Financial Market Dynamics: Superdiffusive or not?

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22 July 2016

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August 24, 2016

Abstract

The behavior of stock market returns over a period of 1-60 days has been investigated for S&P 500 and Nasdaq within the framework of nonextensive Tsallis statistics. Even for such long terms, the distributions of the returns are non-Gaussian. They have fat tails indicating long range correlations persist. In this work, a good fit to a Tsallis q-Gaussian distribution is obtained for the distributions of all the returns using the method of Maximum Likelihood Estimate. For all the regions of data considered, the values of the scaling parameter q, estimated from one day returns, lie in the range 1.4 to 1.65. The estimated inverse mean square deviations β show a power law behavior in time with exponent values between -0.91 and -1.1 indicating normal to mildly subdiffusive behavior. Quite often, the dynamics of market return distributions is modelled by a Fokker-Plank (FP) equation either with a linear drift and a nonlinear diffusion term or with just a nonlinear diffusion term. Both of these cases support a q-Gaussian distribution as a solution. The distributions obtained from current estimated parameters are compared with the solutions of the FP equations. For negligible drift term, the inverse mean square deviation β_{FP} from the FP model follows a power law with exponent values between -1.25 and -1.48 indicating superdiffusion. When the drift term is non-negligible, the corresponding β_{FP} does not follow a power law and becomes stationary after a certain characteristic time that depends on the values of the drift parameter and q. Neither of these behaviors is supported by the results of the empirical fit.

Keywords: Tsallis distribution; stock market dynamics; Maximum Likelihood Estimate; nonlinear Fokker-Plank equation; superdiffusion; econophysics

1. Introduction

Many well-known financial models [1] are based on the efficient market hypothesis [2] according to which: a) investors have all the information available to them and they independently make rational decisions using this information, b) the market reacts to all the information available reaching equilibrium quickly, and c) in this equilibrium state the market essentially follows a random walk [3]. In such a system extreme changes are very rare. In reality, however, the market is a complex system that is the result of decisions by interacting agents (e.g., herding behavior), traders who speculate and/or act impulsively on little news, etc. Such a

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collective/chaotic behavior can lead to wild swings in the system, driving it away from equilibrium into the realm of nonlinearity, resulting in a variety of interesting phenomena such as phase transition, critical phenomena such as bubbles, crashes [4], superdiffusion [5] and so on.

The entropy of an equilibrium system following a random walk is given by Shannon entropy [6]. Maximization of this entropy [7] with constraints on the first three moments yields a Gaussian distribution. Therefore, if the stock market follows a pattern of random walk, the corresponding returns should show a Gaussian distribution. However, it is well known [8] that stock market returns, in general, show a more complicated distribution. This is illustrated in Figure 1 which compares the distributions of 1 day and 20 day log returns of S&P 500 and Nasdaq stock markets (1994-2014) with the corresponding Gaussian distributions. The data distributions show sharp peaks in the center and fat tails over many scales, neither of which is captured by the Gaussian distribution. This points to the presence of factors such as long term correlation and nonlinearity. Several studies [9] [10] indicate that these issues can be addressed using statistical methods based on Tsallis entropy [11], which is a generalization of Shannon entropy to nonextensive systems. These methods were originally proposed to study classical and quantum chaos, physical systems far from equilibrium such as turbulent systems, and long range interacting Hamiltonian systems. However, in the last several years, there has been considerable interest in applying these methods to analyze financial market dynamics as well. Such applications fall into the category of econophysics [5].

The Tsallis generalization of Shannon Entropy:

$$S_{sh} = \sum_i P_i \, ln(1/P_i) \tag{1}$$

to nonextensive systems is given by:

$$S_q = \sum_i P_i \, ln_q(1/P_i) \tag{2}$$

where P_i is the probability density function at the ith sample under the condition $\sum_i P_i = 1$ and the q logarithm $ln_q(x)$ is given by:

$$ln_q(x) = (x^{1-q} - 1)/(1-q) \tag{3}$$

q is a universal parameter, but its value can change from system to system.

Substituting (3) in (2), we get:

$$S_q = \left(1 - \sum_i P_i^q\right)/(q-1) \tag{4}$$

It is important to note that unlike Shannon Entropy, Tsallis entropy is <u>not additive</u> which points to its applicability to correlated systems.

Considering the continuous case for a random variable Ω, one can show [12] that the maximization of S_q with respect to P under the following constraints:

$$\int_{-\infty}^{\infty} P(\Omega) d\Omega = 1 \tag{5a}$$

$$\langle(\Omega - \bar{\Omega})\rangle_q = \int_{-\infty}^{\infty} (\Omega - \bar{\Omega}) \; P^q(\Omega) d\Omega = 0 \tag{5b}$$

$$\langle(\Omega - \bar{\Omega})^2\rangle_q = \int_{-\infty}^{\infty} (\Omega - \bar{\Omega})^2 \; P^q(\Omega) d\Omega = \sigma_q^2 \tag{5c}$$

gives the Tsallis distribution:

$$P_q(\Omega) = \frac{1}{Z_q} \left[1 + (q-1)\beta(\Omega - \bar{\Omega})^2\right]^{1/(1-q)} \tag{6}$$

Z_q is the normalization given by:

$$Z_q = \int [1 + (q-1)\beta(\Omega - \bar{\Omega})^2]^{1/(1-q)} \, d\Omega \tag{7}$$

Here β is the Lagrange multiplier of the constraint (5c) and is given by:

$$\beta = 1/\left(2\sigma_q^2 \, Z_q^{q-1}\right)$$

It is straightforward to show that:

$$Z_q = C_q/\sqrt{\beta} \tag{8}$$

$$C_q = \sqrt{\pi} \frac{\Gamma\left(\frac{1}{q-1} - \frac{1}{2}\right)}{\sqrt{q-1} \, \Gamma\left(\frac{1}{q-1}\right)} \tag{9}$$

Here $\bar{\Omega}$ is the mean value of $\{\Omega_i\}$. Γ is the gamma function. Note that:

a) In the limit $q \to 1$, it can be shown that the Tsallis entropy and the corresponding distribution go to the Shannon entropy and the Gaussian distribution respectively.
b) Unlike the Gaussian distribution case, the regular variance is not defined for all q. It is given by:

$$\sigma^2 = 1/(5-3q)\beta \qquad\qquad q < 5/3 \tag{10}$$

Let us now look at the evolution of $P_q(\Omega)$ across time scales. It has been shown [12] [13] that a solution to a nonlinear diffusion (Fokker-Plank) equation of the form:

$$\partial P(\Omega,t)/\partial t \;=\; -\partial\,[f(\Omega)P(\Omega,t))]/\partial\Omega \;+\; \frac{D}{2}\,\partial^2 P(\Omega,t)^{\nu}/\partial\Omega^2 \qquad (11)$$

is:

$$P(\Omega,t) \;=\; \frac{1}{Z_q(t)}\,[1+(q-1)\beta(t)(\Omega-\,\bar{\Omega})^2\,]^{1/(1-q)} \qquad (12)$$

Here the drift function term $f(\Omega)$ is assumed to be:

$$f(\Omega) = a - b\Omega$$

The probability density function (PDF) given by (12) satisfies (11) under the following conditions:

$$q = \;2-\;\nu$$

$$\left(\frac{1}{\nu+1}\right)\,\partial Z_q^{\nu+1}/\partial t\;+bZ_q^{\nu+1} - 2D\,\nu\left(\beta(0)Z_q^2(0)\right) = 0 \qquad (13)$$

$$\left[Z_q(t)/Z_q(0)\right]^2 = \;\beta(0)/\beta(t) \qquad (14)$$

$$\frac{d\bar{\Omega}}{dt} \;= a - b\,\bar{\Omega} \qquad (15)$$

From (12) – (15), it is straightforward to show:

$$\beta(t)^{-(3-q)/2} = [2\,(2-q)D/b]\;C_q{}^{(q-1)/2}\left[1-\;e^{-t/\tau}\right] \qquad (16)$$

Here $\tau = 1/(b(3-q))$ is the characteristic time and $C_q = \beta(0)\,Z_q^2(0)$ is constant in time. A comparison of (14) with (8) shows that the norm is conserved. In deriving (12) – (16), a boundary condition $P(\Omega,0) = \delta(\Omega)$ (implies $\beta(0) = \;\infty$) is used.

If the drift term is negligible ($b \to 0$), $t \ll \tau$, the exponential in (16) can be expanded up to linear term. In this case, β is given by:

$$\beta(t) \propto t^{-2/(3-q)} \qquad (17)$$

independent of the drift parameter *b*.

For $q > 1$, the absolute value of the exponent of t in (17) is greater than 1. This means that the mean square deviation $(1/\beta)$ of Ω follows a power law in time with exponent greater than 1. In an anomalously diffusive system, the mean square deviation scales as t^{η}. It is superdiffusive if $\eta > 1$, subdiffusive if $\eta < 1$, and normal if $\eta = 1$. Therefore, according to the analysis above, for negligible drift term, the stock market returns should show a superdiffusive character. We will denote the β for superdiffusion and drift + diffusion cases as β_{sd} and β_{dd} respectively.

In reality, are the stock market returns superdiffusive? Analysis of very short term stock returns (1-60 minutes) shows [14] [15] that these high frequency returns are indeed superdiffusive. However, there have been several works [16] extending the superdiffusive model to longer term stock returns (1 day – several months) and in particular option pricing. Figure 1 shows that even longer term returns have fat tails. However, this by itself does not necessarily imply superdiffusion. The objective of this paper is to investigate how well the β from the Fokker-Plank equation (11) agree with those estimated from the long term stock returns data.

Computing the Tsallis distribution of returns involves accurate estimation of the parameters q and β. The usual way is to fit the Tsallis distribution to the data distribution using a combination of linear regression and least square optimization techniques [17] [18]. These methods depend crucially on the binning of data histograms. The tail regions of the data (Figure 1), which are important in the parameter estimation, have relatively fewer samples and this is further reduced by binning. Statisticians [19] have long applied Maximum Likelihood Estimation (MLE) method to estimate the parameters of the Pareto distribution, which for certain parameter choices gives a q-distribution. As shown and discussed in several references [20] [21], under some general conditions, MLE is a consistent estimator, in the sense that for large number of samples N, the estimated parameters approach the true values in a probabilistic sense. It is asymptotically normal, unbiased and consistent, which means that the distribution of errors between the estimated and true values is Gaussian with zero mean and covariance given by I^{-1}/N, where I is the Fisher Information matrix. Further, the variance of the estimator attains the lower limit of Cramer-Rao inequality [22]. Hence, one can calculate the standard errors of the estimated parameters {i} as $\sqrt{I^{-1}{}_{ii}/N}$. Similar conclusions were drawn by Shalizi [23] who applied the method to q-exponential distributions. In this paper MLE is used to estimate the parameters of a q-Gaussian distribution.

The paper is organized as follows. In Section 2, the MLE equations for a q-Gaussian distribution will be discussed. Section 3 deals with the application to market data (S&P 500 and Nasdaq) and comparison of the estimated $\beta(t)$ with the those obtained from the FP equation (11) both for superdiffusion (17) and drift + diffusion (16) cases. Conclusions will be given in Section 4.

2. **Maximum Likelihood Estimation for q-Gaussian Distribution**

(a) Parameter estimation: In the Maximum Likelihood Estimation method, the parameters of a probability density function P, having N samples, are estimated by maximizing the objective function:

$$F = \sum_i ln(P_i) \tag{18}$$

For q-Gaussian distribution:

$$F = -N\, ln(Z_q) + (1/(1-q))\, ln\left[\sum_i (1 + (q-1)\beta\ \Omega_i^2)\right] \tag{19}$$

Here, the variable Ω is assumed to be standardized. Making a change of variables:

$$\alpha = 1/(q-1), \qquad \kappa = \beta/\alpha \tag{20}$$

the objective function becomes:

$$F = -N\, ln(Z_q) - \alpha\, ln\left[\sum_i (1 + \kappa\ \Omega_i^2)\right] \tag{21}$$

where the normalization Z_q in terms of the new variables is given by:

$$Z_q = \sqrt{(\pi/\kappa)}\ \left(\Gamma(\alpha - 1/2)/\Gamma\,(\alpha)\right) \tag{22}$$

Maximizing F with respect to α and κ gives:

$$[\psi(\hat{\alpha}) - \psi(\hat{\alpha} - 1/2)]\ =\ \overline{\log(1 +\ \hat{\kappa}\,\Omega^{\,2})} \tag{23a}$$

$$1/2\hat{\kappa}\ =\ \hat{\alpha}\ \left(\overline{\Omega^2/\left(1 +\ \hat{\kappa}\,\Omega^2\right)}\right) \tag{23b}$$

Here ψ is the digamma function. The bar denotes the mean value. $\hat{\alpha}$ and $\hat{\kappa}$ denote the estimated values of α and κ. In the limit $q \to 1$, (23b) gives $1/\beta\, = 2\,\overline{\Omega^2}$.

Since (23b) depends on α explicitly, it can be eliminated from (23a), so that:

$$[\psi(f(\hat{\kappa})) - \psi(f(\hat{\kappa}) - 1/2)]\ =\ \overline{\log(1 +\ \kappa\,\Omega^{\,2})} \tag{24}$$

where:

$$f(\hat{\kappa}) =\ (1/2\hat{\kappa})\left[\overline{\Omega_i^2/\left(1 + \Omega_i^2/\hat{\kappa}\right)}\right]^{-1}$$

Note that (24) depends only on $\hat{\kappa}$. But it is nonlinear and hence has to be solved numerically. Once $\hat{\kappa}$ is estimated using (24), $\hat{\alpha}$ can be estimated using (23b). The parameters q and β can then be computed from (20). We will denote the q and β so estimated by q and $\hat{\beta}$.

In solving (23b) and (24), the range of q is fixed between 1.1 – 1.66 by requiring that we look for solutions with q > 1 and distributions with finite variance as given in (10). A reasonable initial guess for $1/\beta$ is the variance of the returns. For delays longer than 1 day, the initial guess for $1/\beta$ can be scaled as some function of the delay.

(b) Error estimation: The errors in $\hat{\alpha}$ and $\hat{\kappa}$ (hence q and $\hat{\beta}$) estimates can be calculated using the Fisher Information matrix I which can be either the measured information matrix:

$$I_{kl}^{(m)} = \sum_i \frac{\partial log(P_i)}{\partial \varphi_k} \frac{\partial log(P_i)}{\partial \varphi_l} \tag{25}$$

or the expectation value:

$$I_{kl}^{(e)} = \langle \frac{\partial log(P)}{\partial \varphi_k} \frac{\partial log(P)}{\partial \varphi_l} \rangle \tag{26}$$

Here, φ_i (i = 1…m) are the parameters of the distribution P and the expectation value is taken with P. The standardized errors for parameter estimates are then given by the diagonal elements of I^{-1} evaluated at the estimated values. Therefore, the errors S in $\hat{\alpha}$ and $\hat{\kappa}$ are:

$$S(\hat{\alpha}) = \sqrt{I_{\hat{\alpha}\hat{\alpha}}^{-1}/N}$$

$$S(\hat{\kappa}) = \sqrt{I_{\hat{\kappa}\hat{\kappa}}^{-1}/N} \tag{27}$$

Note that $I^{(m)}$is data dependent and $I^{(e)}$ is only model dependent. As shown in the Appendix:

$$I_{\alpha\kappa}^{(e)} = \begin{bmatrix} I_{\alpha\alpha} & I_{\alpha\kappa} \\ I_{\kappa\alpha} & I_{\kappa\kappa} \end{bmatrix}$$

where:

$$I_{\alpha\alpha} = \psi_1(\alpha - 1/2) - \psi_1(\alpha) \tag{28a}$$

$$I_{\alpha\kappa} = I_{\kappa\alpha} = \frac{1}{2\kappa\alpha} \tag{28b}$$

$$I_{\kappa\kappa} = \left(\frac{1}{4\kappa^2}\right)\frac{(2\alpha-1)}{(\alpha+1)} \tag{28c}$$

and ψ_1 is the tri-gamma function.

The errors in q and $\hat{\beta}$ can be obtained from those of $\hat{\alpha}$ and $\hat{\kappa}$ using the transformations (20).

3. Results

The data chosen for our analysis are S&P 500 and Nasdaq daily (close of the day) stock prices. The stock prices, which are de-trended with CPI to remove inflation trends, are displayed in Figure 2. We will consider the period after 1991 (about a year before the time when electronic trading over the internet was launched), since the character of the stock price variation changes dramatically after that. The time series shows a non-stationary character with wild fluctuations. The data for analysis is divided into two regions bounded by vertical dotted lines. Regions 1 and 2 cover the dot-com bubble period and the crash of 2008 respectively. Region 3 is reserved for testing and prediction purposes.

The variables used for the estimation of q and β are the standardized log returns $\Omega(t,t_0)$ for delay t:

$$\Omega(t,t_0) = \left(y(t,t_0) - \mu_t\right)/\sigma_1 \tag{29}$$

computed for several starting times t_0 over the period of interest. Here

$$y(t,t_0) = \log(S(t_0+t)) - \log(S(t_0))$$

S is the stock value, μ_t is the mean of $y(t)$, and σ_1 is the standard deviation for 1 day log returns. With this choice, $\bar{\Omega} = 0$. As discussed in Section 2, q and β are both estimated from 1 day standardized log returns. For delays greater than 1, q is kept constant and only β is estimated so that a comparison can be made with those from the FP equations (11). The errors in q and $\hat{\beta}$ are calculated using (28) and the transformation (20). For comparison, the errors from the measured Fisher Information matrix were also computed. The difference in errors in the parameters from the two methods is less than 0.3%.

Figures 3 and 4 show the comparison of the Tsallis distributions, from the estimated parameters, with the data distributions for regions 1 and 2 respectively. Also shown are the corresponding Gaussian distributions. Except for large positive returns of some longer delays, the estimated Tsallis distributions fit very well the data PDF. Note that our model distribution is symmetric. But as the delay increases, the data is skewed towards large negative returns. The data for large positive returns is much sparser than that for large negative returns. Hence a better fit is obtained for large negative returns than for large positive returns. The value of q is greater than 1 in all cases, pointing to the non-Gaussian character. The higher value of q for Nasdaq than that for S&P 500 in Figure 3 points to a stronger correlated (herding) behavior in trading tech stocks during the dot-com period. During the crash of 2008, however, S&P 500 shows a stronger correlation, as indicated by the higher value of q than that for Nasdaq (Figure 4).

The variation of $\hat{\beta}$ with the delay t, along with error bars, are shown in Figure 5 on a log-log scale. The error in $\hat{\beta}$ is largest (~5%) for t = 1, when both q and β are estimated. For other values of t it is less than 3%. The straight line character of the plots shows that:

$$\hat{\beta} \propto t^{\lambda}$$

with λ between -0.91 and -1.1. This points to a normal to mildly subdiffusive behavior.

A comparison of $\hat{\beta}$ with β_{dd} and β_{sd} is shown in Figure 6. Note that the computation of β_{dd} depends on the drift parameter b and the diffusion parameter D. These were estimated as follows. The drift parameter b was estimated by fitting the ratio $\hat{\beta}(t)/\hat{\beta}(1)$ to the corresponding ratio of β_{dd}. Once b is estimated, the diffusion parameter D is obtained by setting $\beta_{dd}(1) = \widehat{\beta}(1)$. The values of b and D and the corresponding characteristic times $\tau = 1/(b(3-q))$ are given in Table 1. For values of $t < \tau$, β_{dd} shows an almost power law behavior with an exponent value less than that of β_{sd} and closer to that of $\hat{\beta}$. However, for $t > \tau$, β_{dd} changes its slope and approaches a stationary value. Therefore τ should be considered as the upper time limit for the validity of the drift + diffusion model.

Comparisons of the distributions of data with Tsallis distributions computed with $\hat{\beta}$, β_{dd} and β_{sd} are shown in Figures 7 and 8. For smaller delays, there is good agreement between all the model distributions and the data. However, as the delay increases, the distributions from both the drift + diffusion and the superdiffusion models start deviating from the empirical fit and the data distributions, with the superdiffusion model deviating the most both for small and large returns.

4. Conclusions

Investigations of the behavior of the S&P 500 and Nasdaq stock market long term returns, over a period which includes both the dot-com period of 2000 and the crash of 2008, show that the distributions of the returns are non-Gaussian and fat-tailed even for as long a term as 1-60 days. This points to the persistence of long term correlations even for such long delays. The distributions can be modelled well with a Tsallis distribution, the parameters (q, β) of which have been estimated using the Maximum Likelihood Estimation method. The values of q are greater than 1 for all the regions considered, with high values for the dot-com bubble and the crash of 2008 periods. However, the inverse mean square deviation β shows a power law behavior with an exponent value very close to 1.

In several earlier works generalizing market returns to non-Gaussian distributions [16], the dynamics is assumed to be described by a nonlinear Fokker-Plank equation with only a nonlinear diffusion term. A solution to this equation is a Tsallis type distribution. In this model, the β variation, for a constant $q > 1$, follows a power law in time with the magnitude of the exponent greater than 1, pointing to superdiffusion. However, as discussed above, the present analysis of long term market returns shows that, even though the distributions can be modelled with a Tsallis distribution with $q > 1$, the parameter β falls approximately as $1/t$, indicating normal diffusion. In fact, as the time delay increases, the distributions computed from the superdiffusion model deviate considerably from the corresponding data distributions.

The FP equation (11) supports a Tsallis type distribution as a solution when a drift term is included in addition to the diffusion term. But the variation of β with time is not a power law. In addition, it approaches a stationary value for times greater than the characteristic time $\tau = 1/\big(b * (3 - q)\big)$. It should however be noted that for $t < \tau$, the model with the drift + diffusion terms yields distributions that agree with the data distributions better than those from superdiffusion model.

The present investigations show that the stock market dynamics, for longer delays such as considered in the present work, cannot be adequately modelled with a Fokker-Plank equation that has a linear drift and a nonlinear diffusion term as given in (11). What is needed is a dynamical equation that yields solution close to Tsallis distribution, but shows normal diffusion.

Acknowledgements

Many thanks to Sherman Page for a critical reading of the manuscript.

Appendix: Expected Fisher Information Matrix for q-Gaussian PDF

In terms of the transformed parameters α and κ given in (20), the expected Fisher Information matrix (26) is given by:

$$I_{\alpha\kappa}^{(e)} = \begin{bmatrix} \langle \frac{\partial log(P)}{\partial \alpha} \frac{\partial log(P)}{\partial \alpha} \rangle & \langle \frac{\partial log(P)}{\partial \alpha} \frac{\partial log(P)}{\partial \kappa} \rangle \\ \langle \frac{\partial log(P)}{\partial \kappa} \frac{\partial log(P)}{\partial \alpha} \rangle & \langle \frac{\partial log(P)}{\partial \kappa} \frac{\partial log(P)}{\partial \kappa} \rangle \end{bmatrix}$$

$$= -\begin{bmatrix} \langle \frac{\partial^2 log(P)}{\partial \alpha^2} \rangle & \langle \frac{\partial^2 log(P)}{\partial \alpha \partial \kappa} \rangle \\ \langle \frac{\partial^2 log(P)}{\partial \kappa \partial \alpha} \rangle & \langle \frac{\partial^2 log(P)}{\partial \kappa^2} \rangle \end{bmatrix} \tag{A1}$$

Using (6) – (9) and (20) and noting that P is normalized, it is straightforward to show that:

$$I_{\alpha\alpha} = -\langle \frac{\partial^2 \log(P)}{\partial \alpha^2} \rangle = \psi_1(\alpha - 1/2) - \psi_1(\alpha) \tag{A2}$$

$$I_{\alpha\kappa} = I_{\kappa\alpha} = -\langle \frac{\partial^2 log(P)}{\partial \alpha \partial \kappa} \rangle = \frac{1}{2\kappa\alpha} \tag{A3}$$

$$I_{\kappa\kappa} = -\langle \frac{\partial^2 log(P)}{\partial \kappa^2} \rangle = \left(\frac{1}{4\kappa^2}\right) \frac{(2\alpha-1)}{(\alpha+1)} \tag{A4}$$

Here, ψ_1 is the tri-gamma function. In deriving (A3) and (A4), the following expectation values are needed:

$$\langle \frac{\Omega^2}{(1+\kappa\, \Omega^2)} \rangle = \frac{1}{2\kappa\alpha}$$

$$\langle \frac{\Omega^4}{(1+\kappa\, \Omega^2)^2} \rangle = \left(\frac{3}{4\kappa^2}\right) \frac{1}{\alpha(\alpha+1)}$$

The Fisher matrix $I_{q\beta}^{(e)}$, needed to compute the standard errors in q and β, can be obtained from $I_{\alpha\kappa}^{(e)}$ using the transformation:

$$I_{q\beta}^{(e)} = J\, I_{\alpha\kappa}^{(e)}\, \tilde{J} \tag{A5}$$

where J is the Jacobian. From (20), it is straightforward to show:

$$J = \begin{bmatrix} -\alpha^2 & \kappa\alpha \\ 0 & 1/\alpha \end{bmatrix} \tag{A6}$$

References

[1] F. Black and M. Scholes, *The Pricing of Options and Corporate Liabilities*, The Journal of Political Economy, Vol. 81, No. 3, 637-654 (1973).

[2] E.F. Fama, *The Behavior of Stock-Market Prices*, The Journal of Business, Vol. 38, No. 1, 34-105 (1965).

[3] L. Bachelier, *Theory of Speculation*, In: The Random Character of Stock Market Prices, P Cootner ed., MIT Press, Cambridge, MA, 1964. (Originally published in French in 1900).

[4] D. Sornette, *Critical Market Crashes*, Physics Reports, Vol. 378, No. 1, 1-98 (2003).

[5] R.N. Mantegna and H.E. Stanley, An Introduction to Econophysics: Correlations and Complexity in Finance, Cambridge University Press, Cambridge, 2000.

[6] C.E. Shannon, *A Mathematical Theory of Communication,* The Bell System Technical Journal, Vol. 27, No. 3, 379-423, (1948).

[7] E.T. Jaynes, Papers on Probability, Statistics and Statistical Physics, Reidel, London, 1983.

[8] B.B. Mandelbrot, Fractals and Scaling in Finance, Springer, New York, 1997.

[9] C. Tsallis, C. Anteneodo, L. Borland and R. Osorio, *Nonextensive statistical mechanics and economics*, Physica A, Vol. 324, No. 1-2, 89-100 (2003).

[10] R. Osorio, L. Borland and C. Tsallis, *Distributions of High-Frequency Stock-Market Observables*, In: Nonextensive Entropy: Interdisciplinary Applications, C. Tsallis and M. Gell-Mann eds., Oxford University Press, New York, 2004.

[11] C. Tsallis, Introduction to Nonextensive Statistical Mechanics, Springer, New York, 2000.

[12] C. Tsallis and D.J. Bukman, *Anomalous diffusion in the presence of external forces: Exact time-dependent solutions and their thermostatistical basis,* Physical Review E, Vol. 54, No. 3, 2197 (1996).

[13] D.H. Zanette, *Statistical-thermodynamical foundations of anomalous diffusion,* arXiv:cond-mat/9905064v1 (1999).

[14] F. Michael and M.D. Johnson, *Financial Market Dynamics*, arXiv:cond-mat/108017 (2001).

[15] A.A.G. Cortines and R. Riera, *Non-extensive Behavior of a Stock Market Index at Microscopic Time Scales*, Physica A, Vol. 377, No. 1, 181-192 (2007).

[16] L. Borland, *A theory of non-Gaussian option pricing*, Quantitative Finance, Vol. 2, No. 6, 415-431 (2002).

[17] G.L.Ferri, M.F.Reynoso Savio and A.Plastino, *Tsallis' q-triplet and the ozone layer*, Physica A, Vol. 389, No. 9, 1829-1833 (2010).

[18] G.P. Pavlos, L.P. Karakatsanis, M.N. Xenakis, D. Sarafopoulos, and E.G. Pavlos, *Tsallis statistics and magnetospheric self-organization*, Physica A, Vol. 391, No. 11, 3069–3080 (2012).

[19] B.C. Arnold, Pareto Distributions, International Cooperative Publishing House, Fairland, Maryland, 1983.

[20] E.J.G. Pitman, Some Basic Theory for Statistical Inference, Chapman and Hall, London, 1979.

[21] O.E. Barndorff-Nielsen and D.R. Cox, Inference and Asymptotics, Chapman and Hall, London, 1995.

[22] H. Cramér, Mathematical Methods of Statistics. Almqvist and Wiksells, Uppsala, 1945.

[23] C.R. Shalizi, *Maximum Likelihood Estimation for q-Exponential (Tsallis) Distributions*, arXiv: math/0701854v2 (2007).

[24] V. Schwämmle, E.M.F. Curado, and F.D. Nobre, *Dynamics of normal and anomalous diffusion in nonlinear Fokker-Planck equations*, European Physical Journal B, Vol. 70, No. 1, 107–116 (2009).

Figures

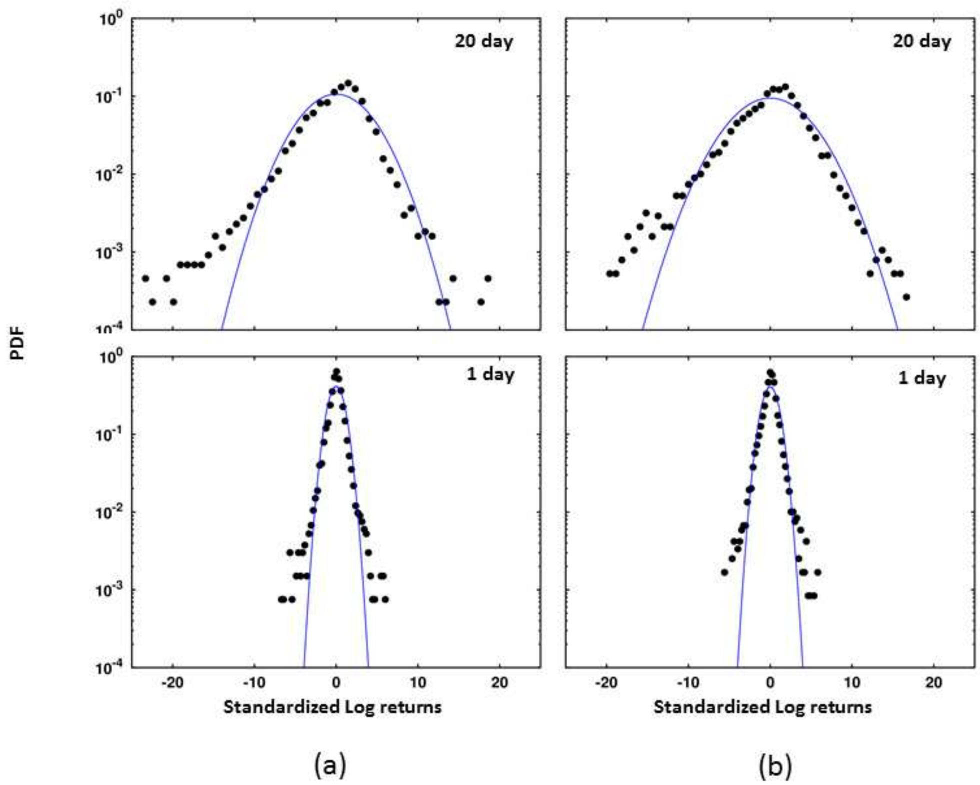

Figure 1. Comparison of the distributions of standardized log returns (as given in (25)) with the Gaussian distributions (solid blue line) having the same mean and standard deviation as the data (black dots). (a) S&P 500 for 2 Jan 1994 – 31 Dec 2013 and (b) Nasdaq over the same period.

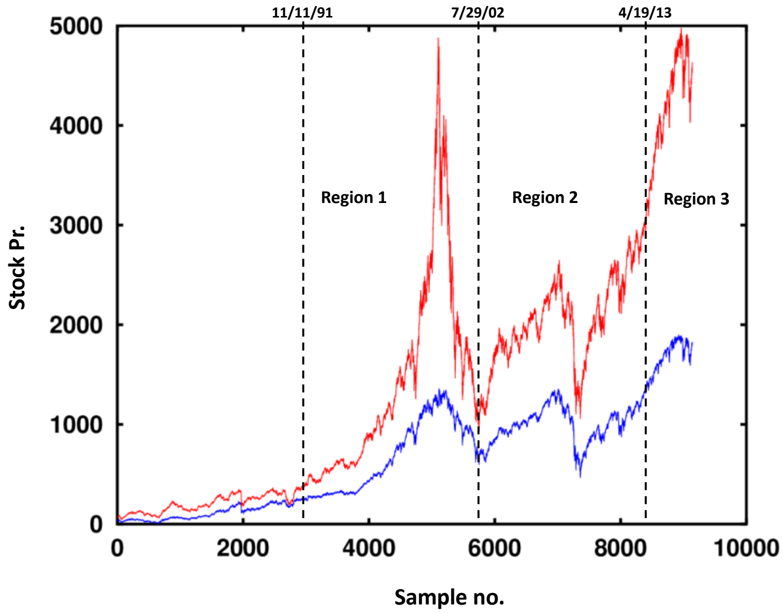

Figure 2. S&P 500 and Nasdaq stock prices for 2 Jan 1980 – 31 March 2016. Region 1 (11 Nov 1991 – 29 Jul 2002) and region 2 (30 Jul 2002 – 4 Sep 2013) are chosen for analysis and region 3 (5 Sep 2013 – 31 Mar 2016) for testing. Blue – S&P 500. Red – Nasdaq.

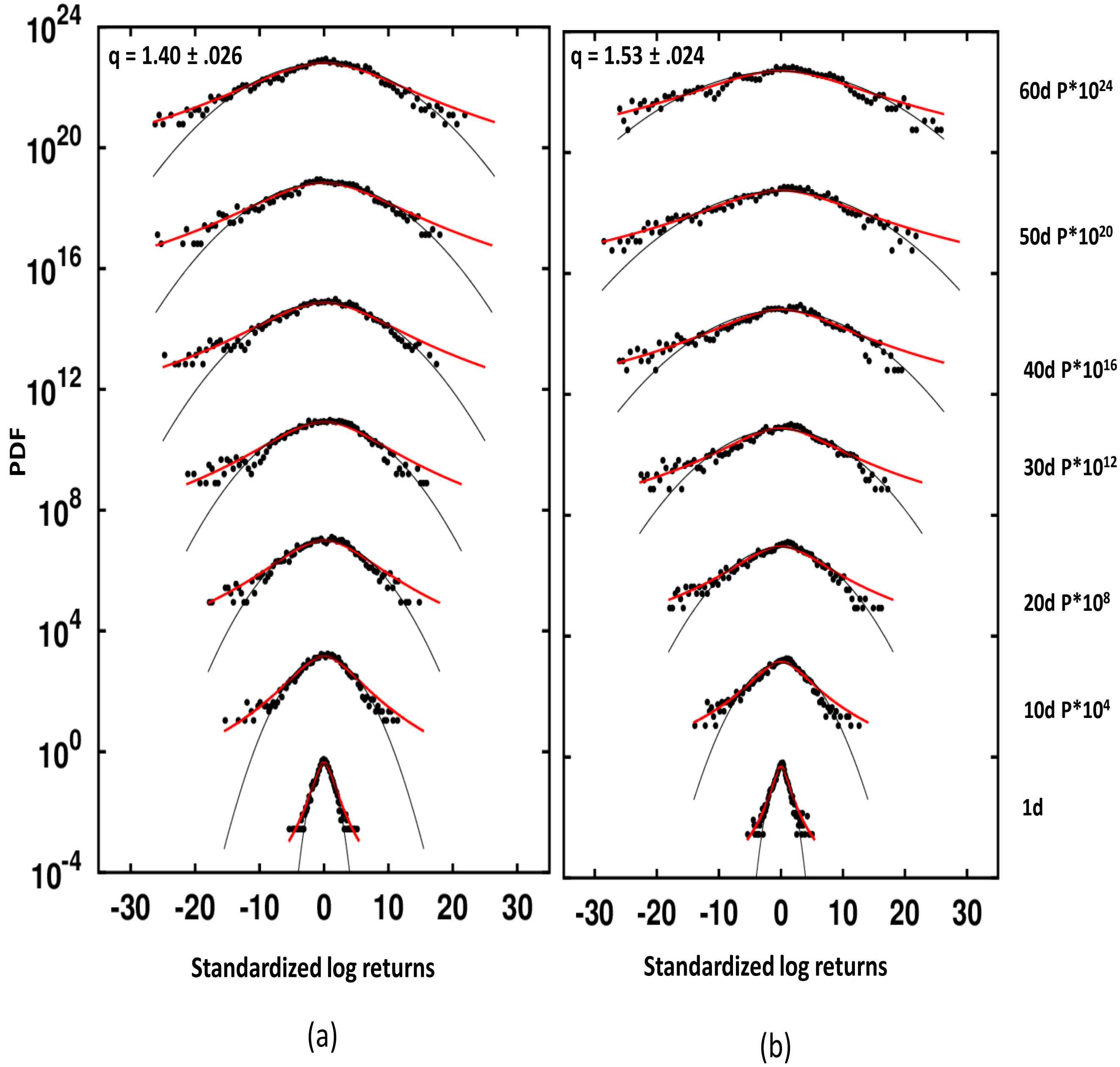

Figure 3. Comparison of the estimated Tsallis distributions with the data distributions for region 1 (11 Nov 1991 – 29 Jul 2002). Red – Estimated. Black – Gaussian. The delays corresponding to the distributions are given on the right hand side of the figure. The distributions for each delay are shifted by multiplying the corresponding PDF with the factors shown on the right hand side, next to the delays. (a) S&P 500 and (b) Nasdaq.

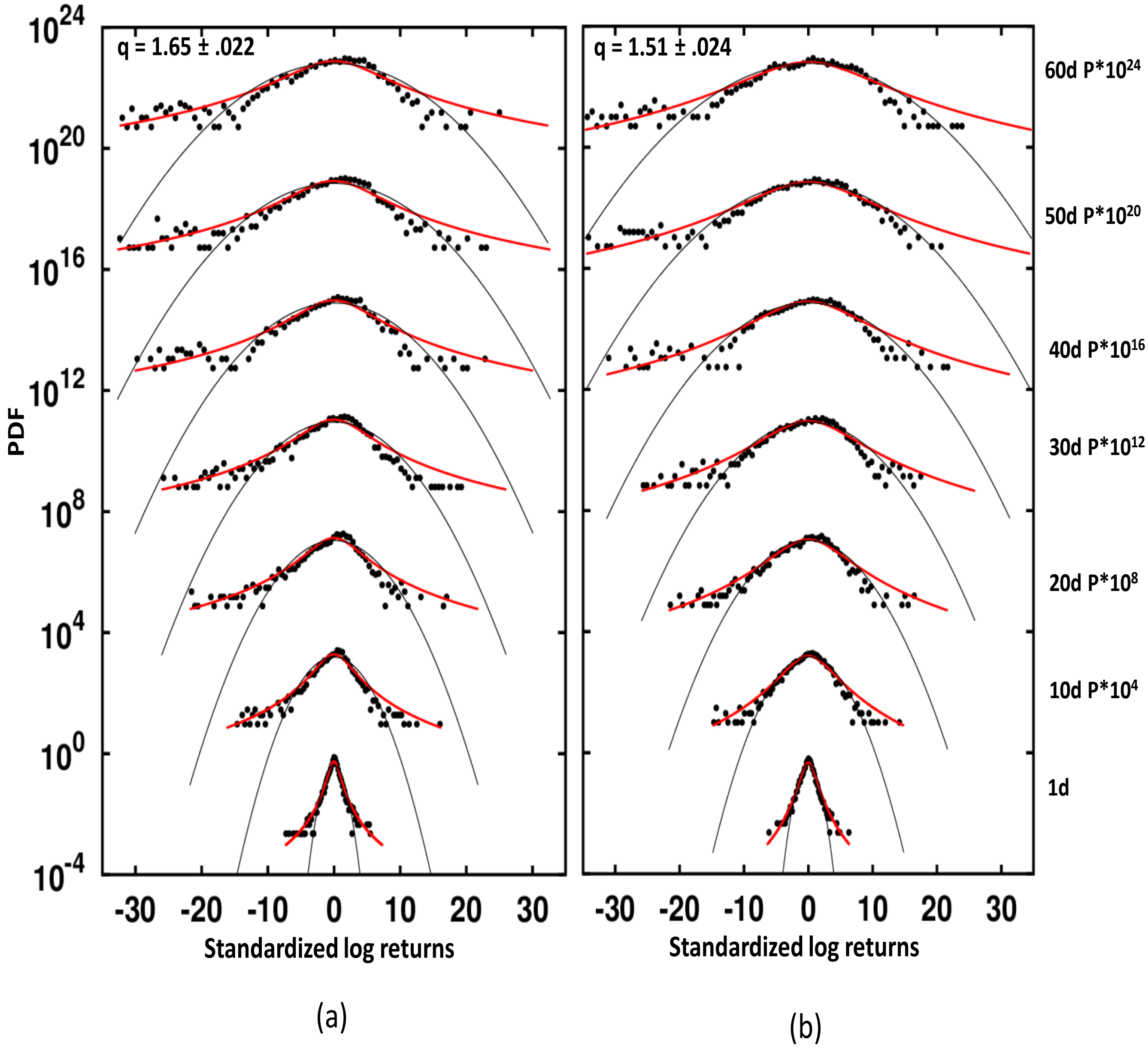

Figure 4. Same as Figure 3, for region 2 (30 Jul 2002 – 4 Sep 2013).

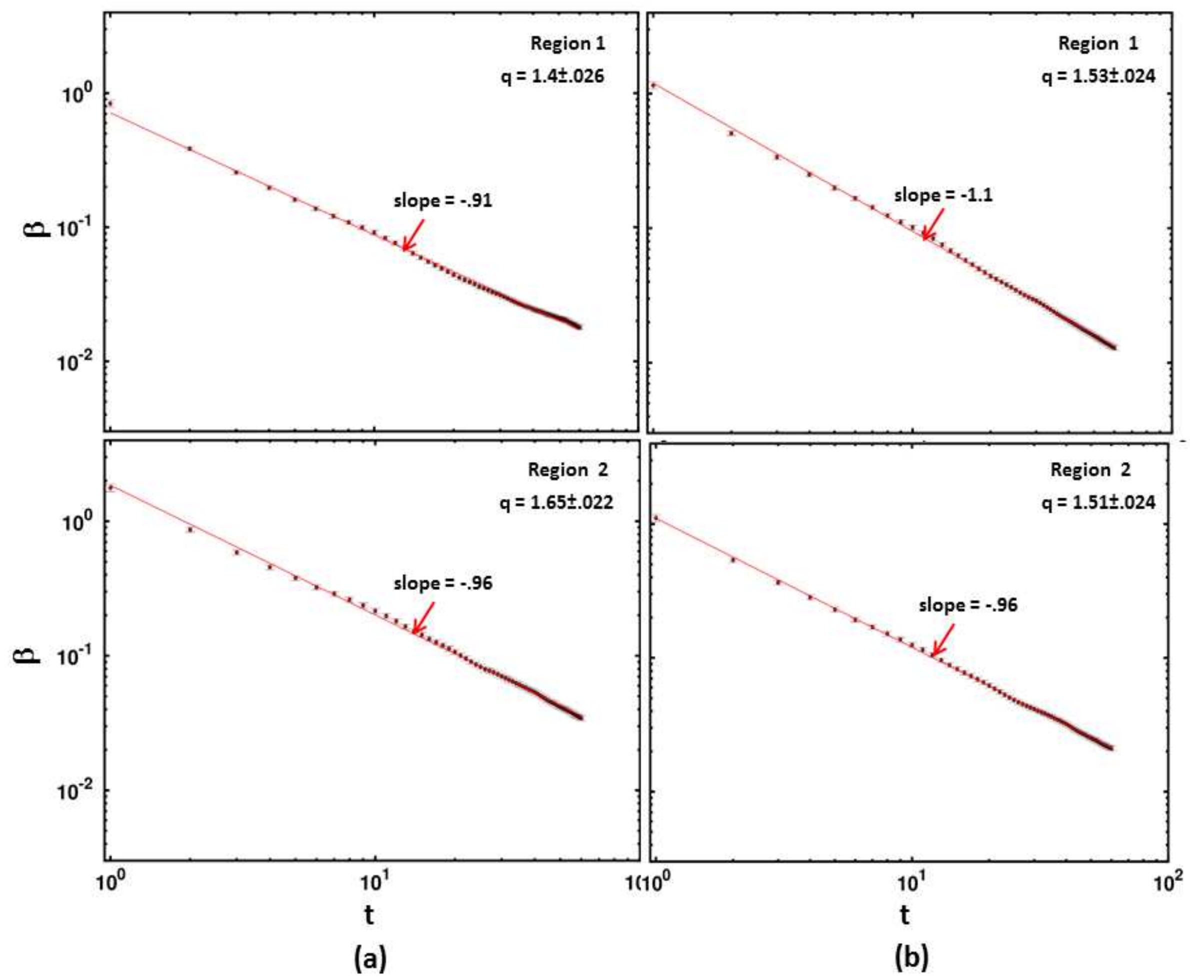

Figure 5. Variation of the estimated $\hat{\beta}$ with the delay t for regions 1 and 2. The error bars for $\hat{\beta}$ are also shown. The solid red line is the linear fit. (a) S&P 500 and (b) Nasdaq.

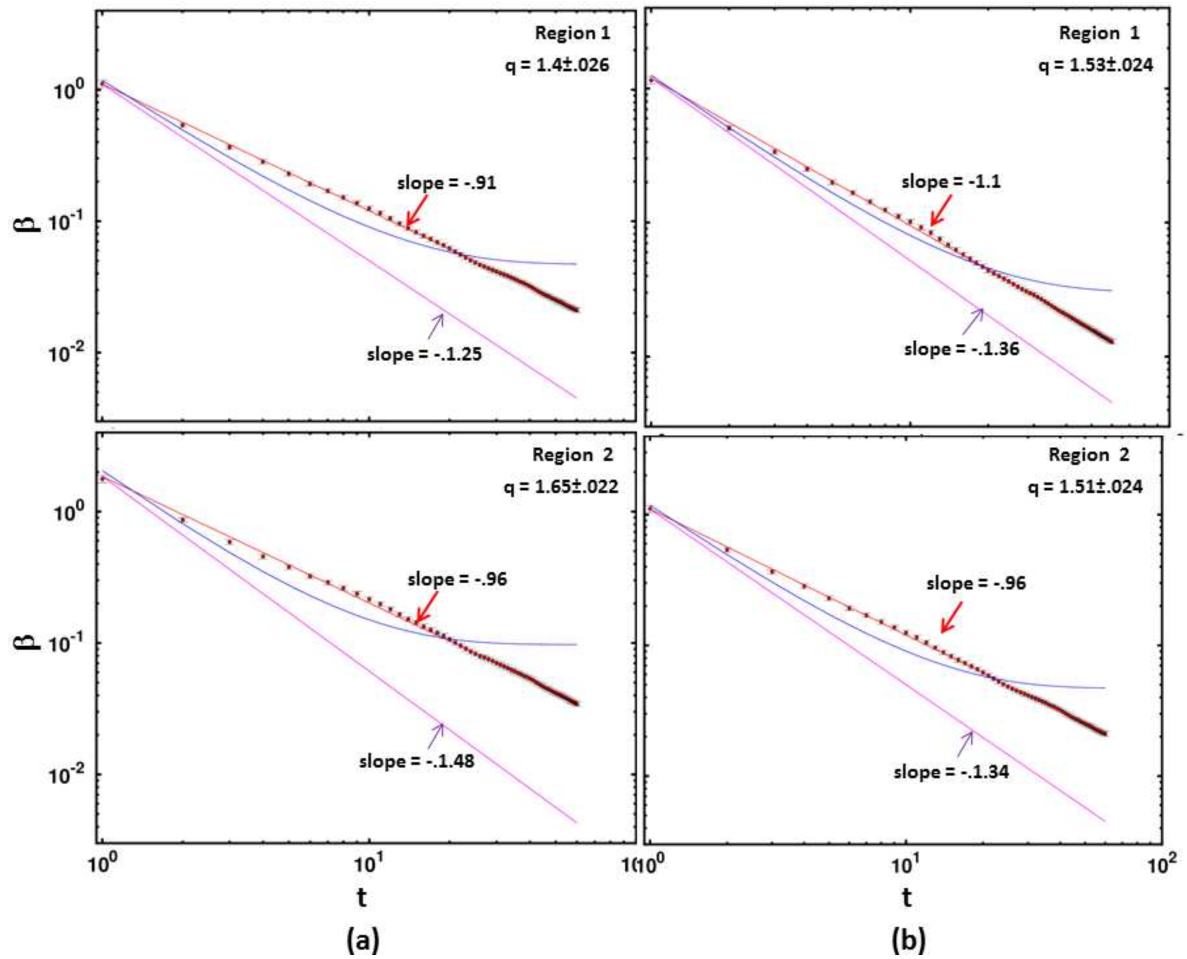

Figure 6. Comparison of $\hat{\beta}$ with β_{dd} and β_{sd}. Red – estimated, Blue – drift + diffusion (β_{dd}), and Magenta – superdiffusion (β_{sd}). The solid red line is the linear fit to $\hat{\beta}$ vs. t. (a) S&P 500 and (b) Nasdaq.

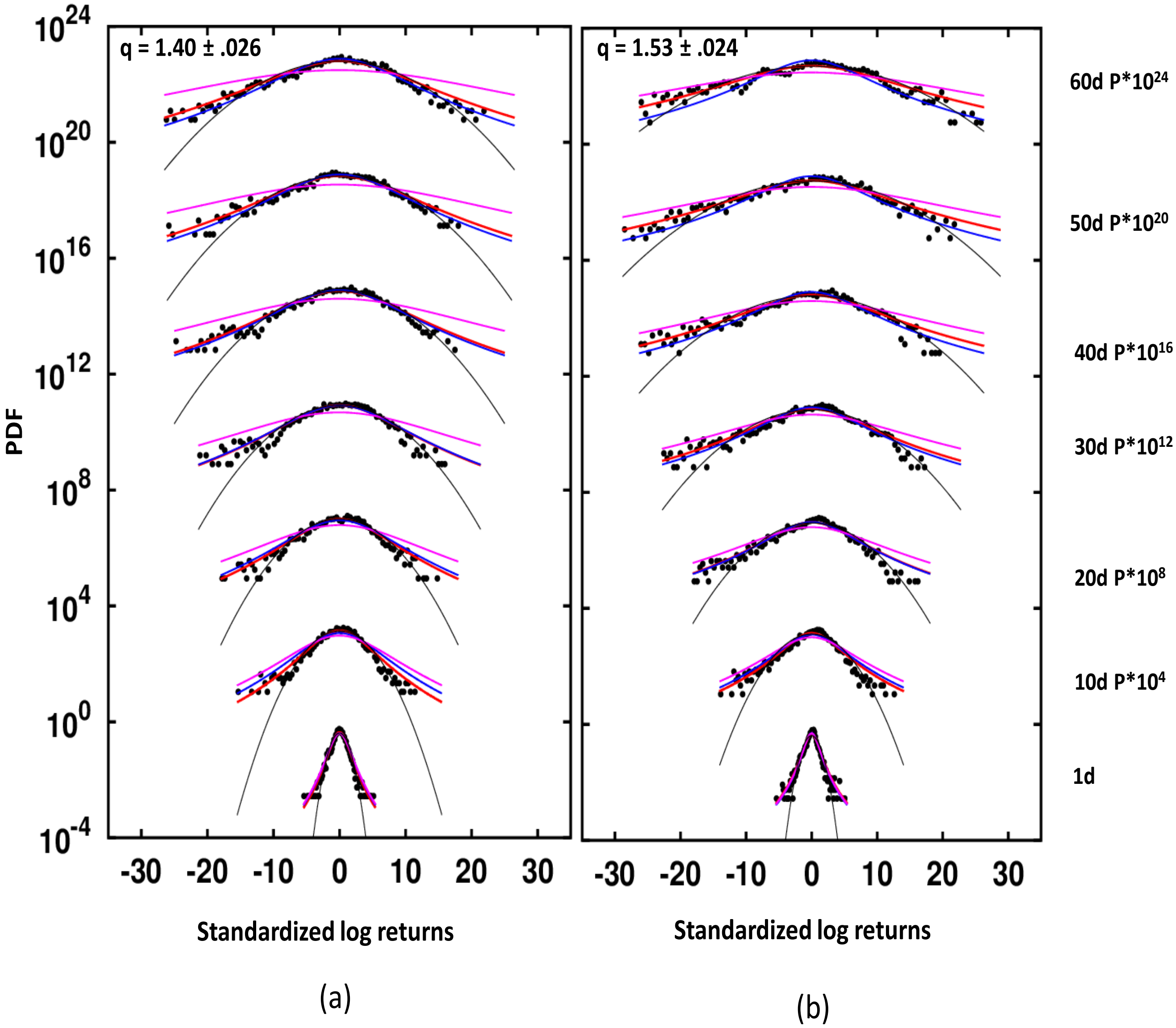

Figure 7. Comparison of the Tsallis distributions from estimated $\hat{\beta}$, β_{dd} and β_{sd} with the data distributions for region 1. Red – Estimated, Blue – drift + diffusion, Magenta – superdiffusion, and Black – Gaussian. (a) S&P 500 and (b) Nasdaq.

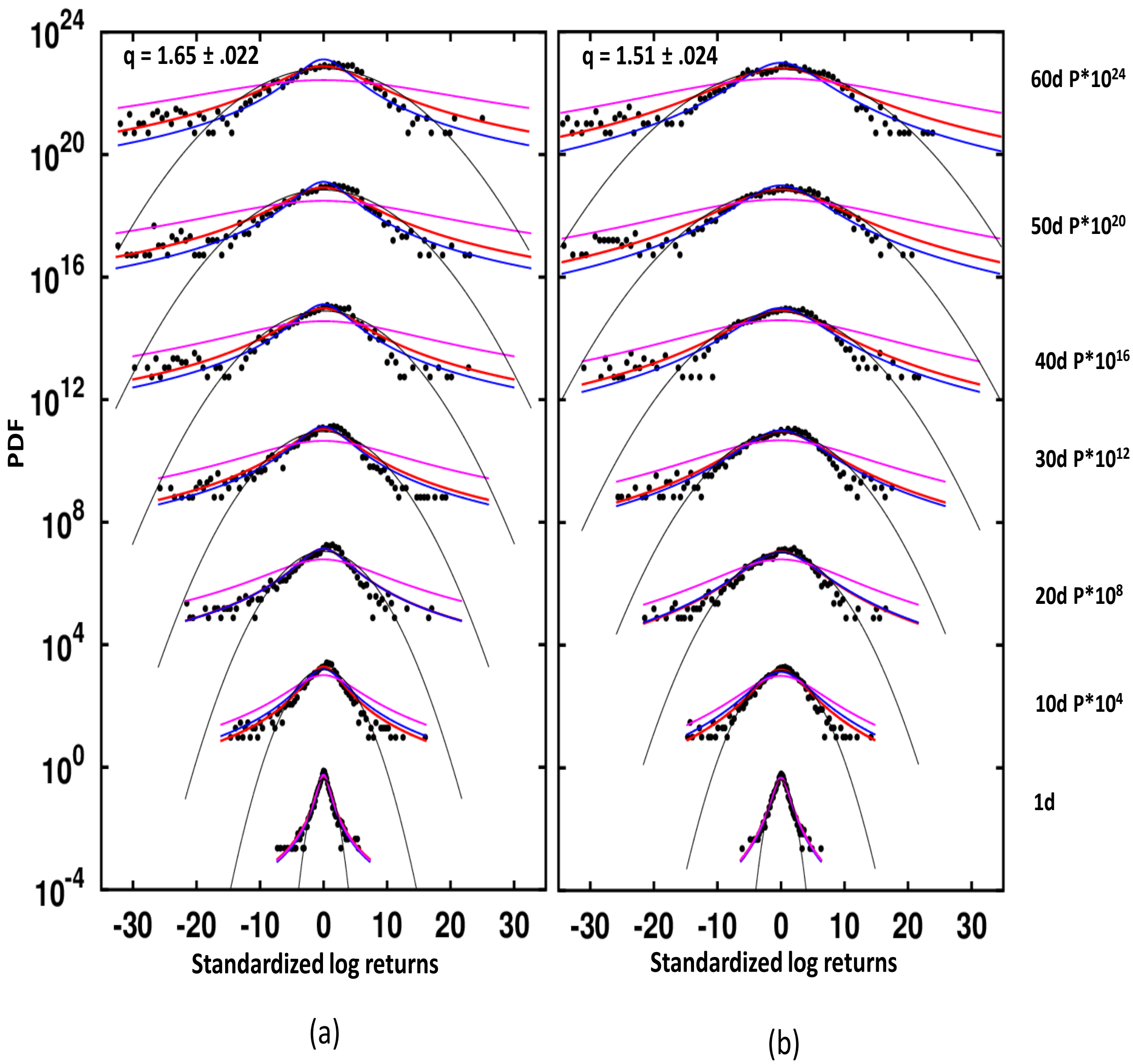

Figure 8. Same as Figure 7 for region 2.

Index	Region	b	D	Tau
S&P 500	1	.043∓.0024	.506	14.31
Nasdaq	1	.046∓.0031	.412	14.87
S&P 500	2	.101∓.0068	.393	7.29
Nasdaq	2	.064∓.0047	.423	10.47

Table 1. The estimated values of the drift parameter b, the diffusion parameter D and the characteristic time (in days) $\tau = 1/(b(3-q))$.