzzle" asks why people do not pay down debt, when receiving low interest rates on deposits, while simultaneously paying higher interest rates on credit card debt.

"Charge cards" can be widely accepted for goods purchases, but do not charge interest, since the debt must be paid off by the end of the period. To be a "credit card," the card must provide a line of credit to the card holder with interest charged on purchases not paid off by the end of the period. For example, American Express provides both charge cards and credit cards. The first credit card was provided by Bank of America. There now are four sources of credit card services in the United States: Visa, Mastercard, Discover, and American Express. From American Express, we use only their credit card account services, not their charge cards. We use data from only those four sources, in accordance with the credit card industry's conventional definition of "credit card."

We let u_t be the representative consumer's current intertemporal utility function at time t over the T-period planning horizon. We assume that u_t is weakly separable in each period's consumption of goods and monetary assets, so that u_t can be written in the form

$$\begin{aligned} u_t &= u_t(\mathbf{m}_t, \dots, \mathbf{m}_{t+T};\ \mathbf{c}_t, \dots, \mathbf{c}_{t+T};\ \mathbf{x}_t, \dots, \mathbf{x}_{t+T};\ A_{t+T}) \\ &= U_t(v(\mathbf{m}_t, \mathbf{c}_t), v_{t+1}(\mathbf{m}_{t+1}, \mathbf{c}_{t+1}), \dots, v_{t+T}(\mathbf{m}_{t+T}, \mathbf{c}_{t+T}); \\ &\quad V(\mathbf{x}_t), V_{t+1}(\mathbf{x}_{t+1}), \dots, V_{t+T}(\mathbf{x}_{t+T}); A_{t+T}), \end{aligned} \tag{1}$$

for some monotonically increasing, linearly homogeneous, strictly quasiconcave functions, $v, v_{t+1}, \dots, v_{t+T}, V, V_{t+1}, \dots, V_{t+T}$.The function U_t also is monotonically increasing, but not necessarily linearly homogeneous. Note that $\mathbf{c}_t$, not $\mathbf{y}_t$, is in the utility function. The reason is that $\mathbf{y}_t$ includes rotation balances, $\mathbf{z}_t$, resulting from purchases in prior periods. To include $\mathbf{y}_t$ in the utility function would introduce a form of double counting into our aggregation theory by counting prior transactions services more than once. Those carried forward balances provided transactions services in previous periods and were therefore in the utility function for that period. Keeping those balances in the utility function for the current period would imply existence of a different kind of services from the transactions and liquidity services we are seeking to measure.

Dual to the functions, V and $V_s (s = t + 1, \dots, t + T)$, there exist current and planned true cost of living indexes, $p_t^* = p(\mathbf{p}_t)$ and $p_s^* = p_s^*(\mathbf{p}_s)(s = t + 1, \dots, t + T)$. Those indexes, which are the consumer goods unit cost functions, will be used to deflate all nominal quantities to real quantities, as in the definitions of m_{is}, c_{js}, and A_s above.

Assuming replanning at each t, we write the consumer's decision problem during each period $s(t \leq s \leq t + T)$ within the planning horizon to be to choose $(\mathbf{m}_t, \dots, \mathbf{m}_{t+T};\ \mathbf{c}_t, \dots, \mathbf{c}_{\mathrm{t+T}};\ \mathbf{x}_t, \dots, \mathbf{x}_{t+T};\ A_{t+T}) \geq \mathbf{0}$ to

$$\max u_t(\mathbf{m}_t, \dots, \mathbf{m}_{t+T};\ \mathbf{c}_t, \dots, \mathbf{c}_{t+T};\ \mathbf{x}_t, \dots, \mathbf{x}_{t+T};\ A_{t+T}),$$

subject to

$$\begin{aligned}\mathbf{p}_s'\mathbf{x}_s = w_s L_s &+ \sum_{i=1}^{n} \left[(1 + r_{i,s-1}) p_{s-1}^* m_{i,s-1} - p_s^* m_{is}\right] \\ &+ \sum_{j=1}^{k} \left[p_s^* c_{js} - (1 + e_{j,s-1}) p_{s-1}^* c_{j,s-1}\right] \\ &+ \sum_{j=1}^{k} \left[p_s^* z_{js} - \left(1 + \overline{e}_{j,s-1}\right) p_{s-1}^* z_{j,s-1}\right] + [(1 + R_{s-1}) p_{s-1}^* A_{s-1} \\ &- p_s^* A_s].\end{aligned} \tag{2}$$

Planned per capita total balances in credit type j during period s are then $y_{js} = c_{js} + z_{js}$.

Equation (2) is a flow of funds identity, with the right hand side being funds available to purchase consumer goods during period *s*. On the right hand side, the first term is labor income. The second term is funds absorbed or released by rolling over the monetary assets portfolio, as explained in Barnett (1980). The third term is particularly important to this paper. That term is the net change in credit card debt during period *s* from purchases of consumer goods, while the fourth term is the net change in rotating credit card debt. The fifth term is funds absorbed or released by rolling over the stock of the benchmark asset, as explained in Barnett (1980). The third term on the right side is specific to current period credit card purchases, while the fourth term is not relevant to the rest of our results, since z_{js} is not in the utility function. Hence z_{js} is not relevant to the user cost prices, conditional decisions, or aggregates in the rest of this paper.

Let

$$\rho_s = \begin{cases} 1, & if\ s = t, \\ \prod_{u=t}^{s-1}(1 + R_u), & if\ t + 1 \le s \le t + T. \end{cases} \tag{3}$$

We now derive the implied Fisherine discounted wealth constraint. The derivation procedure involves recursively substituting each flow of funds identity into the previous one, working backwards in time, as explained in Barnett (1980). The result is the following wealth constraint at time t:

$$\begin{aligned} &\sum_{s=t}^{t+T}\left(\frac{\mathbf{p}'_s}{\rho_s}\right)\mathbf{x}_s + \sum_{s=t}^{t+T}\sum_{i=1}^{n}\left[\frac{p_s^*}{\rho_s} - \frac{p_s^*(1+r_{is})}{\rho_{s+1}}\right]m_{is} + \sum_{i=1}^{n}\frac{p_{t+T}^*(1+r_{i,t+T})}{\rho_{t+T+1}}m_{i,t+T} + \frac{p_{t+T}^*}{\rho_{t+T}}A_{t+T} \\ &+\sum_{s=t}^{t+T}\sum_{j=1}^{k}\left[\frac{p_s^*(1+e_{js})}{\rho_{s+1}} - \frac{p_s^*}{\rho_s}\right]c_{js} + \sum_{s=t}^{t+T}\sum_{j=1}^{k}\left[\frac{p_s^*(1+\overline{e}_{js})}{\rho_{s+1}} - \frac{p_s^*}{\rho_s}\right]z_{js} - \sum_{j=1}^{k}\frac{p_{t+T}^*(1+e_{j,t+T})}{\rho_{t+T+1}}c_{j,t+T} \\ &-\sum_{j=1}^{k}\frac{p_{t+T}^*(1+\overline{e}_{j,t+T})}{\rho_{t+T+1}}z_{j,t+T} = \sum_{s=t}^{t+T}\left(\frac{w_s}{\rho_s}\right)L_s + \sum_{i=1}^{n}(1+r_{i,t-1})p_{t-1}^*m_{i,t-1} + (1+R_{t-1})A_{t-1}p_{t-1}^* \\ &-\sum_{j=1}^{k}(1+e_{j,t-1})p_{t-1}^*c_{j,t-1} - \sum_{j=1}^{k}(1+\overline{e}_{j,t-1})p_{t-1}^*z_{j,t-1}. \end{aligned} \tag{4}$$

It is important to understand that (4) is directly derived from (2) without any additional assumptions. As in Barnett (1978, 1980), we see immediately that the nominal user cost (equivalent rental price) of monetary asset holding m_{is} $(i = 1,2, \dots, n)$ is

$$\pi_{is} = \frac{p_s^*}{\rho_s} - \frac{p_s^*(1 + r_{is})}{\rho_{s+1}}.$$

So the current nominal user cost price, π_{it}, of m_{it} reduces to

$$\pi_{it} = \frac{p_t^*(R_t - r_{it})}{1 + R_t}. \tag{5}$$

Likewise, the nominal user cost (equivalent rental price) of credit card transactions services, c_{js} $(j = 1,2, \dots, k)$, is

$$\tilde{\pi}_{jt} = \frac{p_s^*(1 + e_{js})}{\rho_{s+1}} - \frac{p_s^*}{\rho_s}.$$

Finally, the current period nominal user cost, $\tilde{\pi}_{jt}$, of c_{jt} reduces to

$$\tilde{\pi}_{jt} = \frac{p_t^*(1 + e_{jt})}{1 + R_t} - p_t^* \tag{6}$$

$$= \frac{p_t^*(e_{jt} - R_t)}{1 + R_t}. \tag{7}$$

Equation (7) is a new result central to most that follows in this paper.[9] The corresponding real user costs are

$$\pi_{js}^* = \frac{\pi_{is}}{p_s^*} \tag{8a}$$

and

$$\tilde{\pi}_{js}^* = \frac{\tilde{\pi}_{jt}}{p_s^*}. \tag{8b}$$

Equation (6) is particularly revealing. To consume the transactions services of credit card type j, the consumer borrows p_t^* dollars per unit of goods purchased at the start of the period during which the goods are consumed, but repays the credit

[9] The same user cost formula applies in the infinite planning horizon case, but the derivation is different. The derivation applicable in that case is in the Appendix.

card company $p_t^*(1+e_{jt})$ dollars at the end of the period. The lender will not provide that one period loan to the consumer unless $e_{jt} > R_t$, because of the ability of the lender to earn R_t without making the unsecured credit card loan. The assumption that consumers do not have access to higher expected yields on secured assets than the benchmark rate does not apply to firms providing unsecured liabilities, such as credit card firms. Hence the user cost price in (7) is nonnegative.[10]

Equivalently, equation (7) can be understood in terms of the delay between the goods purchase date and the date of repayment of the loan to the credit card company. During the one period delay, the consumer can invest the cost of the goods purchase at rate of return R_t. Hence the net real cost to the consumer of the credit card loan, per dollar borrowed, is $e_{jt} - R_t$. Multiplication by the true cost of living index in the numerator of (7) converts to nominal dollars and division by $1 + R_t$ discounts to present value within the time period.

3. Conditional Current Period Allocation

We define $\mathcal{J}_t^*$ to be real, and $\mathcal{J}_t$ nominal, expenditure on augmented monetary services --- augmented to include the services of contemporaneous credit card transactions charges. The assumptions on homogeneous blockwise weak separability of the intertemporal utility function, (1), are sufficient for consistent two-stage budgeting. See Green (1964, theorem 4). In the first stage, the aggregated representative consumer selects real expenditure on augmented monetary services, $\mathcal{J}_t^*$, and on aggregate consumer goods for each period within the planning horizon, along with terminal benchmark asset holdings, A_{t+T}.

[10] Our model is of the representative consumer, aggregated over all credit card holders. In an extension to heterogeneous agents, we would separate out consumers who repay the credit card company soon enough to avoid interest on the loan. That possibility could be viewed as a special case of our current model, in which the consumer repays immediately. In that special case, there is no discounting between purchase and repayment, and no interest is charged. The services of the credit card company become a free good with user cost price of zero. The credit card debt then disappears from the flow of funds equation, (2), since the credit cards provide no net services to the economy, and serve as instantaneous intermediaries in payment of goods purchased with money. If this were the case for the representative consumer, aggregated over all consumers, the model would be of a charge card, not a credit card. As explained in section 2, we do not consider charge cards or store cards, only credit cards.

In the second stage, $\mathcal{J}_t^*$ is allocated over demands for the current period services of monetary assets and credit cards. That decision is to select $\mathbf{m}_t$ and $\mathbf{c}_t$ to

$$\max v(\mathbf{m}_t, \mathbf{c}_t), \tag{9}$$

subject to

$$\boldsymbol{\pi^*}_t'\mathbf{m}_t + \tilde{\boldsymbol{\pi}}^*{}_t'\mathbf{c}_t = \mathcal{J}_t^*, \tag{10}$$

where$\mathcal{J}_t^*$ is expenditure on augmented monetary services allocated to the current period in the consumer's first-stage decision.

The rotating balances, z_{js}, from previous periods, not used for transactions this period, add a flow of funds term to the constraints, (2), but do not appear in the utility function. As a result, z_{js} does not appear in the utility function, (9), or on the left side of equation (10), but does affect the right side of (10). To implement this theory empirically, we need data on total credit card transactions volumes each period, c_{js}, not just the total balances in the accounts, $c_{js} + z_{js}$.[11] While those volumes are much more difficult to find than credit card balances, we have been able to acquire those current period volumes from the annual reports of the four credit card companies. For details on available sources, see Barnett and Su (2016).

4. Aggregation Theory

[11] Credit card companies provide a line of credit to consumers, with interest and any late payments added after the due date. New purchases are added as debt to the balance after the due date has passed. Many consumers having balances, z_{js}, pay only the "minimum payment" due. That decision avoids a late charge, but adds the unpaid balance to the stock of debt and boosts the interest due. Depending upon the procedure for aggregating over consumers, the interest rate on c_{js} could be different from the interest rate on z_{js}, with the former interest rate being the one that should be used in our user cost formula.

The exact quantity aggregate is the level of the indirect utility produced by solving problem ((9),(10)):

$$\mathcal{M}_t = \max\ \{v(\mathbf{m}_t, \mathbf{c}_t) : \boldsymbol{\pi}_t' \mathbf{m}_t + \tilde{\boldsymbol{\pi}}_t' \mathbf{c}_t = \mathcal{J}_t\} \tag{11}$$

$$= \max\ \{v(\mathbf{m}_t, \mathbf{c}_t) : \boldsymbol{\pi}^{*}{}_t' \mathbf{m}_t + \tilde{\boldsymbol{\pi}}^{*}{}_t' \mathbf{c}_t = \mathcal{J}_t^*\},$$

where we define $\mathcal{M}_t = \mathcal{M}(\mathbf{m}_t, \mathbf{c}_t) = v(\mathbf{m}_t, \mathbf{c}_t)$ to be the "augmented monetary aggregate" --- augmented to aggregate jointly over the contemporaneous services of money and credit cards. The category utility function v is the aggregator function we assume to be linearly homogeneous in this section. Dual to any exact quantity aggregate, there exists a unique price aggregate, aggregating over the prices of the goods or services. Hence there must exist an exact nominal price aggregate over the user costs $(\boldsymbol{\pi}_t, \tilde{\boldsymbol{\pi}}_t)$. As shown in Barnett (1980,1987), the consumer behaves relative to the dual pair of exact monetary quantity and price aggregates as if they were the quantity and price of an elementary good. The same result applies to our augmented monetary quantity and dual user cost aggregates.

One of the properties that an exact dual pair of price and quantity aggregates satisfies is Fisher's factor reversal test, which states that the product of an exact quantity aggregate and its dual exact price aggregate must equal actual expenditure on the components. Hence, if $\Pi(\boldsymbol{\pi}_t, \tilde{\boldsymbol{\pi}}_t)$ is the exact user cost aggregate dual to $\mathcal{M}_t$, then $\Pi(\boldsymbol{\pi}_t, \tilde{\boldsymbol{\pi}}_t)$ must satisfy

$$\Pi(\boldsymbol{\pi}_t, \tilde{\boldsymbol{\pi}}_t) = \frac{\mathcal{J}_t}{\mathcal{M}_t}. \tag{12}$$

Since (12) produces a unique solution for $\Pi(\boldsymbol{\pi}_t, \tilde{\boldsymbol{\pi}}_t)$, we could use (12) to define the price dual to $\mathcal{M}_t$. In addition, if we replace $\mathcal{M}_t$ by the indirect utility function defined by (11) and use the linear homogeneity of v, we can show that $\Pi = \Pi(\boldsymbol{\pi}_t, \tilde{\boldsymbol{\pi}}_t)$, defined by (12), does indeed depend only upon $(\boldsymbol{\pi}_t, \tilde{\boldsymbol{\pi}}_t)$, and not upon $(\mathbf{m}_t, \mathbf{c}_t)$ or $\mathcal{J}_t$. See Barnett (1987) for a version of the proof in the case of monetary assets. The conclusion produced by that proof can be written in the form

$$\Pi(\boldsymbol{\pi}_t, \tilde{\boldsymbol{\pi}}_t) = [\underset{(\mathbf{m}_t, \mathbf{c}_t)}{max}\{v(\mathbf{m}_t, \mathbf{c}_t): \boldsymbol{\pi}_t'\mathbf{m}_t + \tilde{\boldsymbol{\pi}}_t'\mathbf{c}_t = 1\}]^{-1}, \tag{13}$$

which clearly depends only upon $(\boldsymbol{\pi}_t, \tilde{\boldsymbol{\pi}}_t)$.

Although (13) provides a valid definition of Π, there also exists a direct definition that is more informative and often more useful. The direct definition depends upon the cost function E, defined by

$$E(v_0, \boldsymbol{\pi}_t, \tilde{\boldsymbol{\pi}}_t) = \underset{(\mathbf{m}_t, \mathbf{c}_t)}{min}\{\boldsymbol{\pi}_t'\mathbf{m}_t + \tilde{\boldsymbol{\pi}}_t'\mathbf{c}_t: v(\mathbf{m}_t, \mathbf{c}_t) = v_0\},$$

which equivalently can be acquired by solving the indirect utility function equation (11) for $\mathcal{J}_t$ as a function of $\mathcal{M}_t = v(\mathbf{m}_t, \mathbf{c}_t)$ and $(\boldsymbol{\pi}_t, \tilde{\boldsymbol{\pi}}_t)$. Under our linear homogeneity assumption on v, it can be proved that

$$\begin{aligned} \Pi(\boldsymbol{\pi}_t, \tilde{\boldsymbol{\pi}}_t) &= E(1, \boldsymbol{\pi}_t, \tilde{\boldsymbol{\pi}}_t) \\ &= \min_{(\mathbf{m}_t, \mathbf{c}_t)}\{\boldsymbol{\pi}_t'\mathbf{m}_t + \tilde{\boldsymbol{\pi}}_t'\mathbf{c}_t: v(\mathbf{m}_t, \mathbf{c}_t) = 1\}, \end{aligned} \tag{14}$$

Which is often called the unit cost or price function.

The unit cost function is the minimum cost of attaining unit utility level for $v(\mathbf{m}_t, \mathbf{c}_t)$ at given user cost prices $(\boldsymbol{\pi}_t, \tilde{\boldsymbol{\pi}}_t)$. Clearly, (14) depends only upon$(\boldsymbol{\pi}_t, \tilde{\boldsymbol{\pi}}_t)$. Hence by (12) and (14), we see that $\Pi(\boldsymbol{\pi}_t, \tilde{\boldsymbol{\pi}}_t) = {}^{\mathcal{J}_t}/_{\mathcal{M}_t} = E(1, \boldsymbol{\pi}_t, \tilde{\boldsymbol{\pi}}_t)$.

5. Preference Structure over Financial Assets

5.1. Blocking of the Utility Function

While our primary objective is to provide the theory relevant to joint aggregation over monetary and credit card services, subaggregation separately over monetary asset services and credit card services can be nested consistently within the joint aggregates. The required assumption is blockwise weak separability of money and credit within the joint aggregator function. In particular, we would then assume the existence of functions $\tilde{\upsilon}$, g_1, g_2, such that

$$v(\mathbf{m}_t, \mathbf{c}_t) = \tilde{\upsilon}\big(g_1(\mathbf{m}_t), g_2(\mathbf{c}_t)\big), \tag{15}$$

with the functions g_1 and g_2 being linearly homogeneous, increasing, and quasiconcave.

We have nested weakly separable blocks within weakly separable blocks to establish a fully nested utility tree. As a result, an internally consistent multi-stage budgeting procedure exists, such that the structured utility function defines the quantity aggregate at each stage, with duality theory defining the corresponding user cost price aggregates.

In the next section we elaborate on the multi-stage budgeting properties of decision ((9),(10)) and the implications for quantity and price aggregation.

5.2. Multi-stage Budgeting

Our assumptions on the properties of v are sufficient for a two-stage solution of the decision problem ((9),(10)), subsequent to the two-stage intertemporal solution that produced ((9),(10)). The subsequent two-stage decision is exactly nested within the former one.

Let $M_t = M(\mathbf{m}_t)$ be the exact aggregation-theoretic quantity aggregate over monetary assets, and let $C_t = C(\mathbf{c}_t)$ be the exact aggregation-theoretic quantity aggregate over credit card services. Let $\Pi_m^* = \Pi_m(\boldsymbol{\pi}_t^*)$ be the real user costs aggregate (unit cost function) dual to $M(\mathbf{m}_t)$, and let $\Pi_c^* = \Pi_c(\tilde{\boldsymbol{\pi}}_t^*)$ be the user costs aggregate dual to $C(\mathbf{c}_t)$. The first stage of the two-stage decision is to select M_t and C_t to solve

$$\max_{(\mathbf{m}_t, \mathbf{c}_t)} \tilde{\upsilon}(M_t, C_t) \tag{16}$$

subject to

$$\Pi_m^* M_t + \Pi_c^* C_t = \mathcal{J}_t^*.$$

From the solution to problem (16), the consumer determines aggregate real expenditure on monetary and credit card services, $\Pi_m^* M_t$and $\Pi_c^* C_t$.

In the second stage, the consumer allocates $\Pi_m^* M_t$ over individual monetary assets, and allocates $\Pi_c^* C_t$ over services of individual types of credit cards. She does so by solving the decision problem:

$$\max_{\mathbf{m}_t} g_1(\mathbf{m}_t), \tag{17}$$

subject to

$$\boldsymbol{\pi}_t^{*\prime} \mathbf{m}_t = \Pi_m^* M_t.$$

Similarly, she solves

$$\max_{\mathbf{c_t}} g_2(\boldsymbol{c}_t), \tag{18}$$

subject to

$$\tilde{\boldsymbol{\pi}}_t^{*\prime} \mathbf{c}_t = \Pi_c^* C_t.$$

The optimized value of decision (17)'s objective function, $g_1(\boldsymbol{m}_t)$, is then the monetary aggregate, $M_t = M(\mathbf{m}_t)$, while the optimized value of decision (18)'s objective function, $g_2(\mathbf{c}_\mathrm{t})$, is the credit card services aggregate, $C_t = C(\mathbf{c}_t)$.

Hence,

$$M_t = \max \ \{g_1(\mathbf{m}_t) \colon \boldsymbol{\pi}_t^{*\prime} \mathbf{m}_t = \Pi_m^* M_t\} \tag{19}$$

and

$$C_t = \max \ \{g_2(\mathbf{c}_t) \colon \tilde{\boldsymbol{\pi}}_t^{*\prime} \mathbf{c}_t = \Pi_c^* C_t\}. \tag{20}$$

It then follows from (11) and (15) that the optimized values of the monetary and credit card quantity aggregates are related to the joint aggregate in the following manner:

$$\mathcal{M}_t = \tilde{\upsilon}(M_t, C_t). \tag{21}$$

6. The Divisia Index

We advocate using the Divisia index, in its Törnqvist (1936) discrete time version, to track $\mathcal{M}_t = \mathcal{M}(\mathbf{m}_t, \mathbf{c}_t)$, as Barnett (1980) has previously advocated for tracking $M_t = M(\mathbf{m}_t)$. If there should be reason to track the credit card aggregate separately, the Törnqvist-Divisia index similarly could be used to track $C_t = C(\mathbf{c}_t)$. If there is reason to track all three individually, then after measuring M_t and C_t, the joint aggregate $\mathcal{M}_t$ could be tracked as a two-good Törnqvist-Divisia index using (21), rather as an aggregate over the $n + k$ disaggregated components, $(\mathbf{m}_t, \mathbf{c}_t)$. The aggregation theoretic procedure for selecting the $n + m$ component assets is described in Barnett (1982).

6.1. The Linearly Homogeneous Case

It is important to understand that the Divisia index (1925,1926) in continuous time will track any aggregator function without error. To understand why, it is best to see the derivation. The following is a simplified version based on Barnett (2012, pp. 290-292), adapted for our augmented monetary aggregate, which aggregates jointly over money and credit card services. The derivation is equally as relevant to separate aggregation over monetary assets or credit cards, so long as the prices in the indexes are the corresponding user costs, ((5),(7)). Although Francois Divisia (1925, 1926) derived his consumer goods index as a line integral, the simplified approach below is mathematically equivalent to Divisia's original method.

At instant of continuous time, t, consider the quantity aggregator function, $\mathcal{M}_t = \mathcal{M}(\mathbf{m}_t, \mathbf{c}_t) = v(\mathbf{m}_t, \mathbf{c}_t)$, with components $(\mathbf{m}_t, \mathbf{c}_t)$, having user cost prices $(\boldsymbol{\pi}_t, \tilde{\boldsymbol{\pi}}_t)$.

Let $\mathbf{m}_t^a = (\mathbf{m}_t', \mathbf{c}_t')'$and $\boldsymbol{\pi}_t^a = (\boldsymbol{\pi}_t', \widetilde{\boldsymbol{\pi}}_t')'$. Take the total differential of $\mathcal{M}$ to get

$$d\mathcal{M}(\mathbf{m}_t^a) = \sum_{i=1}^{n+k} \frac{\partial \mathcal{M}}{\partial m_{it}^a}\, dm_{it}^a. \tag{22}$$

Since $\partial \mathcal{M}/\partial m_{it}$ contains the unknown parameters of the function $\mathcal{M}$, we replace each of those marginal utilities by $\lambda \pi_{it}^a = \partial \mathcal{M}/\partial m_{it}$ which is the first-order condition for expenditure constrained maximization of $\mathcal{M}$, where λ is the Lagrange multiplier, and π_{it}^a is the user-cost price of m_{it}^a at instant of time t.

We then get

$$\frac{d\mathcal{M}(\mathbf{m}_t^a)}{\lambda} = \sum_{i=1}^{n+k} \pi_{it}^a dm_{it}^a, \tag{23}$$

which has no unknown parameters on the right-hand side.

For a quantity aggregate to be useful, it must be linearly homogeneous. A case in which the correct growth rate of an aggregate is clearly obvious is the case in which all components are growing at the same rate. As required by linear homogeneity, we would expect the quantity aggregate would grow at that same rate. Hence we shall assume $\mathcal{M}$ to be linearly homogeneous.

Define $\Pi^a(\boldsymbol{\pi}_t^a)$ to be the dual price index satisfying Fisher's factor reversal test, $\Pi^a(\boldsymbol{\pi}_t^a)\mathcal{M}(\mathbf{m}_t^a) = \boldsymbol{\pi}_t^{a\prime}\mathbf{m}_t^a$. In other words, define $\Pi^a(\boldsymbol{\pi}_t^a)$ to equal $\boldsymbol{\pi}_t^{a\prime}\mathbf{m}_t^a / \mathcal{M}(\mathbf{m}_t^a)$, which can be shown to depend only upon $\boldsymbol{\pi}_t^a$, when $\mathcal{M}$ is linearly homogeneous. Then the following lemma holds.

Lemma 1: *Let λ be the Lagrange multiplier in the first order conditions for solving the constrained maximization ((9),(10)), and assume that v is linearly homogeneous. Then*

$$\lambda = \frac{1}{\Pi^a(\boldsymbol{\pi}_t^a)}$$

Proof: See Barnett (2012, p. 291). ■

From Equation (23), we therefore find the following:

$$\Pi^a(\boldsymbol{\pi}_t^a)d\mathcal{M}(\mathbf{m}_t^a) = \sum_{i=1}^{n+k} \pi_i^a dm_i^a. \tag{24}$$

Manipulating Equation (24) algebraically to convert to growth rate (log change) form, we find that

$$dlog\ \mathcal{M}(\mathbf{m}_t^a) = \sum_{i=1}^{n+k} \omega_{it}\ dlog\ m_i^a, \tag{25}$$

where $\omega_{it} = \pi_i^a m_i^a / \boldsymbol{\pi}_t^{a\prime}\mathbf{m}_t^a$ is the value share of m_i^a in total expenditure on the services of $\mathbf{m}_t^a$. Equation (25) is the Divisia index in growth rate form. In short, the growth rate of the Divisia index, $\mathcal{M}(\mathbf{m}_t^a)$, is the share weighted average of the growth rates of the components.[12] Notice that there were no assumptions at all in the derivation about the functional form of $\mathcal{M}$, other than existence (*i.e.*, weak separability within the structure of the economy) and linear homogeneity of the aggregator function.

If Divisia aggregation was previously used to aggregate separately over money and credit card services, then equation (25) can be replaced by a two-goods Divisia index aggregating over the two subaggregates, in accordance with equation (21).

[12] While widespread empirical results are not yet available for the augmented Divisia monetary aggregate, $\mathcal{M}(\mathbf{m}_t^a)$, extensive empirical results are available for the un-augmented Divisia monetary aggregates, $M(\mathbf{m}_t)$. See, e.g., Barnett (2012), Barnett and Chauvet (2011a,b), Barnett and Serletis (2000), Belongia and Ireland (20141,b,c), and Serletis and Gogas (2014).

6.2. The Nonlinearly Homogeneous Case

For expositional simplicity, we have presented the aggregation theory throughout this paper under the assumption that the category utility functions, v, g_1, and g_2, are linearly homogeneous. In the literature on aggregation theory, that assumption is called the "Santa Claus" hypothesis, since it equates the quantity aggregator function with the welfare function. If the category utility function is not linearly homogeneous, then the utility function, while still measuring welfare, is not the quantity aggregator function. The correct quantity aggregator function is then the distance function in microeconomic theory. While the utility function and the distance function both fully represent consumer preferences, the distance function, unlike the utility function, is always linearly homogenous. When normalized, the distance function is called the Malmquist index.

In the latter case, when welfare measurement and quantity aggregation are not equivalent, the Divisia index tracks the distance function, not the utility function, thereby continuing to measure the quantity aggregate, but not welfare. See Barnett (1987) and Caves, Christensen, and Diewert (1982). Hence the only substantive assumption in quantity aggregation is blockwise weak separability of components. Without that assumption there cannot exist an aggregate to track.

6.3. Discrete Time Approximation to the Divisia Index

If $(\mathbf{m}_t, \mathbf{c}_t)$ is acquired by maximizing (9) subject to (10) at instant of time t, then $v(\mathbf{m}_t, \mathbf{c}_t)$ is the exact augmented monetary services aggregate, $\mathcal{M}_t$, as written in equation (11). In continuous time, $\mathcal{M}_t = v(\mathbf{m}_t, \mathbf{c}_t)$ can be tracked without error by the Divisia index, which provides $\mathcal{M}_t$ as the solution to the differential equation

$$\frac{dlog\,\mathcal{M}_t}{dt} = \sum_{i=1}^{n} \omega_{it} \frac{dlog\, m_{it}}{dt} + \sum_{j=1}^{k} \widetilde{\omega}_{jt} \frac{dlog\, c_{jt}}{dt}\,, \qquad (26)$$

in accordance with equation (25). The share ω_{it} is the expenditure share of monetary asset *i* in the total services of monetary assets and credit cards at instant of time *t*,

$$\omega_{it} = \pi_{it} m_{it} / (\boldsymbol{\pi}'_t \mathbf{m}_t + \widetilde{\boldsymbol{\pi}}'_t \mathbf{c}_t),$$

while the share $\widetilde{\omega}_{it}$ is the expenditure share of credit card services, *i*, in the total services of monetary assets and credit cards at instant of time *t*,

$$\widetilde{\omega}_{it} = \tilde{\pi}_{it} c_{it} / (\boldsymbol{\pi}'_t \mathbf{m}_t + \widetilde{\boldsymbol{\pi}}'_t \mathbf{c}_t).$$

Note that the time path of $(\mathbf{m}_t, \mathbf{c}_t)$ must continually maximize (9) subject to (10), in order for (26) to hold.

In discrete time, however, many different approximations to (25) are possible, because ω_{it} and $\widetilde{\omega}_{it}$ need not be constant during any given time interval. By far the most common discrete time approximations to the Divisia index is the Törnqvist-Theil approximation (often called the Törnqvist (1936) index or just the Divisia index in discrete time). That index can be viewed as the Simpson's rule approximation, where *t* is the discrete time period, rather than an instant of time:

$$\begin{aligned} &log\,\mathcal{M}(\mathbf{m}_t^a) - log\,\mathcal{M}(\mathbf{m}_{t-1}^a) \\ &\qquad = \sum_{i=1}^{n} \overline{\omega}_{it} \big(\log m_{it} - \log m_{i,t-1}\big) \\ &\qquad + \sum_{i=1}^{k} \overline{\widetilde{\omega}}_{it} \big(\log c_{it} - \log c_{i,t-1}\big), \end{aligned} \qquad (27)$$

where $\overline{\omega}_{it} = (\omega_{it} + \omega_{i,t-1})/2$ and $\overline{\widetilde{\omega}}_{it} = (\widetilde{\omega}_{it} + \widetilde{\omega}_{i,t-1})/2$.

A compelling reason exists for using the Törnqvist index as the discrete time approximation to the Divisia index. Diewert (1976) has defined a class of index numbers, called "superlative" index numbers, which have particular appeal in producing discrete time approximations to aggregator functions. Diewert defines a superlative index number to be one that is exactly correct for some quadratic approximation to the aggregator function, and thereby provides a second order local approximation to the unknown aggregator function. In this case the aggregator function is $\mathcal{M}(\mathbf{m}_t, \mathbf{c}_t) = v(\mathbf{m}_t, \mathbf{c}_t)$. The Törnqvist discrete time approximation to the continuous time Divisia index is in the superlative class, because it is exact for the translog specification for the aggregator function. The translog is quadratic in the logarithms. If the translog specification is not exactly correct, then the discrete Divisia index (27) has a third-order remainder term in the changes, since quadratic approximations possess third-order remainder terms.

With weekly or monthly monetary asset data, the Divisia monetary index, consisting of the first term on the right hand side of (27), has been shown by Barnett (1980) to be accurate to within three decimal places in measuring log changes in $M_t = M(\mathbf{m}_t)$ in discrete time. That three decimal place error is smaller than the roundoff error in the Federal Reserve's component data. We can reasonably expect the same to be true for our augments Divisia monetary index, (27), in measuring the log change of $\mathcal{M}_t = \mathcal{M}(\mathbf{m}_t, \mathbf{c}_t)$.

7. Risk Adjustment

In index number theory, it is known that uncertainty about future variables has no effect on contemporaneous aggregates or index numbers, if preferences are intertemporally separable. Only contemporaneous risk is relevant. See, e.g., Barnett (1995). Prior to Barnett, Liu, and Jensen (1997)), the literature on index number theory assumed that contemporaneous prices are known with certainty, as is reasonable for consumer goods. But Poterba and Rotemberg (1987) observed that contemporaneous user cost prices of monetary assets are not known with certainty, since interest rates are not paid in advance. As a result, the need existed to extend the field of index number theory to the case of contemporaneous risk.

For example, the derivation of the Divisia index in Section 6.1 uses the perfect certainty first-order conditions for expenditure constrained maximization of $\mathcal{M}$, in a manner similar to Francois Divisia's (1925,1926) derivation of the Divisia index for consumer goods. But if the contemporaneous user costs are not known with certainty, those first order conditions become Euler equations. This observation motivated Barnett, Liu, and Jensen (1997)) to repeat the steps in the Section 6.1 derivation with the first order conditions replaced by Euler equations. In this section, we analogously derive an extended augmented Divisia index using the Euler equations that apply under risk, with utility assumed to be intertemporally strongly separable. The result is a Divisia index with the user costs adjusted for risk in a manner consistent with the CCAPM (consumption capital asset price model).[13]

The approach to our derivation of the extended index closely parallels that in Barnett, Liu, and Jensen (1997), Barnett and Serletis (2000, ch. 12), and Barnett (2012, Appendix D) for monetary assets alone. But our results, including credit card services, are likely to result in substantially higher risk adjustments than the earlier results for monetary assets alone, since interest rates on credit card debt are much higher and much more volatile than on monetary assets.

7.1 The Decision

Define Y to be the consumer's survival set, assumed to be compact. The decision problem in this section will differ from the one in section 2 not only by introducing risk, but also by adopting an infinite planning horizon. The consumption possibility set, $S(s)$, for period s is the set of survivable points, $(\mathbf{m}_s, \mathbf{c}_s, \mathbf{x}_s, A_s)$ satisfying equation (2).

The benchmark asset A_s provides no services other than its yield, R_s. As a result, the benchmark asset does not enter the consumer's contemporaneous utility

[13] Regarding CCAPM, see Lucas (1978), Breeden (1979), and Cochrane (2000).

function. The asset is held only as a means of accumulating wealth. The consumer's subjective rate of time preference, ξ, is assumed to be constant. The single-period utility function, $u(\mathbf{m}_t, \mathbf{c}_t, \mathbf{x}_t)$, is assumed to be increasing and strictly quasi-concave.

The consumer's decision problem is the following.

Problem 1. Choose the deterministic point $(\mathbf{m}_t, \mathbf{c}_t, \mathbf{x}_t, A_t)$ and the stochastic process $(\mathbf{m}_s, \mathbf{c}_s, \mathbf{x}_s, A_s)$, $s = t + 1, \ldots, \infty$, to maximize

$$u(\mathbf{m}_t, \mathbf{c}_t, \mathbf{x}_t) + E_t[\sum_{s=t+1}^{\infty} \left(\frac{1}{1+\xi}\right)^{s-t} u(\mathbf{m}_s, \mathbf{c}_s, \mathbf{x}_s)], \tag{28}$$

Subject to $(\mathbf{m}_s, \mathbf{c}_s, \mathbf{x}_s, A_s) \in S(s)$ for $s = t$, t+1, … , ∞, and also subject to the transversality condition

$$\lim_{s\to\infty} E_t \left(\frac{1}{1+\xi}\right)^{s-t} A_s = 0. \tag{29}$$

7.2 Existence of an Augmented Monetary Aggregate for the Consumer

We assume that the utility function, u, is blockwise weakly separable in $(\mathbf{m}_s, \mathbf{c}_s)$ and in $\boldsymbol{x}_s$. Hence, there exists an augmented monetary aggregator function, $\mathcal{M}$, consumer goods aggregator function, X, and utility functions, F and H, such that

$$u(\mathbf{m}_s, \mathbf{c}_s, \mathbf{x}_s) = F[\mathcal{M}(\mathbf{m}_s, \mathbf{c}_s), X(\mathbf{x}_s)]. \tag{30}$$

We define the utility function V by $V(\mathbf{m}_s, \mathbf{c}_s, X_s) = F[\mathcal{M}(\mathbf{m}_s, \mathbf{c}_s), X_s]$, where aggregate consumption of goods is defined by $X_s = X(\boldsymbol{x}_s)$. It follows that the exact augmented monetary aggregate is

$$\mathcal{M}_s = \mathcal{M}(\mathbf{m}_s, \mathbf{c}_s). \tag{31}$$

The fact that blockwise weak separability is a necessary condition for exact aggregation is well known in the perfect-certainty case. If the resulting aggregator function also is linearly homogeneous, two-stage budgeting can be used to prove that the consumer behaves as if the exact aggregate were an elementary good, as in section 5.2. Although two-stage budgeting theory is not applicable under risk, $\mathcal{M}(\boldsymbol{m}_s, \boldsymbol{c}_s)$ remains the exact aggregation-theoretic quantity aggregate in a well-defined sense, even under risk.[14]

The Euler equations that will be of the most use to us below are those for monetary assets and credit card services. Those Euler equations are

$$E_s\left[\frac{\partial V}{\partial m_{is}} - \rho\frac{p_s^*(R_s - r_{is})}{p_{s+1}^*}\frac{\partial V}{\partial X_{s+1}}\right] = 0 \qquad (32a)$$

and

$$E_s\left[\frac{\partial V}{\partial c_{js}} - \rho\frac{p_s^*\left(e_{js} - R_s\right)}{p_{s+1}^*}\frac{\partial V}{\partial X_{s+1}}\right] = 0 \qquad (32\text{b})$$

for all $s \geq t$, $i = 1, \ldots, n$, and $j = 1, \ldots, k$, where $\rho = 1/(1+\xi)$ and where p_s^* is the exact price aggregate that is dual to the consumer goods quantity aggregate X_s.

Similarly, we can acquire the Euler equation for the consumer goods aggregate, X_s, rather than for each of its components. The resulting Euler equation for X_s is

$$E_s\left[\frac{\partial V}{\partial X_s} - \rho\frac{p_s^*(1 + R_s)}{p_{s+1}^*}\frac{\partial V}{\partial X_{s+1}}\right] = 0. \qquad (32c)$$

For the two available approaches to derivation of the Euler equations, see the Appendix.

7.3 The Perfect-Certainty Case

[14]See Barnett (1995) and the appendix in Barnett, Liu, and Jensen (1997).

In the perfect-certainty case with finite planning horizon, we have already shown in section 2 that the contemporaneous nominal user cost of the services of m_{it} is equation (5) and the contemporaneous nominal user cost of credit card services is equation (7). We have also shown in section 6 that the solution value of the exact monetary aggregate, $\mathcal{M}(\mathbf{m}_t, \mathbf{c}_t) = \mathcal{M}(\mathbf{m}_\mathrm{t}^\mathrm{a})$, can be tracked without error in continuous time by the Divisia index, equation (25).

The flawless tracking ability of the index in the perfect-certainty case holds regardless of the form of the unknown aggregator function, $\mathcal{M}$. Aggregation results derived with finite planning horizon also hold in the limit with infinite planning horizon. See Barnett (1987, section 2.2). Hence those results continue to apply. However, under risk, the ability of equation (25) to track $\mathcal{M}(\mathbf{m}_t, \mathbf{c}_t)$ is compromised.

7.4 New Generalized Augmented Divisia Index

7.4.1 User Cost Under Risk Aversion

We now find the formula for the user costs of monetary services and credit card services under risk.

Definition 1. *The contemporaneous risk-adjusted real user cost price of the services of m_{it}^a is $\wp_{it}^a$, defined such that*

$$\wp_{it}^a = \frac{\frac{\partial V}{\partial m_{it}^a}}{\frac{\partial V}{\partial x_t}}, i = 1,2, \dots, n + k.$$

The above definition for the contemporaneous user cost states that the real user cost price of an augmented monetary asset is the marginal rate of substitution between that asset and consumer goods.

For notational convenience, we convert the nominal rates of return, r_{it}, e_{jt} and R_t, to real total rates, $1 + r_{it}^*$, $1 + e_{jt}^*$ and $1 + R_t^*$ such that

$$1 + r_{it}^* = \frac{p_t^*(1 + r_{it})}{p_{t+1}^*}, \tag{33a}$$

$$1 + e_{jt}^* = \frac{p_t^*(1 + e_{jt})}{p_{t+1}^*}, \tag{33b}$$

$$1 + R_t^* = \frac{p_t^*(1 + R_t)}{p_{t+1}^*}, \tag{33c}$$

where r_{it}^*, e_{jt}^*, and R_t^* are called the real rates of excess return. Under this change of variables and observing that current-period marginal utilities are known with certainty, Euler equations (32a), (32b), and (32c) become

$$\frac{\partial V}{\partial m_{it}} - \rho E_t\left[(R_t^* - r_{it}^*)\frac{\partial V}{\partial X_{t+1}}\right] = 0, \tag{34}$$

$$\frac{\partial V}{\partial c_{jt}} - \rho E_t\left[\left(e_{jt}^* - R_t^*\right)\frac{\partial V}{\partial X_{t+1}}\right] = 0, \tag{35}$$

and

$$\frac{\partial V}{\partial X_t} - \rho E_t\left[(1 + R_t^*)\frac{\partial V}{\partial X_{t+1}}\right] = 0. \tag{36}$$

We now can provide our user cost theorem under risk.

Theorem 1 (a). *The risk adjusted real user cost of the services of monetary asset i under risk is* $\mathscr{p}_{it}^m = \pi_{it} + \psi_{it}$, *where*

$$\pi_{it} = \frac{E_t R_t^* - E_t r_{it}^*}{1 + E_t R_t} \tag{37}$$

and

$$\psi_{it} = \rho(1 - \pi_{it}) \frac{Cov\left(R_t^*, \frac{\partial V}{\partial X_{t+1}}\right)}{\frac{\partial V}{\partial X_t}} - \rho \frac{Cov\left(r_{it}^*, \frac{\partial V}{\partial X_{t+1}}\right)}{\frac{\partial V}{\partial X_t}}. \tag{38}$$

(b). *The risk adjusted real user cost of the services of credit card type j under risk is* $\wp_{jt}^c = \tilde{\pi}_{jt} + \tilde{\psi}_{jt}$, *where*

$$\tilde{\pi}_{jt} = \frac{E_t e_{jt}^* - E_t R_t^*}{1 + E_t R_t} \tag{39}$$

and

$$\tilde{\psi}_{jt} = \rho \frac{Cov\left(e_{jt}^*, \frac{\partial V}{\partial X_{t+1}}\right)}{\frac{\partial V}{\partial X_t}} - \rho\left(1 + \tilde{\pi}_{jt}\right) \frac{Cov\left(R_t^*, \frac{\partial V}{\partial X_{t+1}}\right)}{\frac{\partial V}{\partial X_t}}. \tag{40}$$

Proof. See the Appendix. ■

Under risk neutrality, the covariances in (38) and (40) would all be zero, because the utility function would be linear in consumption. Hence, the user cost of monetary assets and credit card services would reduce to $\pi_{i,t}$ and $\tilde{\pi}_{j,t}$ respectively, as defined in equation (37) and (39). The following corollary is immediate.

Corollary 1 to Theorem 1. Under risk neutrality, the user cost formulas are the same as equation (5) and (7) in the perfect-certainty case, but with all interest rates replaced by their expectations.

7.4.2 Generalized Augmented Divisia Index Under Risk Aversion

In the case of risk aversion, the first-order conditions are Euler equations. We now use those Euler equations to derive a generalized Divisia index, as follows.

Theorem 2. *In the share equations,* $\omega_{it} = \pi_{it}^a m_{it}^a / \boldsymbol{\pi}_t^{a\prime} \mathbf{m}_t^a$, *we replace the user costs,* $\boldsymbol{\pi}_t^a = (\boldsymbol{\pi}_t', \tilde{\boldsymbol{\pi}}_t')'$, *defined by (5) and (7), by the risk-adjusted user costs,* $\wp_{it}^a$, *defined by Definition 1, to produce the risk adjusted shares,* $\mathscr{s}_{it} = \wp_{it}^a m_{it}^a / \sum_{j=1}^{n+k} \wp_{jt}^a m_{jt}^a$. *Under our weak-separability assumption,* $V(\mathbf{m}_s, \mathbf{c}_s, X_s) = F[\mathcal{M}(\mathbf{m}_s, \mathbf{c}_s), X_s]$, *and our assumption that the monetary aggregator function,* $\mathcal{M}$, *is linearly homogeneous, the following generalized augmented Divisia index is true under risk:*

$$dlog\ \mathcal{M}_t = \sum_{i=1}^{n+k} \mathscr{s}_{it} dlog\ m_{it}^a. \tag{41}$$

Proof. See the Appendix. ■

The exact tracking of the Divisia monetary index is not compromised by risk aversion, as long as the adjusted user costs, $\pi_{it} + \psi_{it}$ and $\tilde{\pi}_{jt} + \tilde{\psi}_{jt}$, are used in computing the index. The adjusted user costs reduce to the usual user costs in the case of perfect certainty, and our generalized Divisia index (41) reduces to the usual Divisia index (25). Similarly, the risk-neutral case is acquired as the special case with $\psi_{it} = \tilde{\psi}_{jt} = 0$, so that equations (37) and (39) serve as the user costs. In short, our generalized augmented Divisia index (41) is a true generalization, in the sense that the risk-neutral and perfect-certainty cases are strictly nested special cases. Formally, that conclusion is the following.

Corollary 1 to Theorem 2. *Under risk neutrality, the generalized Divisia index (41) reduces to (25), where the user costs in the formula are defined by (37) and (39).*

7.5 CCAPM Special Case

As a means of illustrating the nature of the risk adjustments, $\psi_{i,t}$ and $\tilde{\psi}_{j,t}$, we consider a special case, based on the usual assumptions in CAPM theory of either quadratic utility or Gaussian stochastic processes. Direct empirical use of Theorems 1 and 2, without any CAPM simplifications, would require availability of prior econometric estimates of the parameters of the utility function, V, and of the subjective rate of time discount. Under the usual CAPM assumptions, we show in this section that empirical use of Theorems 1 and 2 would require prior estimation of only one property of the utility function: the degree of risk aversion, on which a large body of published information is available.

Consider first the following case of utility that is quadratic in consumption of goods, conditionally on the level of monetary asset and credit card services.

Assumption 1. Let V have the form

$$V(\mathbf{m}_t, \mathbf{c}_t, X_t) = F[\mathcal{M}(\mathbf{m}_t, \mathbf{c}_t), X_t] = A[\mathcal{M}(\mathbf{m}_t, \mathbf{c}_t)]X_t - \frac{1}{2}B[\mathcal{M}(\mathbf{m}_t, \mathbf{c}_t)]X_t^2, \quad (42)$$

where A is a positive, increasing, concave function and B is a nonnegative, decreasing, convex function.

The alternative assumption is Guassianity, as follows:

Assumption 2. Let $(r_{it}^*, e_{jt}^*, X_{t+1})$ be a trivariate Gaussian process for each asset $i = 1, \ldots, n$, and credit card service, $j = 1, \ldots, k$.

We also make the following conventional CAPM assumption:

Assumption 3. The benchmark rate process is deterministic or already risk-adjusted, so that R_t^* is the risk-free rate.

Under this assumption, it follows that

$$Cov\left(R_t^*, \frac{\partial V}{\partial X_{t+1}}\right) = 0.$$

We define $H_{t+1} = H(\mathcal{M}_{t+1}, X_{t+1})$ to be the well-known Arrow-Pratt measure of absolute risk aversion,

$$H(\mathcal{M}_{t+1}, X_{t+1}) = -\frac{E_t[V'']}{E_t[V']}, \tag{43}$$

where $V' = \partial V(\mathbf{m}_{t+1}^a, X_{t+1})/\partial X_{t+1}$ and $V'' = \partial^2 V(\mathbf{m}_{t+1}^a, X_{t+1})/\partial X_{t+1}^2$. In this definition, risk aversion is measured relative to consumption risk, conditionally upon the level of augmented monetary services produced by $\mathcal{M}_{t+1} = \mathcal{M}(\mathbf{m}_t, \mathbf{c}_t)$. Under risk aversion, H_{t+1} is positive and increasing in the degree of absolute risk aversion. The following lemma is central to our Theorem 3.

Lemma 2. *Under Assumption 3 and either Assumption 1 or Assumption 2, the user-cost risk adjustments, ψ_{it} and $\tilde{\psi}_{jt}$, defined by (38) and (40), reduce to*

$$\psi_{it} = \frac{1}{1+R_t^*} H_{t+1} cov(r_{it}^*, X_{t+1}) \tag{44a}$$

and

$$\tilde{\psi}_{jt} = -\frac{1}{1+R_t^*} H_{t+1} cov\left(e_{jt}^*, X_{t+1}\right). \tag{44b}$$

Proof. See the Appendix. ■

The following theorem identifies the effect of the risk adjustment on the expected own interest rates in the user cost formulas.

Theorem 3. *Let $\hat{H}_t = H_{t+1} X_t$. Under the assumptions of Lemma 2, we have the following for each asset $i = 1, \dots, n$, and credit card service, $j = 1, \dots, k$.*

$$\wp_{it}^m = \frac{E_t R_t^* - (E_t r_{it}^* - \phi_{it})}{1 + E_t R_t^*}, \tag{45}$$

where

$$\phi_{it} = \hat{H}_t \, Cov\left(r_{it}^*, \frac{X_{t+1}}{X_t}\right), \tag{46}$$

and

$$\wp_{jt}^c = \frac{(E_t e_{jt}^* - \tilde{\phi}_{jt}) - E_t R_t^*}{1 + E_t R_t^*}, \tag{47}$$

where

$$\tilde{\phi}_{jt} = \hat{H}_t \, Cov\left(e_{jt}^*, \frac{X_{t+1}}{X_t}\right). \tag{48}$$

Proof. See the Appendix. ■

As defined, $\hat{H}_t$ is a time shifted Arrow-Pratt relative risk aversion measure. Theorem 3 shows that the risk adjustment on the own interest rate for a monetary asset or credit card service depends upon relative risk aversion, $\hat{H}_t$, and the covariance between the consumption growth path, X_{t+1}/X_t, and the real rate of excess return earned on a monetary asset, r_{it}^*, or paid on a credit card service, e_{jt}^*.

7.6 Magnitude of the Adjustment

In accordance with the large and growing literature on the equity premium puzzle, the CCAPM risk adjustment term is widely believed to be biased downward.[15] A promising explanation may be the customary assumption of intertemporal separability of utility, since response to a change in an interest rate may not be fully reflected in contemporaneous changes in consumption. Hence the contemporaneous covariance in the CCAPM "beta" correction may not take full account of the effect of an interest rate change on life style. An approach to risk adjustment without assumption of intertemporal separability was developed for monetary aggregation by Barnett and Wu (2005).

8. Data Sources

The credit card transactions services are measured by the transactions volumes summed over four sources: Visa, MasterCard, American Express, and Discover. Our theory does not apply to debit cards or to store cards or to charge cards not providing a line of credit or not widely acceptable for retail purchases. We acquired the volumes from their annual reports and seasonally adjusted them by the Census X-13ARIMA-SEATS program. The start date is the quarter during which those credit card firms went public and the annual reports became available. The contemporaneous transactions volumes do not include the carried forward rotating balances resulting from transactions during prior periods.[16] The credit card interest rates imputed to the representative consumer are from the Federal Reserve Board's data on all commercial bank credit card accounts, including those not charged interest, since paid off within the month.[17] The benchmark rate on monetary assets owned by consumers is the one used by the CFS in its current Divisia monetary

[15]See, e.g., Campbell and Cochrane (1999), Cochrane (2000), Kocherlakota (1996), Marshall (1997), Mehra and Prescott (1985).

[16] Credit limits are not considered, since we do not have a way to untangle the effect of those constraints on contemporaneous transactions volumes from the effect on the carried forward rotating balances associate with previous period transactions.

[17]This interest rate includes credit card accounts not assessed interest, and hence is lower than the Federal Reserve's supplied interest rates on accounts assessed interest. This imputation includes only explicit interest paid, averaged over all credit card accounts.
[Further research, using a heterogeneous agents approach, is planned on this imputation. In that future research, implicit interest rate could include annual credit card fees and the increased product prices on goods purchased primarily with credit cards, as well as the explicit interest rate paid on contemporaneous transactions volumes by the approximately 80% of credit card holders who maintain rotating balances.]

aggregates releases.[18] All other component quantities and interest rates are as used in the CFS Divisia monetary aggregates at:

http://www.centerforfinancialstability.org/amfm.php

In the near future, the CFS plans to add to that site our augmented Divisia monetary aggregates, $\mathcal{M}_t = \mathcal{M}(\mathbf{m}_t, \mathbf{c}_t)$, as defined in equations 11 and 21, including credit card services. Monthly updates will be provided to the public by the CFS through monthly releases. The monthly updates will also be provided by Bloomberg to its terminal users. Our extensive search for relevant sources of credit card data are provided in detail in Barnett and Su (2016), which documents our decisions about credit card data sources. All details about data sources and data decisions regarding monetary asset components and interest rates are provided in Barnett, Liu, Mattson, and van den Noort (2013). We use only sources available to the public.[19]

The resulting augmented Divisia monetary services aggregates, $\mathcal{M}_t = \mathcal{M}(\mathbf{m}_t, \mathbf{c}_t)$, satisfy the existence conditions for a structural economic variable in a macroeconomic model. Hence those aggregates can be used as the quantity of monetary services in a demand for money equation, or as a monetary intermediate target or long run anchor in a monetary rule, or in any other econometric or policy application requiring a macroeconomic model containing the monetary service flow as a structural variable.

Alternatively, money can be used as an indicator of the state of the economy. For example, new-Keynesian nominal GDP targeting policies require monthly measures of nominal GDP, although data on nominal GDP are available only quarterly. The usefulness of Divisia monetary aggregates in Nowcasting monthly nominal GDP has been established by Barnett, Chauvet, and Leiva-Leon (2016). Indicator uses of monetary data are free from the controversies that have surrounded uses of money as a policy target. In the next section, we produce an

[18] That imputed interest rate is the short term (2 to 31 days) rate of interest on bank loans.

[19] The CFS sweep adjusts demand deposits. During periods when available from the Federal Reserve, the CFS uses the reported sweep adjustments. When not available, the CFS uses an econometric model to approximate the sweep adjustment. Although sweep adjustment is important at the M1 level of aggregation, the sweep adjustment has insignificant effect on the broader aggregates, since sweeps are largely internalized within those aggregates.

indicator optimized augmented monetary aggregate, $\mathcal{M}_t^* = \mathcal{M}_t^*(\mathbf{m}_t, \mathbf{c}_t)$. Since this aggregate is application specific, its existence condition is different from the one used above to produce the augmented Divisia monetary aggregates. Unlike the augmented Divisia monetary aggregates, $\mathcal{M}_t = \mathcal{M}(\mathbf{m}_t, \mathbf{c}_t)$, which are statistical index numbers in the superlative index number class, the indicator optimized aggregates, $\mathcal{M}_t^* = \mathcal{M}_t^*(\mathbf{m}_t, \mathbf{c}_t)$, are econometrically estimated aggregator functions, not statistical index numbers. The estimated aggregator function is time dependent because of the real time estimation used in the nowcasting.

9. Nowcasting Nominal GDP

In this section we turn to the use of our data as indicators, rather than as policy targets or as structural variables in the macroeconomy. We find that the information contained in credit card transaction volumes is a valuable addition to the indicator set in formal nowcasting of nominal GDP. A consequence is a directly derived indicator-optimized augmented aggregator function over monetary and credit card services. This aggregator function uniquely captures the contributions of monetary and credit card services as indicators of nominal GDP in the nowcasting.

An important contribution to the literature on nowcasting is Giannone, Reichlin, and Small (2008). Their approach, based on factor analysis, has proved to be very successful. Barnett, Chauvet, and Leiva-Leon (2016) propose an alternative methodology based on confirmatory factor analysis and find that Divisia monetary aggregates are particularly valuable indicators within the resulting set of optimal indicators. Barnett and Tang (2016) compared the factor analysis approach of Giannone, Reichlin, and Small (2008) and Barnett, Chauvet, and Leiva-Leon (2016) with alternative nowcasting approaches, and find that the factor analysis approaches are usually best and benefit substantially from inclusion of the CFS Divisia monetary aggregates among its indicators.

In this paper, we investigate the further gains from inclusion of credit card transactions volumes in the Nowcasting. We also produce and explore the derived indicator optimized aggregates, $\mathcal{M}_t^* = \mathcal{M}_t^*(\mathbf{m}_t, \mathbf{c}_t)$.

9.1. The Model

In this paper we use data on credit card transaction volumes along with the optimal indicators found by Barnett, Chauvet, and Leiva-Leon (2016) to provide a model useful to yield accurate nowcasts of monthly Nominal GDP. Accordingly, as indicators we use growth rates of quarterly Nominal GDP, $y_{1,t}$, monthly Industrial Production, $y_{2,t}$, monthly Consumer Price Index, $y_{3,t}$, a monthly Divisia monetary aggregate measure, $y_{4,t}$, and a monthly credit card transaction volume, $y_{5,t}$, to estimate the following Mixed Frequency Dynamic Factor model:

$$\begin{bmatrix} y_{1,t} \\ y_{2,t} \\ y_{3,t} \\ y_{4,t} \\ y_{5,t} \end{bmatrix} = \begin{bmatrix} \gamma_1[(1/3)f_t + (2/3)f_{t-1} + f_{t-2} + (2/3)f_{t-3} + (1/3)f_{t-4}] \\ \gamma_2 f_t \\ \gamma_3 f_t \\ \gamma_4 f_t \\ \gamma_5 f_t \end{bmatrix} + \begin{bmatrix} (1/3)v_{1,t} + (2/3)v_{1,t-1} + v_{1,t-2} + (2/3)v_{1,t-3} + (1/3)v_{1,t-4} \\ v_{2,t} \\ v_{3,t} \\ v_{4,t} \\ v_{5,t} \end{bmatrix}. \tag{49}$$

The model separates out, into the unobserved factor, f_t, the common cyclical fluctuations underlying the observed variables. The idiosyncratic movements are captured by the terms, $v_{i,t}$, for $i = 1,2, \dots ,5$. The factor loadings, γ_i, measure the sensitivity of the common factor to the observed variables. The dynamics of the factor and idiosyncratic components are given by

$$f_t = \phi_1 f_{t-1} + \cdots + \phi_p f_{t-p} + e_t, \qquad e_t \sim N(0,1) \tag{50}$$

$$v_{i,t} = \varphi_{i1} v_{i,t-1} + \cdots + \varphi_{iQ_i} v_{i,t-Q_i} + \varepsilon_{i,t}, \qquad \varepsilon_{i,t} \sim N\left(0, \sigma_{\varepsilon_i}^2\right), \text{ for } i = 1, \dots ,5. \tag{51}$$

Following Stock and Watson (1989), the model assumes that f_t and $v_{i,t}$ are mutually independent at all leads and lags for all $n = 5$ variables.

The model in equations (49)-(51) can be cast into a measurement equation and transition equation yielding the following state-space representation

$$\mathbf{y}_t = \mathbf{H}\mathbf{F}_t + \boldsymbol{\xi}_t, \quad \boldsymbol{\xi}_t \sim i.i.d.N(\mathbf{0}, \mathbf{R}) \tag{52}$$

$$\mathbf{F}_t = \mathbf{G}\mathbf{F}_{t-1} + \boldsymbol{\zeta}_t, \quad \boldsymbol{\zeta}_t \sim i.i.d.N(\mathbf{0}, \mathbf{Q}). \tag{53}$$

We apply the Kalman filter to extract optimal inferences on the state vector, $\mathbf{F}_t$, which contains the common factor of interest, f_t, and the idiosyncratic terms, $v_{i,t}$.

Following Mariano and Murasawa (2003), we modify the state-space model to incorporate into the system missing observations, which are frequently present when performing nowcasts in real-time. The modification consists of substituting each missing observation with a random draw $\beta_t \sim N(0, \sigma_\beta^2)$. This substitution keeps the matrices conformable, without affecting the estimation of the model parameters, in accordance with the rule:

$$y_{i,t}^* = \begin{cases} y_{i,t}^* & \text{if } y_{i,t} \text{ observed} \\ \beta_t & \text{otherwise} \end{cases}, \qquad \mathbf{H}_{i,t}^* = \begin{cases} \mathbf{H}_i & \text{if } y_{i,t} \text{ observed} \\ \mathbf{0}_{1k} & \text{otherwise} \end{cases}$$

$$\xi_{i,t}^* = \begin{cases} 0 & \text{if } y_{i,t} \text{ observed} \\ \beta_t & \text{otherwise} \end{cases}, \qquad R_{i,t}^* = \begin{cases} 0 & \text{if } y_{i,t} \text{ observed} \\ \sigma_\beta^2 & \text{otherwise} \end{cases}$$

where $\mathbf{H}_{i,t}^*$ is the i-th row of a matrix $\mathbf{H}^*$, which has k columns, and $\mathbf{0}_{1k}$ is a k row vector of zeros. Hence, the modified measurement equation of the state-space model remains as

$$\mathbf{y}_t^* = \mathbf{H}_t^*\mathbf{F}_t + \boldsymbol{\xi}_t^*, \quad \boldsymbol{\xi}_t^* \sim i.i.d.N(\mathbf{0}, \mathbf{R}_t^*). \tag{54}$$

The output is an optimal estimator of the dynamic factor, constructed using information available through time t. As new information becomes available, the filter is applied to update the state vector on a real-time basis.

9.2. In-Sample Analysis

We empirically evaluate the predictive ability of the information contained in credit card volumes to produce the most accurate nowcasts of nominal GDP growth, when credit card transactions volumes are included into the optimal indicator set found by Barnett, Chauvet, and Leiva-Leon (2016). One of the indicators in that set is the current CFS Divisia monetary aggregates, unaugmented by inclusion of credit card data. We perform pairwise comparisons between models that include credit card information and models that do not. In the former case, the indicator set includes four variables, while in the latter case the indicator set includes five variables. Both sets include the same CFS unaugmented Divisia monetary aggregates, $M_t = M(\mathbf{m}_t)$, as defined in equation 19, among its optimal indicators. We first examine the predictive ability of both models, with and without credit card information as a fifth indicator, by performing an in-sample analysis. We consider the sample period from November 2003 until May 2015 as a result of the availability of the needed data. For the in-sample analysis, we estimate the model only once for the full sample. From November 2003 to June 2006, there are some missing observations of some variables, but this does not present a problem, since the nowcasting model allows dealing with missing observations using the Kalman filter. Regular data availability for all relevant variables begins in July 2006, when the credit card companies' data became available in annual reports.

The first two columns of Table 1 report the full sample Mean Square Errors (MSE) associated with the models containing each of the two indicator sets. The table shows that models containing both CFS Divisia monetary aggregates and credit card transactions volumes produce lower MSE than models containing only Divisia monetary aggregates, $M_t = M(\mathbf{m}_t)$ among the other three indicators. This applies at any of the four levels of disaggregation, M1, M2, M3, and M4. Next, we compute the

MSE only for the years associated with the Great Recession (2008-2009), reported in the last two columns of Table 1. The results show that the models including credit card information produce lower MSE than the models omitting such information in nowcasting of nominal GDP growth.

Table 1. In-Sample Mean Squared Errors

	FULL SAMPLE		GREAT RECESSION	
	CFS	Augmented	CFS	Augmented
DM1	0.16	**0.17**	0.33	**0.30**
DM2	0.18	**0.17**	0.36	**0.31**
DM3	0.16	**0.15**	0.32	**0.26**
DM4	0.18	**0.15**	0.39	**0.25**

Note. The table reports the mean squared errors associated with each model for the entire sample period, November 2003 - May 2015, and for the Great Recession years, January 2008 - December 2009. The CFS column includes the CFS Divisia monetary aggregates, $M_t = M(\mathbf{m}_t)$, among the Barnett, Chauvet, and Leiva-Leon (2016) optimal indicator set, but without inclusion of credit card transaction volumes, while the Augmented column includes credit card transactions volumes among the indicators as a fifth independent indicator.

To provide a deeper exploration about the role that each indicator plays in the construction of nominal GDP predictions, we follow the line of Banbura and Rustler (2007) and decompose each forecast into the relative contribution of each indicator, with emphasis on the Divisia monetary aggregate, $M_t = M(\mathbf{m}_t)$, and credit card transactions volume. In doing so, we substitute the prediction error, $\boldsymbol{\xi}^*_{t|t-1}$, and the predicted state, $\mathbf{F}_{t|t-1}$, into the updating equation of the Kalman filter, yielding

$$\mathbf{F}_{t|t} = (\mathbf{I} - \mathbf{K}^*_t \mathbf{H}^*_t)\mathbf{G}\mathbf{F}_{t-1|t-1} + \mathbf{K}^*_t \mathbf{y}^*_t, \qquad (55)$$

where the Kalman gain is denoted by $\mathbf{K}^*_t = \mathbf{P}_{t|t-1}(\mathbf{H}^{*'}_t(\mathbf{H}^*_t \mathbf{P}_{t|t-1}\mathbf{H}^{*'}_t + \mathbf{R}^*_t))$, and the predicted variance of the state vector is given by $\mathbf{P}_{t|t-1} = \mathbf{G}\mathbf{P}_{t-1|t-1}\mathbf{G} + \mathbf{Q}$. When the Kalman filter approaches its steady state, the updated state vector can be decomposed into a weighted sum of observations

$$\mathbf{F}_{t|t} = \sum_{j=0}^{\infty} \mathbf{Z}^*_{jt} \mathbf{y}^*_{t-j}\ , \qquad (56)$$

where $\mathbf{Z}_t^*(L) = (\mathbf{I} - (\mathbf{I} - \mathbf{K}_t^*\mathbf{H}_t^*)\mathbf{GL})^{-1}\mathbf{K}_t^*$, and each element of the matrix $\mathbf{Z}_t^*(L)$ measures the effects of unit changes in the lags of individual observations on the inference of the state vector $\mathbf{F}_{t|t}$. Therefore, the matrix $\mathbf{Z}_t^*(1)$ contains the cumulative impacts of the individual observations in the inference of the state vector. For further details about this decomposition, see Banbura and Rustler (2007). Accordingly, the vector containing the cumulative impact of each indicator on the forecast of nominal GDP growth can be calculated as follows

$$\boldsymbol{\omega}_t = \mathbf{H}_1\left(\frac{1}{3}\mathbf{z}_{1t}^* + \frac{2}{3}\mathbf{z}_{2t}^* + \mathbf{z}_{3t}^* + \frac{2}{3}\mathbf{z}_{4t}^* + \frac{1}{3}\mathbf{z}_{5t}^*\right) + \left(\frac{1}{3}\mathbf{z}_{7t}^* + \frac{2}{3}\mathbf{z}_{8t}^* + \mathbf{z}_{9t}^* + \frac{2}{3}\mathbf{z}_{10t}^* + \frac{1}{3}\mathbf{z}_{11t}^*\right), \quad (57)$$

where, $\mathbf{z}_{1t}^*$, is the *i*-th row of $\mathbf{Z}_t^*(1)$.

The average cumulative forecast weights, $\boldsymbol{\omega}_t$, associated with each indicator are reported in Table 2 for all the models under consideration. The results show that, on average, one third of the contribution is associated with previous releases of nominal quarterly GDP itself. Such information is primary in the model, but is only observed once per quarter. Regarding the monthly indicators, Industrial Production is the indicator that contributes the most to nominal GDP growth predictions, followed by the Divisia monetary aggregates. The indicator that provides the least contribution across models is often the Consumer Price Index, CPI. However, when credit card information is included, it shows a significantly greater forecast contribution than the unaugmented CFS Divisia monetary aggregates or the Consumer Price Index. This conclusion is independent of the aggregation level of the monetary measure. These results corroborate that the in sample predictive ability of the optimal combination, including both Divisia monetary aggregates and credit-card volumes, outperforms models that exclude credit card information.

Table 2. Cumulative Forecast Weight of Each Indicator

	NGDP	IP	CPI	DIVISIA	CREDIT
DM1 CFS	0.33	0.59	0.03	0.05	--
DM1 Augmented	0.33	0.34	0.05	0.03	0.25
DM2 CFS	0.33	0.58	0.03	0.06	--
DM2 Augmented	0.33	0.34	0.04	0.04	0.24
DM3 CFS	0.33	0.63	0.04	0.01	--
DM3 Augmented	0.33	0.35	0.05	0.01	0.26
DM4 CFS	0.33	0.60	0.03	0.03	--
DM4 Augmented	0.33	0.37	0.04	0.02	0.24

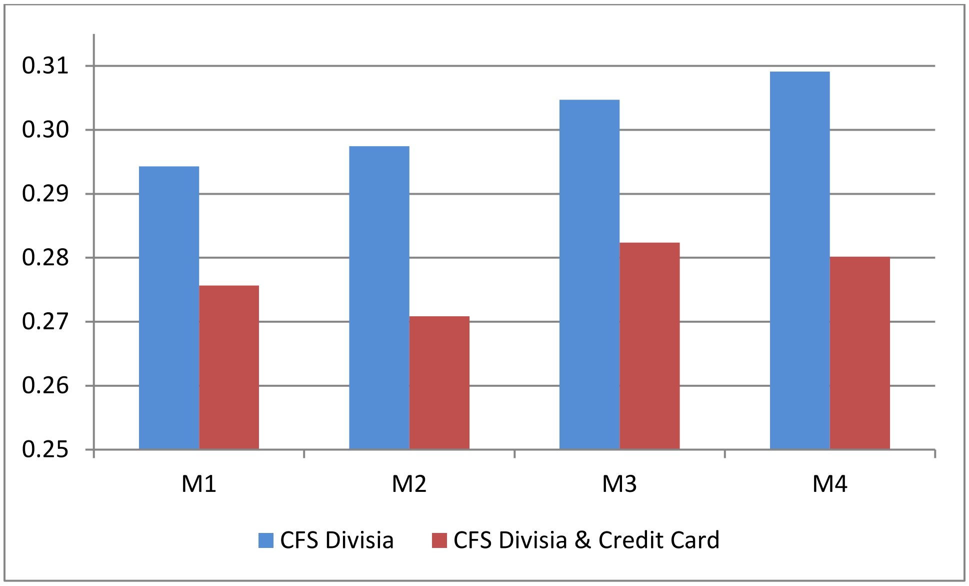
Note. The table reports the cumulative forecast weights, averaged over time, for the entire sample. As in table 1, the CFS rows include the CFS Divisia monetary aggregates among the Barnett, Chauvet, and Leiva-Leon (2016) optimal indicator set, but without inclusion of credit card transaction volumes, while the Augmented rows include credit card transactions volumes among the indicators as a fifth independent indicator. In both cases, the Divisia column is the CFS unaugmented Divisia monetary aggregate, $M_t = M(\mathbf{m}_t)$, defined in equation 19.

9.3. Real Time Analysis

For the initial estimation of the model in real time analysis, we use data from November 2003 to September 2007, yielding 47 observations. Hence, our nowcasting evaluation sample is the remaining observations from October 2007 to May 2015, yielding 92 observations. The samples have been chosen based on two criteria, (i) to guarantee that the estimation sample represents one third of the total available sample, and (ii) to incorporate the Great Recession episode in the evaluation sample, since it is of particular interest.[20] For every month of the evaluation sample, we re-estimate the model parameters, compute the nowcast of the target variable, and compare it with the first release of nominal GDP to construct mean squared errors.

[20]We also tried with different partitions of the sample, but the results remained qualitatively unchanged.

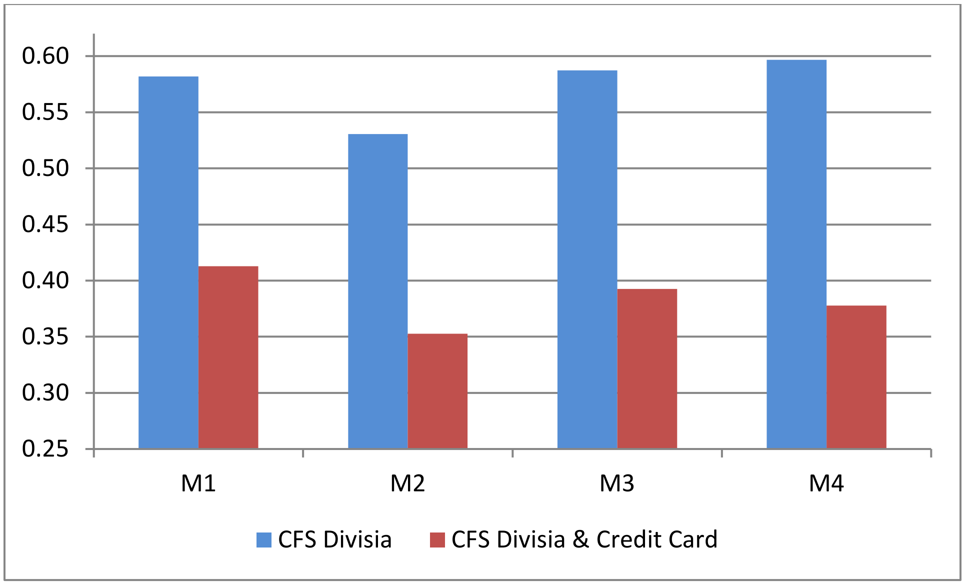
Figure 1: Mean Square Error Comparison (Full sample)



With each model, the MSE associated with the real-time nowcasts are shown in Figure 1 for the entire evaluation sample. The figure shows that models incorporating credit card information provide a significantly lower MSE than the models not incorporating such information. Optimal weighting between credit card transactions volumes and Divisia monetary aggregates improves the accuracy in producing real-time nowcasts of Nominal GDP. The superiority of the extended models, which include credit card information, over the un-extended models omitting that information can be observed at all four levels of aggregation, and particularly for the M2 monetary aggregates.

Additionally, we perform the same evaluations, but only focusing on the subsample containing the years of the Great Recession. The motivation for doing this analysis relies on comparing the ability of the extended and un-extended models to track Nominal GDP dynamics during recessionary periods, associated with macroeconomic instabilities and higher uncertainty. Figure 2 shows the mean squared errors associated with real-time nowcasts computed with each model for the evaluation sample, containing the years of 2008 and 2009. The results corroborate the significant superiority of the extended over unextended models in nowcasting Nominal GDP during contractionary episodes.

Figure 2: Mean Square Error Comparison (Great Recession)



The model is re-estimated at every period of time during which new information is available, to simulate real-time conditions. We thereby investigate potential changes in the contemporaneous relationship between each indicator in the model and the extracted factor used to produce real-time nowcasts of nominal GDP growth. This information allows us to examine in detail the comovement between each indicator and the signals used to forecast nominal GDP during periods of instabilities, such as the Great Recession. In Table 1, the first row at each level of aggregation is for the four indicator model, while the second row is for the five indicator model.

The upper part of Table 3 reports the full sample average of the recursively estimated factor loadings for each indicator and for each model. The results show a positive and strong comovement between Industrial Production and the common factor, and a positive but weak comovement between Consumer Price Index and the common factor, with stronger comovement in the case of the five factor model. Regarding the CFS Divisia monetary aggregates, the results show relatively weak and sometimes negative comovement with the common factor. In the five factor models, credit card transactions volumes show very strong comovement with the common factor, even stronger than the comovement of quarterly nominal GDP with the common factor. Clearly the four factor model is missing important indicator information.

To assess the comovements during the Great Recession period, we compute the average recursive loadings for the period January 2008 to December 2009 and report them in the lower part of Table 3. The comovement between each indicator and the common factor across models presents a similar pattern to the one obtained with the full sample averages, with one notable exception. With both the four indicator and the five indicator models, the Consumer Price Index experiences a negative relationship with the common factor, providing countercyclical signals to nowcasts of nominal GDP growth. Again the credit-card transactions volumes experience positive and strong comovement with the common factor, and hence show the ability to improve the accuracy of signals in nowcasting nominal GDP growth during periods of instability.

Table 3. Out of Sample Recursive Loadings

	NGDP	IP	CPI	DIVISIA	CREDIT
Full sample period					
DM1 CFS	0.19	0.39	0.09	-0.10	--
DM1 CFS & CREDIT	0.22	0.42	0.15	-0.14	0.38
DM2 CFS	0.20	0.38	0.07	-0.13	--
DM2 CFS & CREDIT	0.22	0.41	0.14	-0.17	0.36
DM3 CFS	0.18	0.38	0.08	0.02	--
DM3 CFS & CREDIT	0.21	0.41	0.16	0.03	0.38
DM4 CFS	0.19	0.39	0.06	-0.11	--
DM4 CFS & CREDIT	0.21	0.41	0.14	-0.12	0.36
Great Recession period					
DM1 CFS	0.21	0.43	-0.04	-0.05	--
DM1 CFS & CREDIT	0.24	0.48	0.00	-0.08	0.29
DM2 CFS	0.25	0.39	-0.08	-0.01	--
DM2 CFS & CREDIT	0.25	0.46	-0.03	-0.06	0.25
DM3 CFS	0.21	0.42	-0.05	0.00	--
DM3 CFS & CREDIT	0.23	0.48	-0.01	0.01	0.31
DM4 CFS	0.23	0.44	-0.09	-0.16	--
DM4 CFS & CREDIT	0.24	0.47	-0.01	-0.14	0.26

Note. The table reports the average out of sample recursively estimated factor loading. The upper part of the table focuses on the entire sample November 2003 - May 2015, while the lower part of the table focuses on the Great Recession years, January 2008 - December 2009.

10. Indicator Optimized Augmented Aggregate

As explained in the previous section, the nowcasts can be transformed into weighted averages of the indicators, with the weights being the vector $\boldsymbol{\omega}_t$ provided in Table 2. The nowcasting-derived indicator-optimized aggregate, $\mathcal{M}_t^* = \mathcal{M}_t^*(\mathbf{m}_t, \mathbf{c}_t)$, is the weighted averages of the CFS Divisia monetary aggregate and the credit card transactions volume. The weights of those two components are in fourth and fifth columns of Table 2. The estimated aggregator function, $\mathcal{M}_t^*(.)$, is time dependent, since the weights, $\boldsymbol{\omega}_t$, are time dependent.[21] The detailed procedure for

[21] In principle, it might be possible to factor a non-time-dependent function solely of $(\mathbf{m}_t, \mathbf{c}_t)$ out of the nowcasting equation. But because of the deep nonlinearity of that equation in $(\mathbf{m}_t, \mathbf{c}_t)$ and the recursive real time nature of the nowcasting estimation, it would be impossible to solve for that aggregator function in algebraic closed form. The extreme difficulty of solving for that function

computing the weights in Table 2 and the indicator optimized aggregate, $\mathcal{M}_t^* = \mathcal{M}_t^*(\mathbf{m}_t, \mathbf{c}_t)$, is provided in the appendix VI.

It is important to observe that if the CFS Divisia monetary aggregate is replaced by $\mathcal{M}_t^*$ computed in that manner, then all of the results in Tables 1, 2, and 3 for five indicators are equally and exactly applicable to the nowcasting with four indicators. As evident from those tables, replacing the CFS Divisia monetary aggregates, M_t, by $\mathcal{M}_t^*$ produces very large gains in indicator information with four indicators in each case. No indicator information is lost by the aggregation, $\mathcal{M}_t^* = \mathcal{M}_t^*(\mathbf{m}_t, \mathbf{c}_t)$, since that optimized augmented indicator is uniquely nowcasting indicator exact.

All of the figures below display three graphs: (1) nominal quarterly measured GDP growth, (2) growth of the CFS Divisia monetary aggregates, $M_t = M(\mathbf{m}_t)$, (3) growth of the indicator optimized augmented monetary aggregates, $\mathcal{M}_t^* = \mathcal{M}_t^*(\mathbf{m}_t, \mathbf{c}_t)$. Although the nowcasts and the monetary aggregates are available monthly, the plots below are quarterly, since GDP data are available only quarterly.

The following observations follow from the figures. The fluctuations in the credit-card augmented Divisia monetary aggregates lead the conventional Divisia monetary aggregates at all four levels of aggregation. The credit-card augmented Divisia monetary aggregates better correlate with nominal GDP than the conventional Divisia monetary aggregates do. The credit-card augmented Divisia monetary aggregates more accurately reflect the Great Recession time period than the conventional Divisia monetary aggregates do.

Although the broadest aggregates, DM3 and DM4, more accurately and completely measure the economy's flow of monetary services, the transmission of policy to the aggregates is somewhat slower for the distant substitutes for money than for the assets in DM1 and DM2.

It is evident from these results why the new credit-card augmented Divisia monetary aggregates improve so dramatically in Tables 1 and 2 upon the performance of the nominal GDP nowcasting approach developed by Barnett, Chauvet, and Leiva-Leon (2016). That approach successfully incorporates the

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numerically, if the function exists, would have no benefit, since $\mathcal{M}_t^*(\mathbf{m}_t, \mathbf{c}_t)$ is indicator optimal and loses no information in the nowcasting.

conventional CFS Divisia monetary aggregates among its significant indicators, with improved performance compared with use of the official simple sum monetary aggregates in the same nowcasting procedure.

10.1. Average Quarterly Growth Rates

Figure 3: M1 Average Quarterly Growth Rates (2007Q4-2015Q1)

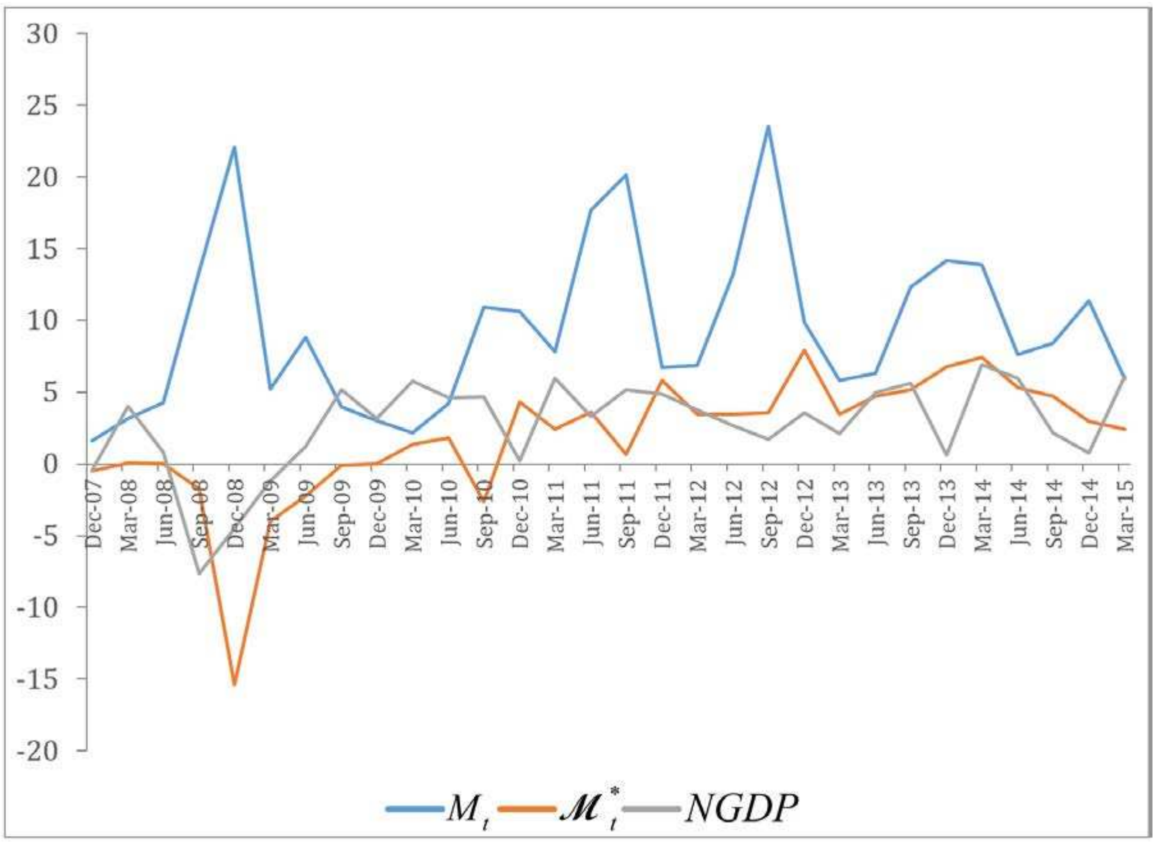


Figure 4: M2 Average Quarterly Growth Rates (2007Q4 – 2015Q1)

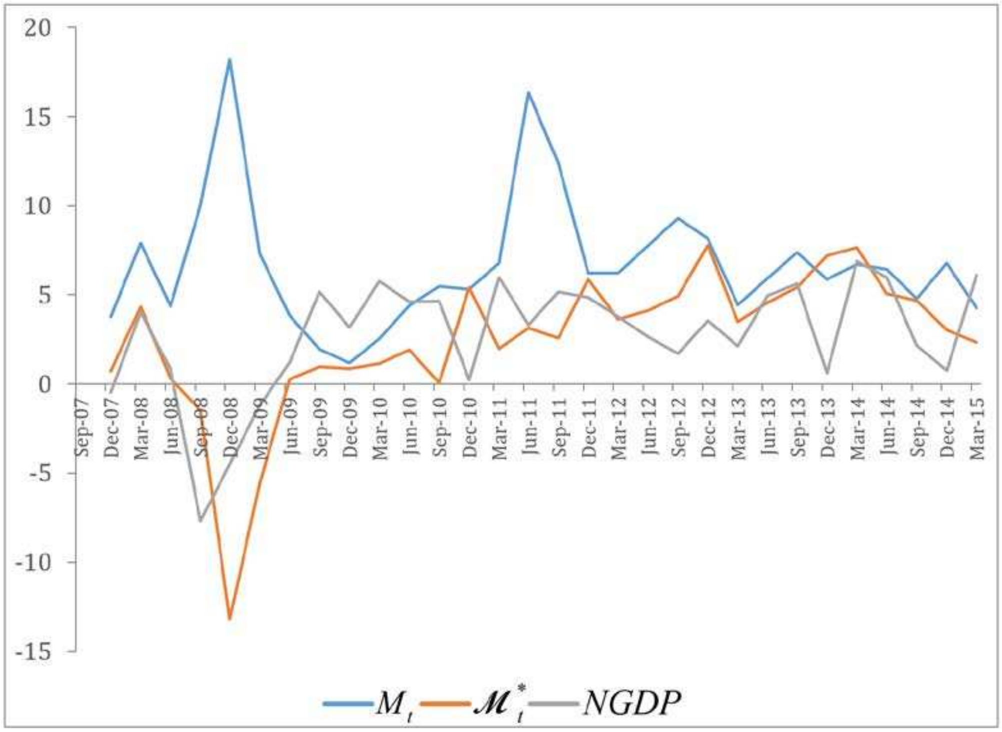


Figure 5: M3 Average Quarterly Growth Rates (2007Q4 – 2015Q1)

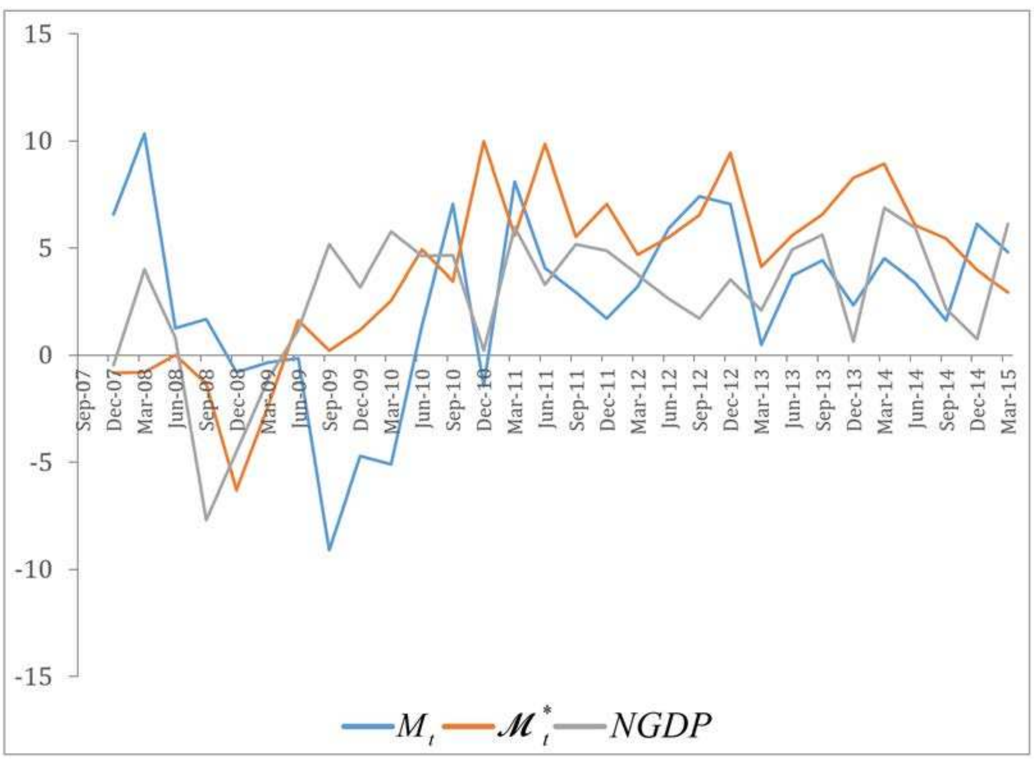
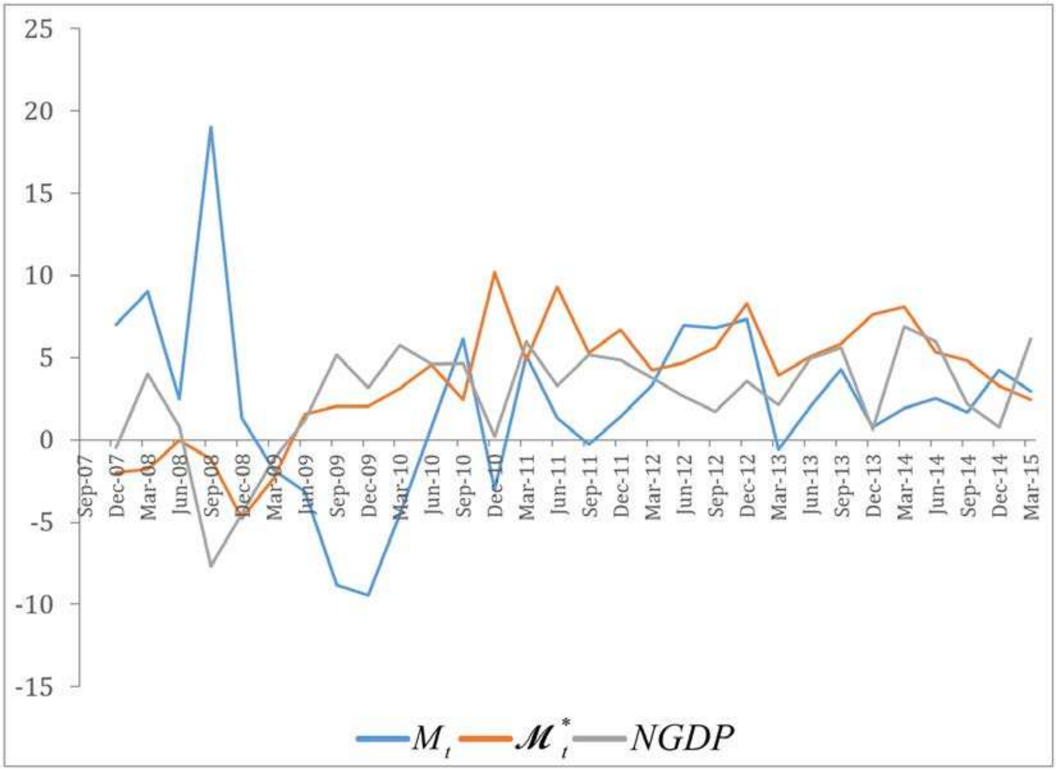


Figure 6: M4 Average Quarterly Growth Rates (2007Q4 – 2015Q1)



10.2. Quarterly Year-over-Year Growth Rates

Figure 7: M1 Quarterly Year-over-Year Growth Rates (2007Q4 – 2015Q1)

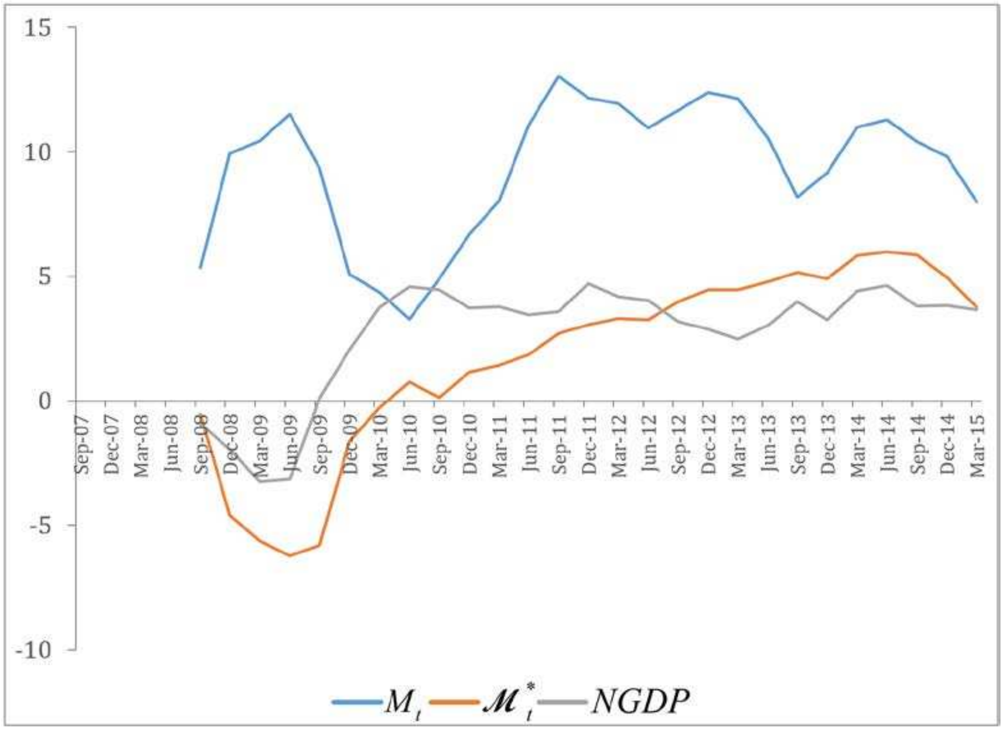


Figure 8: M2 Quarterly Year-over-Year Growth Rates (2007Q4 – 2015Q1)

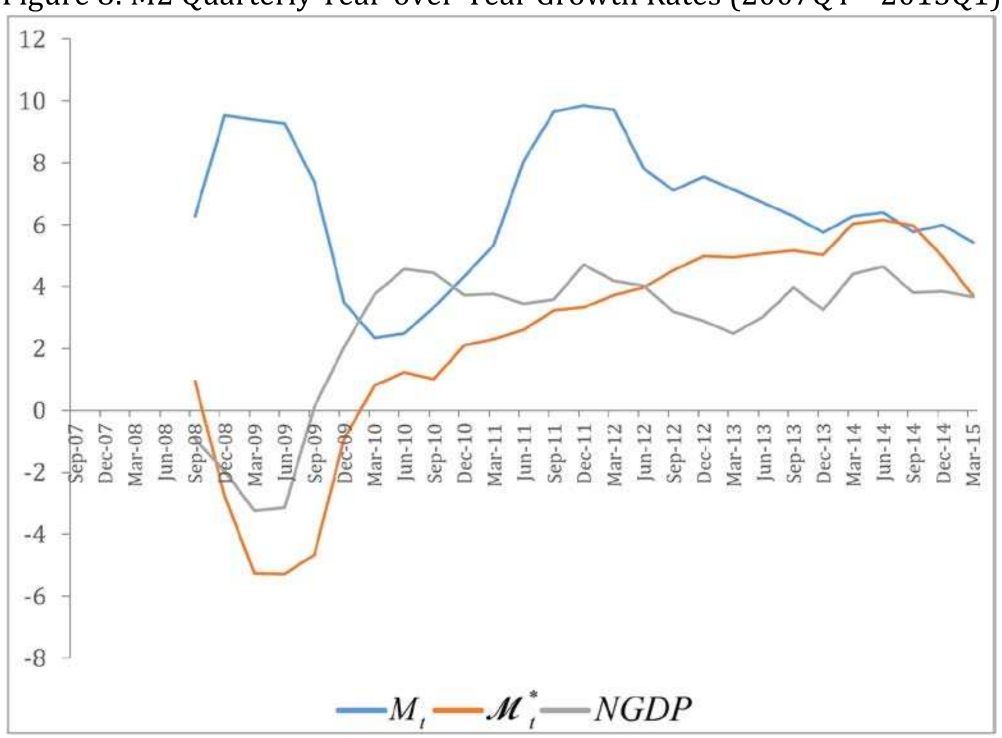


Figure 9: Quarterly M3 Year-over-Year Growth Rates (2007Q4 – 2015Q1)

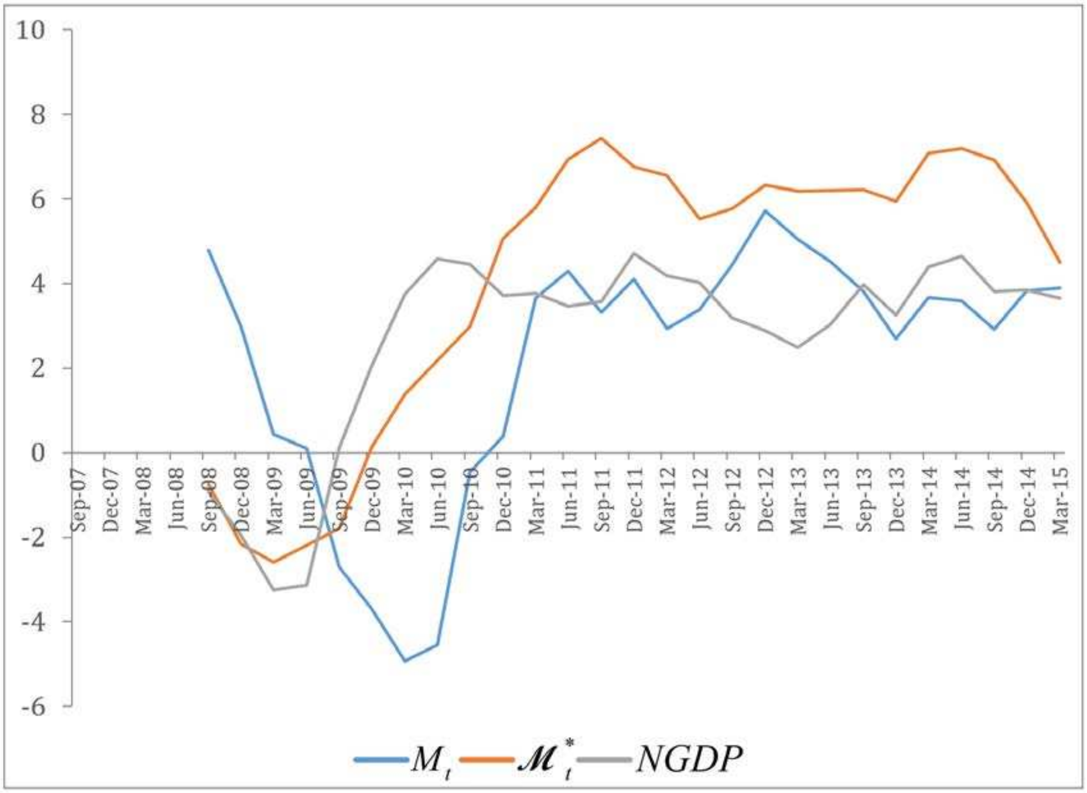
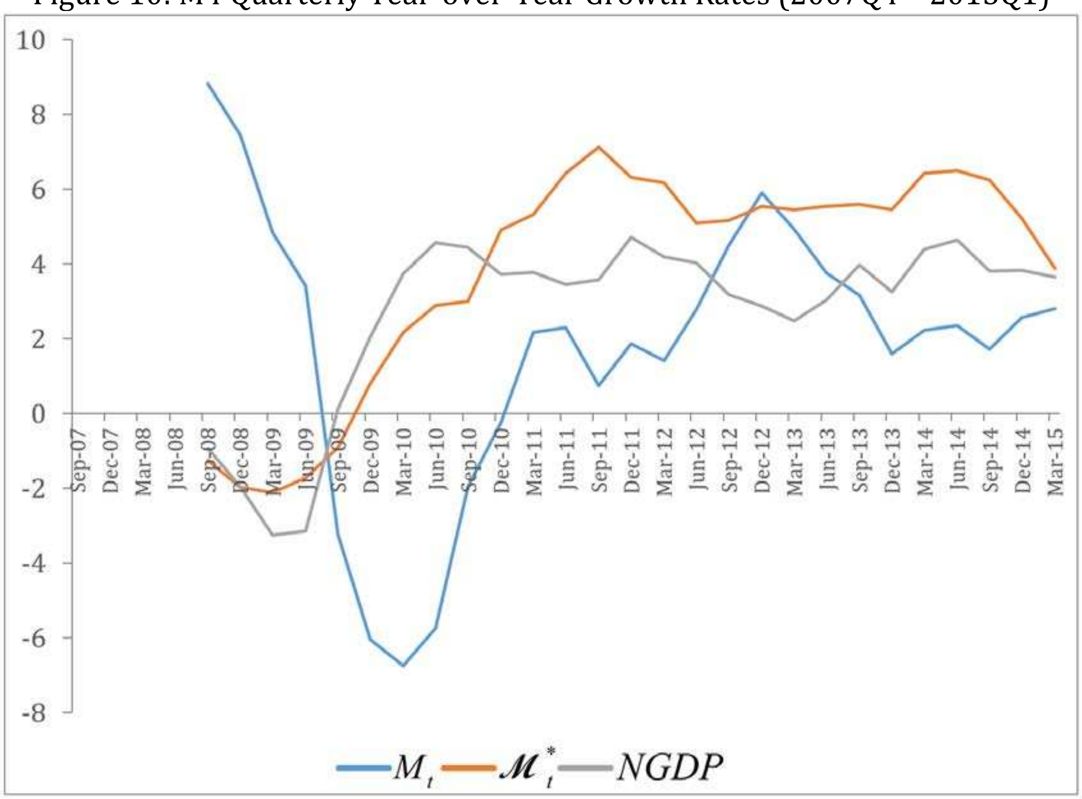


Figure 10: M4 Quarterly Year-over-Year Growth Rates (2007Q4 – 2015Q1)



11. Conclusions

Many economists have wondered how the transactions services of credit cards could be included in monetary aggregates. The conventional simple sum accounting approach precludes solving that problem, since accounting conventions do not permit adding liabilities to assets. But economic aggregation and index number theory measure service flows, independently of whether from assets or liabilities. We have provided theory solving that long overlooked problem both for use as a structural economic variable and as an indicator. The aggregation theoretic exact approach to aggregation as a structural variables provides our credit card-augmented aggregate, $\mathcal{M}_t = \mathcal{M}(\mathbf{m}_t, \mathbf{c}_t)$, while the indicator optimized augmented aggregate, uniquely derived from our nowcasting model, produces our aggregate, $\mathcal{M}_t^* = \mathcal{M}_t^*(\mathbf{m}_t, \mathbf{c}_t)$. In the former case, the aggregate is defined to be weakly separable within the structure of the economy, while in the latter approach the aggregate is defined to be separable within the nowcasting equation. The former approach is relevant to any application requiring a measure of monetary services within the structure of the economy, while the latter approach is application specific and only relevant for use as an indicator.

We have provided the solution under various levels of complexity in terms of theory, econometrics, and data availability. We have implemented the theory under risk neutrality. We have further provided the extension of our theory to CCAPM risk adjustment under risk aversion. The new aggregates will be provided to the public in monthly releases by the Center for Financial Stability (CFS) in NY City and also to Bloomberg terminal users. The CFS is now providing the unaugmented aggregates, $M_t = M(\mathbf{m}_t)$, and will soon be providing both of the credit card augmented aggregates: the structural augmented aggregates, $\mathcal{M}_t = \mathcal{M}(\mathbf{m}_t, \mathbf{c}_t)$, and indicator optimized aggregates, $\mathcal{M}_t^* = \mathcal{M}_t^*(\mathbf{m}_t, \mathbf{c}_t)$.

A more demanding approach would remove the CCAPM assumption of intertemporal separability, in accordance with Barnett and Wu (2005). Adapting that advanced approach to our augmented aggregates, including credit card services, remains a topic for future research. While excluding credit card services, the currently available CFS Divisia monetary aggregates have been found to be reasonably robust to introduction of risk, variations of the benchmark rate, introduction of taxation of interest rates, and other such refinements.[22] But such simplifications might not be the case with our augmented monetary aggregates, because of the high and volatile interest rates on credit card balances.

In previous research, Barnett, Chauvet, and Leiva-Leon (2016) have found that the CFS un-augmented Divisia monetary aggregate, $M_t = M(\mathbf{m}_t)$, is a valuable indicator in a four factor nowcasting model of nominal GDP. In this research, we have found that our new augmented Divisia monetary aggregates, $\mathcal{M}_t^* = \mathcal{M}_t^*(\mathbf{m}_t, \mathbf{c}_t)$, have substantially greater indicator value than $M_t = M(\mathbf{m}_t)$. Considering the current interest in the possibility of nominal GDP targeting, requiring the existence of monthly nominal GDP nowcasts, the augmented aggregates merit serious consideration. Although the greater indicator value is evident from the time series plots, we have provided the formal nowcasting results to confirm the evidence from the plots.

An extensive literature exists on policy relevance of the Divisia monetary aggregates. See, e. g., Barnett (2012), Belongia and Ireland (2014; 2015a,b), Barnett and Chauvet (2011a), Serletis and Rahman (2013), and Serletis and Gogas (2014). Much of that literature could be strengthened further by use of the soon to be available credit-card augmented CFS Divisia monetary aggregates.

[22] While those refinements slightly change the un-augmented Divisia monetary aggregates, those changes are negligible relative to the gap between the simple sum monetary aggregate path and the corresponding Divisia monetary aggregate path. See, e.g., the online library of relevant research and the Divisia monetary aggregates databases at the Center for Financial Stability (www.centerforfinancialstability.org/amfm.php).

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APPENDICES

(I) Derivation of the User Cost Formula for Credit Card Services, Equation (7), in the Infinite Lifetimes Case under Perfect Certainty:

From equation 2, the flow of funds identities, for $s = t, t+1, \ldots, \infty$, are

$$\begin{aligned}\mathbf{p}_s'\mathbf{x}_s = w_s L_s &+ \sum_{i=1}^{n}\left[\left(1+r_{i,s-1}\right)p_{s-1}^{*}m_{i,s-1} - p_s^{*}m_{is}\right] \\ &+ \sum_{j=1}^{k}\left[p_s^{*}c_{js} - \left(1+e_{j,s-1}\right)p_{s-1}^{*}c_{j,s-1}\right] \\ &+ \left[(1+R_{s-1})p_{s-1}^{*}A_{s-1} - p_s^{*}A_s\right].\end{aligned} \tag{A.1}$$

The intertemporal utility function

$$u(\mathbf{m}_t, \mathbf{c}_t, \mathbf{x}_t) + E_t\left[\sum_{s=t+1}^{\infty}\left(\frac{1}{1+\xi}\right)^{s-t} u(\mathbf{m}_s, \mathbf{c}_s, \mathbf{x}_s)\right]$$

under perfect certainty is

$$\sum_{s=t}^{\infty}\left(\frac{1}{1+\xi}\right)^{s-t} u(\mathbf{m}_s, \mathbf{c}_s, \mathbf{x}_s). \tag{A.2}$$

Let $\Im$ be the Lagrangian for maximizing intertemporal utility subject to the sequence of flow of funds identities for $s = t, \ldots, \infty$, and let λ_t be the Lagrange multiplier for the t'th constraint. Then the following are the first order conditions for maximizing (A.2) subject to the sequence of constraints, (A.1).

$$\frac{\partial \Im}{\partial A_t} = -\lambda_{t+1}(1+R_t)p_t^{*} + \lambda_t p_t^{*} = 0, \tag{A.3}$$

$$\frac{\partial \Im}{\partial x_{it}} = \frac{\partial u}{\partial x_{it}} - \lambda_t p_{it} = 0, \tag{A.4}$$

$$\frac{\partial \Im}{\partial m_{it}} = \frac{\partial u}{\partial m_{it}} - \lambda_t p_t^* + \lambda_{t+1}(1 + r_{it}) p_t^* = 0, \tag{A.5}$$

$$\frac{\partial \Im}{\partial c_{jt}} = \frac{\partial u}{\partial c_{jt}} + \lambda_t p_t^* - \lambda_{t+1}(1 + e_{jt}) p_t^* = 0. \tag{A.6}$$

From equation (A.3), we have

$$-\lambda_{t+1}(1 + R_t) + \lambda_t = 0. \tag{A.7}$$

Substitute equation (A.7) into (A.6) to eliminate λ_{t+1}, we get

$$\frac{\partial u}{\partial c_{jt}} = -\lambda_t p_t^* + \frac{\lambda_t}{1 + R_t} p_t^* (1 + e_{jt}). \tag{A.8}$$

Rearranging we get the first order condition that identifies $\tilde{\pi}_{jt}$ as the user cost price of credit card services:

$$\frac{\partial u}{\partial c_{jt}} = \lambda_t \tilde{\pi}_{jt}, \tag{A.9}$$

where

$$\tilde{\pi}_{jt} = p_t^* \frac{e_{jt} - R_t}{1 + R_t}. \qquad \blacksquare \tag{A.10}$$

(II) Derivation of Euler Equations for Credit Card Services, Equation (35):

The following are the Euler equations provided in the paper as equations (34), (35), and (36):

$$\frac{\partial V}{\partial m_{it}} - \rho E_t \left[(R_t^* - r_{it}^*) \frac{\partial V}{\partial X_{t+1}} \right] = 0, \tag{A.11}$$

$$\frac{\partial V}{\partial c_{jt}} - \rho E_t\left[\left(e_{jt}^* - R_t^*\right)\frac{\partial V}{\partial X_{t+1}}\right] = 0, \tag{A.12}$$

$$\frac{\partial V}{\partial X_t} - \rho E_t\left[(1 + R_t^*)\frac{\partial V}{\partial X_{t+1}}\right] = 0. \tag{A.13}$$

for all $s \geq t, i = 1, \ldots, n$, and $j = 1, \ldots, k$, where $\rho = 1/(1+\xi)$ and where p_s^* is the exact price aggregate that is dual to the consumer goods quantity aggregate X_s.

Equation (A.11) was derived in Barnett (1995, Sec 2.3) using Bellman's method. An alternative approach to that derivation using calculus of variations was provided by Poterba and Rotemberg (1987). Equation (A.12) follows by the same approach to derivation, using either Bellman's method or calculus of variations. We are not providing the lengthy derivation of (A.12) in this appendix, since the steps in the Bellman method approach for this class of models are provided in detail in Barnett and Serletis (2000, pp. 201-204).

(III) Proof of Theorem 1

Theorem 1 (a). *The risk adjusted real user cost of the services of monetary asset i under risk is* $\wp_{it}^m = \pi_{it} + \psi_{it}$, *where*

$$\pi_{it} = \frac{E_t R_t^* - E_t r_{it}^*}{1 + E_t R_t^*} \tag{A.14}$$

and

$$\psi_{it} = \rho(1 - \pi_{it})\frac{Cov\left(R_t^*, \frac{\partial V}{\partial X_{t+1}}\right)}{\frac{\partial V}{\partial X_t}} - \rho\frac{Cov\left(r_{it}^*, \frac{\partial V}{\partial X_{t+1}}\right)}{\frac{\partial V}{\partial X_t}}. \tag{A.15}$$

(b). *The risk adjusted real user cost of the services of credit card type j under risk is* $\wp_{jt}^c = \tilde{\pi}_{jt} + \tilde{\psi}_{jt}$, *where*

$$\tilde{\pi}_{jt} = \frac{E_t e_{jt}^* - E_t R_t^*}{1 + E_t R_t^*} \tag{A.16}$$

and

$$\tilde{\psi}_{jt} = \rho \frac{Cov\left(e_{jt}^*, \frac{\partial V}{\partial X_{t+1}}\right)}{\frac{\partial V}{\partial X_t}} - \rho\left(1 + \tilde{\pi}_{jt}\right) \frac{Cov\left(R_t^*, \frac{\partial V}{\partial X_{t+1}}\right)}{\frac{\partial V}{\partial X_t}} . \tag{A.17}$$

Proof. For the analogous proof in the case of monetary assets only, relevant to part (a), see Barnett, Liu, and Jensen (1997), Barnett and Serletis (2000, ch. 12), or Barnett (2012, Appendix D). We provide the proof of part (b) for the extended case including credit. There are two approaches to proving this important theorem, the direct approach and the indirect approach. We provide both approaches, beginning with the indirect approach.

By definition (1) in the paper, we have for the credit card services user cost price

$$\wp_{jt}^c = \left. \frac{\partial V}{\partial c_{jt}} \middle/ \frac{\partial V}{\partial X_t} \right. . \tag{A.18}$$

Defining $\tilde{\psi}_{jt}$ to be $\tilde{\psi}_{jt} = \wp_{jt}^c - \tilde{\pi}_{jt}$, it follows that

$$\frac{\partial V}{\partial c_{jt}} = (\tilde{\pi}_{jt} + \tilde{\psi}_{jt}) \frac{\partial V}{\partial X_t} .$$

Substituting equations (A.12) and (A.13) into this equation, we get

$$\rho E_t\left[\left(e_{jt}^* - R_t^*\right)\frac{\partial V}{\partial X_{t+1}}\right] = (\tilde{\pi}_{jt} + \tilde{\psi}_{jt})\rho E_t\left[\left(1+R_t^*\right)\frac{\partial V}{\partial X_{t+1}}\right].$$

Using the expectation of the product of correlated random variables, we have

$$E_t\left(e_{jt}^* - R_t^*\right)E_t\left(\frac{\partial V}{\partial X_{t+1}}\right) + Cov\left(e_{jt}^* - R_t^*, \frac{\partial V}{\partial X_{t+1}}\right)$$
$$= \left\{\left[\frac{E_t e_{jt}^* - E_t R_t^*}{1+E_t R_t^*}\right] + \tilde{\psi}_{jt}\right\}\left\{E_t\left(1+R_t^*\right)E_t\left(\frac{\partial V}{\partial X_{t+1}}\right) + Cov\left(R_t^*, \frac{\partial V}{\partial X_{t+1}}\right)\right\}.$$

Multiplying $\left(1+E_t R_t^*\right)$ through on both sides of the equation, we get:

$$\left(1+E_t R_t^*\right)E_t\left(e_{jt}^* - R_t^*\right)E_t\left(\frac{\partial V}{\partial X_{t+1}}\right) + \left(1+E_t R_t^*\right)Cov\left(e_{jt}^* - R_t^*, \frac{\partial V}{\partial X_{t+1}}\right)$$
$$= \left[E_t\left(e_{jt}^* - R_t^*\right) + \left(1+E_t R_t^*\right)\tilde{\psi}_{jt}\right]\left\{E_t\left(1+R_t^*\right)E_t\left(\frac{\partial V}{\partial X_{t+1}}\right) + Cov\left(R_t^*, \frac{\partial V}{\partial X_{t+1}}\right)\right\}.$$

Manipulating the algebra, we have

$$E_t\left(e_{jt}^* - R_t^*\right)E_t\left(\frac{\partial V}{\partial X_{t+1}}\right) + \left(E_t R_t^*\right)E_t\left(e_{jt}^* - R_t^*\right)E_t\left(\frac{\partial V}{\partial X_{t+1}}\right) + Cov\left(e_{jt}^* - R_t^*, \frac{\partial V}{\partial X_{t+1}}\right)$$
$$+ \left(E_t R_t^*\right)Cov\left(e_{jt}^* - R_t^*, \frac{\partial V}{\partial X_{t+1}}\right)$$
$$= \left[E_t\left(e_{jt}^* - R_t^*\right) + \left(1+E_t R_t^*\right)\tilde{\psi}_{jt}\right]\left\{E_t\left(\frac{\partial V}{\partial X_{t+1}}\right) + \left(E_t R_t^*\right)E_t\left(\frac{\partial V}{\partial X_{t+1}}\right) + Cov\left(R_t^*, \frac{\partial V}{\partial X_{t+1}}\right)\right\},$$

and hence

$$E_t\left(e_{jt}^*-R_t^*\right)E_t\left(\frac{\partial V}{\partial X_{t+1}}\right)+\left(E_tR_t^*\right)E_t\left(e_{jt}^*-R_t^*\right)E_t\left(\frac{\partial V}{\partial X_{t+1}}\right)+Cov\left(e_{jt}^*-R_t^*,\frac{\partial V}{\partial X_{t+1}}\right)$$
$$+\left(E_tR_t^*\right)Cov\left(e_{jt}^*-R_t^*,\frac{\partial V}{\partial X_{t+1}}\right)$$
$$=E_t\left(e_{jt}^*-R_t^*\right)E_t\left(\frac{\partial V}{\partial X_{t+1}}\right)+E_t\left(e_{jt}^*-R_t^*\right)\left(E_tR_t^*\right)\left(E_t\left(\frac{\partial V}{\partial X_{t+1}}\right)\right)+E_t\left(e_{jt}^*-R_t^*\right)Cov\left(R_t^*,\frac{\partial V}{\partial X_{t+1}}\right)$$
$$+\left(1+E_tR_t^*\right)\tilde{\psi}_{jt}\left\{E_t\left(\frac{\partial V}{\partial X_{t+1}}\right)+\left(E_tR_t^*\right)\left(E_t\left(\frac{\partial V}{\partial X_{t+1}}\right)\right)+Cov\left(R_t^*,\frac{\partial V}{\partial X_{t+1}}\right)\right\}.$$

Notice that by equation (A.13),

$$\frac{\partial V}{\partial X_t}=\rho E_t\left[\left(1+R_t^*\right)\frac{\partial V}{\partial X_{t+1}}\right]$$
$$=\rho\left\{E_t\left(\frac{\partial V}{\partial X_{t+1}}\right)+\left(E_tR_t^*\right)\left(E_t\left(\frac{\partial V}{\partial X_{t+1}}\right)\right)+Cov\left(R_t^*,\frac{\partial V}{\partial X_{t+1}}\right)\right\}.$$

Substituting this back into the prior equation, we have

$$E_t\left(e_{jt}^*-R_t^*\right)E_t\left(\frac{\partial V}{\partial X_{t+1}}\right)+\left(E_tR_t^*\right)E_t\left(e_{jt}^*-R_t^*\right)E_t\left(\frac{\partial V}{\partial X_{t+1}}\right)+Cov\left(e_{jt}^*-R_t^*,\frac{\partial V}{\partial X_{t+1}}\right)$$
$$+\left(E_tR_t^*\right)Cov\left(e_{jt}^*-R_t^*,\frac{\partial V}{\partial X_{t+1}}\right)$$
$$=E_t\left(e_{jt}^*-R_t^*\right)E_t\left(\frac{\partial V}{\partial X_{t+1}}\right)+E_t\left(e_{jt}^*-R_t^*\right)\left(E_tR_t^*\right)\left(E_t\left(\frac{\partial V}{\partial X_{t+1}}\right)\right)+E_t\left(e_{jt}^*-R_t^*\right)Cov\left(R_t^*,\frac{\partial V}{\partial X_{t+1}}\right)$$
$$+\left(1+E_tR_t^*\right)\tilde{\psi}_{jt}\left(\frac{1}{\rho}\frac{\partial V}{\partial X_t}\right).$$

Simplifying the equation, we get

$$Cov\left(e_{jt}^*-R_t^*,\frac{\partial V}{\partial X_{t+1}}\right)+\left(E_tR_t^*\right)Cov\left(e_{jt}^*-R_t^*,\frac{\partial V}{\partial X_{t+1}}\right)$$
$$=E_t\left(e_{jt}^*-R_t^*\right)Cov\left(R_t^*,\frac{\partial V}{\partial X_{t+1}}\right)+\left(1+E_tR_t^*\right)\tilde{\psi}_{jt}\left(\frac{1}{\rho}\frac{\partial V}{\partial X_t}\right).$$

Recall that by equation (A.16),

$$\tilde{\pi}_{jt} = \frac{E_t e_{jt}^* - E_t R_t^*}{1 + E_t R_t^*}.$$

Substituting this equation back into the prior equation, we have

$$Cov\left(e_{jt}^* - R_t^*, \frac{\partial V}{\partial X_{t+1}}\right) + \left(E_t R_t^*\right) Cov\left(e_{jt}^* - R_t^*, \frac{\partial V}{\partial X_{t+1}}\right)$$
$$= \tilde{\pi}_{jt}\left(1 + E_t R_t^*\right) Cov\left(R_t^*, \frac{\partial V}{\partial X_{t+1}}\right) + \left(1 + E_t R_t^*\right)\tilde{\psi}_{jt}\left(\frac{1}{\rho}\frac{\partial V}{\partial X_t}\right).$$

Rearranging the equation, we have

$$\left(1 + E_t R_t^*\right) Cov\left(e_{jt}^* - R_t^*, \frac{\partial V}{\partial X_{t+1}}\right) = \tilde{\pi}_{jt}\left(1 + E_t R_t^*\right) Cov\left(R_t^*, \frac{\partial V}{\partial X_{t+1}}\right) + \left(1 + E_t R_t^*\right)\tilde{\psi}_{jt}\left(\frac{1}{\rho}\frac{\partial V}{\partial X_t}\right),$$

so that

$$Cov\left(e_{jt}^* - R_t^*, \frac{\partial V}{\partial X_{t+1}}\right) = \tilde{\pi}_{jt} Cov\left(R_t^*, \frac{\partial V}{\partial X_{t+1}}\right) + \tilde{\psi}_{jt}\left(\frac{1}{\rho}\frac{\partial V}{\partial X_t}\right).$$

Hence, it follows that

$$\begin{aligned}
\tilde{\psi}_{jt} &= \rho \frac{Cov\left(e_{jt}^* - R_t^*, \frac{\partial V}{\partial X_{t+1}}\right)}{\frac{\partial V}{\partial X_t}} - \rho\tilde{\pi}_{jt} \frac{Cov\left(R_t^*, \frac{\partial V}{\partial X_{t+1}}\right)}{\frac{\partial V}{\partial X_t}} \\
&= \rho \frac{Cov\left(e_{jt}^*, \frac{\partial V}{\partial X_{t+1}}\right)}{\frac{\partial V}{\partial X_t}} - \rho \frac{Cov\left(R_t^*, \frac{\partial V}{\partial X_{t+1}}\right)}{\frac{\partial V}{\partial X_t}} - \rho\tilde{\pi}_{jt} \frac{Cov\left(R_t^*, \frac{\partial V}{\partial X_{t+1}}\right)}{\frac{\partial V}{\partial X_t}} \\
&= \rho \frac{Cov\left(e_{jt}^*, \frac{\partial V}{\partial X_{t+1}}\right)}{\frac{\partial V}{\partial X_t}} - \rho\left(1 + \tilde{\pi}_{jt}\right) \frac{Cov\left(R_t^*, \frac{\partial V}{\partial X_{t+1}}\right)}{\frac{\partial V}{\partial X_t}}.
\end{aligned}$$

The alternative direct approach to proof is the following.
By equation (A.13), we have

$$\frac{\partial V}{\partial X_t} = \rho E_t \left[\left(1+R_t^*\right) \frac{\partial V}{\partial X_{t+1}} \right]$$

$$= \rho\left(1+E_t R_t^*\right)\left(E_t\left(\frac{\partial V}{\partial X_{t+1}} \right)\right) + \rho Cov\left(R_t^*, \frac{\partial V}{\partial X_{t+1}} \right).$$

Rearranging, we get

$$\rho\left(1+E_t R_t^*\right)\left(E_t\left(\frac{\partial V}{\partial X_{t+1}} \right)\right) = \frac{\partial V}{\partial X_t} - \rho Cov\left(R_t^*, \frac{\partial V}{\partial X_{t+1}} \right),$$

and hence

$$\rho E_t\left(\frac{\partial V}{\partial X_{t+1}} \right) = \frac{1}{1+E_t R_t^*}\left[\frac{\partial V}{\partial X_t} - \rho Cov\left(R_t^*, \frac{\partial V}{\partial X_{t+1}} \right)\right]. \qquad \text{(A.19)}$$

But from (A.12), we have

$$\frac{\partial V}{\partial c_{jt}} = \rho E_t \left[\left(e_{jt} - R_t^*\right) \frac{\partial V}{\partial X_{t+1}} \right].$$

From the expectation of the correlated product, we then have

$$\frac{\partial V}{\partial c_{jt}} = \rho E_t\left(e_{jt} - R_t^*\right) E_t\left(\frac{\partial V}{\partial X_{t+1}} \right) + \rho Cov\left[\left(e_{jt} - R_t^*\right), \frac{\partial V}{\partial X_{t+1}} \right],$$

so that

$$\frac{\partial V}{\partial c_{jt}} = \rho E_t\left(e_{jt} - R_t^*\right) E_t\left(\frac{\partial V}{\partial X_{t+1}} \right) + \rho Cov\left(e_{jt}, \frac{\partial V}{\partial X_{t+1}} \right) - \rho Cov\left(R_t^*, \frac{\partial V}{\partial X_{t+1}} \right). \qquad \text{(A.20)}$$

Now substitute equation (A.19) into equation (A.20), to acquire

$$\frac{\partial V}{\partial c_{jt}} = \frac{E_t\left(e_{jt} - R_t^*\right)}{1 + E_t R_t^*}\left[\frac{\partial V}{\partial X_t} - \rho Cov\left(R_t^*, \frac{\partial V}{\partial X_{t+1}}\right)\right] + \rho Cov\left(e_{jt}, \frac{\partial V}{\partial X_{t+1}}\right) - \rho Cov\left(R_t^*, \frac{\partial V}{\partial X_{t+1}}\right)$$
$$= \tilde{\pi}_{jt}\left[\frac{\partial V}{\partial X_t} - \rho Cov\left(R_t^*, \frac{\partial V}{\partial X_{t+1}}\right)\right] + \rho Cov\left(e_{jt}, \frac{\partial V}{\partial X_{t+1}}\right) - \rho Cov\left(R_t^*, \frac{\partial V}{\partial X_{t+1}}\right).$$

Multiplying and dividing the right side by $\frac{\partial V}{\partial X_t}$, we get

$$\frac{\partial V}{\partial c_{jt}} = \frac{\partial V}{\partial X_t}\left\{\tilde{\pi}_{jt} - \rho\tilde{\pi}_{jt}\frac{Cov\left(R_t^*, \frac{\partial V}{\partial X_{t+1}}\right)}{\frac{\partial V}{\partial X_t}} + \rho\frac{Cov\left(e_{jt}, \frac{\partial V}{\partial X_{t+1}}\right)}{\frac{\partial V}{\partial X_t}} - \rho\frac{Cov\left(R_t^*, \frac{\partial V}{\partial X_{t+1}}\right)}{\frac{\partial V}{\partial X_t}}\right\}$$
$$= \frac{\partial V}{\partial X_t}\left\{\tilde{\pi}_{jt} + \rho\frac{Cov\left(e_{jt}, \frac{\partial V}{\partial X_{t+1}}\right)}{\frac{\partial V}{\partial X_t}} - \rho\left(1 + \tilde{\pi}_{jt}\right)\frac{Cov\left(R_t^*, \frac{\partial V}{\partial X_{t+1}}\right)}{\frac{\partial V}{\partial X_t}}\right\}.$$

Define $\tilde{\psi}_{jt}$ by

$$\tilde{\psi}_{jt} = \rho\frac{Cov\left(e_{jt}, \frac{\partial V}{\partial X_{t+1}}\right)}{\frac{\partial V}{\partial X_t}} - \rho\left(1 + \tilde{\pi}_{jt}\right)\frac{Cov\left(R_t^*, \frac{\partial V}{\partial X_{t+1}}\right)}{\frac{\partial V}{\partial X_t}}.$$

Then we have

$$\frac{\frac{\partial V}{\partial c_{jt}}}{\frac{\partial V}{\partial X_t}} = \tilde{\pi}_{jt} + \tilde{\psi}_{jt},$$

so that

$$\wp^{c}_{jt} = \tilde{\pi}_{jt} + \tilde{\psi}_{jt}. \qquad \blacksquare$$

(IV) Proof of Lemma 2:

Assumption 1. Let V have the form

$$V(\mathbf{m}_t, \mathbf{c}_t, X_t) = F[\mathcal{M}(\mathbf{m}_t, \mathbf{c}_t), X_t] = A[\mathcal{M}(\mathbf{m}_t, \mathbf{c}_t)]X_t - \frac{1}{2}B[\mathcal{M}(\mathbf{m}_t, \mathbf{c}_t)]X_t^2, \quad (\mathrm{A.21})$$

Where A is a positive, increasing, concave function and B is a nonnegative, decreasing, convex function.

Assumption 2. Let $\left(r_{it}^*, e_{jt}^*, X_{t+1}\right)$ be a trivariate Gaussian process for each asset $i = 1, \dots, n$, and credit card service, $j = 1, \dots, k$.

Assumption 3. The benchmark rate process is deterministic or already risk-adjusted, so that R_t^* is the risk-free rate.

Under this assumption, it follows that

$$Cov\left(R_t^*, \frac{\partial V}{\partial X_{t+1}}\right) = 0.$$

Define $H_{t+1} = H(\mathcal{M}_{t+1}, X_{t+1})$ to be the well-known Arrow-Pratt measure of absolute risk aversion,

$$H(\mathcal{M}_{t+1}, X_{t+1}) = -\frac{E_t[V'']}{E_t[V']}\,, \tag{A.22}$$

Where $V' = \partial V(\boldsymbol{m}_{t+1}^{a}, X_{t+1})/\partial X_{t+1}$ and $V'' = \partial^2 V(\boldsymbol{m}_{t+1}^{a}, X_{t+1})/\partial X_{t+1}^{2}$.

Lemma 2. *Under Assumption 3 and either Assumption 1 or Assumption 2, the user-cost risk adjustments, ψ_{it} and $\tilde{\psi}_{jt}$, defined by (A.15) and (A.17), reduce to*

$$\psi_{it} = \frac{1}{1+R_t^*} H_{t+1} cov(r_{it}^*, X_{t+1}) \tag{A.23}$$

and

$$\tilde{\psi}_{jt} = -\frac{1}{1+R_t^*} H_{t+1} cov\left(e_{jt}^*, X_{t+1}\right). \tag{A.24}$$

Proof. For the analogous proof in the case of monetary assets only, relevant to equation (44a), see Barnett, Liu, and Jensen (1997), Barnett and Serletis (2000, ch. 12), or Barnett (2012, Appendix D). We provide the proof of equation (A.24) for the extended case including credit.

Under Assumption 3, the benchmark asset is risk-free, so that

$$Cov\left(R_t^*, \frac{\partial V}{\partial X_{t+1}}\right) = 0\,.$$

By equation (A.17),

$$\tilde{\psi}_{jt} = \rho \frac{Cov\left(e_{jt}^*, \dfrac{\partial V}{\partial X_{t+1}}\right)}{\dfrac{\partial V}{\partial X_t}} - \rho\left(1+\tilde{\pi}_{jt}\right)\frac{Cov\left(R_t^*, \dfrac{\partial V}{\partial X_{t+1}}\right)}{\dfrac{\partial V}{\partial X_t}}$$

$$= \rho \frac{Cov\left(e_{jt}^*, \dfrac{\partial V}{\partial X_{t+1}}\right)}{\dfrac{\partial V}{\partial X_t}}.$$

But by equation (A.13),

$$\frac{\partial V}{\partial X_t} = \rho E_t\left[\left(1+R_t^*\right)\frac{\partial V}{\partial X_{t+1}}\right],$$

So

$$\tilde{\psi}_{jt} = \rho \frac{Cov\left(e_{jt}^*, \dfrac{\partial V}{\partial X_{t+1}}\right)}{\rho\left(1+R_t^*\right)E_t\left(\dfrac{\partial V}{\partial X_{t+1}}\right)}$$

$$= \frac{Cov\left(e_{jt}^*, \dfrac{\partial V}{\partial X_{t+1}}\right)}{\left(1+R_t^*\right)E_t\left(\dfrac{\partial V}{\partial X_{t+1}}\right)}. \tag{A.25}$$

Under Assumption 1,

$$\frac{\partial V}{\partial X_t} = A\left[\mathcal{M}\left(\mathbf{m}_t, \mathbf{c}_t\right)\right] - B\left[\mathcal{M}\left(\mathbf{m}_t, \mathbf{c}_t\right)\right]X_t .$$

Hence,

$$\frac{\partial^2 V}{\partial X_t^{\ 2}} = -B\left[\mathcal{M}\left(\mathbf{m}_t, \mathbf{c}_t\right)\right].$$

Shifting one period forward, those two equations become

$$\frac{\partial V}{\partial X_{t+1}} = V' = A - BX_{t+1}$$

and

$$\frac{\partial^2 V}{\partial {X_t}^2} = V'' = -B\,.$$

Substituting into equation (A.25), we get

$$\begin{aligned}
\tilde{\psi}_{jt} &= \frac{Cov\left(e_{jt}^*,\ A - BX_{t+1}\right)}{\left(1+R_t^*\right)E_t\left(\dfrac{\partial V}{\partial X_{t+1}}\right)} \\
&= \frac{-B}{1+R_t^*}\frac{Cov\left(e_{jt}^*,\ X_{t+1}\right)}{E_t\left(V'\right)} \\
&= \frac{1}{1+R_t^*}\frac{E\left(V''\right)}{E_t\left(V'\right)}Cov\left(e_{jt}^*,\ X_{t+1}\right) \\
&= -\frac{1}{1+R_t^*}H_{t+1}Cov\left(e_{jt}^*,\ X_{t+1}\right).
\end{aligned}$$

Alternatively, consider Assumption 2. We then can use Stein's lemma, which says the following.[23] Suppose (X,Y) are multivariate normal. Then

$$Cov(g(X),Y) = E(g'(X))Cov(X,Y).$$

In that formula, let $g(X) = \dfrac{\partial V}{\partial X_{t+1}}$, $X = X_{t+1}$, and $Y = e_{jt}^*$. Then from Stein's lemma, we have

$$Cov\left(e_{jt}^*,\ \frac{\partial V}{\partial X_{t+1}}\right) = E_t\left(\frac{\partial^2 V}{\partial X_{t+1}^2}\right)Cov\left(X_{t+1}, e_{jt}^*\right).$$

Substituting into (A.25), we get

[23] For Stein's lemma, see Stein (1973), Ingersoll (1987, p. 13, eq. 62) or Rubinstein (1976).

$$\tilde{\psi}_{jt} = \frac{E_t\left(\dfrac{\partial^2 V}{\partial X_{t+1}^2}\right) Cov\left(X_{t+1}, e_{jt}^*\right)}{\left(1+R_t^*\right) E_t\left(\dfrac{\partial V}{\partial X_{t+1}}\right)}.$$

Using the definitions of V', V'', and H_{t+1}, we have

$$\tilde{\psi}_{jt} = -\frac{1}{1+R_t^*} H_{t+1} Cov\left(e_{jt}^*,\ X_{t+1}\right).$$ ■

(V) Proof of Theorem 3:

Theorem 3. *Let $\hat{H}_t = H_{t+1} X_t$. Under the assumptions of Lemma 2, we have the following for each asset $i = 1, \ldots, n$, and credit card service, $j = 1, \ldots, k$,*

$$\wp_{it}^{m} = \frac{E_t R_t^* - (E_t r_{it}^* - \phi_{it})}{1 + E_t R_t^*}, \tag{A.26}$$

where

$$\phi_{it} = \hat{H}_t\, Cov\left(r_{it}^*, \frac{X_{t+1}}{X_t}\right), \tag{A.27}$$

and

$$\wp_{jt}^{c} = \frac{(E_t e_{jt}^* - \tilde{\phi}_{jt}) - E_t R_t^*}{1 + E_t R_t^*}, \tag{A.28}$$

where

$$\tilde{\phi}_{jt} = \hat{H}_t\, Cov\left(e_{jt}^*, \frac{X_{t+1}}{X_t}\right). \tag{A.29}$$

Proof. For the proof in the case of monetary assets only, relevant to equations (A.26) and (A.27), see Barnett, Liu, and Jensen (1997), Barnett and Serletis (2000, ch. 12), or Barnett (2012, Appendix D). We here provide the proof of equations (A.28) and (A.29) for the extended case including credit.

From part b of Theorem 1,

$$\wp_{jt}^{c} = \frac{E_t e_{jt}^{*} - E_t R_t^{*}}{1 + E_t R_t} + \tilde{\psi}_{jt} .$$

Letting $\hat{H}_t = H_{t+1} X_t$. and using Lemma 2, we get

$$\begin{aligned}
\wp_{jt}^{c} &= \frac{E_t e_{jt}^{*} - E_t R_t^{*}}{1 + E_t R_t} - \frac{H_{t+1} Cov\left(e_{jt}^{*},\ X_{t+1}\right)}{1 + E_t R_t^{*}} \\
&= \frac{E_t e_{jt}^{*} - E_t R_t^{*}}{1 + E_t R_t} - \frac{H_{t+1} X_t Cov\left(e_{jt}^{*},\ \dfrac{X_{t+1}}{X_t}\right)}{1 + E_t R_t^{*}} \\
&= \frac{E_t e_{jt}^{*} - E_t R_t^{*}}{1 + E_t R_t} - \frac{\hat{H}_t Cov\left(e_{jt}^{*},\ \dfrac{X_{t+1}}{X_t}\right)}{1 + E_t R_t^{*}}.
\end{aligned}$$

Define $\tilde{\phi}_{j,t} = \hat{H}_t Cov\left(e_{jt}^{*},\ \dfrac{X_{t+1}}{X_t}\right)$ to get

$$\begin{aligned}
\wp_{jt}^{c} &= \frac{E_t e_{jt}^{*} - E_t R_t^{*}}{1 + E_t R_t} - \frac{\tilde{\phi}_{jt}}{1 + E_t R_t^{*}} \\
&= \frac{(E_t e_{jt}^{*} - \tilde{\phi}_{jt}) - E_t R_t^{*}}{1 + E_t R_t^{*}}.
\end{aligned}$$

■

(VI) Computation of the Weights, $\boldsymbol{\omega}_t$, in Table 2 and the Indicator Optimized Augmented Aggregates, $\mathcal{M}_t^* = \mathcal{M}_t^*(\mathbf{m}_t, \mathbf{c}_t)$, in Figures 3 - 10:

The nowcasting model, defined in equations (49) - (51), contains information about the growth rates of Nominal GDP, $y_{1,t}$, Industrial Production, $y_{2,t}$, Consumer Price Index, $y_{3,t}$, CFS Divisia monetary aggregate, $y_{4,t}$, and credit card transactions volume, $y_{5,t}$. To facilitate convergence in the Maximum likelihood optimization, the data are standardized before entering the model. Define $y_{i,t}^*$ for *i = 1, ... , 5*, to be the standardized variables.

We then estimate the model in a recursive real-time manner. At each period of time, *t*, that the model is re-estimated, it produces nowcasts of standardized Nominal GDP, defined as $\hat{y}_{1,t}^*$, during the evaluation sample, $t = \tau, \dots, T$, where τ is October 2007 and T is May 2015. The standardized nowcasts can be re-standardized to recover the original units of measure to obtain $\hat{y}_{1,t}$, which are our nowcasts.

The weight, $\omega_{i,t}$, associated with each variable i of the model in the construction of the standardized nominal GDP nowcast, $\hat{y}_{1,t}^*$, can be computed from equation (55). Notice that the weights, $\omega_{i,t}$, can vary over time. There are two reasons: (*i*) the weights depend on the state-space model matrices, which change over time depending on the set of information available at time t, and (*ii*) the model is re-estimated at every period of time to obtain updated parameter estimates. Accordingly, we acquire a vector of weights, one for each variable,

$$\boldsymbol{\omega}_t = (\omega_{1,t}, \omega_{2,t}, \omega_{3,t}, \omega_{4,t}, \omega_{5,t})'.$$

It is important to recognize that (*i*) these weights are endogenously computed inside the nowcasting model, (*ii*) they depend on the matrices of the state space model, equations (52) - (53), and (*iii*) they are time-varying.

Once we obtain the vector $\boldsymbol{\omega}_t$, we focus our attention on two elements of that vector: the weight, $\omega_{4,t}$, associate with the CFS Divisia aggregate, $M_t = M(\mathbf{m}_t)$, and the weight, ω_{5t}, associated with the Credit Card Volumes. We then can aggregate over the two of them to acquire the indicator-optimized credit-card-augmented aggregate,

$$\mathcal{M}_t^a = \left(\omega_{4,t}y_{4,t} + \omega_{5,t}y_{5,t}\right).$$

Since $y_{4,t}$ and $y_{5,t}$ are expressed in growth rates, $\mathcal{M}_t$ is also in the same growth rate units of measure. To recover the levels of the indicator-optimized aggregate, we apply the following transformation

$$\mathcal{M}_t^* = \mathcal{M}_{t-1}^*\left(\frac{\mathcal{M}_t^a}{100} + 1\right),$$

with the initial $\mathcal{M}_0^*$ set equal to the corresponding value of the CFS Divisia index, M_t = $M(\mathbf{m}_t)$, for September 2007.