An algorithmic approach to one-round electoral systems

Gutiérrez, José Manuel and Müller, Wolfgang C.

Universidad de Salamanca, Universität Wien

26 August 2016

AN ALGORITHMIC APPROACH TO ONE-ROUND ELECTORAL SYSTEMS

JOSÉ MANUEL GUTIÉRREZ[1] AND WOLFGANG C. MÜLLER[2]

[1] *Universidad de Salamanca. Spain*
[2] *Universität Wien. Austria*

Abstract. A family of algorithms provides a formalization of how the basic one-round electoral systems — highest average and largest remainders, single transferable vote and single non-transferable vote systems — proceed in transforming votes into seats. In this way the basic one-round electoral systems are parametrized with the four parameters n (size of the constituency), m (size of the nomination lists), c_k (a factor providing the electoral formula) and l (signed election threshold). The parametrization reveals that the most important electoral systems have a common basic structure.

1. Introduction

The study of electoral systems has developed into a rich and diverse field of research with important contributions coming from several disciplines including political science, philosophy, mathematics, and economics. Such studies, inter alia, address issues of classification, investigate the direct and indirect effects of different electoral systems, and evaluate existing and ideal type electoral systems against a large number of normatively or analytically relevant criteria. All these research questions are conditional on the understanding of the actual working of electoral systems. Consequently, the literature is rich in work highlighting how different systems translate votes into seats. While comparative studies typically calculate the effects of variation in the electoral systems' individual components (v. Gallagher and Mitchell [3], Gallagher and Mitchell [4], Lijphart [7], Nohlen [8], Rae [9], Taagepera and Shugart [11]), country studies mostly demonstrate how these components interact to produce the electoral outcomes (v. Bowler and Grofman [1], Grofman and Lijphart [5] and the country chapters in Gallagher and Mitchell [2]). Common to most such endeavour is detailed description of institutional detail. Clearly, this is essential for understanding individual cases, illustrating the empirical range of various manifestations of the individual components of electoral systems, and reaching out to political practitioners. In this paper we take a different perspective at describing how electoral systems work that puts the emphasis on parsimony.

The political science literature beginning with the seminal study of Rae [9] has parsimoniously reduced the complexity of real world electoral systems to their core components. Given its focus on real world effects, it has, however, eschewed the formalization of how electoral systems work. Yet, although it is not a classic research question per se, formalization is important as the best way to avoid ambiguity. Providing formalization of how electoral systems work is the first goal of the present paper. Formalization also allows for seeing communalities of different classes of electoral systems. While the literature generally treats Rae's *categorical electoral systems* (i.e., closed list systems) and *ordinal electoral systems* (where the voter can rank in the ballot paper the candidates, irrespective of their party, in any order he/she wishes) as systems with fundamentally different mechanisms to allocate seats, formalization allows to bringing these types into one common framework.

We approach algorithmically the basic types of electoral system in the election of national parliaments. Our focus is on structural characteristics of the electoral systems, disregarding detail and exceptions (e.g. to allow national minority representation). In a general (pre-mathematical) sense, algorithms are precise procedures designed to solve a problem, be it locating a book in a library or, as in our case, allocating a given number of parliamentary seats among the contenders of elections. An algorithm has to satisfy several criteria including universality (it must work in all specific applications), definiteness (its steps must be clearly defined in their sequence and content), and finiteness (it must terminate after a finite number of steps and deliver a result).

By reducing how electoral systems transmite votes into seats to an algorithmic scheme, it becomes clear how every particular system is determined by the choice of very few parameters. In this sense, our analysis is in the wake of the seminal contribution of Rae [9], where the stylized facts of reality are profiled, and the famous three "electoral law variables" (ballot structure, electoral district size and electoral formula) are considered.

In this paper we formalize these three variables in mathematical terms, adding a forth variable, the election threshold, already a "secondary variable" in Rae [9]. By introducing a *general scheme of algorithms*, the four resulting variables, adequately modified, become parameters determining all the basic one-round electoral systems. Hereafter under the "basic one-round electoral systems" the following are referred to: (1) the standard closed list systems, i.e. highest average and largest remainders (including first-past-the-post); (2) single transferable vote systems (including preferential voting); (3) single non-transferable vote systems.

A parametrization is more than a classification. In a parametrization of electoral systems, if the values of all the parameters are given, an electoral system is univocally determined. Thus the common structure of these electoral systems is revealed, and both the theoretical and the empirical work are made easier. The comparison between parametric a non-parametric statistics illustrates sufficiently to which extent parametrization facilitates scientific analysis and evaluation.

Can the basic one-round electoral systems be parametrized? Have they some common structure? The answer to both questions is affirmative.

The general framework and notation are introduced in Section 2. Section 3 profiles briefly the four parameters. The general scheme of algorithms is explained in Section 4: it encompasses all the procedures applied in the basic one-round electoral systems to transform votes into seats.

Section 5 considers the two standard sorts of closed list systems, highest average and largest remainders, as determined by the parameters.

The parameters determining ordinal single transferable vote systems are dealed with in Section 6. In a closed list electoral system (including the first-past-the post system), the ballots contain a limited amount of information, which does not seem in principle difficult to process to obtain the overall result of the vote. But if the ballots contain more information, as in ordinal systems, how the manifold preferences of the voters can be processed to obtain a result? A simplification (or "interpretation") of the preferences expressed in the ballot papers has to be implemented, in a way pre-determined by the electoral system. Two ideas, in fact developed from closed list systems, appear in the general scheme of algorithms. On the one hand, the candidates ranked lower in a ballot paper "weigh" less in the counting. On the other hand, the candidates are eliminated through thresholds: in closed list systems there is often a threshold of entry (if a list is below this threshold, all the members of the list are eliminated); in ordinal systems we introduce an election threshold acting in a more complex way (based on lowering the "weight" of the candidates ranked lower). In contrast with closed list systems, in ordinal systems, with the ballots having more information, we have somebody to resort to if a candidate is eliminated: the second choice is considered in the ballots where the first choice candidate has been eliminated (and so on if the second choice is eliminated...).

The article considers two structural parameters and two counting parameters. The two structural parameters are n (size of the constituency) and m (size of the nomination lists). In the nomination of candidates the political parties can take more or less decision power, and the voter be left with less or more of it; as explained below, parameter m (a positive integer) measures this characteristic of an electoral system. We shall also introduce the two counting parameters c_k (factor providing the electoral formula) and l (signed (i.e. with the sign + or the sign -) election threshold). The general scheme of algorithms is so written that when the four parameters are fixed, a concrete electoral system is determined. The article shows that all the basic one-round electoral systems are represented as resulting from values of these parameters.

2. General framework

We consider a constituency with n seats. There is a set S of eligible candidates; we suppose that the number of candidates, $|S|$, is greater than or equal to the magnitude of the constituency: $|S| \geq n$.

Each ballot contains an ordered list of at most $|S|$ candidates. Let S^* be the set of (ordered) lists formed by at most $|S|$ different elements of S; according to the relevant electoral system, perhaps not all these lists are admissible to be chosen by the voters. Let $T \subseteq S^*$ be the set of those lists that are allowed to be written on a ballot. For example, in a closed list system, only the lists proposed by the parties that have met the requirements for candidacy are in T.

As an example, suppose a constituency where 3 candidates must be chosen ($n = 3$). Further, 12 candidates take part in the election, belonging to 4 parties (3 candidates for each party); there are no independent candidates. Thus there are 12 elements in S. In the case of an ordinal electoral system, the 3 candidates of each party may be chosen individually, and the elector is free to place in his/her ballot paper 12 names, irrespective of party, in any order. Therefore T has 12! elements, a huge number. In contrast, in the case of a closed list system, T has only 4 elements: the voter may choose only among 4 lists of candidates, already prepared by the political parties. As for the design of the electoral system, processing the declared wishes of the voters is easier when they have 4 options than when they have 12!; in the end, the electoral system must provide as a result just 3 names. Another question is whether the voters have the time and willingness to decide among 12! options. In closed list systems, a large amount of the decision power passes from the voters to the political parties.

The electoral system determines whether the candidates may stand for election individually or they are to be nominated within set lists of candidates. Let m be the number of candidates of such set nomination lists as prescribed by the electoral system. The voter must include complete nomination lists as blocks in the ballot paper. In practice, either $m = 1$ (each candidate stands individually) or $m = n$ (the nomination lists must have as many candidates as seats to be assigned).

The smaller is m, then the larger is T, and the more information is provided by the voter (by choosing among more options). The maximum information is provided if $m = 1$ (and thus $T = S^*$); for this to be practicable, the magnitude of the constituency should be small. The minimum information corresponds to closed lists.

After the election, the set B of valid ballots is obtained. After counting, the final outcome of the election in the constituency is the set E of elected candidates.

We summarize the basic notation:

- n: number of seats of the constituency
- S: set of eligible candidates (we assume $|S| \geq n$)
- m: number of candidates in each nomination list
  (now $T \subseteq S^*$ is the set of admissible contents of ballots, where S^* is the set of (ordered) lists formed by at most $|S|$ different elements of S)
- B: set of valid ballots.
- E: set of elected candidates.

The counting process is iterative; at the end of each iteration one candidate is either selected or eliminated. At the beginning E contains no candidate (i.e. $E = \emptyset$). During the implementation of the algorithm, E increases (as the candidates are selected) and S decreases (as the candidates are selected or eliminated). Thus if candidate k is selected, then it is added to E and removed from S:

$$E \leftarrow E \cup \{k\},\ S \leftarrow S \sim \{k\}$$

If candidate h is eliminated, it is removed from S:

$$S \leftarrow S \sim \{h\}$$

In each iteration only the first candidate in the ballot (*after all the candidates already selected or eliminated are disregarded*) is considered for selection. Given an eligible candidate $s \in S$, we denote by $B_s \subseteq B$ the set of those ballots in which s is the top candidate among those still in S (i.e. among those not yet selected or eliminated); note that B_s may change in every iteration, as S changes.

Every ballot $b \in B$ has a weight ρ_b. This weight is initially 1, and then the weight decreases as candidates listed in the ballot are selected.

In each iteration we begin by counting the votes z_s of every candidate $s \in S$, i.e. the weighted number of ballots in which s is the top candidate (among those not yet selected or eliminated):

$$z_s \leftarrow \sum_{b \in B_s} \rho_b \text{ , for every } s \in S$$

(If $B_s = \emptyset$, it applies the usual convention that the value of the empty sum is zero).

3. Parameters

There are two sorts of parameters:

(i) Structural parameters, namely:

(1) n, the size of the constituency. It is a positive integer. It corresponds to the electoral district size of Rae [9].

(2) m, the size of the nomination lists. It is a positive integer (in practice, 1 or n). The larger is m, the more information is provided in the nomination process (and the less information is left to be provided by the ballot paper).

From m results $T \subseteq S^*$, the set of admissible contents of the ballots, which corresponds to the ballot structure of Rae [9]. The ballots are ordered lists of candidates. T indicates which lists of at most $|S|$ candidates are allowed to be included in a ballot. The set of feasible lists can be small (closed list systems) or large (ordinal systems), i.e., the ballot provides less or more information.

(ii) Counting parameters, determining the procedure to assign the seats from the number of ballots where the candidates appear and their positions in these ballots. We consider here two parameters (see next section):

(3) c_k, the factor reducing the "weight" of the candidates ranked lower than candidate k in the ballot paper. It is a function of k and (perhaps) of the weighted number of ballots z_k. It corresponds to the electoral formula of Rae [9].

(4) l, the election threshold, with a preceding sign. We leave it positive ($l > 0$) in closed list systems, and add a minus sign in ordinal systems (thus $l < 0$); if $l = 0$ no threshold is applied. All in all, the threshold is the absolute value $|l|$ of l. In Rae [9] the election threshold was not considered among the three main variables of electoral systems, but among the secondary variables.

4. The general scheme of algorithms

In each iteration of the algorithms, the only information taken from each ballot is the first-choice (the first element of the list, after all the candidates already selected or eliminated are disregarded). Thus the candidate k with the highest weighted number of ballots is pre-selected:

$$k \leftarrow \arg\max \{z_s, s \in S\}$$

(Some pre-determined procedure must be used to break the ties, if any, i.e. in case that $|\arg\max \{z_s, s \in S\}| > 1$).

Once candidate k is determined, the possibility of elimination of candidates is considered. In closed list systems there is often a threshold (represented by parameter l); if a list is below this threshold, all the members of the list are eliminated. In ordinal single transferable vote systems we introduce an election threshold (represented, with a minus sign, by parameter l), acting in a more complex way (based on lowering the "weight" of the candidates ranked lower). But an essential point is that in ordinal systems if a candidate is eliminated this does not mean that all the ballot papers including this candidate become ineffectual. The ballots contain more information than in closed list systems. In the ballots where the first choice candidate has been eliminated, the second choice is considered instead in single transferable vote systems (and so on if the second choice is eliminated...).

Formally, for every electoral system there is a (signed) threshold parameter l, positive (closed list systems), zero (no threshold is applied) or negative (ordinal single transferable vote systems). A threshold may be applied to the total number of ballots $|B_k|$ or to the weighted sum z_k; in the first case the threshold is l and in the second $-l$. The threshold l is applied on $|B_k|$ if $l > 0$, and the threshold $-l$ is applied on z_k if $l < 0$. If $|B_k| < l$, then candidate k is eliminated:

$$S \leftarrow S \sim \{k\}$$

If this not the case, the $-l$ test follows. If $z_k < -l$, then the candidate h with the lowest weighted number of ballots

$$h \leftarrow \arg\min \{z_s, s \in S\}$$

is eliminated. (Also here some pre-determined procedure must be used to break the ties in case that $|\arg\min \{z_s, s \in S\}| > 1$).

If the signed threshold l does not eliminate any candidate, then k is selected. At the end of each iteration one candidate is either selected or eliminated. After candidate k has been selected, the only thing to be done in the iteration is to decrease the weights ρ_b of all the ballots b in B_k (those ballots in which k is the top candidate). If we call c_k the reduction factor, then

$$\rho_b \leftarrow c_k \rho_b \text{ , for every } b \in B_k$$

Thus the *reduction factor* c_k, with $0 \leq c_k \leq 1$, is the factor reducing the "weight" of the candidates ranked lower than candidate k in the ballot papers (of B_k). Expressed in an alternative manner, a ballot has a "strength" that weakens as it "gets its way", and c_k measures the "sapping by fruition"; c_k larger (near 1) means that a smaller "price" is paid for having elected candidate k. All in all, the candidates ranked lower in a ballot paper "weigh" less. As an example, in the *quota electoral systems* (both the closed list and ordinal ones; see below)

$$c_k = \frac{z_k - q}{z_k} \tag{1}$$

where the quota q characteristic of each system is defined from the total number $|B|$ of valid ballots cast; so $q = |B|/(n+1)$ is the Droop quota, $q = |B|/n$ is the Hare quota and $q = |B|/(n+2)$ is the Imperiali quota.

In the electoral process three steps are to be considered: the nomination of candidates, the vote and the count transforming votes into seats. In the nomination of candidates the political parties can take more or less decision power, and the voter be left with less or more of it. The simple parameter m measures this characteristic of an electoral system. A scheme of a family of algorithms follows; as it is written, a concrete algorithm is fixed when the values of the parameters n, c_k and l are given (therefore when the four parameters n, m, c_k and l are given, a concrete one-round electoral system is determined.). Each algorithm defines the counting process of a one-round electoral system. Every time an algorithm is run, it is applied to a particular *instance* of the problem of selecting among candidates; the instance is determined by the two data S and B. The outcome of the algorithm is the set of elected candidates E.

GENERAL SCHEME OF ALGORITHMS

STRUCTURAL PARAMETERS: n
COUNTING PARAMETERS: l, c_k
DATA OF THE INSTANCE: S, B
OUTCOME: E

STEP 1 (inicialization)

$E \leftarrow \emptyset$;
$\rho_b \leftarrow 1$ for every $b \in B$

STEP 2 (counting and pre-selection)
if $S \neq \emptyset$ **then** $z_s \leftarrow \sum_{b \in B_s} \rho_b$ for every $s \in S$ **else** END;
$k \leftarrow \arg\max \{z_s, s \in S\}$;
if $z_k = 0$ **then** END

STEP 3 (elimination)
if $|B_k| < l$
then $S \leftarrow S \sim \{k\}$;
go to step 2;
if $z_k < -l$
then $h \leftarrow \arg\min \{z_s, s \in S\}$;
$S \leftarrow S \sim \{h\}$;
go to step 2

STEP 4 (selection)
$E \leftarrow E \cup \{k\}$; $S \leftarrow S \sim \{k\}$;
$\rho_b \leftarrow c_k \rho_b$ for every $b \in B_k$;
if $|E| < n$ **then** go to step 2 **else** END

Note that at the end of each iteration one candidate is either selected or eliminated, and thus the algorithms terminate after a finite number of steps. If the signed threshold l is too constraining, there is obviously the possibility that not all the n seats are filled.

If $l \geq 0$, then only the first n elements of the ballot paper play a role, and so the ballot papers may consist of lists of at most n candidates.

5. Closed list electoral systems

The parametrization covers all the standard closed list electoral systems, where voters may only vote for closed lists proposed by political parties and the electoral formula uses either highest average or largest remainders (first-past-the-post is a particular case, with $n = 1$). For these electoral systems $m = n$ and $l \geq 0$. Thus the ballot papers consist of lists of n candidates (at least n, since $m = n$; at most n, since $l \geq 0$). There may be a threshold or not (i.e., either $l > 0$ or $l = 0$). We shall see that the adequate parameter c_k can be determined for every standard closed list electoral system.

Candidate k is identified by the party list i where they figure and their rank j in that list, where $i = 1, .., |T|$ and $j = 1, ..., n$; accordingly we can write c_{ij} instead of c_k. Two possibilities are to be considered:

- The reduction factor c_{ij} depends only on j. The systems based on the *highest average* method (see, e.g., Gallagher and Mitchell [4]) are obtained just setting the parameter c_{ij} with the adequate values. For example, in the D'Hondt system (sequence of "divisors" 1, 2, 3, 4, 5, ...)

(2)
$$c_{ij} = \frac{j}{j+1} \ ,$$

in the Sainte-Laguë system (sequence of "divisors" 1, 3, 5, 7, 9, ...)

$$c_{ij} = \frac{2j-1}{2j+1} \ ,$$

in the Imperiali system (sequence of "divisors" 1, 1.5, 2, 2.5, 3, ...)

$$c_{ij} = \frac{j+1}{j+2} \ ,$$

and in the so-called Danish system (sequence of "divisors" 1, 4, 7, 10, 13, ...)

$$c_{ij} = \frac{3j-2}{3j+1}$$

- The reduction factor c_{ij} depends on the rank j and on the number of votes y_i obtained by the list. This is the situation of the quota electoral systems (see (1)), which in the case of closed lists are called *largest remainders* systems. Here

$$c_{ij} = \frac{y_i - jq}{y_i - (j-1)q} \tag{3}$$

Note that highest average systems are defined only for closed list voting, whereas quota systems do not have this restriction.

We introduce some general notation for closed list electoral systems (with $m = n$ and $l \geq 0$). As a motivation, consider the usual table to allocate the seats by D'Hondt system (without threshold), in a numerical example (with $n = 10$) taken from Gallagher and Mitchell [4]:

	Votes (V)	$(1/2) \cdot V$	$(1/3) \cdot V$	$(1/4) \cdot V$	$(1/5) \cdot V$	Seats
Socialist Party	$\boxed{34{,}000}$	$\boxed{17{,}000}$	$\boxed{11{,}333}$	$\boxed{8{,}500}$	6,800	4
Center-right Party	$\boxed{25{,}000}$	$\boxed{12{,}500}$	$\boxed{8{,}333}$			3
Liberal Party	$\boxed{15{,}000}$	7,500				1
Green Party	$\boxed{12{,}000}$	6,000				1
Radical Right Party	$\boxed{10{,}000}$	5,000				1
Regionalist Party	4,000					
TOTAL	100,000					10

This table, taking aside the last row and the last column, provides a 6×5 matrix, making up the first 5 columns of an instance of the *matrix of the electoral system*, as we shall call it. The matrix is calculated stepwise by the algorithm. The n largest entries of the matrix (here boxed) are also worked out; only the entries of the matrix that may be relevant for this purpose are calculated.

Let $\tau := |T|$ be the number of competing lists. Any candidate s is identified by the party list i where they figure and their rank j in that list, where $i = 1, .., \tau$ and $j = 1, ..., n$; accordingly we can write z_{ij} instead of z_s. Note that in closed list electoral systems, the value of z_s defined in Step 2 can be *trivial* (i.e. defined by an empty sum) or non-trivial; once it is defined non-trivially, it does not change in subsequent iterations. Suppose that $l = 0$ (or that the candidates to be eliminated by the threshold l have already been excluded). Given a closed list electoral system (algorithm) and a distribution of votes $y = (y_1, y_2, ..., y_\tau)$, where y_i is the number of votes obtained by party i , we define the *matrix of the electoral system (for* y*)*, $Y = (y_{ij})$, $i = 1, ..., \tau$, $j = 1, ..., n$ as follows: y_{ij} is the non-trivial value (as provided by the algorithm) of z_{ij} (whether the non-trivial values are actually determined in the implementation of a particular instance depends on when the algorithm terminates). Thus the first column is just vector y (so $y_{i1} := y_i$ for every party $i = 1, ..., \tau$), and the rest of the columns are given by

$$y_{ij} := c_{i(j-1)} \cdot c_{i(j-2)} \cdot ... \cdot c_{i2} \cdot c_{i1} \cdot y_i \text{ , for } i = 1, ..., \tau, \ j = 2, ..., n \tag{4}$$

The algorithm finds the n highest entries of the matrix Y, and the corresponding candidates are selected to represent the constituency.

The matrices of closed list electoral systems are used in Gutiérrez [6] to study in a unified way whether an electoral system is more or less favourable to large or small parties.

Let us consider matrix Y in these two cases: the highest average systems and the largest remainders systems. In highest average systems the reduction factor c_{ij} depends only on the rank j in the party list, and thus the columns of Y are proportional to the first column (the distribution of votes y)[1]. On the other hand, in largest remainders systems every column of Y is the result of adding a constant to all the elements of the first column[2]. Note that the two sorts of standard closed list electoral systems, highest average and largest remainders systems, represent the two simplest and more "natural" ways of deriving the matrix Y from vector y, the former multiplicative and the latter additive.

We have just seen an example of Y for highest average systems. As an example for largest remainders systems, consider the following table (not the usual one appearing in books) to allocate the seats by the Hare quota according to the algorithm (wihout threshold), with the same data (for the D'Hondt system) as above:

	Votes (V)	V$-q$	V$-2q$	V$-3q$	Seats
Socialist Party	$\boxed{34{,}000}$	$\boxed{24{,}000}$	$\boxed{14{,}000}$	4,000	3
Center-right Party	$\boxed{25{,}000}$	$\boxed{15{,}000}$	$\boxed{5{,}000}$		3
Liberal Party	$\boxed{15{,}000}$	$\boxed{5{,}000}$			2
Green Party	$\boxed{12{,}000}$	2,000			1
Radical Right Party	$\boxed{10{,}000}$	0			1
Regionalist Party	4,000				
TOTAL	100,000				10

This table, taking aside the last row and the last column, provides the first 4 columns of an instance of the matrix of the electoral system (only the relevant entries are calculated).

In real world closed list systems, often a threshold is imposed: only the candidates of the lists with a number of votes greater than $l > 0$ are considered (e.g., 3% in Spain[3] (*Congreso de los Diputados*), 4% in Austria (*Nationalrat*) or 5% in Germany (*Bundestag*)). On the other hand, it is sometimes possible for the voters to intervene in the setting of the order of candidates in the party list through an additional voting

[1]In highest average systems $c_{ij} =: \kappa_j$, $i = 1, ..., \tau$, and

$$y_{ij} = \kappa_{j-1} \cdot ... \cdot \kappa_2 \cdot \kappa_1 \cdot y_i \text{ , for } i = 1, ..., \tau, \ j = 2, ..., n$$

[2]From (4) and (3), after an easy calculation, it results for largest remainders systems determined by quota q that

$$y_{ij} := y_i - (j-1)q \text{ , for } i = 1, ..., \tau, \ j = 2, ..., n$$

[3]Considering the magnitude of the constituencies, this threshold has very limited significance in the Spanish system (and it can be effective only in the constituencies of Madrid and Barcelona).

process; this is the case in Germany, where only after considering the so-called "first vote" the order of the candidates in the lists of the parties is established[4].

6. Ordinal electoral systems

In ordinal electoral systems the voter can rank in the ballot paper the candidates, irrespective of their party, in any order he/she wishes. The parametrization covers all single transferable vote systems (preferential voting is a particular case), which are the main ordinal systems. In principle, there is a single transferable vote system for every possible quota q; normally the Droop quota is used (see below).

The higher the threshold is, the more candidates are eliminated. The political effect of the threshold is more nuanced in ordinal systems than in closed list systems. We consider ordinal single transferable vote systems. On the one hand, the higher the threshold is, the more candidates with low preferences in the ballots are eliminated, and this goes in principle against small parties. On the other hand, when a candidate is eliminated, the second choices in the ballots where this candidate was the first option come to the fore (in contrast with the threshold of closed list systems, where the ballots of the parties failing at the threshold simply become ineffectual). The final effect depends on the distribution of second choices (third choices, etc). Middle sized parties were predicted to benefit from the introduction of the simplest of this sort of electoral systems, the defeated proposal of *alternative voting* in Britain (in force in Australia under the name of *preferential voting*), with single member constituencies and absolute majority as threshold (see Sanders et al. [10]).

For single transferable vote systems $m = 1$ and $l < 0$. In fact, we shall explain how the single transferable vote system with quota q is determined by the parameters $l = -q$ and

$$c_k = \frac{z_k - q}{z_k}$$

Note that the quota is used to define both c_k and the threshold $-l$. In order to avoid being too abstract in the explanation, we consider the Irish single transferable vote system in some detail . Here the voter provides the maximum of information: $T = S^*$ (certainly the voters may refrain from listing all the candidates). The magnitude n of the constituencies is 3, 4 or 5. Now $l = -q$, where $q = |B|/(n+1)$ is the Droop quota. In every iteration the selection of a candidate is attempted; if no candidate can be selected in this iteration, one candidate is eliminated. If k is selected, the weight ρ_b of the ballots in the set B_k (formed by the ballots in which k is the top candidate among those still in S) is decreased by the factor $c_k = \frac{z_k - q}{z_k}$. Alternatively, without using weights, a random sample (of the corresponding size) of ballots of B_k is taken, and the rest of the ballots of B_k are discarded. Either simple sampling or stratified sampling (with proportional allocation) can be applied; in the latter case the strata are defined according to the candidate (in S) following k in the ballots. In the long run, weighting (called "Gregory method") and sampling are equivalent; in practice stratified sampling is used in Ireland.

[4]The first vote is applied to single-seat constituencies (first-past-the post). The candidates so elected for each party are now to be considered at the top of the corresponding party list (whether they were also candidates in the party list or not). Those elected with the first vote become members of the Bundestag even if they are not elected with the vote for the party lists (the "second vote"); the size of the Bundestag is increased accordingly (excess seats or "Überhangmandate").

Consider the following table, with $n = 3$ (the data are taken from Gallagher and Mitchell [4]). In the first iteration, no candidate can be selected and O'Riordan is eliminated ($-l = 8352$). Now his name plays no further part in all ballots, and thus the 3,796 ballots where he is in the top position are "transferred" according to the second preferences shown on them (in 287 ballots only the name of O'Riordan comes up; these are "non-transferable ballots"). In the second iteration, Crowley is selected. Now $c_{CROWLEY} = \frac{166}{8518}$, and the algorithm assigns the weight $\frac{166}{8518}$ to all ballots where Crowley is the first preference ($\rho_b \leftarrow c_{CROWLEY} \cdot 1$); alternatively, as in the table, 166 of these ballots are chosen with stratified sampling (the apportionment for the 4 strata is given by the distribution of the 8,518 Crowley's ballots: 5,644 for Creed, 564 for Moynihan, 1,232 for Roche , and 1,078 non-transferable), and the remaining 8,352 are put aside of the counting. Creed is selected in the third iteration, and Moynihan in the fourth one.

	1st PREF.	TRANSF.		TRANSF.	
Creed	7,037	+1,292	8,349	+110	$\boxed{8{,}469}$
Crowley	7,431	+1,087	$\boxed{8{,}518}$	-166	
Moynihan	7,777	+566	8,343	+11	$\boxed{8{,}354}$
O'Riordan	3,796	-3,796			
Roche	7,343	+564	7,907	+24	7,931
NON-TRANSF.		+287	287	+21	308
TOTAL	33,404				

In fact, when a candidate is selected a variant of the procedure above is used in Ireland. Stratified sampling is not implemented in B_k, but in a subset $V \subseteq B_k$. Thus when Crowley is selected, V is formed by those ballots transferred to Crowley after the elimination of O'Riordan, excluding those having no preference after the names of O'Riordan and Crowley (non-transferable). The distribution of these 1,087 ballots is: 783 for Creed, 72 for Moynihan, 145 for Roche , and 87 non-transferable. The proportions of Creed, Moynihan and Roche of the 1,087-87=1,000 ballots provide now the apportionment for the 3 strata, as shown in the table below.

	1st PREF.	TRANSF.		TRANSF.	
Creed	7,037	+1,292	8,349	+130	$\boxed{8{,}479}$
Crowley	7,431	+1,087	$\boxed{8{,}518}$	-166	8,352
Moynihan	7,777	+566	8,343	+12	$\boxed{8{,}355}$
O'Riordan	3,796	-3,796			
Roche	7,343	+564	7,907	+24	7,931
NON-TRANSF.		+287	287	+0	287
TOTAL	33,404				

When in the algorithm selection and elimination are carried out, usually non-transferable ballots appear. For simplicity, we have supposed that there are enough transferable ballots for all seats to be allocated with the algorithm.

7. Conclusions

This article has introduced a family of algorithms; it provides a formalization of how the "basic one-round electoral systems" proceed in transforming votes into seats. In this way the basic one-round electoral systems, including the standard closed list systems and the single transferable vote ordinal systems, are parametrized with the four parameters n (size of the constituency), m (size of the nomination lists), c_k (a factor providing the electoral formula) and l (signed election threshold).

Other electoral systems beyond the standard closed list systems and the single transferable vote ordinal systems are within the parametrization. For example, the single non-transferable vote results by setting $m = 1$, $l = 0$ and $c_k = 0$. (Thus only the first element of the ballot paper plays a role, and consequently only one name has to appear in the ballot).

Statistical inference provides a good example of the advantages and disadvantages of parametrization. On the one hand, in parametric statistics the theoretical analysis and the empirical testing are made easier. On the other hand, these very advantages make tempting to assume hypotheses allowing to work parametrically, even when the topic under consideration does not justify sufficiently these hypotheses.

Here we see that it is not necessary to formulate additional assumptions in order to parametrize the basic one-round electoral systems[5]. The parametrization reveals that the usual electoral systems have a common basic structure. They differ indeed in the amount of information gathered in the ballot paper and the consequent effect of the thresholds. In closed list systems, if a candidate is eliminated by the threshold, all the members of the list are also eliminated, whereas in single transferable vote ordinal systems, with the ballots having more information, we have somebody to resort to if a candidate is eliminated: the candidate in the ballot paper following the one crossed out.

In the electoral process three steps are to be considered: the nomination of candidates, the vote and the count transforming votes into seats. In the nomination of candidates the political parties can take more or less decision power, and the voter be left with less or more of it. The simple parameter m measures this characteristic of an electoral system. The parametrization of two-round electoral systems and that of the nomination of candidates by primaries (closed, open or blanket) may be a further research topic.

References

[1] Bowler, S.; Grofman, B. (eds.) (2000): *Elections in Australia, Ireland, and Malta under the Single Transferable Vote. Reflections on an Embedded Institution.* University of Michigan Press (Ann Arbor).

[2] Gallagher, M.; Mitchell, P. (eds.) (2005): *The Politics of Electoral Systems.* Oxford University Press (Oxford).

[3] Gallagher, M.; Mitchell, P. (2005): "Introduction to Electoral Systems", in *The Politics of Electoral Systems*, by M. Gallagher and P. Mitchel (eds.), 3-23. Oxford University Press (Oxford).

[5]The mathematically orientated reader may recall that in order to speak, e.g., of parametric statistics the indexing parameter must be a finite-dimensional vector (in p-dimensional Euclidean space); i.e. the set of possible values of the parameters is a subset of $\mathbb{R}^p$, for some nonnegative integer p. The parameters n, m and l are certainly one-dimensional. On the other hand, c_k is a function of k and (perhaps) the weighted number of ballots z_k. However, the only functions considered in basic one-round electoral systems are the quota function (1), the divisors function of the type (2), and the zero constant function. In the first case, the function can be determined by the one-dimensional parameter q. In the second case, the argument of the function is just the rank j of the candidate k in the closed list, and the function is a quotient of two affine functions.

[4] Gallagher, M.; Mitchell, P. (2005): "Appendix A: The Mechanics of Electoral Systems", in *The Politics of Electoral Systems*, by M. Gallagher and P. Mitchel (eds.), 579-597. Oxford University Press (Oxford).

[5] Grofman, B.; Lijphart, A. (eds.) (2002): *The Evolution of Electoral and Party Systems in the Nordic Countries*. Agathon Press (New York).

[6] Gutiérrez, J.M. (2015): "Majorization Comparison of Closed List Electoral Systems Through a Matrix Theorem", *Annals of Operations Research*, **235**, 807-814.

[7] Lijphart, A. (1994): *Electoral Systems and Party Systems*. Oxford University Press (Oxford).

[8] Nohlen, D. (2004): *Wahlrecht und Parteiensystem*. 4th edition (1st edition 1990). Verlag Barbara Budrich (Opladen).

[9] Rae, D.W. (1971): *The Political Consequences of Electoral Laws*. 2nd edition (1st edition 1967). Yale University Press (New Haven).

[10] Sanders; D.; Clarke, H.D.; Stewart, M.C.; Whiteley, P. (2011): "Simulating the Effects of the Alternative Vote in the 2010 UK General Election", *Parliamentary Affairs*, **64**, 5-23.

[11] Taagepera, R.; Shugart, M.S. (1989): *Seats and Votes. The Determinants of Electoral Systems*. Yale University Press (New Haven).

José Manuel Gutiérrez. Facultad de Economía y Empresa. Universidad de Salamanca. Salamanca 37007. Spain,

Wolfgang C. Müller. Institut für Staatswissenschaft. University of Vienna. Rooseveltplatz 3/1, 1090 Wien. Austria.