Vintage Capital in the AK growth model: a Dynamic Programming approach. Extended version.

Fabbri, Giorgio and Gozzi, Fausto

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G. Fabbri[*] and F. Gozzi[†]

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Abstract

This paper deals with an endogenous growth model with vintage capital and, more precisely, with the AK model proposed in [18]. In endogenous growth models the introduction of vintage capital allows to explain some growth facts but strongly increases the mathematical difficulties. So far, in this approach, the model is studied by the Maximum Principle; here we develop the Dynamic Programming approach to the same problem by obtaining sharper results and we provide more insight about the economic implications of the model. We explicitly find the value function, the closed loop formula that relates capital and investment, the optimal consumption paths and the long run equilibrium. The short run fluctuations of capital and investment and the relations with the standard AK model are analyzed. Finally the applicability to other models s also discussed.

Key words: Endogenous growth, Vintage capital, AK model, Dynamic programming.

Journal of Economic Literature: E22, E32, O40.

AMS classification: 49L20, 49K25, 34K35.

Contents

[*]Dipartimento di Scienze Economiche ed Aziendali, Università *LUISS - Guido Carli* Roma, e-mail: gfabbri@luiss.it

[†]Dipartimento di Scienze Economiche ed Aziendali, Università *LUISS - Guido Carli* Roma, e-mail: fgozzi@luiss.it

Introduction

In this work we develop the Dynamic Programming (DP in the following) approach to study a continuous time endogenous growth model with vintage capital. We focus on the AK model proposed by Boucekkine, Puch, Licandro and Del Rio in [18] (see e.g. [15], [12] for related models) which is summarized in Section I.1.

In the literature continuous time endogenous growth models with vintage capital are treated by using the Maximum Principle (MP in the following). Here we develop the DP approach to the representative model of [18] getting sharper results. The improvements we obtain mainly come from the fact that we are able to find the value function and solve the optimal control problem in closed loop form, a key feature of the DP approach.

We stress the fact that the novelty of this paper is mainly on the methodological side.

In our opinion the DP approach to continuous time optimal control problems arising in economic theory has not been exploited in its whole power. This is especially true when the model presents some features (like the presence of Delay Differential Equations and/or Partial Differential Equations and/or state-control constraints) that call for the use of infinite dimensional analysis making it harder to treat with the standard theory. However the presence of such features is needed when we want to look at problems with vintage capital, see for instance the quoted papers [18, 15, 12], and also [10, 11], [39],[37, 38] on optimal technology adoption and capital accumulation.

To be clear and honest we must say that in this paper the DP approach works very well thanks to the availability of explicit solutions which happens also in other models (see Section IV.1) but when explicit solutions are not available still something interesting (like the points (II) and (III) below or the qualitative behavior of optimal path) can be said, usually with more technical difficulties. What can be said and the amount of difficulties strongly depend on the structure of the problem under study: in some cases almost everything can be repeated, in some other ones almost nothing, at the present stage (see Section IV.2). We also clarify that we are not saying that the DP is generally better than the MP approach: when the difficulties are hard it is often useful to use an integrated approach developing both the MP and the DP[1]. In Part IV we present a detailed discussion on these points.

The main methodological issues treated in this paper are the following.

(I) (*Explicit form of solutions*).

Providing solutions in explicit form, when possible, helps the analysis of the model. In [18] it is shown that the optimal consumption path has a specific form (i.e. it is an exponential multiplied by a constant Λ) but none is said about the form of Λ, the explicit expression of the capital stock and investment trajectories. Moreover existence of a long run equilibrium for the discounted paths is established but none is said about its form.

Here, using the fact that we can calculate explicitly the value function, we show a more precise result on the optimal consumption path determining the constant Λ and an equation for the optimal trajectories of the capital stock and of the investment. This allows to find explicitly the long run equilibrium of the discounted paths; in particular we can give more precise analysis of the presence of oscillations in the capital and investment stock and in the growth rates comparing the model with the standard AK model. See Section III.1 for further explanations.

(II) (*Admissibility of candidate solutions*).

When state/control constraints are present the necessary conditions of MP are difficult to solve. Often in studying growth models one considers the problem without

[1] For example such an integrated approach is used successfully in [41].

such constraints and then checks if the optimal path for the unconstrained problem satisfy them. This may be a difficult task and in some cases may even be not true. Indeed, in [18] it is not proved that the candidate optimal trajectory of capital and investment is admissible (see the discussion in Subsection 4.3, p. 60 of [18]) so a nontrivial gap remains in the theoretical analysis of the model.

Here we prove that the candidate optimal trajectory is admissible, so fixing such a gap: such difficult task is accomplished by changing the point of view used in [18] (and in many papers on continuous time endogenous growth models) to find the optimal trajectory. See Section III.2 for further explanations.

(III) (*Wider parameter set*).

We work under more general assumptions on the parameters that includes cases which may be still interesting from the economic point of view. These cases are not included in [18] and for this reason the set of parameters for which their theory applies can be empty for some values of $\sigma \in (0, 1)$. See Section III.3 for further explanations.

Concerning the economic interpretation of the methodological results listed above we underline the following.

- We have at hand a power series expansion of the investment and capital path where the dependence of the coefficients on the initial investment path is explicit. This means that the short run fluctuations of investment and capital and of their growth rates (which are driven by replacement echoes) can be analyzed in terms of the deviation of the investment's history from the "natural" balanced growth path (see Subsection III.1.1). Moreover the presence of explicit formulae opens the door to a more precise empirical testing of the model.

- We provide a comparison of the model with the standard AK model with depreciation rate of capital equal to 0. First we see that when the lifetime T of machines goes to infinity the vintage AK model reduces to such a standard AK model. Moreover we show that in the vintage AK model the quantity that we call "equivalent capital" (see Subsection III.1.2 for a definition) has a constant growth rate.
  This may explain two qualitative characteristic of the model: first the consumption path has a constant growth rate since the decision of the agent is to consume a constant share of the "equivalent capital" which is the key variable of the system (see the closed loop relation (35)); second the agent adjusts the investments to keep constant the growth rate of the "equivalent capital" (compare (35) and (36)) and this gives rise to the fluctuations in the investment path (due to replacement echoes). In this regard this is not a model of business cycle, as already pointed out in [18].

- In our setting, differently from the standard AK model with zero depreciation rate of capital, a positive investment rate is compatible with a negative long run growth rate. This enlarge the scenarios where the deviation between growth and investment rates can arise (see e.g. the discussion on this given in [18]).

The paper is organized in four parts: the first (Part I) contains a brief description of the model of [18] (Section I.1), a description of our approach to the problem (Section I.2) and (Section I.3) an outline of the related literature. Part II is devoted to the description of the new mathematical results and it is composed of three sections. In Section II.1 we give some preliminary results about the solution of the state equation, the existence of optimal controls and the properties of the value function. The mathematical core of the paper is Section II.2. Here we give, with complete proofs: the precise formulation of the problem in infinite dimension (Subsection II.2.1); the formulation of the Hamilton-Jacobi-Bellman (HJB in the following) equation and its explicit solution (Subsection II.2.2); the closed loop formula for the optimal strategies in explicit form (Subsection II.2.3). In Section II.3 we come back to the original problem proving, as corollaries of the results of Section II.2, our results about the explicit form of the value function (Subsection II.3.1), the explicit closed loop strategies (Subsection II.3.2) and the asymptotic behavior (long run equilibrium, costate dynamics, transversality conditions, balanced growth paths) of the optimal trajectories (Subsection II.3.3). In Part III we discuss the implications of our results in the vintage AK model and make a comparison with the previous ones. In the first three sectios we refer to the methodological points (I)-(II)-(III) raised above, while in Section III.4 we present some numerical results. Part IV is devoted to the description of the possible extensions of the described approach to others models. It contains two sections: the first (Section IV.1) due to the models where an explicit expression for the value function can be given and the second (Section IV.2) that describes what can be done when this does not happen.

The last section concludes the paper. Appendix A is devoted to a quick development of the Dynamic Programming approach to the standard AK model with zero depreciation rate of capital. It is given here partly because we did not find it in the literature (even if it is standard), partly for the commodity of the reader to have a sketch of the DP approach in an easy case and to make more clear the comparison with the present model (done in Subsection III.1.2) and the related comments. Appendix B contains the proofs.

Part I
Outline of the model and of the method

I.1 Description of the model

We deal with the vintage capital model presented in [18] as a representative continuous time endogenous growth model with vintage capital. Vintage capital is a well known topic in the growth theory literature of last ten years (see for instance [60], [1], [49], [43], [12], [15]). Even in a simple setting like the one of AK models the introduction of vintage capital involve the presence of oscillations in the short-run[2] and this is one of the main features that make the model interesting. Indeed the optimal paths in the model of [18] converge asymptotically to a steady state but the transition is complex and involve nontrivial dynamics. So this model can be used to study the contribution of the vintage structure of the capital in the transition and the behavior of the system after economic shocks.

For an in-dept explanation of the model and its background see the Introduction of

[2]The optimal trajectories of the standard AK models are simply exponential without transition towards steady state and this is one of main limits of such models.

[18] or [15]. We report here only its main features. The model presented in [18] is a vintage version of standard AK model with CRRA (Constant Relative Risk Aversion) utility function. (which is recalled in Appendix A in the case of zero depreciation rate of capital).

Obsolescence and deterioration of physical capital are simply modeled assuming that all machines have the same technology and that they have a fixed lifetime T (a constant "scrapping time").

The time is continuous and starts at $t = 0$ (the horizon is infinite as usual in growth models). However, to introduce a delay effect in the model due to the vintage capital structure, we assume that the economy exists at least at time $-T$ and that its behavior between $t = -T$ and $t = 0$ is known. So all variables of the model will be defined on $[-T, +\infty)$. Of course their paths between $t = -T$ and $t = 0$ will be considered data of the problem so we will define equations and constraints for $t \geq 0$.

We denote by $k(t)$, $i(t)$ and $c(t)$ the stock of capital, the investment and the consumption at time $t \geq -T$. All of them are nonnegative. The AK technology is the following: the aggregate production at time t is denoted by $y(t)$ and it satisfies, for $t \geq 0$

$$y(t) = a \int_{t-T}^{t} i(s)\mathrm{d}s \qquad a > 0. \tag{1}$$

Interpreting the integral in the right hand side as the capital we then have, for $t \geq 0$, $y(t) = ak(t)$. We have the following accounting relation, for $t \geq 0$

$$ak(t) = y(t) = i(t) + c(t) \tag{2}$$

so the non-negativity of all variables is equivalent to ask that, for $t \geq 0$

$$i(t), c(t) \in [0, y(t)] = [0, ak(t)]. \tag{3}$$

If the investment function $i(\cdot)$ is assumed to be sufficiently regular (e.g. continuous), then the above relation (1) can be rewritten as a Delay Differential Equation (DEE in the following) for the capital stock

$$\dot{k}(t) = i(t) - i(t - T) \tag{4}$$

with initial datum $k(0)$ given as function of the past investments by

$$k(0) = \int_{-T}^{0} i(s)\mathrm{d}s. \tag{5}$$

Given the above relations, the only initial datum needed to set up the model is the past of the investment strategy $i(\cdot)$[3]: we will denote it by $\bar{\imath}(\cdot)$. This datum is a function from $[-T, 0)$ to $\mathbb{R}$ so it belongs to a space of functions which is infinite dimensional. Since we want to work in Hilbert spaces (which are in some sense the best possible kind of infinite dimensional spaces one can work with) we assume that $\bar{\imath}(\cdot) \in L^2([-T, 0); \mathbb{R}^+)$ i.e.

[3] We use throughout the paper the notation $(\cdot)$ to denote a function, e.g. $i(\cdot)$; this notation will be suppressed when it is clear that we are dealing with a function (e.g. in Section II.2 where we develop the infinite dimensional setting).

to the space of all functions from $[-T, 0)$ to $\mathbb{R}^+$ that are Lebesgue measurable and square integrable[4].

The equilibrium is the solution of the problem of maximizing, over all investment-consumption strategies that satisfy the above constraints (1), (2), (3), the functional of CRRA (Constant Relative Risk Aversion) type

$$\int_0^{+\infty} e^{-\rho t} \frac{c(t)^{1-\sigma}}{1-\sigma} \mathrm{d}t \tag{6}$$

where $\rho > 0$, $\sigma > 0$ (and $\sigma \neq 1$). More general set of parameters ρ and σ (e.g. $\sigma = 1$ or some cases when $\rho \leq 0$) can be treated without big effort; we avoid this for simplicity. From the mathematical point of view this model is an optimal control problem. The state variable is the capital k, the control variables are the consumption c and the investment i, the state equation is the DDE (4) with the initial condition (5) (which is somehow unusual, see the following discussion and Notation II.1.1 for more explanations); the objective functional is (6). A control strategy $c(\cdot), i(\cdot)$ defined for $t \geq 0$ is admissible if it satisfies for every such t the constraints (2) and (3). Since the two control functions $i(\cdot)$ and $c(\cdot)$ are connected by the relation (2) then we can eliminate the consumption $c(\cdot)$ from the mathematical formulation of the problem. So the only control function is $i(\cdot)$ giving the present investment (as said above its 'history' in the interval $[-T, 0)$ is the initial datum $\bar{\imath}(\cdot)$). Similarly to $\bar{\imath}(\cdot)$ we assume that $i(\cdot) \in L^2_{\mathrm{loc}}([0, +\infty), \mathbb{R}^+)$ i.e. to the space of all functions from $[0, +\infty)$ to $\mathbb{R}^+$ that are Lebesgue measurable and square integrable on all bounded intervals[5].

Given an initial datum $\bar{\imath}(\cdot)$ and an investment strategy $i(\cdot)$ we denote by $k_{\bar{\imath},i}(\cdot)$ the associated solution (see Section II.1 for its explicit form) of the state equation (4)-(5). The strategy $i(\cdot)$ will be called admissible if it satisfies the constraints (coming from (3)):

$$0 \leq i(t) \leq a k_{\bar{\imath},i}(t) \qquad \forall t \geq 0. \tag{7}$$

(note that such constraints involve both the present value of the state and of the control: so they are called *state-control constraints*).

Now, using (2), we write the associated intertemporal utility from consumption as

$$J(\bar{\imath}(\cdot); i(\cdot)) \stackrel{def}{=} \int_0^{\infty} e^{-\rho s} \frac{(a k_{\bar{\imath},i}(t) - i(t))^{1-\sigma}}{(1-\sigma)} \mathrm{d}s$$

(note that we have explicitly written in the functional the dependence on the initial datum $\bar{\imath}(\cdot)$).

Our problem is the one of maximizing the functional $J(\bar{\imath}(\cdot); i(\cdot))$ over all admissible investment strategies $i(\cdot)$.

It must be noted that the model reduces to the standard AK model with zero depreciation rate of capital when the delay T (i.e. the "scrapping time") is $+\infty$[6].

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[4]For the study of the economic implication of the model it would be enough to consider data $\bar{\imath}(\cdot)$ that are piecewise continuous. Since any piecewise continuous function on $[-T, 0)$ is also a square integrable function our setting includes all interesting cases.

[5]The sign 'loc' in the name of the space simply means that we do not ask integrability over all interval $[0, +\infty)$ since this would be too restrictive (e.g. constant nonzero strategies would not be included).

[6]Indeed in such a case $k(t) = \int_{-\infty}^t i(s)ds = k(0) + \int_0^t i(s)ds$ and so the DDE (4)-(5) becomes the Ordinary Differential Equation of the standard AK model with zero depreciation rate of capital.

I.2 The Dynamic Programming Approach

The DP approach to optimal control problems can be summarized in four main steps (see for instance Fleming and Rishel [40]) for the DP in the finite dimensional case and Li and Yong [56] for the DP in infinite dimension). Before to explain how to perform them in the vintage AK model we clarify that the two main difficulties of it:

- The state equation is a DDE while the DP approach is formulated for the case when the state equation is an ODE. To overcome this difficulty one way (not the only one, for other approaches on can see e.g. Kolmanovskii and Shaikhet [55]) is to rewrite the DDE as an ODE in an infinite dimensional space which become the new state space. This can be done in our case using the techniques developed by Delfour, Vinter and Kwong (see Subsection II.2.1 below for explanation and Section I.3 for references). It must be noted that the resulting infinite dimensional control problem is harder than the ones mainly treated in the literature (see e.g. [56]) due to the unboundedness of the control operator and the non-analyticity of the semigroup involved (see again Subsection II.2.1).

- The pointwise constraints (7) involve both the state and the control (state-control constraints). Their presence makes the problem much more difficult. Indeed for such problems in infinite dimension there is no well established theory available up to now. Only few results in special cases, (different from the one treated here), can be found. This fact is at the basis of the theoretical problem contained in the paper [18] and mentioned at point (II) in the introduction: show that the candidate optimal trajectory satisfies the pointwise constraints (7) (see Section III.2 for more on this).

The key tool to overcome such difficulties will be the explicit solution of the HJB equation as explained in the four points below.

(i) First of all, given an initial datum $\bar{\iota}(\cdot) \in L^2([-T,0);\mathbb{R}^+)$ we define the set of admissible strategies given $\bar{\iota}(\cdot)$ as $\mathcal{I}_{\bar{\iota}} = \left\{ i(\cdot) \in L^2_{\text{loc}}([0,+\infty);\mathbb{R}^+) : \ i(t) \in [0, ak_{\bar{\iota},i}(t)], \ a.e. \right\}$ and then the value function as $V(\bar{\iota}(\cdot)) = \sup_{i(\cdot)\in\mathcal{I}_{\bar{\iota}}} \left\{ \int_0^\infty e^{-\rho s} \frac{(ak_{\bar{\iota},i}(t)-i(t))^{1-\sigma}}{(1-\sigma)} \mathrm{d}s \right\}$. The first step of DP approach recommends to write the DP Principle and the HJB equation for the value function. The DP Principle (see for instance [40] for the finite dimensional case and [56] for the infinite dimensional one) gives a functional equation that is always satisfied by the value function. Since this functional equation is not easily treatable one usually considers its infinitesimal version, the HJB equation. We do not write it here as we will be using, for technical reasons, a setting where the initial data will be both $\bar{\iota}(\cdot)$ and $k(0)$ ignoring the relation (5) that connects them. So in Section II.2 we will consider an artificial value function depending on $\bar{\iota}(\cdot)$ and $k(0)$ and write and solve the HJB equation for it (see Subsection II.2.1). After this, in Section II.3 we will go back to the value function defined here.

(ii) The second step of DP approach is now to solve the HJB equation. We will find explicitly a solution of the HJB equation and prove that it is the value function (see Propositions II.2.11 and II.3.1). The only other examples of explicit solution of the HJB equation in infinite dimension involve, for what we know, linear state equations and quadratic functionals (see Section I.3 below for references).

Note that this HJB equation cannot be treated with the results of the existing literature. This is due, as previously said, to the presence of the state/control constraint, to the unboundedness of the control operator and the non-analyticity of the semigroup given by the solution operator of the state equation (see the discussion in Subsection II.2.2 for more details).

(iii) The third step will be then to write the closed loop (feedback) formula. This means to write a formula that gives the present value of the optimal control as function only of the present value of the state. In this case the state is infinite dimensional and it is composed, for each $t \geq 0$, by the present value of the capital $k(t)$ and by the past (at time t) of the investment strategy $\{i(t+s), s \in [-T, 0)\}$. So the closed loop formula will give the present value of the investment $i(t)$ as a function of the present value of the state and of the past of the investment itself (see equation (31) for the feedback in infinite dimension and equation (34) for its DDE version). This formula will be given in term of the value function and so, using its explicit expression found in step (ii), also the closed loop formula will be given in explicit form. For details see Theorem II.2.15 for the result in infinite dimensions and Proposition II.3.2 for the DDE version.

(iv) The closed loop formula will be then substituted into the state equation (4)-(5) to get an equation for the optimal state trajectory (the so-called Closed Loop Equation). Such equation will be a DDE, as recalled at point (iii) above, and explicit solutions cannot be given in general. However it allows to study the behavior of the optimal paths and to perform numerical simulations. For details see Theorem II.3.4 and Subsections II.3.3, III.1.1.

I.3 The literature on DDEs and on Dynamic Programming in infinite dimensions

For DDEs a recent, interesting and accurate reference is the book of Diekmann, van Gils, Verduyn, Lunel and Walther [33].

The original idea of writing delay system using a Hilbert space setting was first due to Delfour and Mitter [31], [32]. Variants and improvements were proposed by Delfour [28], [26], [27], Vinter and Kwong [64], Delfour and Manitius [29], Ichikawa [47] (see also the references and the precise systematization of the argument in Chapter 4 of Bensoussan, Da Prato, Delfour and Mitter [13]). Using this idea the optimal control of DDEs becomes an optimal control problem in infinite dimension. When the state equation is linear and the objective is quadratic (the so-called linear quadratic (LQ) case) and without state or state control constraints the HJB equation reduces to a simpler operator (matrix in finite dimension) equation: the so-called Riccati equation. This case is then considered a special one and is deeply studied in the literature: see, for the DDEs case Vinter and Kwong [64], Ichikawa [47], Delfour, McCalla and Mitter [30] and Kolmanovskii and co-authors [54], [53], [55]. Kolmanovskii and co-authors have also given sufficient optimality conditions in term of the value function but they did not solve HJB equation except for the LQ case (without state or state/control constraints) and under suitable assumptions (see [54] ch.14 or [53] ch 6 or [55] ch. 2.). Out of the LQ case the infinite dimensional HJB equation coming from the optimal control of DDEs is much more difficult to deal with.

The study of the HJB equation in Hilbert spaces, started with the papers of Barbu and Da Prato (see [5], [6], [7]) is a large and diversified research field. We recall that one usually wants to find "classical" solutions of HJB equations (i.e. solutions that are differentiable in time and state) since this allows to get a more treatable closed loop form of the optimal strategy. This is not always possible. So there is a stream in the literature that studies in which cases classical solutions exists and there is another stream that studies the existence of "weak" solutions (i.e. solutions that are not differentiable)[7]. In this paper we are looking for "classical" solutions[8]. Up to now, to our knowledge, the existence of such solutions for the HJB equation in cases where the state equation is a DDE has not been studied in the literature (apart from the LQ case)[9].

For the study of optimal control of DDEs without DP we mention (beyond the papers of Boucekkine and co-authors [15, 17, 18]), in the so-called overtaking literature, the works of Carlson and co-authors [20, 22, 21] and Zaslavski [65, 66, 67]. Here some existence results of overtaking optimal solutions, turnpike properies and optimal conditions (using the MP) are proved but in a class of problems that does not include our case (due to the presence of the delay in the control and state control constraints).

The presence of state and/or state control constraints, in an optimal control problem, creates various difficulties already in finite dimensions (see e.g. [59] and [45] for MP and [8] for DP). Concerning DP, the state control constraints oblige to define the value function in a proper subset of the whole state space so the HJB equation becomes a PDE coupled with boundary conditions. Such boundary conditions are non-standard (see e.g. [8]) and many results (like verification theorems) are more difficult to get. In the infinite dimensional case the study of PDEs with boundary condition is just at the beginning since many tools, like the Sobolev spaces with respect to the Lebesgue measure, could not be used (see e.g. [19], [56] and [25] for more details).

Part II
New mathematic results on the model

II.1 Preliminary results on the control problem

We first introduce a notation useful to rewrite more formally equation (4)-(5) as in (9) below.

Notation II.1.1. *We call* $\bar{\imath}\colon [-T,0) \to \mathbb{R}^+$ *the initial datum,* $i\colon [0,+\infty) \to \mathbb{R}^+$ *the control strategy and* $\tilde{\imath}\colon [-T,+\infty) \to \mathbb{R}^+$ *the function*

$$\tilde{\imath}(s) = \begin{cases} \bar{\imath}(s) & s \in [-T,0) \\ i(s) & s \in [0,+\infty). \end{cases} \tag{8}$$

[7]The right concept of weak solution is the one of viscosity solution, introduced by Crandall and Lions in the finite dimensional case and then applied to infinite dimension by the same authors, see [23] for an introduction to the topic and further references.

[8]Since the definition of solution we use is adapted to the features of the problem it is not exactly the classical one, see on this Subsection II.2.2.

[9]In the economic literature the study of infinite dimensional optimal control problems that deals with vintage/heterogeneous capital is a quite recent tool but of growing interest: see for instance [11], [39], and [38].

The state equation is now written as the DDE (on $\mathbb{R}^+$)

$$\begin{cases} \dot{k}(t) = \tilde{\imath}(t) - \tilde{\imath}(t-T), \ \forall t \geq 0, \\ \tilde{\imath}(s) = \bar{\imath}(s) \ \ for \ s \in [-T, 0), \quad k(0) = \int_{-T}^{0} \bar{\imath}(s)\mathrm{d}s, \end{cases} \tag{9}$$

where $T \in \mathbb{R}$ is a positive constant, $\bar{\imath}(\cdot)$ and $k(0)$ are the initial conditions. We will assume $\bar{\imath}(\cdot) \geq 0$ and $\bar{\imath}(\cdot) \not\equiv 0$. Moreover $\bar{\imath}(\cdot) \in L^2([-T,0);\mathbb{R}^+)$. For every $i(\cdot) \in L^2_{\text{loc}}([0,+\infty);\mathbb{R}^+)$ and every $\bar{\imath}(\cdot) \in L^2([-T,0);\mathbb{R}^+)$ the DDE (9) admits a unique locally absolutely continuous solution given by

$$k_{\bar{\imath},i}(t) \overset{def}{=} \int_{(t-T)\vee 0}^{t} i(s)\mathrm{d}s \ + \int_{(t-T)\wedge 0}^{0} \bar{\imath}(s)\mathrm{d}s = \int_{(t-T)}^{t} \tilde{\imath}(s)\mathrm{d}s \tag{10}$$

The functional to maximize is $J(\bar{\imath}(\cdot); i(\cdot)) \overset{def}{=} \int_0^\infty e^{-\rho s} \frac{(ak_{\bar{\imath},i}(t) - i(t))^{1-\sigma}}{(1-\sigma)}\mathrm{d}s$ over the set $\mathcal{I}_{\bar{\imath}} \overset{def}{=} \{i(\cdot) \in L^2_{\text{loc}}([0,+\infty);\mathbb{R}^+) \ : \ i(t) \in [0, ak_{\bar{\imath},i}(t)] \text{ for a.e. } t \in \mathbb{R}^+\}$. Here a and σ are strictly positive constants with $\sigma \neq 1$. The choice of $\mathcal{I}_{\bar{\imath}}$ implies $k_{\bar{\imath},i}(\cdot) \in W^{1,2}_{\text{loc}}(0,+\infty;\mathbb{R}^+)$ for every $i(\cdot) \in \mathcal{I}_{\bar{\imath}}$.

Remark II.1.2 (On the irreversibility constraint). In the definition of $\mathcal{I}_{\bar{\imath}}$ we have imposed two control constraints for each $t \geq 0$: the first is of course $(ak_{\bar{\imath},i}(t) - i(t)) \geq 0$ that means exactly that the consumption cannot be negative; the second is $i(t) \geq 0$, i.e. irreversibility of investments. It may be possible to consider a wider set of control strategies without imposing irreversibility but only the positivity of the capital: $k(t) \geq 0$ (or some weaker "no Ponzi game" condition). This is done e.g. in the standard AK model recalled in Appendix A. There are some arguments to believe that irreversibility is a more natural choice in our delay setting. First of all in the vintage model $i(t)$ is the investment in new capital and so the irreversibility assumption is natural from the economic point of view. Moreover we can observe that, unlike the non-delay case, $i(t) \geq 0$ does not imply a growth of the capital (see Section III.3 on this). Finally if this constraint hold on the datum $\bar{\imath}(\cdot)$ (as we assume) the set of admissible strategies is always nonempty. If we take only the constraints $k(t) \geq 0$ and $(ak_{\bar{\imath},i}(t) - i(t)) \geq 0$ then there are examples of initial data $\bar{\imath}(\cdot)$ (not always positive) with $k(0) \geq 0$ such that the set of admissible trajectories is empty (for instance $\bar{\imath}(s) = 2\chi_{[-T,-T/2]}(s) - 2\chi_{(-T/2,0)}(s)$ for $s \in [-T,0)$) where χ_A is the indicator function of the set A. ∎

We will name *Problem (**P**)* the problem of finding an optimal control strategy i.e. to find an $i^*(\cdot) \in \mathcal{I}_{\bar{\imath}}$ such that:

$$J(\bar{\imath}(\cdot); i^*(\cdot)) = V(\bar{\imath}(\cdot)) \overset{def}{=} \sup_{i(\cdot)\in\mathcal{I}_{\bar{\imath}}} \left\{ \int_0^\infty e^{-\rho s} \frac{(ak_{\bar{\imath},i}(t) - i(t))^{1-\sigma}}{(1-\sigma)}\mathrm{d}s \right\}. \tag{11}$$

We now give a preliminary study of the problem concerning the asymptotic behavior of admissible trajectories, the finiteness of the value function, the existence of optimal strategies and the positivity of optimal trajectories.

II.1.1 Asymptotic behavior of admissible trajectories

To find conditions ensuring the finiteness of the value function we need first to study the asymptotic behavior of the admissible trajectories, in particular to determine which is the maximum asymptotic growth rate of the capital.

We introduce, as in the standard AK model (see Appendix A page 33), a first restriction on the parameters that ensure the finiteness of the value function V at every initial datum $\bar{\iota}(\cdot)$.

Proposition II.1.3. *Given an initial datum $\bar{\iota}(\cdot) \in L^2([-T,0);\mathbb{R}^+)$ and a control $i(\cdot) \in L^2_{\text{loc}}([0,+\infty);\mathbb{R}^+)$, we have that the solution $k_{\bar{\iota},i}(\cdot)$ of (9) is dominated at any time $t \geq 0$ by the solution $k^M(\cdot)$ obtained taking the same initial datum $\bar{\iota}(\cdot)$ and the admissible control defined by the feedback relation $i^M(t) = ak^M(t)$ for all $t \geq 0$ (that is the maximum of the range of admissibility).*

Observe now that, by its definition, $k^M(\cdot)$ is the unique solution of

$$\begin{cases} \dot{k^M}(t) = \tilde{\imath}^M(t) - \tilde{\imath}^M(t-T) \\ \tilde{\imath}^M(s) = \bar{\iota}(s) \;\; for \; s \in [-T,0), \quad k^M(0) = \int_{-T}^0 \bar{\iota}(r)\mathrm{d}r > 0 \end{cases} \tag{12}$$

and then for $t \geq T$, $k^M(t) = h(t)$ where $h(\cdot)$ the unique solution of

$$\dot{h}(t) = a(h(t) - h(t-T)) \; for \; t \geq T, \qquad h(s) = k^M(s) \; for \; s \in [0,T) \tag{13}$$

For equation (13) we can apply standard statements on DDEs as follows. We define the *characteristic equation* of the DDE (13) as

$$z = a(1 - e^{-zT}), \qquad z \in \mathbb{C} \tag{14}$$

The characteristic equation is defined for general linear DDE as described in [33] (page 27). In our case, by a convexity argument, we can easily prove the following result.

Proposition II.1.4. *There exists exactly one strictly positive root of (14) if and only if $aT > 1$. Such root ξ belongs to $(0,a)$. If $aT \leq 1$ then the only root with non negative real part is $z = 0$.*

Since, as we will see in Proposition II.1.6 and Remark II.1.7, the maximum characteristic root give the maximum rate of growth of the solution, to rule out the cases where growth cannot occur it is natural to require the following.

Hypothesis II.1.5. $aT > 1$.

We will assume from now on that Hypothesis II.1.5 holds. Note that, assuming Hypothesis II.1.5 we have

$$\begin{array}{lll} g \in (0,\xi) & \Longrightarrow & g < a(1-e^{-gT}) \\ g \in (-\infty,0) \cup (\xi,+\infty) & \Longrightarrow & g > a(1-e^{-gT}). \end{array} \tag{15}$$

Proposition II.1.6. *Let Hypothesis II.1.5 hold true. Given an initial datum $\bar{\iota}(\cdot) \in L^2([-T,0);\mathbb{R}^+)$ with $\bar{\iota}(\cdot) \not\equiv 0$ and a control $i(\cdot) \in L^2_{\text{loc}}([0,+\infty);\mathbb{R}^+)$, we have that for every $\varepsilon > 0$*

$$\lim_{t\to+\infty} \frac{k^M(t)}{e^{(\xi+\varepsilon)t}} = 0.$$

Remark II.1.7 (On the Hypothesis II.1.5). Hypothesis II.1.5 has a clear economic meaning: if there are no strictly positive root we can see, as in Proposition II.1.3, that the maximal growth of the capital stock[10] is not positive since the stock of capital always goes to zero. So positive growth would be excluded from the beginning. Moreover Hypothesis II.1.5 is verified when we take the limit of the model as T goes to $+\infty$ which is "substantially" the standard AK model with zero depreciation rate of capital. In this case we will have $\xi \to a$. ∎

The above Proposition II.1.6 is what we need to analyze the finiteness of the value function. Before to proceed with it we give a refinement of Proposition II.1.4 that give a more detailed analysis of the solutions of characteristic equation (14) and so of the solution of equation (13) that will be useful later, see the proof of Proposition II.3.5 and Subsection III.1.1.

Proposition II.1.8. *Assuming Hypothesis II.1.5 we can state that:*

(a) The characteristic equation (14) has only simple roots.

(b) There are exactly 2 real roots of (14), i.e. ξ *and* 0.

(c) There is a sequence $\{\lambda_k,\ k = 1, 2, ...\} \subset \mathbb{C}$ *such that* $\{\lambda_k, \overline{\lambda}_k\ k = 1, 2, ...\}$ *are the only complex and non real roots of (14).*

For each k *we have* $T \cdot Im\lambda_k \in (2k\pi, (2k + 1)\pi)$.

The real sequence $\{Re\lambda_k,\ k = 1, 2, ...\}$, *is strictly negative and strictly decreasing to* $-\infty$. *Finally*

$$Re\lambda_1 < \xi - a. \tag{16}$$

II.1.2 Finiteness of the value function

We now introduce the following assumption that, given Hypothesis II.1.5, will be a sufficient condition for the finiteness of the value function for every initial datum[11].

Hypothesis II.1.9. $\rho > \xi(1 - \sigma)$.

From now on we will assume that Hypotheses II.1.5 and II.1.9 hold. Now, thanks to Proposition II.1.3 and Hypothesis II.1.9 we can exclude two opposite cases: on one hand, when $\sigma < 1$, the existence of some $\bar{\iota}(\cdot)$ in which $V(\bar{\iota}(\cdot)) = +\infty$ (Proposition II.1.10), on the other hand, when $\sigma > 1$, the existence of some $\bar{\iota}(\cdot)$ in which $V(\bar{\iota}(\cdot)) = -\infty$ (Proposition II.1.11).

Proposition II.1.10. $V(\bar{\iota}(\cdot)) < +\infty$ *for all* $\bar{\iota}(\cdot)$ *in* $L^2([-T, 0); \mathbb{R}^+)$.

Proof. For $\sigma > 1$ it is obvious since $J(\bar{\iota}(\cdot); i(\cdot)) \le 0$ always. For $\sigma \in (0, 1)$ we observe that for every $i(\cdot) \in L^2_{\text{loc}}([0, +\infty); \mathbb{R}^+)$,

$$J(\bar{\iota}(\cdot); i(\cdot)) \le \frac{1}{1-\sigma} \int_0^{+\infty} e^{-\rho t} (ak_{\bar{\iota},i}(t))^{1-\sigma} dt \le \frac{1}{1-\sigma} \int_0^{+\infty} e^{-\rho t} (ak^M(t))^{1-\sigma} dt.$$

so from the definition of the value function, Proposition II.1.3 and Hypothesis II.1.9, the claim follows. □

[10] That occurs re-investing all capital.

[11] Indeed in the standard AK model with zero depreciation rate of capital such a condition with $\xi = a$ is also necessary, see e.g. [42]. In our case a similar result can be proved but we avoid it for simplicity.

Proposition II.1.11. *If $\bar{\imath}(\cdot) \in L^2([-T,0);\mathbb{R}^+)$ and $\bar{\imath}(\cdot) \not\equiv 0$ there exists a control $\theta(\cdot) \in \mathcal{I}_{\bar{\imath}}$ such that $J(\bar{\imath}(\cdot);\theta(\cdot)) > -\infty$.*

II.1.3 Existence of optimal strategies

We now state and prove the existence of optimal paths.

Proposition II.1.12. *An optimal control exists in $\mathcal{I}_{\bar{\imath}}$, i.e. we can find in $\mathcal{I}_{\bar{\imath}}$ an admissible strategy $i^*(\cdot)$ such that $V(\bar{\imath}(\cdot)) = J(\bar{\imath}(\cdot); i^*(\cdot))$.*

II.1.4 Strict positivity of optimal trajectories

We can now prove the strict positivity of optimal trajectories that we will use in Section II.3. Note that we have already proved the strict positivity of the capital path k^M in the proof of Proposition II.1.6.

Proposition II.1.13. *Let $\bar{\imath}(\cdot)$ be in $L^2([-T,0);\mathbb{R}^+)$ and $\bar{\imath}(\cdot) \not\equiv 0$ and let $i^*(\cdot) \in \mathcal{I}_{\bar{\imath}}$ be an optimal strategy then $k_{\bar{\imath},i^*}(t) > 0$ for all $t \in [0,+\infty)$.*

II.2 Writing and solving the infinite dimensional problem

II.2.1 Rewriting *problem P* in infinite dimensions

Given $t \geq 0$ we indicate the "history" of investments at time t with $\tilde{\imath}_t$ defined as:

$$\tilde{\imath}_t\colon [-T,0] \to \mathbb{R}, \qquad \tilde{\imath}_t(s) = \tilde{\imath}(t+s) \tag{17}$$

The capital stock can then be rewritten as $k(t) = \int_{-T}^0 \tilde{\imath}_t(s)\mathrm{d}s$ and so the DDE (9) can be rewritten as

$$\dot{k}(t) = B(\tilde{\imath}_t), \qquad (k(0), \tilde{\imath}_0) = (\int_{-T}^0 \bar{\imath}(s)\mathrm{d}s,\ \bar{\imath}) \tag{18}$$

where B is the continuous linear map $B\colon C([-T,0];\mathbb{R}) \to \mathbb{R}$ defined as $B(f) = f(0) - f(-T)$. Equation (18) has a pointwise meaning only if the control is continuous but always has an integral sense (as in (10)).
The link between the initial condition for $k(t)$ and $\tilde{\imath}_t$ (that is $k(0) = \int_{-T}^0 \tilde{\imath}_0(s)\mathrm{d}s$) has a clear economic meaning but is, so to speak, nonstandard from a mathematical point of view. We "suspend" it in this section and will reintroduce it in Section II.3 when we will find the optimal feedback for *problem* (**P**). So we consider now initial data given by $(k_0, \bar{\imath})$ where k_0 and $\bar{\imath}$ have no relationship. Our problem becomes a bit more general:

$$\dot{k}(t) = B(\tilde{\imath}_t), \qquad (k(0), \tilde{\imath}_0) = (k_0,\ \bar{\imath}) \tag{19}$$

Its solution is

$$k_{k_0,\bar{\imath},i}(t) = k_0 - \int_{-T}^0 \bar{\imath}(s)\mathrm{d}s + \int_{t-T}^t \tilde{\imath}(s)\mathrm{d}s \tag{20}$$

Clearly for every $t \geq 0$, $k_{\int_{-T}^0 \bar{\imath}(s)\mathrm{d}s,\bar{\imath},i}(t) = k_{\bar{\imath},i}(t)$ as defined in equation (10). Now we introduce the infinite dimensional space in which we re-formulate the problem, it is:

$$M^2 \stackrel{def}{=} \mathbb{R} \times L^2([-T,0);\mathbb{R})$$

A generic element x of M^2 will be denoted as a couple (x^0, x^1). The scalar product on M^2 will be the one on a product of Hilbert spaces i.e. $\langle (x^0, x^1), (z^0, z^1)\rangle_{M^2} \stackrel{def}{=} x^0 z^0 + \langle x^1, z^1\rangle_{L^2}$ for every $(x^0, x^1), (z^0, z^1) \in M^2$. We introduce the operator A on M^2:

$$\begin{cases} D(A) \stackrel{def}{=} \{(\psi^0, \psi^1) \in M^2 \ : \ \psi^1 \in W^{1,2}(-T, 0; \mathbb{R}), \ \psi^0 = \psi^1(0)\} \\ A \colon D(A) \to M^2, \qquad A(\psi^0, \psi^1) \stackrel{def}{=} (0, \frac{\mathrm{d}}{\mathrm{d}s}\psi^1) \end{cases}$$

Abusing of notation it is also possible to confuse, on $D(A)$, $\psi^1(0)$ with ψ^0 and redefine

$$B \colon D(A) \to \mathbb{R}, \qquad B(\psi(0), \psi) = B\psi = \psi(0) - \psi(-T) \in \mathbb{R}$$

Notation II.2.1. *We will indicate with F the application*

$$F \colon L^2([-T, 0); \mathbb{R}) \to L^2([-T, 0); \mathbb{R}), \qquad F(z)(s) \stackrel{def}{=} -z(-T - s) \tag{21}$$

and with R the application

$$R \colon L^2([-T, 0); \mathbb{R}) \to \mathbb{R}, \qquad R \colon z \mapsto \int_{-T}^{0} z(s)\mathrm{d}s.$$

Definition II.2.2. *Given initial data $(k_0, \bar{\imath})$ we set for simplicity $y = (k_0, F(\bar{\imath})) \in M^2$ (that will be the initial datum in the Hilbert setting). Given $\bar{\imath} \in L^2([-T, 0); \mathbb{R}^+)$, $i \in L^2_{\mathrm{loc}}([0, +\infty); \mathbb{R}^+)$, $k_0 \in \mathbb{R}$ and $k_{k_0, \bar{\imath}, i}(t)$ as in (20) we define the* structural state[12] *of the system the couple $x_{y,i}(t) = (x^0_{y,i}(t), x^1_{y,i}(t)) \stackrel{def}{=} (k_{k_0,\bar{\imath},i}(t), F(\tilde{\imath}_t))$. In view of what we have said $x^0_{y,i}(t) \in \mathbb{R}$ and $x^1_{y,i}(t) \in L^2([-T, 0); \mathbb{R})$ and so $x_{y,i}(t) \in M^2$.*

Remark II.2.3 (On the structural state). The structural state, also called *Vinter-Kwong state*, is useful in a very general setting, for example when $k'(t)$ also depends on "the history" of k and on a measurable $f(t)$. In every problem the structural state appears in a different form but it is always a new couple (y^0, y^1) (obtained by original state and control variables using the so call "structural operator") that is solution of a simpler equation in M^2 (see Delfour [28] or Vinter and Kwong [64] for details). Here we have used the notations of Bensoussan, Da Prato, Delfour, Mitter ([13] page. 234). From now on (in this section) we will use the structural state to describe the state of the system.
■

Theorem II.2.4. *Assume that $\bar{\imath} \in L^2([-T, 0); \mathbb{R}^+)$, $i \in L^2_{\mathrm{loc}}([0, +\infty); \mathbb{R}^+)$, $k_0 \in \mathbb{R}$ $y = (k_0, F(\bar{\imath}))$, then, for every $T > 0$, the structural state $x_{y,i}(t) = (x^0_{y,i}(t), x^1_{y,i}(t)) = (k_{k_0,\bar{\imath},i}(t), F(\tilde{\imath}_t))$ is the unique solution in*

$$\Pi \stackrel{def}{=} \left\{ f \in C(0, T; M^2) \ : \ \frac{d}{\mathrm{d}t} j^* f \in L^2(0, T; D(A)') \right\} \tag{22}$$

to the equation:

$$\frac{d}{\mathrm{d}t} j^* x(t) = A^* x(t) + B^* i(t), \quad t > 0, \qquad x(0) = y = (k_0, F(\bar{\imath})) \tag{23}$$

where j^*, A^* and B^* are the dual maps of the continuous linear operators[13] *$j \colon D(A) \hookrightarrow M^2$, $A \colon D(A) \to M^2$, $B \colon D(A) \to \mathbb{R}$.*

[12]See [13] and [64]. There are also alternative ways of defining the state; they can be found in [13] and [47].

[13]Here j is simply the embedding, $D(A)$ is equipped with the graph norm and $D(A)'$ is the topological dual of $D(A)$.

Proof. This is part of a more general theory. The proof can be found in Bensoussan, Da Prato, Delfour, Mitter ([13] Theorem **5.1** page. 258) □

Remark II.2.5 (On the adjoint of the operators A and B). A^* is the adjoint of the linear operator A and so it is linear and continuous from M^2 to $D(A)' = \mathcal{L}(D(A), \mathbb{R})$. The explicit expression of $A^*(\psi^0, \psi^1)$ for the couples in which ψ^1 is differentiable is

$$A^*(\psi^0, \psi^1)[(\varphi^0, \varphi^1)] = \psi^1(0)\varphi^1(0) - \psi^1(-T)\varphi^1(-T) - \int_{-T}^{0} \frac{d}{\mathrm{d}s}\psi^1(s)\varphi^1(s)\mathrm{d}s$$

for all $(\varphi^0, \varphi^1) \in D(A)$. Endowing $D(A)$ with the graph norm we get that A^* is continuous and can be extended on all M^2 by density. The expression for B^* is simpler: $B^* \colon \mathbb{R} \to D(A)'$ is defined as $B^* i = i(\delta_0 - \delta_{-T})$ where δ_0 and δ_{-T} are the Dirac deltas in 0 and $-T$ respectively and they are elements of $D(A)'$. Note that the treatment of the our optimal control problem would be easier if the operator A^* would generate an analytic semigroup and if B^* would be bounded. This is not the case so the problem is more difficult since the known infinite dimensional theory (see e.g. [5]) cannot be applied. ■

Remark II.2.6 (Another choice of the state variables). The state we used is not the only one introduced in the literature to give an infinite dimensional description of Delay Differential Equations. It is, for example, also possible to use an extended state $\tilde{x} \overset{def}{=} (k(t), k_t, i_t)$ in $M^2 \times L^2$. The space is bigger but the state is more intuitive. See Ichikawa [47], and Bensoussan, Da Prato, Delfour, Mitter ([13] chapter 4) for details. ■

We want to formulate an optimal control problem in infinite dimensions that, thanks to results of the previous section, "contains" the *problem* $\boldsymbol{P}$. To do this we need first the following result that extends the existence and uniqueness results of the previous Theorem II.2.4.

Theorem II.2.7. *The equation $\frac{d}{\mathrm{d}t} j^* x(t) = A^* x(t) + B^* i(t)$ for $t > 0$ with initial condition $x(0) = y$ for $y \in M^2$, $i \in L^2_{\mathrm{loc}}([0, +\infty); \mathbb{R})$ has a unique solution in Π (defined in (22)).*

Proof. The proof can be found in Bensoussan, Da Prato, Delfour, Mitter ([13] Theorem **5.1** page. 258). □

Now we can formulate our optimal control problem in infinite dimensions. The state space is M^2, the control space is $\mathbb{R}$, the time is continuous. The state equation in M^2 is given by

$$\frac{d}{\mathrm{d}t} j^* x(t) = A^* x(t) + B^* i(t), \quad t > 0, \qquad x(0) = y \tag{24}$$

for $y \in M^2$, $i \in L^2_{\mathrm{loc}}([0, +\infty); \mathbb{R})$. Thanks to Theorem II.2.7 it has a unique solution $x_{y,i}(t)$ in Π (it extends the structural state defined in Definition II.2.2 only for positive initial data and control), so $t \mapsto x^0_{y,i}(t)$ is continuous and it makes sense to consider the set of controls

$$\mathcal{I}_y \overset{def}{=} \{ i \in L^2_{\mathrm{loc}}([0, +\infty); \mathbb{R}^+) \ : \ i(t) \in [0, a x^0_{y,i}(t)] \ \text{ for a.e. } t \in \mathbb{R}^+ \}$$

The objective functional is $J_0(y; i(\cdot)) \overset{def}{=} \int_0^\infty e^{-\rho s} \frac{(a x^0_{y,i}(t) - i(t))^{1-\sigma}}{(1-\sigma)} \mathrm{d}s$. The value function is then $V_0(y) \overset{def}{=} \sup_{i \in \mathcal{I}_y} J_0(y; i(\cdot))$ if $\mathcal{I}_y \neq \emptyset$ and $V_0(y) \overset{def}{=} -\infty$ if $\mathcal{I}_y = \emptyset$.

Remark II.2.8 (Connection with the starting problem). If we have, for some $\bar{\imath}(\cdot) \in L^2([-T,0);\mathbb{R}^+)$, $y = \big(R(\bar{\imath}), F(\bar{\imath})\big)$, we find $\mathcal{I}_y = \mathcal{I}_{\bar{\imath}}$, $J_0(y;i) = J(\bar{\imath};i)$ and $V_0(y) = V(\bar{\imath})$ and the solution of the differential equation (24) is given by Theorem II.2.4. ∎

II.2.2 The *HJB* equation and its explicit solution

We now describe the Hamiltonians of the system. First of all we introduce the *current value* Hamiltonian: it will be defined on a subset E of $M^2 \times M^2 \times \mathbb{R}$ (the product of state space, co-state space and control space) given by[14]

$$E \stackrel{def}{=} \{(x,P,i) \in M^2 \times M^2 \times \mathbb{R} \ : x^0 > 0,\ i \in [0, ax^0],\ P \in D(A)\}$$

and its form is the following: (note that $\langle i, BP\rangle_{\mathbb{R}}$ is simply the product on $\mathbb{R}$):

$$\mathcal{H}_{CV}(x,P,i) \stackrel{def}{=} \langle x, AP\rangle_{M^2} + \langle i, BP\rangle_{\mathbb{R}} + \frac{(ax^0 - i)^{1-\sigma}}{(1-\sigma)}$$

When $\sigma > 1$ the above is not defined in the points in which $ax^0 = i$. In such points we set then $\mathcal{H}_{CV} = -\infty$. Note that in this way we take $\mathcal{H}_{CV}$ with values in $\overline{\mathbb{R}}$.

We can now define the maximum value Hamiltonian (that we will simply call Hamiltonian) of the system. Let $G \stackrel{def}{=} \{(x,P) \in M^2 \times M^2 \ : \ x^0 > 0,\ P \in D(A)\}$. The Hamiltonian is given by:

$$\mathcal{H}\colon G \to \overline{\mathbb{R}}, \qquad \mathcal{H}\colon (x,P) \mapsto \sup_{i\in[0,ax^0]} \mathcal{H}_{CV}(x,P,i)$$

The *HJB* equation for the value function V is $\rho V(x) - \mathcal{H}(x, DV(x)) = 0$ i.e.

$$\rho V(x) - \sup_{i\in[0,ax^0]} \left\{ \langle x, ADV(x)\rangle_{M^2} + \langle i, BDV(x)\rangle_{\mathbb{R}} + \frac{(ax^0 - i)^{1-\sigma}}{(1-\sigma)} \right\} = 0 \qquad (25)$$

As we have already noted the HJB equation (25) cannot be treated with the results of the existing literature. This is due to the presence of the state/control constraint (i.e. the investments that are possible at time $t \geq 0$ depend on k at the same time t: $i(t) \in [0, ak(t)]$), to the unboundedness of the control operator (i.e. the term $BDV(x^0,x^1)$) and the non-analyticity of the semigroup generated by the operator A^*. To overcome these difficulties we have to give a suitable definition of solution. We will require the following:

(i) the solution of the HJB equation (25) is defined on a open set Ω of M^2 and is C^1 on such a set;

(ii) on a subset $\Omega_1 \subseteq \Omega$, closed in Ω where the trajectories interesting from the economic point of view must remain, the solution has differential in $D(A)$ (on $D(A)$ also the Dirac δ and so B make sense);

(iii) the solution satisfies (25) on Ω_1.

Definition II.2.9. *Let Ω be an open set of M^2 and $\Omega_1 \subseteq \Omega$ a subset closed in Ω. An application $g \in C^1(\Omega;\mathbb{R})$ is a solution of the HJB equation (25) on Ω_1 if $\forall x \in \Omega_1$*

$$(x, Dg(x)) \in G, \qquad \text{and} \qquad \rho g(x) - \mathcal{H}\big(x, Dg(x)\big) = 0.$$

[14] recall that an element $x \in M^2$ is done by two components: x^0 and x^1, so $x = (x^0, x^1)$.

Remark II.2.10 (On the form of the Hamiltonian). If $P \in D(A)$ and $(BP)^{-1/\sigma} \in (0, ax^0]$, by elementary arguments, the function $\mathcal{H}_{CV}(x, P, \cdot)\colon [0, ax^0] \to \mathbb{R}$ admits a unique maximum point given by

$$i^{MAX} = ax^0 - (BP)^{-1/\sigma} \ \in [0, ax^0) \tag{26}$$

and then we can write the Hamiltonian in a simplified form:

$$\mathcal{H}((x^0, x^1), P) = \langle (x^0, x^1), AP \rangle_{M^2} + ax^0 BP + \frac{\sigma}{1-\sigma}(BP)^{\frac{\sigma-1}{\sigma}} \tag{27}$$

We will use (26) to write the solution of the *problem (***P***)* in closed-loop form. ■

We can now give an explicit solution of the HJB equation. First define, for $x \in M^2$, the quantity

$$\Gamma_0(x) \stackrel{def}{=} x^0 + \int_{-T}^{0} e^{\xi s} x^1(s) \mathrm{d}s \tag{28}$$

and then define the set $X \subset M^2$ (which will be the Ω of the Definition II.2.9) as

$$X \stackrel{def}{=} \left\{ x = (x^0, x^1) \in M^2 \ : \ x^0 > 0, \ \Gamma_0(x) > 0 \right\}$$

Finally $\left(\text{calling } \alpha = \dfrac{\rho - \xi(1-\sigma)}{\sigma\xi}\right)$ we define the set $Y \subseteq X$ (which will be the Ω_1 of the Definition II.2.9) as

$$Y \stackrel{def}{=} \left\{ x = (x^0, x^1) \in X \ : \ \Gamma_0(x) \leq \frac{1}{\alpha} x^0 \right\} \tag{29}$$

It is easy to see that X is an open subset of M^2 while Y is closed in X. We are now ready to present an explicit solution of the HJB equation (25) which, in next subsection, will be proved to be the value function under an additional assumption.

Proposition II.2.11. *Under the Hypotheses II.1.5 and II.1.9 the function*

$$v\colon X \to \mathbb{R}, \qquad v(x) \stackrel{def}{=} \nu [\Gamma_0(x)]^{1-\sigma} \tag{30}$$

$$\text{with} \qquad \nu = \left(\frac{\rho - \xi(1-\sigma)}{\sigma} \cdot \frac{a}{\xi} \right)^{-\sigma} \frac{1}{(1-\sigma)} \cdot \frac{a}{\xi}$$

is differentiable in all $x \in X$ and is a solution of the HJB equation (25) on Y in the sense of Definition II.2.9.

The reason why we expect that the value function (and so the solution of the HJB equation) is of the form of v above comes from the following considerations:

- that the value function must be $(1-\sigma)$ homogeneous in the state variable (the "capital" in some sense) due to the structure of the problem;
- that the term $\Gamma_0(x)$ inside the power $(1-\sigma)$ must be connected linearly with the amount of capital.

Both the above arguments rely on the similarities between the model studied in this paper and the standard AK model with zero depreciation rate of capital (where there are no corner solutions). See Appendix A.

What is more difficult to guess and to interpret is the form and the meaning of the quantity $\Gamma_0(x)$: in Subsection III.1.2 we will give a possible interpretation of $\Gamma_0(x)$ as "equivalent capital".

Moreover the choice of Y comes from the need of avoiding corner solutions. Indeed we know that in the standard AK model, in presence of corner solutions, the value function is different (see Appendix A). The same would happen here. To prove that v is the value function in next subsection we will need to prove that the closed loop strategies coming from v are admissible and this will be true assuming another restriction on the parameters of the model. This is a key point to solve the theoretical problem of [18] mentioned at point (II) of the Introduction and in Section III.2.

II.2.3 Closed loop in infinite dimensions

We begin with some definitions.

Definition II.2.12. *Given $y \in M^2$ we will call $\phi \in C(M^2)$ an* admissible closed loop strategy related to the initial point y *if the equation.*

$$\frac{d}{dt} j^* x(t) = A^* x(t) + B^*(\phi(x(t))), \quad t > 0, \qquad x(0) = y$$

has an unique solution $x_\phi(t)$ in Π and $\phi(x_\phi(\cdot)) \in \mathcal{I}_y$. We will indicate the set of admissible closed loop strategies related to y with AFS_y.

Definition II.2.13. *Given $y \in M^2$ we will call ϕ an* optimal closed loop strategy related to y *if it is in AFS_y and*

$$V_0(y) = \int_0^{+\infty} e^{-\rho t} \frac{\left(a x_\phi^0(t) - \phi(x_\phi(t))\right)^{1-\sigma}}{(1-\sigma)} \mathrm{d}t.$$

We will indicate the set of optimal closed loop strategies related to y with OFS_y.

We have a solution v of the HJB equation (25) only in a part of the state space (the set Y). To this solution is naturally associated a closed loop formula given by the maximum point of the Hamiltonian (equation (26) where P is the gradient of v). The function v is the value function and the associated closed loop strategies are optimal if and only if they remain in Y[15]. To guarantee this we have to impose another condition on the parameters of the problem. As we will remark in Section III.2 such a hypothesis is reasonable from an economic point of view as it substantially requires to rule out corner solutions.

Hypothesis II.2.14. $\dfrac{\rho - \xi(1-\sigma)}{\sigma} \le a.$

From now on we will assume that Hypotheses II.1.5, II.1.9, II.2.14 hold true.

[15]Indeed to get these results one also need that v verifies a limit condition at infinity (which may be regarded as a kind of transversality condition) but this is ensured by the explicit form of v and by Hypothesis II.1.9, see Remark II.3.7 on this.

Theorem II.2.15. *Given $\bar{\iota} \in L^2([-T,0);\mathbb{R}^+)$ with and $\bar{\iota} \not\equiv 0$, if we call $y = (R(\bar{\iota}), F(\bar{\iota}))$, then the application*

$$\phi\colon M^2 \to \mathbb{R}, \qquad \phi(x) \stackrel{def}{=} ax^0 - \left(\frac{\rho - \xi(1-\sigma)}{\sigma} \cdot \frac{a}{\xi}\right) \Gamma_0(x) \tag{31}$$

is in OFS_y.

From the proof of Theorem II.2.4 we get the explicit expression for the value function V_0:

Corollary II.2.16. *Given any $\bar{\iota} \in L^2([-T,0);\mathbb{R}^+)$ and setting $y = (R(\bar{\iota}), F(\bar{\iota}))$ we have that $V(\bar{\iota}) = V_0(y) = v(y)$ where v is given in Proposition II.2.11.*

From Theorem II.2.15 it follows that the optimal control $i^*\colon \mathbb{R}^+ \to \mathbb{R}$ is in $W^{1,2}_{\text{loc}}(0,+\infty;\mathbb{R}^+)$. Moreover for every $\theta \in \mathbb{N}$ we have $i^*|_{(\theta T,+\infty)}(t) \in W^{\theta,2}_{\text{loc}}(\theta T,+\infty;\mathbb{R}^+)$.

II.3 Back to problem P

We now use the results we found in the infinite dimensional setting to solve the original optimal control *problem **P***.

II.3.1 The explicit form of the value function

First of all observe that, given any initial datum $\bar{\iota}(\cdot) \in L^2([-T,0);\mathbb{R}^+)$ and writing $y = (R(\bar{\iota}), F(\bar{\iota}))$, the quantity $\Gamma_0(y)$ defined in (28) becomes

$$\Gamma(\bar{\iota}(\cdot)) \stackrel{def}{=} \Gamma_0\left(R(\bar{\iota}), F(\bar{\iota})\right) = \int_{-T}^{0} \left(1 - e^{-\xi(T+s)}\right) \bar{\iota}(s)\mathrm{d}s = k(0) - \int_{-T}^{0} e^{-\xi(T+s)}\, \bar{\iota}(s)\mathrm{d}s \tag{32}$$

A comment on the meaning of such a quantity is given in Subsection III.1.2. Now, as a consequence of Corollary II.2.16 we have:

Proposition II.3.1. *Under Hypotheses II.1.5, II.1.9, II.2.14, the explicit expression for the value function V related to* problem **P** *is*

$$V(\bar{\iota}(\cdot)) = \nu[\Gamma(\bar{\iota}(\cdot))]^{1-\sigma} = \nu\left(k(0) - \int_{-T}^{0} e^{-\xi(T+s)}\, \bar{\iota}(s)\mathrm{d}s\right)^{1-\sigma}$$

$$\text{where} \qquad \nu = \left(\frac{\rho - \xi(1-\sigma)}{\sigma} \cdot \frac{a}{\xi}\right)^{-\sigma} \frac{1}{(1-\sigma)} \cdot \frac{a}{\xi}$$

II.3.2 Closed loop optimal strategies for problem P

We now use the closed loop in infinite dimension to write explicitly the closed loop formula and the closed loop equation for problem **P**. First of all we recall that, given $t \geq 0$, $\bar{\iota}(\cdot) \in L^2([-T,0);\mathbb{R}^+)$ and $i(\cdot) \in \mathcal{I}_{\bar{\iota}}$ the "history" $\tilde{\iota}_t(\cdot) \in L^2([-T,0);\mathbb{R}^+)$ is defined as in (17) and we can write

$$\Gamma(\tilde{\iota}_t(\cdot)) = \int_{-T}^{0} \left(1 - e^{-\xi(T+s)}\right) \tilde{\iota}_t(s)\mathrm{d}s = \int_{t-T}^{t} \left(1 - e^{-\xi(T+s)}\right) \tilde{\iota}(s)\mathrm{d}s. \tag{33}$$

We use the * for the optimal investment (and capital) so $\tilde{\imath}^*_t(\cdot) \in L^2([-T,0);\mathbb{R}^+)$ is the history of the optimal investment.

Next we apply Theorem II.2.15 (in particular (31) and (65)) and (10) getting the following result whose proof is immediate.

Proposition II.3.2. *Under Hypotheses II.1.5, II.1.9, II.2.14, given an initial datum $\bar{\imath}(\cdot) \in L^2([-T,0);\mathbb{R}^+)$ in equation (9) the optimal investment strategy $i^*(\cdot)$ and the related capital stock trajectory $k^*(\cdot)$ satisfy for all $t \geq 0$:*

$$i^*(t) \quad = \quad ak^*(t) - \left(\frac{\rho - \xi(1-\sigma)}{\sigma} \cdot \frac{a}{\xi}\right)\Gamma(\tilde{\imath}^*_t(\cdot)). \tag{34}$$

so calling $c^*(t) = ak^*(t) - i^*(t)$ we have

$$c^*(t) \quad = \quad \left(\frac{\rho - \xi(1-\sigma)}{\sigma} \cdot \frac{a}{\xi}\right)\Gamma(\tilde{\imath}^*_t(\cdot)). \tag{35}$$

We now want to find a more useful closed loop formula.

Lemma II.3.3. *Under Hypotheses II.1.5, II.1.9, II.2.14, given an initial datum $\bar{\imath}(\cdot) \in L^2([-T,0);\mathbb{R}^+)$ in equation (9), there exist constants $\Lambda = \Lambda(\bar{\imath}(\cdot)) > 0$, $g \in \mathbb{R}$ (g independent of $\bar{\imath}(\cdot)$) such that the optimal investment strategy $i^*(\cdot)$ for* problem **P** *and the related capital stock trajectory $k^*(\cdot)$ satisfy for all $t \geq 0$:*

$$ak^*(t) - i^*(t) = \Lambda e^{gt} \tag{36}$$

(i.e. the optimal consumption path is of exponential type). Moreover

$$g = \frac{\xi - \rho}{\sigma} \in [\xi - a, \xi) \tag{37}$$

and

$$\Lambda = \left(\frac{\rho - \xi(1-\sigma)}{\sigma} \cdot \frac{a}{\xi}\right)\Gamma(\bar{\imath}(\cdot)) \tag{38}$$

Using the above Lemma II.3.3 we can now write a more useful closed loop formula with the associated closed loop equation.

Theorem II.3.4. *Under Hypotheses II.1.5, II.1.9, II.2.14, given an initial datum $\bar{\imath}(\cdot) \in L^2([-T,0);\mathbb{R}^+)$ in equation (9), the optimal investment strategy for* problem **P** *$i^*(\cdot)$ is connected with the related state trajectory $k^*(\cdot)$ by the following closed loop formula for all $t \geq 0$:*

$$i^*(t) = ak^*(t) - \Lambda e^{gt} \tag{39}$$

where $\Lambda = \Lambda(\bar{\imath}(\cdot))$ is given in (38).

Moreover the optimal investment strategy $i^*(\cdot)$ is the unique solution in $W^{1,2}_{loc}([0,+\infty);\mathbb{R})$ of the following integral equation:

$$\tilde{\imath}^*(t) = a\int_{t-T}^{t} \tilde{\imath}^*(s)\mathrm{d}s - \Lambda e^{gt} \quad t \geq 0, \qquad \tilde{\imath}^*(s) = \bar{\imath}(s), \text{ for } s \in [-T,0). \tag{40}$$

Finally the optimal capital stock trajectory $k^*(\cdot)$ is the only solution in $W^{1,2}_{loc}(0,+\infty;\mathbb{R}^+)$ of the following integral equation:

$$k^*(t) = \int_{(t-T)\wedge 0}^{0} \bar{\imath}(s)ds + \int_{(t-T)\vee 0}^{t} \left[ak(s) - \Lambda e^{gs}\right]ds, \qquad t \geq 0. \tag{41}$$

II.3.3 Growth rates and asymptotic behavior

We have seen that along the optimal path the consumption is exponential. Nevertheless the optimal investment and the capital stock have a more irregular behavior that depends on initial data. We can anyway describe the asymptotic behavior of them. Calling $c^*(t) = ak^*(t) - i^*(t) = \Lambda e^{gt}$ the optimal consumption path we have the following.

Proposition II.3.5. *Under Hypotheses II.1.5, II.1.9, II.2.14, given an initial datum $\bar{\iota}(\cdot) \in L^2([-T,0);\mathbb{R}^+)$ in equation (9), defining, for $t \geq 0$, the optimal detrended paths as:*

$$k_g(t) \stackrel{def}{=} e^{-gt}k^*(t), \qquad i_g(t) \stackrel{def}{=} e^{-gt}i^*(t), \qquad c_g(t) \stackrel{def}{=} e^{-gt}c^*(t)$$

we have that the optimal detrended consumption path $c_g(t) = \big(ak_g(t) - i_g(t)\big)$ is constant and equal to Λ. Moreover there exist positive constants i_B and k_B such that

$$\lim_{t\to+\infty} i_g(t) = i_B \qquad and \qquad \lim_{t\to+\infty} k_g(t) = k_B.$$

We have, when $g \neq 0$, $i_B = \frac{\Lambda}{\frac{a}{g}(1-e^{-gT})-1} > 0$ and $k_B = \frac{1-e^{-gT}}{g} \cdot i_B = \frac{\Lambda}{a - \frac{g}{1-e^{-gT}}} > 0$, while, when $g = 0$, $i_B = \frac{\Lambda}{aT-1} > 0$ and $k_B = T \cdot i_B = \frac{\Lambda T}{aT-1} > 0$.

Remark II.3.6 (On the costate variable in our setting). In the DP approach the costate is (under suitable assumptions) the gradient of the value function along optimal trajectories. In our work we treat the problem in an infinite dimensional setting so the costate is a function of t with infinite dimensional values, more precisely its value at each time t is an element of M^2 that we call $\lambda_0(t)$. It has two parts: $\lambda_0^0(t)$ which is a real number and $\lambda_0^1(t)$ which is a function for each t: the history of $\lambda_0^0(t)$ as introduced in Subsection II.2.1.

Which is the relation between such a costate and the "standard" costate introduced in [18], equations (13) and (14)) that we call it simply λ and is a real valued function?

From the definition given in [18] and from the results we have proven it can be seen (see also Proposition 11 and equation (27) of [18]) that along optimal trajectories

$$\lambda(t) = e^{-\xi t} \cdot \frac{ac_g(t)^{-\sigma}}{\sigma g + \rho}$$

On the other side $\lambda_0(t) = \nabla V_0(R(i_t^*), F(i_t^*))$ and from the explicit form of V_0 given in (30) we find that (see the proof of Proposition II.2.11) $\lambda_0^0(t) = \lambda(t)$, so λ_0^1 is the history of λ. ∎

Remark II.3.7 (On the transversality condition). In the necessary and sufficient conditions proved in [18] the following transversality condition arises $\lim_{t\to\infty} \lambda(t)k(t) = 0$. This condition is verified along optimal trajectories we have found. Indeed, as observed in Proposition II.3.5 and Remark II.3.6, $\lambda(t) = O(e^{-\xi t})$ and $k(t) = O(e^{gt})$.

The transversality conditions play a key role when one approaches an infinite horizon optimal control problem with the MP: they help to select the right trajectory in the state - costate diagram. In our DP approach, apparently, they do not play any active role. This not true: in general the HJB equation does not have a unique solution. When a solution (like our v) of the HJB equation is given one wants to prove a result like Theorem II.2.15 (and the consequent Corollary II.2.16) stating that v is the value function and that the closed loop strategies associated to v (taking the maximum point of the Hamiltonian) are optimal. To accomplish this proof one has to add a "boundary condition" which, in the infinite horizon case, is a limit condition for $t \to +\infty$. In our case it is ($x(t)$ is the closed loop trajectory)

$$\lim_{t \to +\infty} e^{-\rho t} v(x(t)) = 0 \tag{42}$$

as it results from the proof of Theorem II.2.15, formula (71). Without this condition optimality of closed loop strategies cannot be proved and indeed v may not be the value function. We may say that (42) is a kind of transversality condition (see e.g. [58] Proposition 2 and 3 for a study of it in the discrete time case): since it is automatically verified for the explicit solution v of the HJB equation we do not need to impose it, so in our case the role of the transversality conditions is hidden. When we have not explicit solutions, (42) has to be imposed (see Section IV.2 on this). On the connection between the value function and transversality conditions one can see e.g. [58] in the discrete time case and [14, 57] in the continuous time case. ∎

We now look at the existence of Balanced Growth Paths (BGP).

Definition II.3.8. *We will say that an optimal couple for* problem $\mathbf{P}$ (k^*, i^*) *is a* Balanced Growth Path *(BGP) if there exist $a_0, b_0 > 0$, and real numbers a_1, b_1 such that*

$$\tilde{\imath}^*(s) = a_0 e^{a_1 s} \quad for \ s \in [-T, +\infty) \qquad k^*(s) = b_0 e^{b_1 s} for \ s \in [0, +\infty).$$

Proposition II.3.9. *Under Hypotheses II.1.5, II.1.9, II.2.14, the only BGPs of the model are the trajectories of the form*

$$\tilde{\imath}^*(s) = a_0 e^{gs}, \quad for \ s \in [-T, +\infty); \qquad k^*(s) = b_0 e^{gs}, for \ s \in [0, +\infty); \tag{43}$$

where $b_0 = k(0)$ and a_0 and b_0 are connected by the relation:

$$b_0 = a_0 \int_{-T}^{0} e^{gs} \mathrm{d}s = \frac{a_0}{g} \left(1 - e^{-gT}\right) \tag{44}$$

Proof. We give only a sketch of the proof avoiding standard calculations.

We know that the optimal discounted investment follows the DDE (80). If we substitute inside such a relation the generic solution $a_0 e^{(a_1 - g)s}$ we find that $a_1 = g$. So the only possible BGPs are the ones described in (43).

We substitute then the solution $a_0 e^{gt}$ in (40) and we find that the solution of the form (43) are optimal. □

Note that in [18], Subsections 4.2, 4.3 it is proved that detrended consumption is constant over time and that balanced growth path are of the form given in Proposition II.3.9. In particular equation (44) is the analogous of equation (19) in [18]. Here we calculate explicitly the constant Λ.

Part III
Application to the Vintage Capital Model

We now discuss the results of Sections II.1, II.2 and II.3 comparing them with the ones of [18], emphasizing the novelties and their economic implications. We proceed by discussing in detail the three methodological points $(I)-(II)-(III)$ raised in the Introduction. We devote a section to each one of them.

III.1 The explicit form of the value function and its consequences in the study of the optimal paths

In [18] it is shown that the detrended co-state path $\hat{\lambda}(t) := \lambda(t)e^{\xi t}$ and the optimal detrended consumption path $c_g(\cdot) := e^{-gt}c^*(t)$ are both constant (depending only on the initial data) but none is said about the explicit expression of the constants. Moreover the value function and its relation with the costate are not considered.

Here the value function is explicitly given (Proposition II.3.1) and using its closed form, we explicitly calculate such constants[16] i.e.

$$\hat{\lambda}(t) \equiv \frac{a}{\xi}\Lambda^{-\sigma} \qquad \text{and} \qquad c_g(t) \equiv \Lambda$$

where Λ is given by (38).

Moreover in [18] it is shown that the optimal detrended investment path $i_g(t) = e^{-gt}i^*(t)$ and the optimal detrended capital path $k_g(t) = e^{-gt}k^*(t)$ converge asymptotically to a constant (respectively i_B and k_B) but nothing is said about their value.

Here, using (38) and the closed loop equations (40)-(41) for the optimal investment and capital trajectories, we determine the explicit form of the constants i_B and k_B, given in Proposition II.3.5. This way the dependence of the long run equilibrium on the initial datum is explicitly calculated and a comparative statics can be easily performed.

In the following two subsections we discuss some implications of such explicit formulae.

III.1.1 The study of short run fluctuations.

The closed loop equations (40)-(41) for the optimal investment and capital cannot be explicitly solved (apart from very special cases) but they turn out to be useful in studying the qualitative properties of $i_g(\cdot)$ and $k_g(\cdot)$ and of their short run growth rates such as the presence of oscillations and of short run deviations between saving rates and growth rates (see [18], Subsection 5.1).

To see this we first make some remarks on the integral equation (40). From Proposition II.3.5 and its proof we know that the optimal investment $i^*(\cdot)$ (that solves the DDE (40)) can be written as

$$i^*(t) = i_B e^{gt} + \sum_{j=1}^{+\infty} e^{Re\lambda_j t}\left[i_j^1 \cos(Im\lambda_j t) + i_j^2 \sin(Im\lambda_j t)\right]$$

where the λ_j is the sequence described in Proposition II.1.8 - (c) giving the complex and non real roots of the characteristic equation ordered with decreasing real part. We have $Re(\lambda_j) < \xi - a \leq g$ for each j and all λ_j's are simple roots. Moreover for each compact interval I the number of λ_j's with real part in I is finite. Finally i_B is known from

[16] To calculate the co-state $\hat{\lambda}(t)$ one has to observe that it is the gradient of the value function as in Remark II.3.7.

Proposition II.3.5 and, with the notation used in the proof of Proposition II.3.5, for $j \in \mathbb{N}$

$$i_j^1 = 2Re\left(\frac{\alpha_j \Lambda g}{g-\lambda_j} + \alpha_j a(\Gamma_j - \Lambda)\right), \qquad i_j^2 = -2Im\left(\frac{\alpha_j \Lambda g}{g-\lambda_j} + \alpha_j a(\Gamma_j - \Lambda)\right),$$

can be calculated from the initial datum $\bar{\iota}$, the characteristic roots $\{\lambda_j\}_{j\in\mathbb{N}}$ and the coefficients $\{\alpha_j\}_{j\in\mathbb{N}}$. The λ_j's and the α_j's can be calculated at least in numerical way (see for example [33], chapters IV and VI).

So we have a *main part* given by $i_B e^{gt}$, that determines the asymptotic behavior, and a *rest*, that gives the short run fluctuations, given by the series. To get a *first order approximation* of the fluctuations in the long run it is enough to take only the term with $Re(\lambda_1)$.

When the initial datum $\bar{\iota}$ is on the steady state $\bar{\iota}_0$ no fluctuation arise so $i_j^1 = i_j^2 = 0$ for each j. Otherwise the size of the coefficients i_j^1, i_j^2 will depend on the deviation from the steady state, $\bar{\iota} - \bar{\iota}_0$, through the terms Λ and Γ_j.

Using equation (41) (or (10)) we can moreover approximate the short run fluctuations of the optimal capital

$$k^*(t) = k_B e^{gt} + \sum_{j=1}^{+\infty} e^{Re\lambda_j t}\left[k_j^1 \cos(Im\lambda_j t) + k_j^2 \sin(Im\lambda_j t)\right].$$

The term k_B is known while k_j^1, k_j^2 (as i_j^1, i_j^2) can be calculated from $\bar{\iota}$, λ_j and α_j. Using the above formulae we can also study the behavior of the output and investment rate ($\frac{y^*(\cdot)'}{y^*(\cdot)} = \frac{k^*(\cdot)'}{k^*(\cdot)}$ and $\frac{i^*(\cdot)'}{i^*(\cdot)}$) in particular through the study of its first order approximation. Finally the above formulae can be a good basis for an empirical testing of the model.

III.1.2 The "equivalent capital" and the convergence to the standard AK model.

We compare the model treated in this paper with the standard one dimensional AK model with zero depreciation rate of capital. The value function is given by the formula

$$V(\bar{\iota}(\cdot)) = \nu[\Gamma(\bar{\iota}(\cdot))]^{1-\sigma} \tag{45}$$

$$\text{where} \qquad \Gamma(\bar{\iota}(\cdot)) \stackrel{def}{=} \int_{-T}^{0}\left(1 - e^{-\xi(T+s)}\right)\bar{\iota}(s)\mathrm{d}s = k(0) - \int_{-T}^{0} e^{-\xi(T+s)}\,\bar{\iota}(s)\mathrm{d}s \tag{46}$$

$$\text{and} \qquad \nu = \left(\frac{\rho - \xi(1-\sigma)}{\sigma}\cdot\frac{a}{\xi}\right)^{-\sigma}\frac{1}{(1-\sigma)}\cdot\frac{a}{\xi} \tag{47}$$

The quantity $\Gamma(\bar{\iota}(\cdot))$ is the initial amount of capital $k(0)$ minus a weighted integral of the initial investment profile $\bar{\iota}(\cdot)$. The weight for $\bar{\iota}(s)$, $s \in [-T, 0)$, is $e^{-\xi(T+s)}$, a term which is the discount, at rate $-\xi$, in the period from 0 to the corresponding scrapping time $T + s$.

We try to interpret it in a simplified case. Take a discrete time model (or a continuous time model with possibly atomic investments and discontinuous capital) where the past investments are all concentrated at $t = 0$. Then $k(0) = \bar{\iota}(0)$ and (46) would be simply $\Gamma(\bar{\iota}(\cdot)) = \left(1 - e^{-\xi T}\right)k(0)$. Observe that in this case $\frac{a}{\xi}\Gamma(\bar{\iota}(\cdot)) = a\int_0^T e^{-\xi r}\,k(0)\mathrm{d}r$. The right hand side may be interpreted as the present value (at time $t = 0$ of the production flow generated by the capital $k(0)$ discounted at rate ξ[17]. Such present value will go to $k(0)$ as $T \to +\infty$. In this context when the capital is infinitely durable its present value is set equal to itself, so the quantity $\frac{a}{\xi}\Gamma$ may be seen as the initial equivalent amount of infinitely durable capital. For $T < +\infty$ such a quantity is strictly less than the capital (except for the degenerate case $\bar{\imath} \equiv 0$). When $T \to +\infty$, such an amount tends to the initial capital $k(0)$.

If we take $t > 0$, the quantity $\Gamma(\tilde{\imath}_t(\cdot))$ (recall that $\tilde{\imath}_t(\cdot)$ is the history of investments at time t, see (17)) is the "equivalent capital" at time t. The feedback formula (35) shows that the consumption is chosen by taking a constant share of $\Gamma(\tilde{\imath}_t(\cdot))$. Moreover Lemma II.3.3, together with formula (35) shows that $\Gamma(\tilde{\imath}_t(\cdot))$ grows at constant rate g.

In view of this we may say that the key variable of the model is the "equivalent capital" which has a constant growth rate g due to the AK nature of the model. The consumption path is simply a constant share of the "equivalent capital" while the investment fluctuates to keep it growing at such a constant rate. So when $T < +\infty$ the "equivalent capital" plays the role of the capital in the standard AK model.

The standard one dimensional AK model with zero depreciation rate of capital can be seen as the limit case of the model treated here when $T = +\infty$. Indeed in such standard AK model the value function V_0 depends on $k(0)$ and is

$$V_0(k(0)) = \nu_0[k(0)]^{1-\sigma} \tag{48}$$

$$\text{where} \qquad \nu_0 = \left(\frac{\rho - a(1-\sigma)}{\sigma}\right)^{-\sigma} \frac{1}{(1-\sigma)} \tag{49}$$

Since $\xi \to a$ as $T \to +\infty$ then we clearly have, for every initial datum $\bar{\imath}(\cdot)$, $\lim_{T\to+\infty} V(\bar{\imath}(\cdot)) = V_0(k(0))$. Similarly, as $T \to +\infty$ we have (calling g_{AK} the growth rate of the optimal paths in the standard AK model with zero depreciation rate of capital),

$$\Lambda(\bar{\imath}(\cdot)) \to \frac{\rho - a(1-\sigma)}{\sigma}k(0), \qquad g = \frac{\xi - \rho}{\sigma} \to \frac{a - \rho}{\sigma} = g_{AK},$$

so the optimal consumption path converges uniformly on compact sets to the one of the standard AK model. Consequently the closed loop formula (39) converges and passing to the limit in equations (40)-(41) we get the same convergence for the optimal investment and capital paths.

It is worth to remark that we are comparing the model treated here with an AK model with zero depreciation rate of capital because is not easy task to connect a vintage capital model to a model with constant and positive depreciation rate of capital.

III.2 The problem of admissibility of the candidate optimal paths

In [18] it is not proved that the candidate optimal trajectory of capital and investment is admissible (see the discussion in Subsection 4.3, p. 60 of [18]) leaving an unsolved gap in the analysis of the model.

Here we can prove that such a candidate optimal trajectory is admissible. Indeed, using the closed loop form given by (34) and the Hypothesis II.2.14 (i.e. $\frac{(\rho-\xi(1-\sigma))}{\sigma} \le a$)

[17] The reason why the discount rate is ξ is the fact that it is the maximum rate of reproduction of capital.

we see, in the proof of Theorem II.2.15, that the optimal investment $i^*(t)$ remains in the interval $(0, ak(t))$ for all $t \geq 0$.

The emergence of this theoretical problem comes from the strategy used in [18] (and in much of the literature on continuous time endogenous growth models) to attack the problem: first consider the problem without taking account of the "difficult" state-control constraint (3) focusing on interior solutions ([18], p.54) and then check afterwards if the optimal paths for the simplified problem also satisfy (3). Of course this may not be true, or, even if it is true as in this case, it may be very hard to check.

In our approach we always take account of (3) and then it cannot happen that we find a non-admissible candidate optimal trajectory. We also focus on interior solutions but we provide an if and only if condition on parameters (Hypothesis II.2.14) for the existence of interior solutions. This can be done explicitly since we know the explicit form of the value function.

To understand better this point one can consider the standard AK model with zero depreciation rate of capital where one adds the constraint $i(t) \geq 0$ for $t \geq 0$. In this case interior solutions arise if and only if $g_{AK} = \sigma^{-1}(a - \rho) > 0$ (i.e. the economy grows at a strictly positive rate on the optimal paths). If this is not the case then the optimal investment path is constantly 0, so also the capital and the consumption are constant.

In the model of this paper interior solutions arise for every nonzero initial datum $\bar{\iota}(\cdot)$ if and only if $g \geq \xi - a$[18] which is exactly (Hypothesis II.2.14) and reduces to $g_{AK} \geq 0$[19] when $T \to +\infty$. Differently from the standard AK model here when Hypothesis II.2.14 does not hold we do not have constant optimal paths: this depends on the shape of the initial investments profile $\bar{\iota}$.

III.3 The assumptions on the parameters

In this paper we work under more general and sharper assumptions on the parameters that include cases which are interesting from the economic point of view. Indeed the hypotheses in [18] are:

$$(H1)\ aT > 1\,; \qquad (H2)\ \rho > (1-\sigma)a\,; \qquad (H3)\ \frac{\rho - \xi}{\sigma} < 0\,.$$

The first (H1) is the same of Hypothesis II.1.5.

The second (H2) is strictly stronger than Hypothesis II.1.9 because $\xi < a$. This means that we can prove the existence and characterize the form of the optimal trajectories in a more general case. Moreover, in the standard AK model with zero depreciation rate of capital, (H2) is an if and only if conditions for the finiteness of the value function and the existence of optimal paths (see Appendix A, formula (52)). Our Hypothesis II.1.9 has "substantially" the same meaning for the AK vintage model. Indeed the maximum rate of growth of capital is a in the standard AK model and ξ in the vintage one and it may be proved that the value function is somewhere infinite when Hypothesis II.1.9 is not satisfied. Note also that the range of existence for the parameter ρ is greater that in the standard AK-model (see assumption (52) of Appendix A) and in the limit for $T \to +\infty$

[18] One may expect that interior solutions arise when a strict inequality is satisfied. This is not the case here, as it comes from the proof of Theorem II.2.15.

[19] This is not the same of $g_{AK} > 0$ that guarantees interior solutions. This comes from the passage to the limit as $T \to +\infty$.

the Hypothesis II.1.9 tends to $\rho - a(1-\sigma) > 0$ that the condition for the one dimensional case.

Concerning assumption (H3) we also see that it is strictly stronger than Hypothesis II.2.14: we can re-write (H3) as $g > 0$ while Hypothesis II.2.14 is $g \geq \xi - a$ so our results also cover cases where negative growth rates arise. Since investments always remain positive the occurrence of strictly negative long run growth rates in the AK vintage capital model increases the number of cases where deviation between growth and investment rates can arise (see [18] for a discussion on this).

It must also be noted that the assumptions (H2) and (H3) are not compatible for certain values of σ. Indeed (H2) means $\rho > (1-\sigma)a$ while (H3) means $\rho < \xi$. So, when $\xi \leq a(1-\sigma)$, i.e. when $\sigma \leq e^{-\xi T}$, (H2) and (H3) are not satisfied together. This means that the results of [18] do not cover cases with small σ.

III.4 Numerical results

The results obtained in previous sections allow also to improve the numerical study of the properties of the model. We consider in the whole subsection the set of parameters chosen in [18] page 61: $a = 0.30$, $\sigma = 8$, $\rho = 0.06$.

The vintage capital model has a wider flexibility with respect to the standard AK model, indeed we can choose the scrapping time T and we can vary the profile of the initial datum $\bar{\iota}(\cdot)$ also maintaining the same initial stock of capital $k_0 = \int_{-T}^{0} \bar{\iota}(s) \mathrm{d}s$. Here

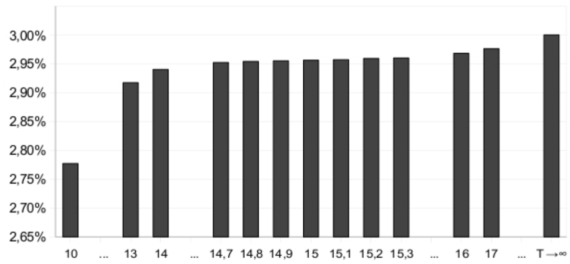

Figure 1: The value of g varying T

we are interested in studying how these new degrees of freedom influence the evolution of the system, namely we will look at how the asymptotic growth rate g varies with the scrapping time T[20] and how the long run levels of the detrended variables k_B, i_B and c_B are influenced by T and by the profile of the initial datum $\bar{\iota}$ keeping the same initial initial capital $k(0)$. Note that the dependence of the long run dynamics on the initial profile is a characteristic of the endogenous growth model we are studying but it is not common to all the vintage capital literature. For example in the exogenous model of Benhabib and Rustichini [12] (in the case of linear utility) the long run dynamics of examples 1, 2 and 3 does not depend on the initial profile, while in example 4-7 and in the non-linear

[20]Recall that, as in the standard AK model, g does not depend on the initial profile $\bar{\iota}(\cdot)$. See the works by Kocherlakota and Yi [51, 52] for an analysis of the dependence of the growth rates on the initial profile for non-vintage capital models.

utility case such a dependence is not studied. Moreover such a dependence does not arise in the Solow vintage capital model [62] and is only touched by Boucekkine, Germain and Licandro in [17]. In our case the results of Subsection II.3.3 allow to find precise results.
Asymptotic growth rate varying T: In Figure 1 we show the relation between the asymptotic growth rate g and the scrapping time T. The quite high value of the σ gives a small elasticity of the g w.r.t. T.
Long-run levels of detrended variables varying the initial profile: Now we consider the dependence of the long run levels of the detrended variables on the initial profile. We fix $T = 15$ as in [18]. To underline the importance of the distribution of the initial capital and not of its total amount we consider different initial profiles with same initial total stock of capital (equal to 1): we consider initial profiles equal to $\bar{\iota}_\eta^T(\cdot)$ where

$$\bar{\iota}_\eta^T : [-T, 0] \to \mathbb{R}, \qquad \bar{\iota}_\eta^T : s \mapsto \frac{e^{\eta s}}{\left(\frac{1}{\eta}\left(1 - e^{-\eta T}\right)\right)}.$$

This for example is the initial profile of the system if it evolved along a BGP (with growth

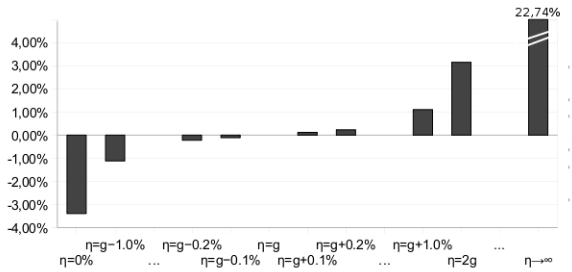

Figure 2: Graph of $\frac{k_B^\eta - k_B^g}{k_B^g}$ varying η

rate equal to η) until time 0. If $g = \eta$ the system continue to follow the BGP-dynamics, otherwise the dynamics is more complex. This second kind of situation happens, for example, when we start from an equilibrium path and we have at time 0 a technological shock[21]. In Figure 2 we see how the asymptotic discounted variable k_B changes varying η, in particular we represent $\frac{k_B^\eta - k_B^g}{k_B^g}$. Note that, since the ratio i_B/k_B (and c_B/k_B) does not depend on the initial datum the graphs for the variables i_B and c_B are the same. We see that small variations in the distribution of the initial capital are significant in the asymptotic discounted capital, for example taking, as in [18], $\eta = 0.0282$ (that is $g - 0.14\%$): we find[22] $\frac{k_B^\eta - k_B^g}{k_B^g} = -0.15\%$.

The dynamics of k_B^η with η is far to be linear: we represented the limit case $\eta \to \infty$ (when all the capital is new, with age 0) and we find that $\frac{k_B^\infty - k_B^g}{k_B^g} = +22.74\%$. On the other side if we compute the limit for $\eta \to -\infty$ (when all the capital has age T) all the asymptotic variables tend, as we expect, to zero.

[21] The introduction of technological shocks (see [51] and [52]) can allow to AK model being consistent with convergence.

[22] We called k_B^η the asymptotic detrended capital when the initial datum is $\bar{\iota}_\eta^T$ and the same for g.

k_B **varying** η **and** T**:** Last we analyze how k_B depends on η and on T. More precisely we see how the graph of Figure 2 changes with T (see Figure 3)[23]. For the three cases

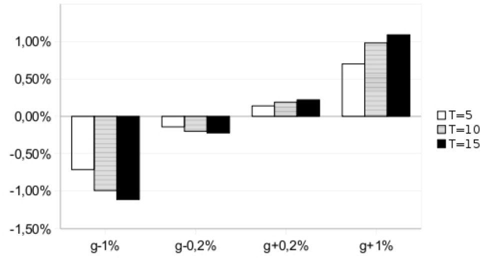

Figure 3: Graph of $\frac{k_B^\eta - k_B^g}{k_B^g}$ varying η and T

$T = 5$, $T = 10$ and $T = 15$ we take the initial profiles $\bar{\iota}_\eta^T(\cdot)$ described above. We see that the elasticity decreases when T become relatively small, in other words the initial condition have less influence on lung run levels. On the other hand we have already seen that for $T \to \infty$ the model tends to the one-dimensional AK-growth model, where the age of capital does not influence the behavior of the system. So we expect that the elasticity of k_B decreases also for high values of T. For our choice of a, ρ and σ the elasticity is maximal around $T = 20$.

Part IV

Applications to other models

We discuss here the possible extension of the ideas of the present work to other models: first when an explicit form of the value function can be found, second when this does not happen.

IV.1 DP when explicit solution can be found

If a model is formulated as an optimal contro problem with an optimal control problem with a linear DDE as state equation and an intertemporal CES or linear utility function, our technique can be applied, but the application is not strightforward at all. Indeed one has to re-prove various results like the conditions needed for the convergence of the utility integral, the admissibility of the optimal feedback and the behaviour of the optimal paths that are difficult and strongly dependent on the set-up of the model (espcially the constraints). The family of problems that can be solved using our techniques covers models already existing in literature including for example the time-to-build model studied in [4] and [2], the model for technological progress, obsolescence and depreciation presented in [16], some of the cases of the model presented in [12]. Moreover, such an approach can

[23]Note that g depends on T as in Figure 1.

be used to find explicit solutions of HJB equations in infinite-dimensional problems non related with DDEs as some cases of the model presented in [39].

IV.2 DP approach when explicit solution is not available

When the solution of the HJB cannot be found the DP method can still be applied and some insights on the model can be obtained, usually with more technical difficulties. What can be said and the amount of difficulties strongly depend on the structure of the problem under study.

To clarify this point we consider two variants of the model studied in the paper.

Consider first the model studied in this paper where we take a generic utility $u(c)$ instead of $u(c) = \frac{c^{1-\sigma}}{1-\sigma}$. Assume $u(\cdot)$ has continuous second derivative and that $u' > 0$, $u'' < 0$. In this case we cannot find an explicit solution of the HJB equation[24]; nevertheless the problem can be treated by the DP approach. The HJB equation can be studied using two approaches: the *strong solutions* approach (initiated by Barbu and Da Prato, see e.g. [5], and applied to this case in [36]) or the *viscosity solutions* approach (initiated by Crandall and Lions, see e.g. [23], and applied to this case in [35]).

With these methods we can prove (with a nontrivial amount of work) that the value function is a solution (in a suitable weak sense that is clarified in such papers) of the HJB equation, possibly unique when we enclose also the "boundary condition" (42) which is indeed a transversality condition as pointed out in Remark II.3.7 . Moreover a verification theorem can be proved that shows a closed loop relation in the same spirit of Proposition II.3.2. Such relation depends on the gradient (or the superdifferential) of the value function and so it cannot be transformed into an explicit form like in Theorem II.3.4. Nevertheless many qualitative results can still be obtained.
We give a detailed description of what can be done. Concerning the results of Chapter II.1:

- We can prove the results of Subsection II.1.1 and Lemma B.1 that do not depend on the form of the instantaneous utility.
- Assuming some specifications on the behavior of $u(c)$ for $c \to \infty$ (for example $u(\cdot)$ with growth of power $(1-\sigma)$ at infinity) we can find that, under conditions similar to the one of Hypothesis II.1.5, the claim of Proposition II.1.10 holds. The same can be done for Proposition II.1.11.
- If $u'(0^+) = +\infty$ is is possible to prove Proposition II.1.13.
- Existence of optimal solution can be proved for a quite general $u(\cdot)$.

The approach used in Section II.2 has to be changed in order to use one of the strong solution or the viscosity solutions approach:

- Instead of explicit solution of the HJB equation we have a theorem that guarantee that the value function V is a "weak" solution of the HJB equation (as in [36, 35]) included the transversality condition (42).

[24] Something could be done following the line of [50] but this is not known at the present stage.

- The set Y is not defined explicitly but in terms of $u(\cdot)$.
- If the value function is proved to have a continuous derivative (it is not obvious but the tools used in [37, 38, 5] can be exploited) the feedback can be given in terms of $V'(\cdot)$ and $u(\cdot)$.
- Proposition II.3.2 and Theorem II.3.4 can be formulated in implicit form obtaining equation similar to (40) and (41) containing $V'(\cdot)$ and $u(\cdot)$.
- We can not prove constant growth rate but it is possible to study qualitatively the asymptotic behavior of $i^*(\cdot)$, $k^*(\cdot)$ and $c^*(\cdot)$ in terms of $V'(\cdot)$ and $u(\cdot)$.
  We show the idea to perform this study. Let $z(\cdot)$ be an optimal discounted variable (for example $i^*(\cdot)$) and suppose that we want to study the existence and the stability of equilibrium points of $z(\cdot)$. We know by the analogous of Proposition II.3.2 and Theorem II.3.4 that $z(\cdot)$ solves the closed loop equation. This is a DDE which can be written, in the infinite dimensional notation (see Definition II.2.12) $z' = A^*z+B^*\phi(z)$ where $\phi(\cdot)$ is the optimal feedback map. $\phi(\cdot)$ now is not known explicitly but we can write $\phi(z) = G(V'(z))$ where $p \mapsto G(p)$ is the function that gives the maximum point of the current value Hamiltonian (which depends on $u(\cdot)$. So z_0 is an equilibrium point if and only if $F(z_0) = A^*z_0 + B^*G(V'(z_0)) = 0$. The existence of such an equilibrium point can be proved using the properties of G and V. The properties of G and V can also be used to study the stability of z_0 that is for example guaranteed if $F'(z_0) < 0$.

We consider now another variant of the model where the technology is not of AK type but it is nonlinear. In this case the associated optimal control problem contains a nonlinear term which makes it more difficult to study. Still the DP approach can be used and, depending on the assumptions on the technology, some insights on the model can be obtained. For example we can take the case where $y(t) = f(k(t))$ for a suitable nonlinear map f increasing and concave. The state equation would be the same as (4) but with the non linear constraint $i(t) \in [0, f(k(t))]$. In this case all is more difficult, but the particular features of f (the monotonicity, the concavity and other sharper assumptions) can be exploited to find results similar to the ones of Section II.1. The properties of f can be also used in order to obtain an infinite dimensional formulation and to study the HJB equation[25] together with the transversality condition (42). Finally the existence of optimal closed loop strategies and the asymptotic behavior of optimal trajectories can be studied with the ideas depicted for the previous case.

Clearly this case is more difficult than the previous one and is less clear to which extent the same results can be proved. In this case it is surely helpful to use also the MP approach (or maybe the Euler equation via calculus of variation) and to see if the integrated approach brings to deeper insights.

Conclusions and further research

In this paper the DP approach has been applied to study the AK model with vintage capital proposed in [18] proving results not available, at the present stage, with other

[25] For example of the technique that can be used one can see [37, 38, 36] for the strong solution approach and [24, 63, 48] for the viscosity solution approach.

approaches.

This is a first step towards the treatment of other and possibly more complex models that take into account e.g. endogenous scrapping times, exogenous fluctuations, markets of old capital goods. We think that in these kind of models, as in the present one, the DP can help to find the solutions and to analyze the core economic implications.

Another line of future research is to test the empirical implications of the model.

A Appendix: The standard AK-model with zero depreciation rate of capital

In this appendix we briefly recall the setup of the classical linear growth model (named *AK-model* with Rebelo [61]) with CRRA (Constant Relative Risk Aversion) utility function and zero depreciation rate of capital. We show how to find the optimal paths with the Dynamic Programming approach. This way the comparison with the AK vintage capital model can be more clear for the reader. Another reason to write this appendix is the fact that, in the classical literature, see e.g. the Barro and Sala-i-Martin's book [9] this model is treated with the maximum principle.

We call $y(t)$ the output level at time t, which is a linear function of the stock of capital $k(t)$: $y(t) = ak(t)$ for some positive constant a. $c(t)$ and $i(t)$ are the consumption and the investment at time t and the system is subject to an accounting equation of the form

$$y(t) = i(t) + c(t).$$

The capital stock follows the state equation (here we use the consumption as control variable, before we have chosen the investment, it is the same in view of the above relation)

$$\begin{cases} \dot{k}(t) = ak(t) - c(t), \\ k(0) = k_0 > 0. \end{cases} \tag{50}$$

We want to maximize (over the set of locally integrable consumption paths) the intertemporal utility function given by

$$\int_0^\infty e^{-\rho s} \frac{c(t)^{1-\sigma}}{1-\sigma} \mathrm{d}t \tag{51}$$

under the constraints $c(t), k(t) \geq 0$ for all $t \geq 0$. We assume

$$\rho - a(1-\sigma) > 0 \tag{52}$$

Note that hypothesis (52) is not only sufficient but also necessary to guarantee that the finiteness of the value function and the existence of optimal strategies (see e.g. on this [42]).

In order to compare in a proper way this standard AK model with the one treated in the paper we analyze separately the case where investments can be negative and the case where we impose positivity of them.

A1. The DP approach for possibly negative investments

Now we see how to perform the steps (i),..., (iv) of the Dynamic Programming approach described in Section I.2 in this one dimensional case.

Step (i): we write the HJB equation of the problem. It appears as

$$\rho v(k) - \sup_{c \geq 0} \left(v'(k)\,(ak - c) + \frac{c^{1-\sigma}}{1-\sigma} \right) = 0.$$

Step (ii): we solve the HJB equation. It is easy to check that the function[26]

$$v(k) = \nu k^{1-\sigma} \tag{53}$$

with $\nu = \frac{1}{1-\sigma}\left(\frac{\rho - a(1-\sigma)}{\sigma}\right)^{-\sigma}$ is a solution of the HJB and it is also the value function of the problem.
Step (iii): we use the value function to solve the optimal control problem in closed loop form. We consider the closed loop relation given by

$$\begin{cases} \phi \colon \mathbb{R}^+ \to \mathbb{R}^+ \\ \phi(k) \stackrel{def}{=} \arg\max_{c \in \mathbb{R}^+} \left(v'(k)\,(ak - c) + \dfrac{c^{1-\sigma}}{1-\sigma} \right) = \\ \qquad\qquad = \big(v'(k)\big)^{-1/\sigma} = \dfrac{\rho - a(1-\sigma)}{\sigma} \cdot k. \end{cases}$$

Using a verification theorem it can be proved that the strategy given by such relation is optimal.
Step (iv): We substitute $c = \phi(k)$ in the state equation:

$$\begin{cases} \dot{k}^*(t) = ak^*(t) - \phi(k^*(t)) = \left(\dfrac{a-\rho}{\sigma}\right) k^*(t), \\ k^*(0) = k_0. \end{cases} \tag{54}$$

So, calling $g_{AK} = \frac{a-\rho}{\sigma}$ the optimal capital and consumption path are:

$$\begin{cases} k^*(t) = e^{g_{AK} t} k_0; \\ c^*(t) = \phi(k^*(t)) = \left(\dfrac{\rho - a(1-\sigma)}{\sigma}\right) e^{g_{AK} t} k_0. \end{cases}$$

and the investment is

$$i^*(t) = ak^*(t) - c^*(t) = g_{AK} e^{g_{AK} t} k_0.$$

Note that we have positive growth rate g_{AK} if and only if $a \geq \rho$. Moreover the optimal investment has always the same sign of the growth rate g_{AK}.

A2. The DP approach for positive investments

We call this case the constrained case while the previous is the unconstrained one. When $a \geq \rho$ the optimal path for the unconstrained case is admissible for the constrained case

[26] It is clear from the structure of the problem that the value function must be $(1-\sigma)$-homogeneous; then the constant ν is calculated substituting into the HJB equation.

too, as $i^*(\cdot)$ is always positive. This does not happen when $a < \rho$. In such case the solution of the HJB is different as the sup is done over $c \in [0, ak]$ instead over $c \geq 0$:

$$\rho v(k) - \sup_{c \in [0,ak]} \left(v'(k)\,(ak - c) + \frac{c^{1-\sigma}}{1-\sigma} \right) = 0.$$

Arguing as in the unconstrained case one finds that the value function is $v(k) = \frac{a}{\rho}(ak)^{-\sigma}$. We perform the step (iii) and (iv). The optimal feedback map is $\phi(k) = ak$ so the optimal paths are constant, i.e. for every $t \geq 0$

$$\begin{cases} k^*(t) = k_0 \\ c^*(t) = ak_0 \\ i^*(t) = 0 \end{cases}$$

Remark A.1. We briefly point out the relations between the assumptions on the AK vintage model and the standard one, recalling that the second is the limit of the first for $T \to +\infty$ (see Sections III.2 and III.3 for comments).

The Hypothesis II.1.5 (i.e. $aT > 1$) means that strictly positive growth is possible and in the one dimensional case reduces to ask $a > 0$.

The Hypothesis II.1.9 (i.e. $\rho > (1-\sigma)\xi$) is substantially an if and only if condition for existence and is the analogous of (52) (indeed for $T \to \infty$ they are the same).

The Hypothesis II.2.14 (i.e. $\frac{\rho - \xi(1-\sigma)}{\sigma} \leq a$) guarantees that the optimal investment strategy is not a corner solution. The analogous assumption in the standard AK model is $a > \rho$. ■

B Appendix: Proofs

In this Appendix we present the main proofs.

Proof of Proposition II.1.3. The statement follows from the integral form of the DDE (equation (10)). Indeed by the admissibility constraints (7) we have, for $t \in [0, T]$,

$$k_{\bar{\iota},i}(t) = \int_{t-T}^{0} \bar{\iota}(s)ds + \int_0^t i(s)ds \leq \int_{t-T}^{0} \bar{\iota}(s)ds + \int_0^t ak_{\bar{\iota},i}(s)ds$$

while the function $k^M(\cdot)$ satisfies, for $t \in [0, T]$, $k^M(t) = \int_{t-T}^0 \bar{\iota}(s)ds + \int_0^t ak^M(s)ds$. Given these the inequality $k_{\bar{\iota},i}(t) \leq k^M(t)$ for $t \in [0, T]$, follows from a straightforward application of the Gronwall inequality (see e.g. [46] page 6).

For $t \in (T, 2T]$ we have, arguing as above $k_{\bar{\iota},i}(t) = \int_{t-T}^T i(s)ds + \int_T^t i(s)ds \leq \int_{t-T}^T ak_{\bar{\iota},i}(s)ds + \int_T^t ak_{\bar{\iota},i}(s)ds$, while the function $k^M(\cdot)$ satisfies, for $t \in (T, 2T]$, $k^M(t) = \int_{t-T}^T ak^M(s)ds + \int_T^t ak^M(s)ds$. Since from the first step we know that $k_{\bar{\iota},i}(t) \leq k^M(t)$ for $t \in [0, T]$ then we have, calling $g(t) = \int_{t-T}^T ak^M(s)ds$ for $t \in (T, 2T]$:

$$k_{\bar{\iota},i}(t) \leq g(t) + \int_T^t ak_{\bar{\iota},i}(s)ds \qquad \text{and} \qquad k^M(t) = g(t) + \int_T^t ak^M(s)ds$$

and then the Gronwall inequality gives the claim for $t \in (T, 2T]$. The claim for every $t \geq 0$ follows by an induction argument on the same line of the above steps. □

Proof of Proposition II.1.6. First we observe that being $\bar{\iota}(\cdot) \not\equiv 0$, $k^M(t)$ is strictly positive for each $t \geq 0$. To prove this it is enough to observe that, for $t \geq 0$, $k^M(t) = \int_{(t-T)\wedge 0}^0 \bar{\iota}(s)ds + \int_{(t-T)\vee 0}^t ak^M(s)ds$ and to argue by contradiction. Moreover, as we said above, for $t \geq T$, $k^M(t) = h(t)$ where $h(\cdot)$ the unique solution of (13). Now the solution $h(t)$ of (13) is continuous on $[T, +\infty)$ (see [13] page 207). Moreover (see [33] page 34) there exist at most $N < +\infty$ (complex) roots $\{\lambda_j\}_{j=1}^N$ of the characteristic equation with real part exceeding ξ and there exist $\{p_j\}_{j=1}^N$ $\mathbb{C}$-valued polynomial such that

$$h(t) = o(e^{(\xi+\varepsilon)t}) + \sum_{j=1}^N p_j(t)e^{\lambda_j t} \qquad for\ t \to +\infty \tag{55}$$

for every $\varepsilon > 0$. Since $k^M(t)$ and so $h(t)$ remain strictly positive for all $t \geq T$, then all the p_j vanish. So we have proved the claim. $\square$

Proof of Proposition II.1.8. First of all we observe that z is a root of (14) if and only if $w = zT$ is a root of

$$w = aT - aTe^{-w}. \tag{56}$$

Then it is enough to apply Theorem 3.2 p. 312 and Theorem 3.12 p.315 of [33]. The only statements which are not contained there are the fact that $Re\lambda_k \to -\infty$ as $k \to +\infty$ and the inequality (16). To see the first observe that, from (56) it follows, calling $\mu_k = T \cdot Re\lambda_k$ and $\nu_k = T \cdot Im\lambda_k$

$$aTe^{-\mu_k} \sin \nu_k = \nu_k \quad \Longrightarrow \quad e^{-\mu_k} > \frac{\nu_k}{aT}$$

and the claim is proved since $\nu_k \to +\infty$ as $k \to +\infty$. The proof of inequality (16) uses elementary arguments but it is a bit long so we give only a sketch of it. First of all by (14) we get that, when $aT > 1$

$$\xi > a\left(1 - \frac{1}{(aT)^2}\right) \tag{57}$$

while, for $aT > 5$

$$\xi > a\left(1 - \frac{1}{(aT)^3}\right). \tag{58}$$

Moreover using (56) we get that $e^{-2\mu_1} - (a - \mu_1)^2 = \nu_1^2 > 4\pi^2$. Now the function $h(\mu) = e^{-2\mu} - (a - \mu)^2$ is strictly decreasing on $(-\infty, 0)$ and using (57) and (58) we get that $h(\xi T - aT) < 4\pi^2$. This gives $\mu_1 < \xi T - aT$ and so the claim. $\square$

Lemma B.1. *Given any initial datum $\bar{\iota}(\cdot) \in L^2([-T, 0); \mathbb{R}^+)$, $\bar{\iota}(\cdot) \not\equiv 0$ there exists an $\varepsilon > 0$ and an admissible control strategy $i(\cdot)$ such that $i(t) = \varepsilon$ for all $t \geq T$. Moreover there exists a $\delta > 0$ such that the control defined by the feedback formula $i_\delta(t) = ak_\delta(t) - \delta$ for all $t \geq 0$ is admissible and $i_\delta(t) \geq \delta > 0$ for all $t \geq 0$.*

Proof. **The idea:** We give a constructive proof in four steps. We first find a small $\alpha > 0$ and a $\beta < T$ such that the (constant) control $i(t) = \varepsilon_1 \stackrel{def}{=} \frac{a\alpha\beta}{4}$ is admissible in the interval $(0, \frac{\beta}{4})$; then we see that such a control can be lengthened defining, on the interval $[\frac{\beta}{4}, T - \frac{\beta}{4})$,

$i(t) = \varepsilon_2 \stackrel{def}{=} \min\{\frac{a^2\alpha\beta^2}{32}, \varepsilon_1\}$. Furthermore we prove that we can extend such a control on $[T - \frac{\beta}{4}, T)$ putting $i(t) = \varepsilon \stackrel{def}{=} min\{\frac{a\varepsilon_2}{2}\frac{T}{4}, \varepsilon_2\}$. Observe that in view of the "minima" in the definitions of $\varepsilon_1, \varepsilon_2, \varepsilon$, the control is decreasing on $[0, T)$. Eventually (fourth step) we see that on the interval $[T, +\infty)$ we can put our control constantly ε. The statement for δ follows from this construction.

The proof: first step: In view of the fact that $\bar{\imath}(\cdot) \not\equiv 0$ we can choose a positive number α such that

$$\beta \stackrel{def}{=} m(\{s \in (-T, 0) \ \ s.t. \ \ \bar{\imath}(s) \geq \alpha\} > 0$$

where m is the Lebesgue measure. So $\int_{t-T}^{0} \bar{\imath}(s)\mathrm{d}s \geq \frac{\alpha\beta}{2}$ for all $t \in (0, \frac{\beta}{2})$ and in particular it is true for $t \in (0, \frac{\beta}{4})$. Now for $t \in [0, \frac{\beta}{4})$ we can put $i(t) = \varepsilon_1 \stackrel{def}{=} \frac{a\alpha\beta}{4} > 0$ obtaining that

$$a \int_{t-T}^{t} \tilde{\imath}(s)\mathrm{d}s \geq a \int_{-T+\frac{\beta}{4}}^{0} \bar{\imath}(s)\mathrm{d}s \geq a\alpha\frac{\beta}{2} > a\alpha\frac{\beta}{4} = i(t)$$

so the strategy is admissible on $[0, \frac{\beta}{4})$. Note that for such a choice of $i(t)$ we have $ak(t) - i(t) \geq \frac{a\alpha\beta}{4}$ for $t \in [0, \frac{\beta}{4})$.

second step: Choosing $i(\cdot)$ in the interval $[0, \frac{\beta}{4})$ as in the first step, and for $t \in [\frac{\beta}{4}, T - \frac{\beta}{4})$, $i(t) = \varepsilon_2 \stackrel{def}{=} \min\{\frac{a^2\alpha\beta^2}{32}, \varepsilon_1\} > 0$ (in view of the previous integral such a constant is in the range of admissible control for all t in the interval) we have that for all $t \in [\frac{\beta}{4}, T - \frac{\beta}{4})$

$$a \int_{t-T}^{t} \tilde{\imath}(s)\mathrm{d}s \geq a \int_{0}^{t} i(s)\mathrm{d}s \geq a\frac{\beta}{4}\frac{a\alpha\beta}{4} = \frac{a^2\alpha\beta^2}{16} > a^2\alpha\frac{\beta^2}{32} \geq i(t)$$

so the strategy is admissible on $[\frac{\beta}{4}, T - \frac{\beta}{4})$. Note that for such a choice of $i(t)$ we have $ak(t) - i(t) \geq \frac{a\alpha\beta^2}{32}$ for $t \in [\frac{\beta}{4}, T - \frac{\beta}{4})$.

third step: In particular we have put $i(t) = \varepsilon_2 > 0$ for $t \in (T/2, T - \frac{\beta}{4})$ and so, with a step similar to the previous one, we can put $i(t) = \varepsilon \stackrel{def}{=} min\{\frac{a\varepsilon_2}{2}\frac{T}{4}, \varepsilon_2\} > 0$ for $t \in [T - \frac{\beta}{4}, 0)$ and we have on such an interval $ak(t) - i(t) \geq \frac{\varepsilon_2}{2}\frac{T}{4}$.

fourth step: In view of the "minima" in the definition of ε_1, ε_2 and ε we have that $\varepsilon \leq \varepsilon_2 \leq \varepsilon_1$ and that $i(t) \geq \varepsilon$ in the interval $[0, T)$. So, choosing $i(t) = \varepsilon$ for all $t \geq T$, we get an admissible control, indeed $\varepsilon > 0$ and

$$a \int_{t-T}^{t} i(s)\mathrm{d}s \geq aT\varepsilon > \varepsilon$$

(the last follows by (H1)), for all $t \geq T$. We have that, on $[T, \infty)$, $ak(t) - i(t) \geq \frac{aT-1}{2}\varepsilon$.

The second statement, related to the δ constant, follows from the previous proof and from the observation we have done during the proof with respect the term $ak(t) - i(t)$. If we consider the strategy of the previous proof we have that

$$ak(t) - i(t) \geq \delta \stackrel{def}{=} \min\{\varepsilon, \frac{aT-1}{2}\varepsilon\}$$

and $i(t) \geq \delta$ for all $t \geq 0$. Now if we consider such a δ the strategy given by the feedback formula $i_\delta(t) = ak_\delta(t) - \delta$ satisfies the inequality $i_\delta(t) > i(t)$ (where $i(\cdot)$ is the strategy defined in first, second and third steps) for all $t \geq 0$ arguing as in the proof of the first statement of Proposition II.1.3. Then we get that $i_\delta(t) \geq i(t) \geq \delta$ for all $t \geq 0$ so it is admissible and the claim is proved. □

Proof of Proposition II.1.11. It is sufficient to take the control $i_\delta(t)$ s.t. $ak(t) - i_\delta(t) = \delta > 0$ found in previous lemma. $\square$

Proof of Proposition II.1.12. The proof is a simple application of a direct method (see also [18]. We will adapt the scheme of Askenazy and Le Van in [3] to our formulation. We will indicate with μ the measure on $\mathbb{R}^+$ given by $\mathrm{d}\mu(t) = e^{(-\varepsilon-\xi)t}\mathrm{d}t$ where $\mathrm{d}t$ is the Lebesgue measure and $\varepsilon > 0$ is fixed. By $L^1(0,+\infty;\mathbb{R};\mu)$, or simply $L^1(\mu)$ we will denote the space of all Lebesgue measurable functions that are integrable with respect to μ. We consider $\mathcal{I}_{\bar\iota}$ as subset of $L^1(\mu)$. We know that on a space of finite measure μ a subset G of $L^1(\mu)$ is relatively (sequentially) compact for the weak topology if and only if: for every $\varepsilon > 0$, there exists a $\delta > 0$ such that for every set I with $\mu(I) < \delta$ and for all $f \in G$ we have $\int_I f(x)\mathrm{d}\mu(x) < \varepsilon$ (this property is also known as *Dunford - Pettis criterion* see for example [34] page 294 Corollary 11). In our case constraint (7), Proposition II.1.3 and Proposition II.1.6 guarantee such a property for $\mathcal{I}_{\bar\iota}$. We choose now a maximizing sequence $i_n(\cdot) \in \mathcal{I}_{\bar\iota}$; thanks to the Dunford-Pettis criterion we can we can find a subsequence $i_{n_m}(\cdot) \in L^1(\mu)$ and $i^*(\cdot) \in L^1(\mu)$ such that $i_{n_m}(\cdot) \rightharpoonup i^*(\cdot) \in L^1(\mu)$. The functional $J(\bar\iota(\cdot);\cdot) : L^1(\mu) \supseteq \mathcal{I}_{\bar\iota} \to \overline{\mathbb{R}}$, that brings any $i(\cdot) \in L^1(\mu)$ to $J(\bar\iota(\cdot); i(\cdot))$, is concave and so it is weakly upper semicontinuous on $L^1(\mu)$ and so $J(\bar\iota(\cdot); i^*(\cdot)) \geq \limsup_{m\to\infty} J(\bar\iota(\cdot); i_{n_m}(\cdot))$. It remain to show that $i^*(\cdot) \in \mathcal{I}_{\bar\iota}$, i.e. $i^*(\cdot)$ satisfies the constraints (7). For the positivity constraint $i_{n_m}(\cdot) \rightharpoonup i^*(\cdot)$ and $i_{n_m}(\cdot) \geq 0$ imply $i^*(\cdot) \geq 0$ since nonnegativity constraints are preserved under weak convergence. Concerning the other constraint we observe that, thanks to (10) we know that $k_{\bar\iota,i_{n_m}}(\cdot) \to k_{\bar\iota,i^*}(\cdot)$ uniformly on the compact sets and so $ak_{\bar\iota,i^*}(t) \geq i^*(t)$ almost everywhere. This also implies that $i^*(\cdot) \in L^2_{\mathrm{loc}}[0,+\infty;\mathbb{R}^+)$. $\square$

Proof of Proposition II.1.13. For simplicity we will drop the $*$ writing $i(\cdot)$ instead of $i^*(\cdot)$ along this proof. If there exist $\bar t \in (0,+\infty)$ such that $k_{\bar\iota,i}(\bar t) = 0$ then by (10) (and a simple Gronwall-type argument) $k_{\bar\iota,i}(t) = 0$ for all $t \geq \bar t$. So if $\sigma > 1$ the statement is a consequence of Proposition II.1.11.

Then suppose that ($\sigma < 1$) and that there exist a first $\bar t > 0$ such that $k_{\bar\iota,i}(\bar t) = 0$. We assume that such a $\bar t$ is greater than $T/2$ but this imposition can be easily overcome (indeed noting that $\bar t > 0$ we can choose $n \in \mathbb{N}$ such that $\bar t > T/n$ and proceed in a similar way). Note that $k_{\bar\iota,i}(\bar t) = 0$ implies $i = 0$ in the set $[\bar t - T, \bar t]$. Thanks to the fact that $k_{\bar\iota,i}(t)$ is positive and continuous until $\bar t$ and that $i = 0$ (or $\tilde\imath = 0$) in the set $[\bar t - T, \bar t]$ we can say that exist $\varepsilon > 0$ such that the measure of the set

$$\Theta^\varepsilon \stackrel{def}{=} \{t \in [\bar t - T/2, \bar t] \ : \ ak_{\bar\iota,i} - i(t) > \varepsilon\}$$

is strictly positive (for the Lebesgue measure m): let be $h = m(\Theta^\epsilon) > 0$. We choose $\varrho < \varepsilon$ and define the new strategy $i_\varrho(\cdot)$:

$$i_\varrho(t) = \begin{cases} i(t) + \varrho \quad = \varrho & for\ t \in \Theta^\varepsilon \\ i(t) & otherwise \end{cases}$$

From the choice of Θ^ε and ϱ we obtain that $i_{\varepsilon,\varrho}(\cdot)$ is in $\mathcal{I}_{\bar\iota}$. The following estimate is valid:

$$J(\bar\iota(\cdot); i_\varrho(\cdot)) = I_1 + I_2 + I_3 + I_4 \stackrel{def}{=} \int_0^{\bar t - T/2} e^{-\rho t} \frac{(ak_{\bar\iota,i}(t) - i(t))^{1-\sigma}}{1-\sigma} \mathrm{d}t +$$

$$+\int_{([\bar{t}-T/2,\bar{t}]-\Theta^\varepsilon)} e^{-\rho t}\frac{(ak_{\bar{\iota},i_\varrho}(t)-i(t))^{1-\sigma}}{1-\sigma}\mathrm{d}t+$$

$$+\int_{\Theta^\varepsilon} e^{-\rho t}\frac{(ak_{\bar{\iota},i_\varrho}(t)-i_\varrho(t))^{1-\sigma}}{1-\sigma}\mathrm{d}t+\int_{\bar{t}}^{\bar{t}+T} e^{-\rho t}\frac{(ak_{\bar{\iota},i_\varrho}(t)-i_\varrho(t))^{1-\sigma}}{1-\sigma}\mathrm{d}t.$$

Moreover we have the following estimates (we use that $i=0$ on the set $[\bar{t}-T,\bar{t}]$):

$$I_2 \geq I_2' \stackrel{def}{=} \int_{([\bar{t}-T/2,\bar{t}]-\Theta^\varepsilon)} e^{-\rho t}\frac{(ak_{\bar{\iota},i}(t))^{1-\sigma}}{1-\sigma}\mathrm{d}t$$

$$I_3 \geq \int_{\Theta^\varepsilon} e^{-\rho t}\frac{(ak_{\bar{\iota},i}(t)-\varrho)^{1-\sigma}}{1-\sigma}\mathrm{d}t \geq \ (linearizing)$$

$$\geq I_3^1 - I_3^2 \stackrel{def}{=} \int_{\Theta^\varepsilon} e^{-\rho t}\frac{(ak_{\bar{\iota},i}(t))^{1-\sigma}}{1-\sigma} - \int_{\Theta^\varepsilon} e^{-\rho t}\varepsilon^{-\sigma}\varrho\mathrm{d}t + o(\varrho).$$

Furthermore:

$$I_4 \geq \int_{\bar{t}}^{\bar{t}+T/2} e^{-\rho t}\frac{(ak_{\bar{\iota},i_\varrho}(t)-i_\varrho(t))^{1-\sigma}}{1-\sigma}\mathrm{d}t \geq \int_{\bar{t}}^{\bar{t}+T/2} e^{-\rho t}\frac{(ah\varrho)^{1-\sigma}}{1-\sigma}\mathrm{d}t$$

So $I_3^2 = a_1\varrho$ and $I_4 \geq a_2\varrho^{1-\sigma}$ where a_1 and a_2 are positive constants independent by ϱ. Summarizing:

$$J(\bar{\iota}(\cdot); i_{\varepsilon,\varrho}(\cdot)) \geq (I_1 + I_2' + I_3^1) + (-I_3^2 + I_4) \geq J(\bar{\iota}(\cdot); i(\cdot)) + (-a_1\varrho + a_2\varrho^{1-\sigma}) + o(\varrho)$$

so for ϱ small enough we have $J(\bar{\iota}(\cdot); i_\varrho(\cdot)) > J(\bar{\iota}(\cdot); i(\cdot))$ and this is a contradiction. □

Proof of Proposition II.2.11. The function v is of course continuous and differentiable in every point of X and its differential in $x=(x^0,x^1)$ is

$$Dv(x) = (\nu(1-\sigma)[\Gamma_0(x)]^{-\sigma}, \{s \mapsto \nu(1-\sigma)[\Gamma_0(x)]^{-\sigma}e^{\xi s}\})$$

So $Dv(x) \in D(A)$ for every $x \in X$. We can also calculate explicitly ADv and BDv getting:

$$ADv(x) = (0, \{s \mapsto \nu(1-\sigma)[\Gamma_0(x)]^{-\sigma}\xi e^{\xi s}\}) \tag{59}$$

$$BDv(x) = \nu(1-\sigma)[\Gamma_0(x)]^{-\sigma}(1-e^{-\xi T}) > 0 \tag{60}$$

so, using the characteristic equation (14)

$$[BDv(x)]^{-1/\sigma} = \left(\frac{\rho-\xi(1-\sigma)}{\sigma}\cdot\frac{a}{\xi}\right)\Gamma_0(x) \tag{61}$$

Form the definition of X we have $[BDv(x)]^{-1/\sigma} > 0$ for $x \in X$. Moreover if $x \in Y$ then

$$\Gamma_0(x) \leq \frac{1}{\alpha}x^0 \tag{62}$$

and then $[BDv(x)]^{-1/\sigma} \leq ax^0$. So we can use Remark II.2.10 and write the Hamiltonian in the form of equation (27). Substituting (59) and (60) in (27) we find, by straightforward calculations, the relation:

$$\rho v(x) - \langle x, ADv(x)\rangle_{M^2} - ax^0 BDv(x) - \frac{\sigma}{1-\sigma}(BDv(x))^{\frac{\sigma-1}{\sigma}} = 0. \tag{63}$$

The claim is proved. □

Proof of Theorem II.2.15. **Part 1.** We prove that $\phi \in AFS_y$.
We claim that

$$\frac{d}{\mathrm{d}t} j^* x_\phi(t) = A^* x_\phi(t) + B^*(\phi(x_\phi(t))), \quad t > 0, \qquad x_\phi(0) = y = (R(\bar\iota), F(\bar\iota)) \tag{64}$$

has a unique solution in Π (defined in 22). Consider first the following integral equation (with unknown i: along this proof we drop the "tilde" sign to avoid heavy notation).

$$i(t) = \left(a - \frac{\rho - \xi(1-\sigma)}{\sigma\xi/a}\right)\left(\int_{t-T}^{t} i(s)\mathrm{d}s\right) - \frac{\rho - \xi(1-\sigma)}{\sigma\xi/a}\int_{-T}^{0} e^{\xi s} F(i_t)(s)\mathrm{d}s, \;\; t \geq 0 \tag{65}$$

with initial datum $i(s) = \bar\iota(s)$ when $s \in [-T, 0)$. Such equation has a solution i which is absolutely continuous solution on $[0, +\infty)$ (see for example [13] page 287 for a proof). We now claim that $i(t) > 0$ for all $t \geq 0$. First we prove that $i(0) > 0$. Indeed

$$i(0) = \int_{-T}^{0} \left[a - \frac{\rho - \xi(1-\sigma)}{\sigma\xi/a}\left(1 - e^{\xi(-T-s)}\right)\right] \bar\iota(s)\mathrm{d}s.$$

Since for every $s \in (-T, 0)$, $1 - e^{\xi(-T-s)} < \frac{\xi}{a}$ (in view of the fact that ξ is a positive solution of equation (14)) then we get by Hypothesis II.1.9 $i(0) > \int_{-T}^{0} \left[a - \frac{\rho-\xi(1-\sigma)}{\sigma}\right] \bar\iota(s)\mathrm{d}s$, so, using Hypothesis II.2.14 we obtain $i(0) > 0$. Now, if there exists a first point $\bar t$ in which the solution is zero then we have:

$$0 = i(\bar t) = \int_{-T}^{0} \left[a - \frac{\rho - \xi(1-\sigma)}{\sigma\xi/a}\left(1 - e^{\xi(-T-s)}\right)\right] i_{\bar t}(s)\mathrm{d}s$$

but, arguing as for $t = 0$, we can see that the right side is > 0 so we have a contradiction.
Now we consider the equation

$$\frac{d}{\mathrm{d}t} j^* x(t) = A^* x(t) + B^*(i(t)), \quad t > 0, \qquad x(0) = y = (R(\bar\iota), F(\bar\iota)). \tag{66}$$

We know, thanks to Theorem II.2.4, that the only solution in Π of such an equation is $x(t) = (\eta(t), F(i_t))$ where $\eta(t)$ is the solution of

$$\begin{cases} \dot z(t) = B(i_t) \\ (z(0), i_0) = (R(\bar\iota),\ \bar\iota) \end{cases} \qquad \left(\textit{that is} \quad \eta(t) = \int_{t-T}^{t} i(s)\mathrm{d}s\right) \tag{67}$$

We claim that $x(t)$ is a solution of (64). Indeed

$$\phi(x(t)) = a\eta(t) - \left(\frac{\rho - \xi(1-\sigma)}{\sigma\xi/a}\right)\left(\int_{-T}^{0} e^{\xi s} F(i_t)(s)\mathrm{d}s + \eta(t)\right) \tag{68}$$

and so (by (65)):

$$\phi(x(t)) = \eta(t)\left(a - \frac{\rho - \xi(1-\sigma)}{\sigma\xi/a}\right) + i(t) - \left(a - \frac{\rho - \xi(1-\sigma)}{\sigma\xi/a}\right)\left(\int_{(t-T)}^{t} i(s)\mathrm{d}s\right)$$

and by (67) we conclude that $\phi(x(t)) = i(t)$ and so $x(t) = x_\phi(t)$ is a solution of (64) and is in Π. Moreover thanks to the linearity of ϕ we obtain that $x_\phi(t)$ is the only solution in Π. We have now to show that $i(\cdot) = \phi(x_\phi(\cdot)) \in \mathcal{I}_y$. The previous steps of the proof gives

$$x_\phi(t) = (x_\phi^0(t), x_\phi^1(t)(\cdot)) = \left(R(i_t), F(i_t)\right)$$

where i is absolutely continuous and so in $L^2_{\mathrm{loc}}([0,+\infty);\mathbb{R})$. We claim that $\phi(x_\phi(t)) = i(t) \in (0, ax^0_\phi(t))$. In view of the fact that $i(t) > 0$ for all $t \geq 0$ it is enough to prove that $i(t) < ax^0_\phi(t)$. Indeed by (65)

$$ax^0_\phi(t) - i(t) = \left(\frac{\rho - \xi(1-\sigma)}{\sigma\xi/a}\right)\left(R(i_t) + \int_{-T}^0 e^{\xi s} F(i_t)(s)\mathrm{d}s\right) \geq$$

$$\geq \left(\frac{\rho - \xi(1-\sigma)}{\sigma\xi/a}\right)\left(\int_{-T}^0 i_t(s)(1 - e^{\xi(-T-s)})\mathrm{d}\right)s > 0.$$

The last inequality is strict due to Hypothesis II.1.9 and to the fact that $i(t) > 0$ for all $t > 0$. So $i(t) < ax^0_\phi(t)$ and we know that ϕ is an admissible feedback strategy related to $y = (R(\bar{\iota}), F(\bar{\iota}))$.

Part 2. We prove now that $\phi \in OFS_y$.
We consider v as defined in Proposition II.2.11. It is easy to see from the first part of the proof that $x_\phi(t)$ remain in Y as defined in (29) and so the Hamiltonian (as in the proof of Proposition II.2.11) can be expressed in the simplified form of equation (27). We introduce the function:

$$v_0(t,x)\colon \mathbb{R} \times X \to \mathbb{R}, \qquad v_0(t,x) \stackrel{def}{=} e^{-\rho t} v(x) \quad (v \textit{ is defined in } (30))$$

Using that $(Dv(x_\phi(t))) \in D(A)$ and that the application $x \mapsto Dv(x)$ is continuous with respect to the norm of $D(A)$, we find:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} v_0(t, x_{\phi(t)}) &= -\rho v_0(t, x_\phi(t)) + \langle D_x v_0(t, x_\phi(t)) | A^* x_\phi(t) + B^* i(t)\rangle_{D(A)\times D(A)'} = \\ &= -\rho e^{-\rho t} v(x_\phi(t)) + e^{-\rho t}\Big(\langle ADv(x_\phi(t)), x_\phi(t)\rangle_{M^2} + \langle BDv(x_\phi(t)), i(t)\rangle_{\mathbb{R}}\Big) \end{aligned} \quad (69)$$

Integrating on $[0,\tau]$ we get:

$$\begin{aligned} &v_0(\tau, x_\phi(\tau)) - v_0(0, x_\phi(0)) = \\ &= \int_0^\tau e^{-\rho t}\Big(-\rho v(x_\phi(t)) + \langle ADv(x_\phi(t)), x_\phi(t)\rangle_{M^2} + \langle BDv(x_\phi(t)), i(t)\rangle_{\mathbb{R}}\Big)\mathrm{d}t. \end{aligned} \quad (70)$$

Now we observe now that, since in the first part of the proof we have seen that $\phi \in AFS_y$, we have that $i(t) \leq ax^0_\phi(t)$. So, by Proposition II.1.3 we know that $i(t) \leq ak^M(t)$ and then

$$\begin{aligned}\Gamma(i_t)^{1-\sigma} &= \left(\int_{-T}^0 \left(1 - e^{-\xi(T+s)}\right)\, i_t(s)\mathrm{d}s\right)^{1-\sigma} \leq \\ &\leq \left(\int_{-T}^0 \left(1 - e^{-\xi(T+s)}\right)\, ak^M(t+s)\mathrm{d}s\right)^{1-\sigma} \leq \left(\int_{-T}^0 ak^M(t+s)\mathrm{d}s\right)^{1-\sigma}\end{aligned}$$

so, by Proposition II.1.6 and Hypothesis II.1.9 we have that

$$v_0(\tau, x_\phi(\tau)) = e^{-\rho t} v(x_\phi(\tau)) = e^{-\rho t}\nu\Gamma(i_t)^{1-\sigma} \xrightarrow{\tau\to 0} 0 \quad (71)$$

So we can pass to the limit in (70) obtaining

$$
\begin{aligned}
& -v(y) = -v_0(0, x_\phi(0)) = \\
& = \int_0^{+\infty} e^{-\rho t}\Big(-\rho v(x_\phi(t)) + \langle ADv(x_\phi(t)), x_\phi(t)\rangle_{M^2} + \langle BDv(x_\phi(t)), i(t)\rangle_{\mathbb{R}}\Big)\mathrm{d}t. \quad (72)
\end{aligned}
$$

Now we write (note that $J_0(y;i) = J_0(y;\phi(x_\phi))$):

$$
v(y) - J_0(y;i) = v(x_\phi(0)) - \int_0^\infty e^{-\rho t}\frac{(ax^0_\phi(t) - \phi(x_\phi)(t))^{1-\sigma}}{(1-\sigma)}\mathrm{d}t =
$$

Then, using (72) (we use Proposition II.1.3 and Hypothesis II.1.9 to guarantee that the integral is finite), we obtain

$$
\begin{aligned}
& = \int_0^\infty e^{-\rho t}\Bigg(\rho v(x_\phi(t)) - \langle ADv(x_\phi(t)), x_\phi(t)\rangle_{M^2} - \langle BDv(x_\phi(t)), i(t)\rangle_{\mathbb{R}}\Bigg)\mathrm{d}t - \\
& - \int_0^\infty e^{-\rho t}\left(\frac{(ax^0_\phi(t) - i(t))^{1-\sigma}}{(1-\sigma)}\right)\mathrm{d}t = \int_0^\infty e^{-\rho t}\Bigg(\rho v(x_\phi(t)) - \langle ADv(x_\phi(t)), x_\phi(t)\rangle_{M^2} - \\
& \qquad - \langle BDv(x_\phi(t)), i(t)\rangle_{\mathbb{R}} - \frac{(ax^0_\phi(t) - i(t))^{1-\sigma}}{(1-\sigma)}\Bigg)\mathrm{d}t = \\
& \qquad = \int_0^\infty e^{-\rho t}\Big(\mathcal{H}(x_\phi(t), Dv(x_\phi(t))) - \mathcal{H}_{CV}(x_\phi(t), Dv(x_\phi(t)), i(t))\Big)\mathrm{d}t \quad (73)
\end{aligned}
$$

The conclusion follows from Remark II.2.8 and by the three observations listed below.

1. Noting that $\mathcal{H}(x_\phi(t), Dv(x_\phi(t))) \geq \mathcal{H}_{CV}(x_\phi(t), Dv(x_\phi(t)), i(t))$ the (73) implies that, for every admissible control i, $v(y) - J_0(y;i) \geq 0$ and then $v(y) \geq V_0(y)$.

2. The original maximization problem is equivalent to the problem of finding a control i that minimizes $v(y) - J_0(y;i)$

3. The feedback strategy ϕ achieves $v(y) - J_0(y;i) = 0$ that is the minimum in view of point 1.

□

Proof of Lemma II.3.3. By Proposition II.3.2, equation (34), along the optimal trajectories we have, for $t \geq 0$:

$$
ak^*(t) - i^*(t) = \left(\frac{\rho - \xi(1-\sigma)}{\sigma}\cdot\frac{a}{\xi}\right)\Gamma(\tilde{i}^*_t(\cdot)).
$$

Now let us note that $\Gamma(\tilde{i}^*_t(\cdot)) = \int_{-T}^0 e^{\xi s} F(\tilde{i}^*_t(\cdot))(s)\mathrm{d}s + k^*(t) = \langle \psi, x(t)\rangle$ where $\psi = (\psi^0, \psi^1) \in M^2$ with $\psi^0 = 1$, $\psi^1(s) = e^{\xi s}$ and $x(t)$ is the structural state as in Definition II.2.2 . We calculate now the derivative of such an expression: it is easy to see that $\psi \in D(A)$. So we have (by Theorem II.2.4)

$$
\frac{d}{dt}\left(\int_{-T}^0 e^{\xi s} F(\tilde{i}^*_t(\cdot))(s)\mathrm{d}s + k^*(t)\right) = \frac{d}{dt}\langle \psi, x(t)\rangle_{M^2} =
$$

(by equation (23) and by the definitions of A and B)

$$= \langle A\psi, x(t)\rangle_{M^2} + \langle B\psi, i^*(t)\rangle_{\mathbb{R}} = \big\langle (0, \xi\psi^1(s)), x(t)\big\rangle_{M^2} + \Big\langle (1-e^{-\xi T}), i^*(t)\Big\rangle_{\mathbb{R}} =$$

(finding $x(t)$, the scalar products and using the (34))

$$= \left[\xi\bigg(\int_{-T}^{0} e^{\xi s} F(\tilde{i}_t^*(\cdot))(s)\mathrm{d}s\bigg)\right] +$$

$$\left[(1-e^{-\xi T})\left(ak^*(t) - \left(\frac{\rho-\xi(1-\sigma)}{\sigma\xi/a}\right)\left(\int_{-T}^{0} e^{\xi s} F\left(\tilde{i}_t^*(\cdot)\right)(s)\mathrm{d}s + k^*\right)\right)\right] =$$

$$= \left(\xi - \frac{\rho-\xi(1-\sigma)}{\sigma}\right)\bigg(\int_{-T}^{0} e^{\xi s} F(\tilde{i}_t^*(\cdot))(s)\mathrm{d}s + k^*(t)\bigg) = g\bigg(\int_{-T}^{0} e^{\xi s} F(\tilde{i}_t^*(\cdot))(s)\mathrm{d}s + k^*(t)\bigg) \tag{74}$$

and so we have the claim. The bounds for g simply follows by Hypotheses II.1.9 and II.2.14. Finally, since, from (36) $\Lambda = ak^*(0) - i^*(0)$ from (34) for $t = 0$ we find (38) observing that $\tilde{i}_0^*(\cdot) = \bar{\imath}(\cdot)$. □

Proof of Proposition II.3.5. The proof of existence of the limits is proved also in [18] using the transversality conditions. Here we use the integral equation (40) and the explicit form of Λ given in (38).

From(40) we can easily find that $i(t)$ satisfies, for $t \geq 0$ the following DDE

$$\begin{cases} i'(t) = a(i(t) - i(t-T)) - \Lambda g e^{gt}, \ \forall t \geq 0, \\ i(s) = \bar{\imath}(s), \ \forall s \in [-T, 0), \quad i(0) = a\int_{-T}^{0} \bar{\imath}(s)\mathrm{d}s - \Lambda, \end{cases} \tag{75}$$

The solution of this linear non homogeneous DDE is the sum of the solution of the associated linear homogeneous DDE plus a convolution term (see [44] page 23). In our case it means that the solution of equation (75) can be written as:

$$i(t) = \int_0^t -\gamma(t-s)\Lambda g e^{gs}\mathrm{d}s + \gamma(t)i(0) - a\int_{-T}^{0} \gamma(t-T-s)\bar{\imath}(s)\mathrm{d}s \tag{76}$$

where $\gamma(t)$ is the solution of the following DDE:

$$\begin{cases} \gamma'(t) = a(\gamma(t) - \gamma(t-T)) \ \forall t \geq 0, \\ \gamma(s) = 0, \ \forall s \in [-T, 0), \quad \gamma(0) = 1 \end{cases} \tag{77}$$

We observe that equation (77) is similar to the DDE we have seen in equation (13). In particular the characteristic equation is the same and it is (like in (14))

$$a(1 - e^{-zT}) - z = 0 \tag{78}$$

From Proposition II.1.8 such a characteristic equation has only simple roots: the only real roots are ξ and 0 and all other complex roots have real part in $(-\infty, \xi - a)$ so, even if $g < 0$ we have $Re\lambda_j < g$ for each $j = 1, 2, ...$ Applying Corollary 6.4 of [33], page 168 we see that the solution of (77) can be written as $\gamma(t) = \alpha_\xi e^{\xi t} + \alpha_0 + \sum_{j=1}^{+\infty}\left[\alpha_j e^{\lambda_j t} + \overline{\alpha}_j e^{\overline{\lambda}_j t}\right]$ where the series converges uniformly on compact subsets of $(0, +\infty)$, α_ξ, α_0 are real numbers and α_j are complex numbers.

We have now only to substitute such an expression in (76). In view of the linearity of (76) with respect γ we can analyze the contribution of three parts of γ in three steps: first we estimate the term due to $\alpha_\xi e^{\xi t}$, second we consider the term α_0 and then the series. We start with $\alpha_\xi e^{\xi t}$: its contribution to $i(t)$ is (in view of (76) is (using (32) and (38))

$$\begin{aligned}
&\int_0^t -\alpha_\xi e^{\xi(t-s)} \Lambda g e^{gs} \mathrm{d}s + a \int_{-T}^0 \bar{\iota}(s) \mathrm{d}s \alpha_\xi e^{\xi t} - \Lambda \alpha_\xi e^{\xi t} - a \int_{-T}^0 \alpha_\xi e^{\xi(t-T-s)} \bar{\iota}(s) \mathrm{d}s = \\
&= e^{\xi t} \alpha_\xi \left(\frac{\Lambda g}{g-\xi} + a\Gamma(\bar{\iota}) - \Lambda \right) + e^{gt} \alpha_\xi \left(-\frac{\Lambda g}{g-\xi} \right) = \\
&= e^{\xi t} \alpha_\xi \left(\frac{\Lambda g}{g-\xi} + a\Lambda \frac{\xi}{a} \frac{\sigma}{\rho - \xi(1-\sigma)} - \Lambda \right) + e^{gt} \alpha_\xi \left(-\frac{\Lambda g}{g-\xi} \right) = \\
&= e^{\xi t} \alpha_\xi \Lambda \left(\frac{\frac{\xi-\rho}{\sigma}}{\frac{\xi-\rho}{\sigma} - \xi} + \frac{\xi\sigma}{\rho - \xi(1-\sigma)} - 1 \right) + e^{gt} \alpha_\xi \left(-\frac{\Lambda g}{g-\xi} \right) = e^{gt} \alpha_\xi \left(-\frac{\Lambda g}{g-\xi} \right)
\end{aligned} \tag{79}$$

Then the part $\alpha_\xi e^{\xi t}$ gives in $i(t)$ a contribution of $e^{gt}\alpha_\xi \left(-\frac{\Lambda g}{g-\xi}\right)$. The contribution of the term α_0 is $-\int_0^t \alpha_0 \Lambda g e^{gs} \mathrm{d}s + a\alpha_0 \int_{-T}^0 \bar{\iota}(s)\mathrm{d}s - \Lambda\alpha_0 - a\alpha_0 \int_{-T}^0 \bar{\iota}(s)\mathrm{d}s = -\alpha_0 \Lambda e^{gt}$. Now to analyze the contribution of the series we use the dominated convergence theorem that allows to exchange the series and the integral. Then for each term $\alpha_j e^{\lambda_j t}$ we can develop the integrals as above obtaining the sum of two terms $-\frac{\alpha_j \Lambda g}{g-\lambda_j} e^{gt} + \left[\frac{\alpha_j \Lambda g}{g-\lambda_j} + \alpha_j a(\Gamma_j - \Lambda)\right] e^{\lambda_j t}$ where $\Gamma_j := \int_{-T}^0 (1 - e^{-\lambda_j (T+s)}) \bar{\iota}(s) \mathrm{d}s$. Since $Re\lambda_j < g$ for each j, then the second term is of order smaller or equal to the first. The same can be done for the terms $\overline{\alpha}_j e^{\overline{\lambda}_j t}$. So the solution can be written in the form $Ce^{gt} + o(e^{gt})$ where $C = -\Lambda g \left[\frac{\alpha_\xi}{g-\xi} + \frac{\alpha_0}{g} + \sum_{j=1}^{+\infty} Re\left(\frac{\alpha_j}{g-\lambda_j}\right)\right]$. This proves the first statement of the Proposition: there exist positive constants i_B and k_B such that $\lim_{t\to+\infty} i_g(t) = i_B$ and $\lim_{t\to+\infty} k_g(t) = k_B$. We now calculate such i_B and k_B. From(40) we find that $i_g(t)$ satisfies, for $t \geq 0$ the following integral equation:

$$i_g(t) = a \int_{-T}^0 e^{gs} i_g(t+s) \mathrm{d}s - \Lambda \tag{80}$$

and then i_B has to satisfy $i_B = a i_B \int_{-T}^0 e^{gs} \mathrm{d}s - \Lambda$, so we use the (80) to find the value of i_B and k_B. Note that the fact that $i_B > 0$ follows, for $g \neq 0$ from the fact that $g < \xi$ (Hypothesis II.1.9) and from (15); for $g = 0$ from Hypothesis II.1.5. □

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