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Continuous-time random walks are pure-jump processes with several applications in physics, but also in insurance, finance and economics. Based on heuristic considerations, a definition is given for the stochastic integral driven by continuous-time random walks. The martingale properties of the integral are investigated. Finally, it is shown how the definition can be used to easily compute the stochastic integral by means of Monte Carlo simulations.

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I. INTRODUCTION

A. The continuous-time random walk

Continuous-time random walks (CTRWs) are pure-jump stochastic processes. They have been introduced by Montroll and Weiss in physics as models for standard and anomalous diffusion when the sojourn time at a site is much greater than the time needed to jump at the new position: Jumps are considered as instantaneous events [1]. Shlesinger wrote a review paper that greatly contributed to popularize CTRWs [2]. More recently, theoretical and empirical studies on CTRWs have been discussed by Klafter and Metzler [3, 4] and by a co-author of the present paper [5]. In a CTRW, if $x(t)$ denotes the position of a diffusing particle at time t, ξ_i denotes a random jump occurring at a random time t_i and $\tau_i = t_i - t_{i-1}$ is the interarrival or waiting time between two jumps, one has

$$x(t) \stackrel{\text{def}}{=} S_{n(t)} \stackrel{\text{def}}{=} \sum_{i=1}^{n(t)} \xi_i \,, \tag{1}$$

where $t_0 = 0$, $x(0) = 0$ and $n(t)$ is a counting random process giving the number of jumps up to time t. Throughout this paper, we assume that

- the jumps ξ_i, $i = 1, 2, \ldots$ are independent and identically distributed (iid) random vectors in $\mathbb{R}^d$, $d = 1, 2, \ldots$ [6];
- the waiting times τ_i, $i = 1, 2, \ldots$ are iid random variables in $\mathbb{R}_+$;
- the families $(\xi_i, i = 1, 2, \ldots)$ and $(\tau_i, i = 1, 2, \ldots)$ are independent.

The third assumption means that we consider so-called uncoupled CTRWs. The first two assumptions entail that the joint distribution of any pair (ξ_i, τ_i) does not depend on i. If, in the uncoupled case, the law of (ξ_i, τ_i) is given by a density function $\varphi(\xi, \tau)$, the independence of ξ_i and τ_i entails that it can be factorized in terms of the marginal probability densities for jumps $w(\xi)$ and waiting times $\psi(\tau)$: $\varphi(\xi, \tau) = w(\xi)\psi(\tau)$. Eq. (1) means that a CTRW is a random sum of independent random variables. The process of the jump times

$$t_n = \sum_{i=1}^{n} \tau_i, \; t_0 = 0, \tag{2}$$

is a renewal point process. Therefore, CTRWs can be seen as compound renewal processes [7–9]. The existence of uncoupled CTRWs can be proved, based on the corresponding theorems of existence for renewal processes and discrete-time random walks. Càdlàg (right-continuous with left limit) realizations of CTRWs can be easily and exactly generated by Monte Carlo simulations and drawn. This is illustrated in Fig. 1. Uncoupled CTRWs are Markovian if and only if the waiting time distribution is exponential, meaning that $\psi(\tau) = \lambda \exp(-\lambda\tau)$ [10, 11]. Uncoupled CTRWs belong to the class of semi-Markov processes [11, 12], i.e. for any $A \subset \mathbb{R}^d$ and $s > 0$ we have

$$P(S_n \in A, \tau_n \le s \,|\, S_0, \ldots, S_{n-1}, \tau_1, \ldots, \tau_{n-1}) = P(S_n \in A, \tau_n \le s \,|\, S_{n-1}) \tag{3}$$

and, if we fix the position $S_{n-1} = y$ of the diffusing particle at time t_{n-1}, the probability on the right will be independent of n. If the law of (ξ_n, τ_n) is given by a density function $\varphi(x, t)$, we can use $S_n = \xi_n + S_{n-1}$ and rewrite this as

$$P(S_n \in A, \tau_n \le s \,|\, S_{n-1}) = \int_0^s \int_A \varphi(x - S_{n-1}, t)\, dx\, dt \tag{4}$$

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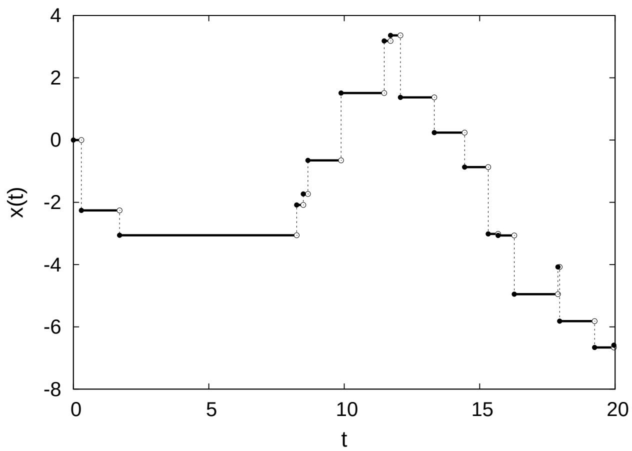

FIG. 1: Realization of a CTRW with exponentially distributed waiting times ($\lambda = 1$) and standard normally distributed jumps ($\mu = 0$ and $\sigma = 1$).

In this case it is possible to write an integral equation for the probability density $p(x, t)$ of finding the particle in position x at time t; this is done in terms of the marginal probability densities of waiting times $\psi(\tau)$ and of jumps $w(\xi)$:

$$p(x,t) = \Psi(t)\delta(x) + \int_{\mathbb{R}^d} w(x-x') \int_0^t \psi(t-t') p(x',t')\, dt'\, dx' \,, \tag{5}$$

where $\Psi(t) = 1 - \int_0^t \psi(t')\, dt'$ is the complementary cumulative distribution function for the waiting times, also called survival function. The solution of Eq. (5), known as Montroll-Weiss equation, can be written in terms of the probability distribution function $P(n, t)$ of the counting process $n(t)$, and the n-fold convolution $w^{*n}(x)$ of $w(\xi)$ as

$$p(x,t) = \sum_{n=0}^{\infty} P(n,t) w^{*n}(x). \tag{6}$$

This result can be derived from Eq. (5) using Fourier and Laplace transforms, a method described in several papers, including the original one by Montroll and Weiss [1]. However, Eq. (6) can also be derived directly by probabilistic considerations. Indeed, Eq. (1) is a random sum of iid random variables. This means that any position x can be reached at time t by a finite number n of jumps. The probability of reaching position x at time t in exactly n jumps is $P(n, t)w^{*n}(x)$. Eq. (6) follows given that these events are mutually exclusive. Note that $P(0, t)w^{*0}(x)$ coincides with the singular term $\Psi(t)\delta(x)$, meaning that the distribution function for x has a jump at position $x = 0$ of height $\Psi(t)$.

CTRWs with exponential waiting times—also called compound Poisson processes (CPP), as in this case $P(n, t) = \exp(-\lambda t)(\lambda t)^n/n!$—are not only Markovian, but they are also Lévy processes. This means that they have independent and time-homogeneous (stationary) increments. In this case, as a consequence of infinite divisibility and Kolmogorov's representation theorem, $p(x, t)$ fully characterizes the stochastic process defined by Eq. (1) [13–15].

B. CTRWs in physics, insurance, finance, and economics

Since the seminal paper by Montroll and Weiss [1], there has been much scientific activity on the application of CTRWs to important physical problems. A line of research investigated anomalous relaxation related to power-law tails of the waiting time distribution as well as the asymptotic behaviour of CTRWs for large times [16–21]. As mentioned above, Klafter and Metzler have extensively reviewed these and subsequent studies [3, 4]. Furthermore, in their book, ben-Avraham and Havlin have discussed the applications to physical chemistry [22]. Here, it is worth mentioning the recent work on the relationship between CTRWs and fractional diffusion that can be traced to papers by Balakrishnan and Hilfer [23, 24] and has been thoroughly discussed in Refs. [25, 26]. Some specific applications include e.g. plasma physics [27] and biopolymers [28, 29].

CTRWs have natural interpretations also in insurance, finance, and economics. Even if well-known in the field of econophysics [5, 30], these interpretations deserve a short summary.

In ruin theory for insurance companies, the jumps ξ_i are interpreted as claims and they are positive random variables; t_i is the instant at which the i-th claim is paid [31].

In mathematical finance, if $P_A(t)$ is the price of an asset at time t and $P_A(0)$ is the price of the same asset at a previous reference time $t_0 = 0$, then $x(t) = \log(P_A(t)/P_A(0))$ represents the log-return (or log-price) at time t. In regulated markets using a continuous double-auction trading mechanism, such as stock markets, prices vary at random times t_i, when a trade takes place, and $\xi_i = x(t_i) - x(t_{i-1}) = \log(P_A(t_i)/P_A(t_{i-1}))$ is the tick-by-tick log-return, whereas $\tau_i = t_i - t_{i-1}$ is the intertrade duration; for more details, see [5, 30, 32] and references therein.

In the theory of economic growth, ξ_i represents a growth shock, $x(t)$ is the logarithm of the size for a firm or of the wealth for an individual, and τ_i is the time interval between two consecutive growth shocks; see [5] and references therein.

C. Motivation for the study of stochastic integrals driven by CTRWs

Given the wide range of applications of CTRWs overviewed in the previous subsection, it is relevant to study diffusive stochastic differential equations where the driving noise is defined in terms of CTRWs:

$$dz = a(z,t)dt + b(z,t)dx\,, \tag{7}$$

where $z(x,t)$ is the unknown random function, $a(z,t)$ and $b(z,t)$ are known functions of z and time t, and dx represents the CTRW measure with respect to which stochastic integrals are defined. In order to give a rigorous meaning to such an expression, some constraints on the properties of CTRWs are necessary. In a recent paper, the theory has been discussed for stochastic integration on time-homogeneous (stationary) CTRWs—i.e., the already mentioned CPPs [33]. Although the theory reported in Ref. [33] was already well known by mathematicians and has been used in finance for option pricing since 1976 [34], that paper contains useful material and is written in a way that is clear and appealing for physicists. Here, inspired by Ref. [33], the theory will be further discussed and developed.

II. STOCHASTIC INTEGRALS

In Ref. [33], the stochastic integral is never explicitly defined. However, starting from the fact that sample paths of a CTRW can be represented by step functions, it is possible to give an explicit formula.

A. Definition

For the definition of the stochastic integral

$$J(t) = \int_0^t y(s)\,dx(s), \tag{8}$$

where $x(t)$ is defined by Eq. (1), $y(t)$ is a further random process (often of the form $y(t) = G(x(t))$ with a suitable function $G(x)$), some heuristic manipulations are useful. Eq. (1) can be written in terms of Heaviside's step function $\theta(t)$, which is 0 for $t < 0$ and 1 for $t \geq 0$:

$$x(t) = \sum_{i=1}^{n(t)} \xi_i \theta(t - t_i). \tag{9}$$

Using the fact that the "derivative" of Heaviside's θ function $\theta(t-t_i)$ is Dirac's δ function $\delta(t-t_i)$, one can write

$$dx(t) = \sum_{i=1}^{n(t)} \xi_i \delta(t - t_i)\,dt. \tag{10}$$

Note that $\delta(t)$ is not a function, but rather a distribution in the sense of Sobolev and Schwartz [35]. Replacing Eq. (10) in Eq. (8) and using the properties of Dirac's δ function, one gets

$$J(t) = \sum_{i=1}^{n(t)} y(t_i)\xi_i. \tag{11}$$

This definition works nicely if the driving noise is a step function and if convergence is not an issue. This observation prompted K. Itô to use martingale convergence theorems to tackle the convergence for a large class of integrators [36]. To do so we have to make sure that $J(t)$ is a martingale whenever $x(t)$ is. For this we have to make the integrand $y(t)$ statistically independent of the "increment" ξ_i and replace $y(t_i)$ in Eq. (11) by $y(t_i^-) = y(t_{i-1})$. This leads to the following definition

$$I(t) = \int_0^t y(s)\,dx(s) = \sum_{i=1}^{n(t)} y(t_i^-)\xi_i = \sum_{i=1}^{n(t)} y(t_{i-1})\xi_i; \tag{12}$$

with such a choice, the integrand becomes *non-anticipating*. An elementary introduction to the concept of non-anticipating function can be found in Ref. [37]. A great advantage of Eq. (12) is that it can be easily implemented by means of Monte Carlo simulations as will be shown in the next section.

However, before that, it is important to study the martingale nature of the stochastic process defined by Eq. (12). In order to define martingales, we need a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$, where $(\mathcal{F}_t)_{t\geq 0}$ is a *filtration*—i.e., an increasing family of sub σ-algebras—representing the information available up to time t. A *martingale* is a stochastic process $X(t)$ for which the expected value $\mathrm{E}[|X(t)|]$ exists for $t \geq 0$ and the conditional average $\mathrm{E}[X(t)\,|\,\mathcal{F}_s]$ is $X(s)$ for all $t \geq s$ [36, 38].

B. Martingale property

Although it is easy to directly simulate the stochastic process defined in Eq. (12), it is not so easy to derive its properties. Each term in the sum depends on the previous ones and the nice properties of convolutions are not so helpful here. However, using the *martingale transform theorem*, it is possible to obtain conditions under which $I(t)$ is a martingale.

Let us consider the natural filtration, that is the σ-algebra generated by the CTRW itself: $\mathcal{F}_s = \sigma(x(t) : t \leq s) = \sigma(\xi_1, \dots, \xi_k; \tau_1, \dots, \tau_k : k \leq n(s)) \stackrel{\text{def}}{=} \mathcal{G}_{n(s)}$. Then $x(t)$ is a martingale with respect to $\mathcal{F}_t$ if, and only if, the mean of the jumps $\mathrm{E}[\xi_i]$ is zero. Denote by (t_i, ξ_i) the time and height of the (finitely many) jumps $i = n(s)+1, \dots, n(t)$ occurring between s and $t > s$. Then

$$\mathrm{E}[x(t)\,|\,\mathcal{F}_s] = x(s) + \sum_{i=n(s)+1}^{n(t)} \mathrm{E}[\xi_i\,|\,\mathcal{F}_s]. \tag{13}$$

Using the semi-Markov property, Eq. (3), we get for $i > n(s)$

$$\mathrm{E}[\xi_i\,|\,\mathcal{F}_s] = \mathrm{E}[\xi_i\,|\,\mathcal{G}_{n(s)}] = \mathrm{E}[\xi_i\,|\,\xi_{n(s)}] = \mathrm{E}[\xi_i] = 0, \tag{14}$$

thanks to the independence of ξ_i and $\xi_1, \ldots, \xi_{n(s)}$. Eq. (13) becomes

$$\mathrm{E}[x(t) \,|\, \mathcal{F}_s] = x(s), \tag{15}$$

which shows that $(x(t))_{t\geq 0}$ is indeed a martingale with respect to its natural filtration.

Note that, for a compound Poisson process with zero-mean jumps, the martingale property can also be derived from the independence of increments defined as $\Delta x(t, \Delta t) = x(t+\Delta t) - x(t)$ for all non-overlapping intervals, whereas in general uncoupled CTRWs do not have independent increments.

Let us now investigate the integral defined in Eq. (12) for a martingale CTRW $x(t)$. If there is an arbitrary but finite number of jumps between s and $t > s$, one has the following:

$$\mathrm{E}[I(t) \,|\, \mathcal{F}_s] = I(s) + \sum_{i=n(s)+1}^{n(t)} \mathrm{E}[y(t_i^-)\xi_i \,|\, \mathcal{G}_{n(s)}]; \tag{16}$$

now, one observes that $\xi_i = x(t_i) - x(t_{i-1})$ and that the random sum in Eq. (16) becomes

$$\sum_{i=n(s)+1}^{n(t)} \mathrm{E}[y(t_i^-)\xi_i \,|\, \mathcal{G}_{n(s)}] = \sum_{i=n(s)+1}^{n(t)} \mathrm{E}[y(t_i^-)(x(t_i) - x(t_{i-1})) \,|\, \mathcal{G}_{n(s)}]. \tag{17}$$

By definition, $y(t_i^-)$ is given by $y(t_{i-1})$ and is therefore $\mathcal{G}_{i-1}$-measurable; this is to say that $y(t_i^-)$ is predictable for the filtration $\mathcal{G}_i$, i.e. the value of $y(t_i^-)$ is known at time t_{i-1}. Whenever for each i the expression $y(t_i^-)(x(t_i) - x(t_{i-1}))$ has a finite absolute mean—e.g., if the process $y(t_i^-)$ is bounded—we have

$$\begin{aligned} &\mathrm{E}[y(t_i^-)(x(t_i) - x(t_{i-1})) \,|\, \mathcal{G}_{n(s)}] \\ &\quad = \mathrm{E}\big[\mathrm{E}[y(t_i^-)(x(t_i) - x(t_{i-1})) \,|\, \mathcal{G}_{i-1}] \,|\, \mathcal{G}_{n(s)}\big] \\ &\quad = \mathrm{E}\big[y(t_i^-)\, \mathrm{E}[(x(t_i) - x(t_{i-1})) \,|\, \mathcal{G}_{i-1}] \,|\, \mathcal{G}_{n(s)}\big] \end{aligned} \tag{18}$$

In the above calculation we have used the fact that $\mathcal{G}_{n(s)}$ is contained in $\mathcal{G}_{i-1}$ as $(i-1) \geq n(s)$, along with the *tower property* and the fact that we can *take out what is known* from beneath the conditional expectation [39]. Since $x(t)$ is a martingale, we have $\mathrm{E}[x(t_i) \,|\, \mathcal{F}_{t_{i-1}}] = x(t_{i-1})$ which means that

$$\mathrm{E}[y(t_i^-)(x(t_i) - x(t_{i-1})) \,|\, \mathcal{G}_{n(s)}] = 0. \tag{19}$$

Consequently, each term in the random sum vanishes and $\mathrm{E}[I(t) \,|\, \mathcal{F}_s] = I(s)$. Summing up, if $x(t)$ is a martingale with respect to $\mathcal{F}_t$ and if the integrand is bounded and predictable, one has that $I(t)$ is also a martingale with respect to $\mathcal{F}_t$.

III. SIMULATIONS

As outlined above, the Monte Carlo simulation of an uncoupled CTRWs is straightforward. If one wants to compute the value $x(t)$, it is sufficient to generate a sequence of $n(t)+1$ iid waiting times τ_i until their sum is greater than t. Then the last waiting time can be discarded and $n(t)$ iid jumps ξ_i can be generated. Their sum is the desired value of $x(t)$. Based on Eqs. (1) and (2), this algorithm was used to generate Fig. 1.

Similarly, an algorithm based on Eq. (12) can be implemented by generating a sequence of $n(t)+1$ iid waiting times τ_i until their sum is greater than t. Then after generating $n(t)$ iid jumps ξ_i, their values can be multiplied by $G(x(t_{i-1}))$ and the results of these multiplications can be summed to obtain $I(t)$. In Fig. 2, a Monte-Carlo-generated histogram for $I(t) = \int_0^t x(s)\, dx(s)$ (i.e., with $y(s) = x(s)$) is given, where $t = 100$ and $x(t)$ is a normal compound Poisson process (NCPP). The simulated NCPP has exponentially distributed waiting times with $\lambda = 1$ and normally distributed jumps with $\mu = 0$ and $\sigma = 1$. For a general NCPP, the probability density of finding the value x at time t is given by

$$p(x,t) = \exp(-\lambda t) \sum_{n=0}^{\infty} \frac{(\lambda t)^n}{n!} \frac{1}{\sqrt{2\pi n}\sigma} \exp\left[-\frac{(x-n\mu)^2}{2n\sigma^2}\right]. \tag{20}$$

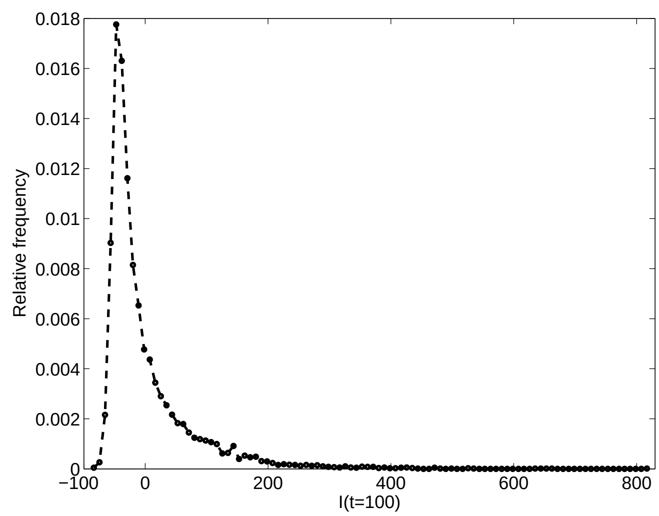

FIG. 2: Histogram of the integral $I(t) = \int_0^t x(s)\, dx(s)$ for exponentially distributed waiting times ($\lambda = 1$) and standard normally distributed jumps ($\mu = 0$ and $\sigma = 1$) and for $t = 100$. The circles represent the results of 10 000 independent realizations of the integral. The dashed line is plotted to guide the eye.

As the NCPP approximates the Bachelier-Wiener process $W(t)$ for $\lambda \to \infty$ and $\sigma \to 0$ with $\lambda\sigma^2 = \sigma_W^2$ [33], when $x(t)$ is an NCPP the integral in Eq. (12) is an approximation of the Itô integral $I_W(t)$. This point is illustrated in Fig. 3 where the histogram of 50 000 values of $I(t = 1) = \int_0^1 x(s)\, dx(s)$ when $\lambda = 10\,000$ and $\sigma = 1/100$ (and $\mu = 0$) is compared to the analytic expression of the probability density for $\sigma_W = 1$ when $I_W(t) = \int_0^t W(s)\, dW(s) = (W^2(t) - t)/2$ and for $t = 1$. The agreement between the Monte Carlo histogram and the analytic formula is excellent. However, a detailed study of convergence properties and bounds is beyond the scope of the present paper.

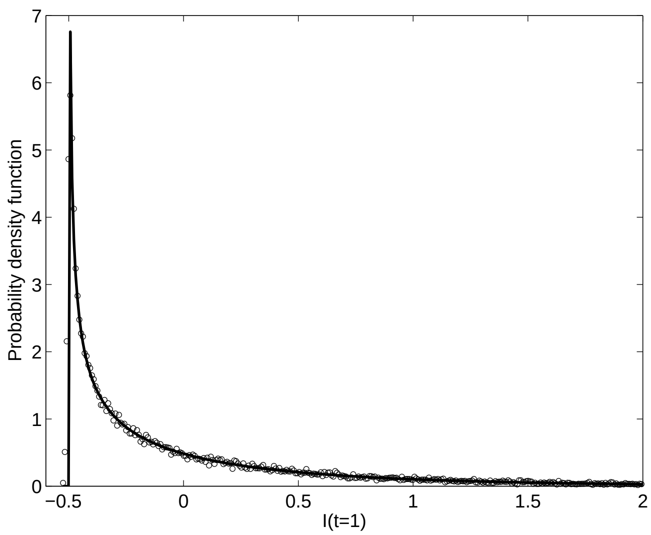

FIG. 3: Comparison between the empirical probability density from Monte Carlo simulation (circles) of $I(t) = \int_0^t x(s)\, dx(s)$ and the analytic probability density for the Itô integral (solid line) $I_W(t) = \int_0^t W(s)\, dW(s) = (W^2(t) - t)/2$, where $W(t)$ is the Bachelier-Wiener process. $x(t)$ is a NCPP with $\lambda = 10\,000$, $\mu = 0$ and $\sigma = 1/100$ yielding $\sigma_W = 1$ for the limiting Bachelier-Wiener process. In this plot $t = 1$ and I_W has the following probability density $p(I_W) = 2\exp[-(2I_W + 1)/2]/\sqrt{2\pi(2I_W + 1)}$.

IV. CONCLUSIONS AND OUTLOOK

This paper is based on the definition of a stochastic integral driven by CTRWs and given in Eq. (12). If the process $x(t)$ that defines the measure used in Eq. (12) is a martingale with respect to its natural filtration, then also $I(t)$ is a martingale. This is a consequence of the martingale transform theorem. It turns out that uncoupled CTRWs with zero-mean jumps are martingales. These results have relevance for applications in insurance and finance as well as in all the fields where martingale methods can help in quantitatively evaluating risks.

Eq. (12) is convenient for Monte Carlo calculations of stochastic integrals. In Section III, it is shown how to use Monte Carlo simulations of the NCPP to effectively approximate the Itô integral based on the Bachelier-Wiener process.

Future work will deal with Monte Carlo simulations for uncoupled and coupled CTRWs where waiting times do not follow the exponential distribution and jumps obey fat-tailed distributions [5, 25, 26, 40, 41].

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