An Alternative Characterization for Iterated Kalai-Smorodinsky-Nash Compromise

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In this paper, we offer for two-person games an alternative characterization of *Iterated Kalai-Smorodinsky-Nash Compromise (IKSNC)*, which was introduced and first characterized by Saglam (2016) for n-person games. We present an axiom called Γ-*Decomposability*, satisfied by any solution that is decomposable with respect to a given reference solution Γ. We then show that the IKSNC solution is uniquely characterized by Γ-Decomposability whenever Γ satisfies the standard axioms of Independence of Equivalent Utility Representations and Symmetry, along with three additional axioms, namely Restricted Monotonicity of Individually Best Extensions, Weak Independence of Irrelevant Alternatives, and Weak Pareto Optimality under Symmetry.

Keywords: Cooperative bargaining; Kalai-Smorodinsky solution; Nash solution

JEL Codes: C71; C78

1 Introduction

In a recent paper, Saglam (2016) proposed a new n-person bargaining solution, called Iterated Kalai-Smorodinsky-Nash Compromise (IKSNC), which reconciles between the well-known solutions of Nash and Kalai-Smorodinsky using no more information than is already contained in these solutions. He also showed that this new solution can be characterized by a single axiom called

Kalai-Smorodinsky-Nash Decomposability, which requires that the outcome of the solution on any bargaining problem can be obtained by first calculating the referential *compromise* point at which each player receives the minimum of the utility payoffs he or she would have received under the Kalai-Smorodinsky and Nash solutions, and then adding this point to the solution of the subproblem admitting it as both the starting and the disagreement point.[1]

The procedure of repeated use of a concept for defining a solution was introduced by Raiffa (1953) in the definition of the discrete (sequential) Raiffa solution. For an alternative characterization of the same solution, this procedure was recently translated by Trockel (2015) into an axiom, called Repeated Application of the Same Solution (RASS), which is a weakening of an earlier axiom of Kalai (1977), called Step by Step Negotiations.[2] The axiom of RASS requires that for any problem including a bargaining set S and a disagreement point d in S, the solution on (S, d) can be obtained by calculating the solution on a reduced problem (S', d') instead, where d' is the solution on the largest hyperplane game (S_d^H, d) with $S_d^H \subseteq S$ and S' is the set of utilities in S not smaller than d'. While the IKSNC solution does not satisfy the axiom of RASS by Trockel (2015), it satisfies, for two-person games, a similar property which we call Γ-*Decomposability*.[3] Given a referential solution Γ, this axiom is satisfied by any solution F if it chooses on any (well-defined) problem S an allocation point that can be obtained by adding the reference solution point $\Gamma(S)$ to the solution of F on the utilities in S that are not smaller than $\Gamma(S)$.

The main result of this paper is that in a two-person bargaining setup the IKSNC solution is the unique solution that satisfies the axiom of Γ-Decomposability when the solution Γ satisfies the axioms of IEUR and SY along with three addi-

[1]The solution of Saglam (2015) can be obtained by the repeated use of an axiom of Rachmilevitch (2014), called Kalai-Smorodinsky-Nash Robustness (KSNR). This axiom requires that each agent receives at least the minimum of the utility payoffs he or she would have received under the Kalai-Smorodinsky and Nash solutions.

[2]This axiom of Kalai (1977) requires that for any two bargaining problems S and S' with $S' \supseteq S$ and the disagreement points normalized to zero, the solution on S' can be obtained by first calculating the solution on S and then taking it to be the starting point for the distribution of the utilities in S' to calculate the solution on this normalized set.

[3]Various forms of decomposability axioms were earlier used by Salonen (1988), Rachmilevitch (2012), Saglam (2014), and Trockel (2014), among others.

tional axioms, namely Restricted Monotonicity of Individually Best Extensions (RMIBE), Weak Independence of Irrelevant Alternatives (WIIA), and Weak Pareto Optimality under Symmetry (WPO-S). Altogether, these five axioms uniquely characterize a solution called *Kalai-Smorodinsky-Nash Compromise (KSNC)*, which selects the aforementioned referential compromise point at each bargaining problem and yields the IKSNC solution when indefinitely repeated. WPO-S requires the solution to lie on the weak Pareto frontier of the bargaining problem whenever it is symmetric. Clearly, this axiom weakens WPO, a common axiom of Kalai-Smorodinsky and Nash solutions not satisfied by the KSNC solution. Likewise, WIIA weakens the axiom of IIA, which is also not satisfied by the KSNC solution. WIIA demands that if a bargaining set S contracts to a subset S' that contains for each player the individually best extension of the solution on S, then for each player the individually best extension of the solution on S and on S' must be the same. Finally, the axiom of RMIBE requests that if a bargaining set S expands to a set S' with the ideal point being unchanged, then the individually best extension of the solution on S' for some player must be weakly Pareto superior to the individually best extension of the solution on S for at least one of the players. In our characterization result, the axiom of WIIA is needed because of the dependence of the KSNC solution on the Nash solution. On the other hand, the axiom of RMIBE accounts for the dependence of the KSNC solution on the Kalai-Smorodinsky solution. However, RMIBE neither implies nor is implied by the Restricted Monotonicity axiom used by Roth (1979) in an alternative characterization of the Kalai-Smorodinsky solution for two-person games.

The paper is organized as follows: Section 2 introduces the basic structures and Section 3 presents our axiomatization results. Finally, Section 4 contains some concluding remarks.

2 Basic Structures

We consider a two-person bargaining problem (simply a problem) denoted by a nonempty subset S of $\mathbb{R}^2_+$, representing von Neumann-Morgenstern utilities attainable through the cooperative actions of two agents. If the agents fail to agree on any point in S, then each of them gets zero utility (i.e., the disagreement utility is normalized yo zero). We also assume that

(a) S is compact and convex and there exists $x \in S$ such that $x > 0$;[4]

(b) for all $x, y \in \mathbb{R}^2_+$, i.e., if $x \in S$ and $x \geq y \geq 0$, then $y \in S$ (comprehensiveness or the possibility of free disposal of utility).

Let Σ^2_0 denote the set of all 0-normalized two-person bargaining problems satisfying the above assumptions. We define a solution F on Σ^2_0 as a mapping from Σ^2_0 to $\mathbb{R}^2_+$ such that for each $S \in \Sigma^2_0$, $F(S) \in S$. The solution by Nash (1950) maps each problem $S \in \Sigma^2_0$ to the point $N(S) = \text{argmax}_{x \in S}\, x_1 x_2$, at which the product of players' payoff gains from agreement is maximized. Below, we will present the set of axioms used by Nash (1950) for an arbitrary solution F on Σ^2_0.

Let the weak and the strong Pareto frontier of any set $S \in \mathbb{R}^2_+$ be respectively defined as $WP(S) = \{x \in S \,|\, y > x \;\text{ implies } y \notin S\}$ and $P(S) = \{x \in S \,|\, y \geq x \text{ and } y \neq x \;\text{ implies } y \notin S\}$.

Weak Pareto Optimality (WPO): If $S \in \Sigma^2_0$, then $F(S) \in WP(S)$.

Any set $S \in \mathbb{R}^2_+$ is said to be symmetric if $(x_1, x_2) \in S$ implies $(x_2, x_1) \in S$.

Symmetry (SY): If $S \in \Sigma^2_0$ and S is symmetric, then $F_1(S) = F_2(S)$.

Let Λ be the set of all $\lambda = (\lambda_1, \lambda_2)$ where each $\lambda_i : \mathbb{R} \to \mathbb{R}$ is a positive affine function, and let $\lambda(S) = \{\lambda(x) : x \in S\}$.

Independence of Equivalent Utility Representations (IEUR): If $S \in \Sigma^2_0$ and $\lambda \in \Lambda$, then $F(\lambda(S)) = \lambda(F(S))$.

Independence of Irrelevant Alternatives (IIA): If $S, S' \in \Sigma^2_0$, $S \supseteq S'$, and $F(S) \in S'$, then $F(S') = F(S)$.

[4] Given two vectors x and y in $\mathbb{R}^2_+$, $x > y$ means $x_i > y_i$ for all $i \in N$ and $x \geq y$ means $x_i \geq y_i$ for $i = 1, 2$.

Nash (1950) showed that his solution is the unique solution that satisfies the axioms of IEUR, IIA, SY, and WPO. In fact, the Nash solution satisfies the strong version of the Pareto optimality, as well.

Pareto Optimality (PO): If $S \in \Sigma_0^2$, then $F(S) \in P(S)$.

While the axioms of IEUR, SY, and WPO are satisfied by many well-known solutions and are therefore called the *standard* axioms in the bargaining literature, IIA has been a *controversial* axiom, having led the researchers to seek for alternative solutions that would satisfy more plausible axioms, possibly in addition to the standard axioms. In that respect, a well-known alternative is the Kalai-Smorodinsky solution (Raiffa, 1953; Kalai and Smorodinsky, 1975), which is based on the ideal (utopia) point of the given bargaining problem. Formally, for any bargaining set S, let $a_i(S)$ denote the maximal utility agent i can expect in S, i.e., $a_i(S) = max\{x_i : x \in S\}$. Then, for any bargaining problem S, the point $a(S) = (a_1(S), a_2(S))$ is called the ideal point. The Kalai-Smorodinsky solution selects in each bargaining set the maximal point on the line segment joining the disagreement point to the ideal point. So, it maps each problem $S \in \Sigma_0^2$ to the point $KS(S) \in WP(S)$ such that $KS_1(S)/KS_2(S) = a_1(S)/a_2(S)$, implying that each player's payoff gain from agreement has the same proportion to his or her ideal payoff gain from agreement. Kalai (1975) showed that this solution is the only solution that satisfies IEUR, SY, WPO, and the following axiom. (In fact, for two-person games the Kalai-Smorodinsky solution satisfies PO, as well.)

Individual Monotonicity (IM): If $S, S' \in \Sigma_0^2$, $S' \supseteq S$, and $a_j(S') = a_j(S)$, then $F_i(S') \geq F_i(S)$ for $i \neq j$.

For two-person problems IM can be replaced by a weaker axiom called Restricted Monotonicity, as shown by Roth (1979). This axiom requires that both players should weakly benefit from an expansion of the bargaining set if the ideal point does not change.

Restricted Monotonicity (RM): If $S, S' \in \Sigma_0^2$, $S' \supseteq S$, and $a(S') = a(S)$, then $F(S') \geq F(S)$.

From the characterization results of the Kalai-Smorodinsky and Nash solutions for two-person games, it should become evident that these two solutions are distinguished from each other only by whether they possess RM or IIA. In situations where it may be necessary to reconcile between the Kalai-Smorodinsky and Nash solutions (or alternatively between the axioms of RM and IIA), one can use the compromise point in Rachmilevitch (2014) to define a benchmark solution. Formally, given any problem S, the said compromise point is the allocation $C(S)$ in S such that $C_i(S) = \min\{KS_i(S), N_i(S)\}$ for every $i \in \{1, 2\}$. Then, consider the solution that maps each problem $S \in \Sigma_0^2$ to the point $C(S)$. We will call this solution **Kalai-Smorodinsky-Nash Compromise (KSNC)**.

Obviously, KSNC is not a desirable solution because it does not satisfy WPO. However, as observed by Saglam (2016), one can iterate KSNC to obtain a limit point on the weak Pareto frontier of a given problem. Formally, given any problem $S \in \Sigma_0^2$ and any point $a \in S$, define $S - \{a\} = \{y \in \mathbb{R}^2 : y = x - a \text{ for some } x \in S\}$. Then, consider the sequence of points $(c^t(S))_{t=0}^{\infty}$ where $c^0(S) = 0$, and $c^t(S) = c^{t-1}(S) + C((S - \{c^{t-1}(S)\}) \cap \mathbb{R}_+^2)$ for each integer $t \geq 1$. (Note that in this definition $c^1(S) = C(S)$.) Clearly, $\lim_{t\to\infty} c^t(S) \in S$. The solution that maps each problem $S \in \Sigma_0^2$ to the point $\lim_{t\to\infty} c^t(S)$ is called, by Saglam (2016), **Iterated Kalai-Smorodinsky-Nash Compromise (IKSNC)**.

3 Results

We will first study the KSNC solution in its relation to the axioms introduced in Section 2. (Below, we denote by $co(S)$ the convex-hull of the set S.)

Remark 1. The KSNC solution satisfies both of the axioms IEUR and SY, but it does not satisfy any of the axioms IIA, RM, and WPO.

Proof. Below, we will check for each axiom in the remark.

IEUR: Let $S \in \Sigma_0^2$. Since the solutions of Nash (N) and Kalai-Smorodinsky (K) both satisfy IEUR, for any vector of positive affine functions $\lambda = (\lambda_1, \lambda_2)$

it is true that $N(\lambda(S)) = \lambda(N(S))$ and $K(\lambda(S)) = \lambda(K(S))$. Then, $C_i(\lambda(S)) = \min\{K_i(\lambda(S)), N_i(\lambda(S))\} = \lambda(\min\{K_i((S)), N_i((S))\} = \lambda(C_i(S))$ for every $i \in \{1, 2\}$. Thus, C satisfies IEUR.

IIA: Let $S = co\{(0,0),(0,1),(1,1),(2,0)\}$. Clearly, $S \in \Sigma_0^2$. It is easy to check that $N(S) = (1,1)$ and $K(S) = (4/3, 2/3)$. Therefore, $C(S) = (1, 2/3)$. Now, consider

$$S' = co\{(0,0),(0,1),(1,1),(4/3,2/3),(4/3,0)\}.$$

Clearly, $S' \in \Sigma_0^2$, $S \supseteq S'$, and $C(S) \in S'$. Also, it is easy to check that $N(S') = (1,1)$ while $K(S') = (8/7, 6/7)$, implying $C(S') = (1, 6/7)$. Since $C(S') \neq C(S)$, we conclude that C does not satisfy IIA.

SY: Let $S \in \Sigma_0^2$ be such that S is symmetric. Since both of the solutions N and K satisfy SY and WPO, $N(S) = K(S)$ and $N_1(S) = N_2(S)$. Then, $C(S) = N(S)$, implying $C_1(S) = C_2(S)$. Thus, C satisfies SY.

RM: Consider the problems

$$S = co\{(0,0),(0,1),(10/7,3/7),(2,0)\}$$

and

$$S' = co\{(0,0),(0,1),(2/3,1),(2,0)\}.$$

Apparently, $S, S' \in \Sigma_0^2$, $S' \supseteq S$, and $a(S') = a(S) = (2,1)$. One can easily check that $K(S') = (6/5, 3/5)$ and $N(S') = (1, 3/4)$, implying $C(S') = (1, 3/5)$. On the other hand, we have $K(S) = (10/9, 5/9)$ and $N(S) = (5/4, 1/2)$, implying $C(S) = (10/9, 1/2)$. It follows that the inequality $C(S') \geq C(S)$ does not hold. So, C does not satisfy RM.

WPO: Reconsider from above $S' = co\{(0,0),(0,1),(2/3,1),(2,0)\} \in \Sigma_0^2$ with $K(S') = (6/5, 3/5)$ and $N(S') = (1, 3/4)$, implying $C(S') = (1, 3/5)$. One can easily check that $y \equiv [K(S') + N(S')]/2 = (44/40, 27/40) \in S'$ and $y > C(S')$. So, $C(S') \notin WP(S)$, implying that C does not satisfy WPO. $\square$

Now, we will axiomatize the KSNC solution after introducing a number of axioms defined for an arbitrary solution F on Σ_0^2. We will first weaken WPO.

Weak Pareto Optimality under Symmetry (WPO-S): If $S \in \Sigma_0^2$ is such that it is symmetric, then $F(S) \in WP(S)$.

For any problem $S \in \Sigma_0^2$, let us denote by $b^i(S, y)$ the best allocation player i can expect in $P(S)$, given that agent $j \neq i$ obtains at least y_j units of utility. We will call the allocation $b^i(S, y)$ *the individually best extension of* y *on* S *for player* i.

Weak Independence of Irrelevant Alternatives (WIIA): If $S, S' \in \Sigma_0^2$, $S \supseteq S'$, and $b^i(S, F(S)) \in S'$ for every $i \in \{1,2\}$, then there exists $j \in \{1,2\}$ such that $b^j(S', F(S')) = b^j(S, F(S))$.

WIIA simply requires that if a bargaining set S contracts to a subset S' that contains for every player the individually best extension of the solution on S, then for some player the individually best extension of the solution on S and S' must be the same. This axiom weakens the independence axiom, IIA, of Nash (1950), as illustrated by the below remarks.

Remark 2. IIA implies WIIA.

Proof. Let F be a solution on Σ_0^2 that satisfies IIA and let $S, S' \in \Sigma_0^2$ be such that $S \supseteq S'$ and $b^i(S, F(S)) \in S'$ for every $i \in \{1,2\}$. Then $F(S) \in S'$ by the comprehensiveness of S'. On the other hand, IIA implies that $F(S') = F(S)$. Then, we have $b^1(S', F(S')) = b^1(S', F(S))$. Note that $b^1(S, F(S)) \in S'$ by assumption and $b^1(S, F(S)) \in P(S)$ by the definition of b^1. Thus, $b^1(S, F(S)) \in S' \cap P(S)$. Moreover, $[S' \cap P(S)] \subseteq P(S')$. So, $b^1(S, F(S)) \in P(S')$. Since we also have $b^1(S', F(S)) \in P(S')$ by the definition of b^1, we must have $b^1(S', F(S)) = b^1(S, F(S))$. It then follows that $b^1(S', F(S')) = b^1(S, F(S))$, implying that F satisfies WIIA. □

Remark 3. WIIA does not imply IIA.

Proof. Let $S, S' \in \Sigma_0^2$ be such that $S \supseteq S'$, and $b^i(S, C(S)) \in S'$ for every $i \in \{1,2\}$. We have $C_i(S) = \min\{KS_i(S), N_i(S)\}$ for every $i \in \{1,2\}$, by the definition of the KSNC solution. Since N satisfies PO, there exists $k \in \{1,2\}$

such that $b^k(S, C(S)) = N(S)$, implying $N(S) \in S'$ due to the assumption about S'. Moreover, $N(S') = N(S)$, since N satisfies IIA. By the definition of C, it follows that there exists $j \in \{1, 2\}$ such that $C_j(S') = N_j(S')$. Along with $N_j(S') = N_j(S)$, this implies $C_j(S') = N_j(S)$. Then, for $k \neq j$ we must have $b^k(S', C(S')) = N(S)$, implying $b^k(S', C(S')) = b^k(S, C(S))$. So, the KSNC solution satisfies WIIA. On the other hand, Remark 1 shows that the KSNC solution does not satisfy IIA. □

Restricted Monotonicity of Individually Best Extensions (RMIBE): If $S, S' \in \Sigma_0^2$, $S' \supseteq S$, and $a(S') = a(S)$, then there exists $j \in \{1, 2\}$ and $k \in \{1, 2\}$ such that $b^j(S', F(S')) \geq b^k(S, F(S))$.

The above axiom requires that if a bargaining set S expands to a set S' with no change in the ideal point, then the individually best extension of the solution on S' for some player must be weakly Pareto superior to the individually best extension of the solution on S for at least one of the players. As will be shown below, this new axiom neither implies nor is implied by the axiom of RM.

Remark 4. RMIBE does not imply RM.

Proof. Let $\hat{S} = co\{(0, 0), (1, 0), (0, 1)\}$. Clearly, $\hat{S} \in \Sigma_0^2$. Let F be a solution on Σ_0^2 such that

$$F(\tilde{S}) = \begin{cases} (3/4, 0) & \text{if } \tilde{S} = \hat{S}, \\ (1/2, 0) & \text{if } \tilde{S} \neq \hat{S}. \end{cases}$$

Step 1: Let $S, S' \in \Sigma_0^2$ be such that $S' \supseteq S$ and $a(S') = a(S)$. If $S' = S$, then the condition for RMIBE trivially holds. So, let $S' \neq S$. First assume that $S = \hat{S}$. Then, $F(S) = (3/4, 0)$, whereas $F(S') = (1/2, 0)$, since $S' \neq \hat{S}$. It follows that $b^1(S', F(S')) = b^1(S, F(S)) = (1, 0)$ since $a(S') = a(S)$ by assumption and $a(S) = a(\hat{S}) = (1, 1)$. Therefore, for $j = 1$ and $k = 1$, the inequality $b^j(S, F(S)) \geq b^k(S', F(S'))$ is satisfied. Now, assume that $S' \neq \hat{S}$. Then, we must have $F(S') = F(S) = (1/2, 0)$, implying $b^1(S', F(S')) \geq b^1(S, F(S))$ since $S' \supseteq S$. So, it is true that for $j = 1$ and $k = 1$, the inequality $b^j(S, F(S)) \geq b^k(S', F(S'))$ is satisfied.

Step 2: We will show that F does not satisfy RM. Let $S = \hat{S}$ and $S' =$

$co\{(0,0),(1,0),(1,1),(0,1)\}$. Clearly, $S, S' \in \Sigma_0^2$, $S' \supseteq S$, and $a(S') = a(S)$. In order F to satisfy RM, the inequality $F(S') \geq F(S)$ must hold. But, we have $F(S') = (1/2, 0)$ and $F(S) = (3/4, 0)$, violating this inequality. □

Remark 5. RM does not imply RMIBE.

Proof. Let $\hat{S} = co\{(0,0),(0,1),(1,0)\}$. (Note that $\hat{S} \in \Sigma_0^2$.) Also let F be a solution on Σ_0^2 such that

$$F(\tilde{S}) = \begin{cases} (0,0) & \text{if } \tilde{S} = \hat{S}, \\ K(\tilde{S}) & \text{if } \tilde{S} \neq \hat{S}. \end{cases}$$

Step 1. Let $S, S' \in \Sigma_0^2$ be such that $S' \supseteq S$ and $a(S') = a(S)$. If $S' = S$, the condition for RMIBE trivially holds, thus let $S' \neq S$. Note that S' cannot be equal to $\hat{S}$. (For otherwise S would also be equal to $\hat{S}$ since $WP(\hat{S}) = P(\hat{S})$.) This implies $F(S') = K(S')$. As to S, we have two possibilities. If $S = \hat{S}$, then $F(S) = (0,0)$. Since $K(S') > (0,0)$, the inequality $F(S') \geq F(S)$ would be satisfied. On the other hand, if $S \neq \hat{S}$, then $F(S) = K(S)$. Along with the fact that $F(S') = K(S')$, this implies $F(S') \geq F(S)$, since K satisfies RM. We have established that the inequality $F(S') \geq F(S)$ always holds. Thus, F satisfies RM.

Step 2. Let $S = \hat{S}$ and $S' = co\{(0,0),(1,0),(3/4,3/4),(0,1)\}$. Apparently, $S, S' \in \Sigma_0^2$, $S' \supseteq S$, and $a(S') = a(S) = (1,1)$. It follows that $F(S) = (0,0)$ and $F(S') = K(S')$, implying $b^1(S, F(S)) = b^1(S,(0,0)) = (1,0)$ and $b^2(S, F(S)) = b^2(S,(0,0)) = (0,1)$. On the other hand, $b^1(S', F(S')) = b^2(S', F(S')) = K(S')$ since K satisfies PO. But, neither $K(S') \geq (1,0)$ nor $K(S') \geq (0,1)$ can hold, since $\{(1,0),(0,1),K(S')\} \subset PO(S')$. It then follows that $b^j(S', F(S'))$ $\geq b^k(S, F(S))$ cannot hold for any $j, k \in \{1,2\}$. Therefore, F does not satisfy RMIBE. □

The following remark implies that the axiom of RMIBE is satisfied by any solution that satisfies both RM and PO (e.g. the Kalai-Smorodinsky solution).

Remark 6. RM and PO together imply RMIBE.

Proof. Let F be a solution that satisfies RM and PO, and consider any

$S, S' \in \Sigma_0^2$ be such that $S' \supseteq S$ and $a(S') = a_i(S)$. If $S' = S$, the condition for RMIBE trivially holds. So, let $S' \neq S$. Note that $b^j(S', F(S')) = F(S')$ for every $j \in \{1,2\}$ and $b^k(S, F(S)) = F(S)$ for every $k \in \{1,2\}$ by PO. Moreover, $F(S') \geq F(S)$ by RM, implying $b^j(S', F(S')) \geq b^k(S, F(S))$ for every $j, k \in \{1,2\}$, which ensures that F satisfies RMIBE. □

Now, we are ready to introduce our first characterization result.

Theorem 1. *A solution satisfies IEUR, RMIBE, SY, WIIA, and WPO-S if and only if it is the KSNC solution.*

Proof. "⇒": Remark 1 shows that the KSNC solution satisfies IEUR and SY. On the other hand, the proof of Remark 3 shows that the KSNC solution satisfies WII, as well. To show that it satisfies WPO-S, consider any symmetric S in Σ_0^2. Since both N and K satisfy SY and WPO, we must have $N(S) = K(S)$, implying $C(S) = N(S)$. Therefore, $C(S) \in WP(S)$, implying WPO-S is satisfied. Finally, to show that the KSNC solution also satisfies RMIBE, let $S, S' \in \Sigma_0^2$ be such that $S' \supseteq S$ and $a(S') = a(S)$. If $S' = S$, the condition of RMIBE trivially holds. So, let $S' \neq S$. By the definition of the solution C, we know that $C_j(S') = K_j(S')$ for some $j \in \{1,2\}$ and $C_k(S) = K_k(S)$ for some $k \in \{1,2\}$. Consider players $m, n \in \{1,2\}$ such that $m \neq j$ and $n \neq k$. Then, we must have $b^m(S', C(S')) = K(S')$ and $b^n(S, C(S)) = K(S)$. Finally, since K satisfies RM, we must have $K(S') \geq K(S)$, implying $b^m(S', C(S')) \geq b^n(S, C(S))$. Thus, the KSNC solution satisfies RMIBE.

"⇐": Pick any solution F on Σ_0^2 that satisfies IEUR, RMIBE, SY, WIIA, and WPO-S. Let $S \in \Sigma_0^2$.

Step 1: Since the solution N satisfies IEUR, there exists a vector of positive affine functions $\lambda' = (\lambda'_1, \lambda'_2)$ such that $(1,1) = N(\lambda'(S))$. Then, let $S' = \lambda'(S)$. Consider $T = co\{(0,0), (0,2), (2,0)\}$. As T is symmetric and F satisfies SY and WPO-S, we have $F(T) = (1,1)$. Then, for every $i \in \{1,2\}$ we have $b^i(T, F(T)) = (1,1)$, while we already know that $(1,1) \in S'$. Since F also satisfies WIIA, there exists $j \in \{1,2\}$ such that $b^j(S', F(S')) = b^j(T, F(T)) = (1,1)$. Using $(1,1) = N(S')$, we then have $b^j(S', F(S')) = N(S')$ for some $j \in \{1,2\}$. Finally, using $S' = \lambda'(S)$ along with the fact that both F and N satisfy IEUR, we can replace $b^j(S', F(S'))$ and $N(S')$ in the above equality with $\lambda'(b^j(S, F(S)))$ and $\lambda'(N(S))$, respectively. This would imply that there exists $j \in \{1, 2\}$ such that $b^j(S, F(S)) = N(S)$.

Step 2: Since the solution K satisfies IEUR, there exists a vector of positive affine functions $\lambda'' = (\lambda''_1, \lambda''_2)$ such that $(1, 1) = K(\lambda''(S))$. Then, let $S'' = \lambda''(S)$. Consider $T = co\{(0, 0), (0, a_2(S'')), (1, 1), (a_1(S''), 0)\}$. From $K_1(S'') = K_2(S'')$, it follows that $a_1(S'') = a_2(S'')$, implying that T is symmetric. Then, $F(T) = (1, 1)$, because F satisfies SY and WPO-S. Note also that $S'' \supseteq T$ and $a(S'') = a(T)$. Since F satisfies RMIBE, there exist $j, k \in \{1, 2\}$ such that $b^j(S'', F(S'')) \geq b^k(T, F(T)) = (1, 1)$. This is equivalent to saying that there exists j such that $b^j(S'', F(S'')) \geq K(S'')$, since $(1, 1) = K(S'')$. Finally, using $S'' = \lambda''(S)$ along with the fact that both F and K satisfy IEUR, we can replace $b^j(S'', F(S''))$ and $K(S'')$ in the last inequality with $\lambda''(b^j(S, F(S)))$ and $\lambda''(K(S))$, respectively. This would imply that there exists $j \in \{1, 2\}$ such that $b^j(S, F(S)) \geq K(S)$. For every $j \in \{1, 2\}$, $b^j(S, F(S)) \in P(S)$ by definition. Moreover, $K(S) \in P(S)$ since K satisfies PO. Thus, we must have $b^j(S, F(S)) = K(S)$ for some $j \in \{1, 2\}$.

Steps 1 and 2 respectively show that $b^j(S, F(S)) = N(S)$ for some $j \in \{1, 2\}$ and $b^k(S, F(S)) = K(S)$ for some $k \in \{1, 2\}$. So, it must be true that $F(S) = C(S)$. □

Obviously, the KSNC solution also satisfies a stronger version of WPO-S, which we call PO-S, since both of the Kalai-Smorodinsky and Nash solutions satisfy the axiom of PO and the bargaining sets are assumed to be convex.

Pareto Optimality under Symmetry (PO-S): If $S \in \Sigma_0^2$ and S is symmetric, then $F(S) \in P(S)$.

Now, we will consider the characterization of the IKSNC solution. However, we first observe the following.

Remark 7. The IKSNC solution satisfies all of the axioms IEUR, SY, and WPO, but it satisfies neither IIA nor RM.

Proof. Since the solutions K and N are different from each other, the IKSNC solution is different from both K and N. From the definition of the IKSNC solution and the fact that the KSNC solution satisfies IEUR and SY, it is clear that the IKSNC solution satisfies all of the standard axioms IEUR, SY, and WPO. However, since the solutions K and N are the unique solutions that respectively satisfy RM and IIA in addition to these standard axioms, the IKSNC solution cannot satisfy RM or IIA. $\square$

Apparently, the IKSNC solution also satisfies PO, as this axiom is satisfied by both Kalai-Smorodinsky and Nash solutions. Now, we consider the following axiom for any solution F on Σ_0^2.

Γ-Decomposability: If Γ is a solution on Σ_0^2, then $F(S) = \Gamma(S) + F((S - \Gamma(S)) \cap \mathbb{R}_+^2)$.

Theorem 2. *There exists a unique solution that satisfies Γ-Decomposability whenever the solution Γ satisfies IEUR, RMIBE, SY, WIIA, and WPO-S. That solution is the IKSNC solution.*

Proof. "Existence": By Theorem 1, the KSNC solution satisfies IEUR, RMIBE, SY, WIIA, and WPO-S. On the other hand, the definition of the IKSNC solution implies that the IKSNC solution satisfies Γ-Decomposability when Γ is equal to KSNC solution. Thus, we have established that there exists a solution that satisfies Γ-Decomposability whenever the solution Γ satisfies IEUR, RMIBE, SY, WIIA, and WPO-S.

"Uniqueness": Let F be a solution that satisfies Γ-Decomposability whenever Γ is a solution on Σ_0^2 satisfying IEUR, RMIBE, SY, WIIA, and WPO-S. By Theorem 1, the solution Γ is unique and equal to the KSNC solution, denoted by C. Note that Γ-Decomposability with $\Gamma = C$ implies $F(S) = C(S) + F((S - C(S)) \cap \mathbb{R}_+^2)$ for any $S \in \Sigma_0^2$. Now, pick any $S \in \Sigma_0^2$, and consider the sequence of problems $(S^t)_{t=0}^{\infty}$ where $S^0 = S$ and $S^t = (S^{t-1} - C(S^{t-1})) \cap \mathbb{R}_+^2$ for every integer $t \geq 1$. It is clear that $F(S^0) = C(S^0) + F((S^0 - C(S^0)) \cap \mathbb{R}_+^2) = C(S^0) + F(S^1)$. Iterating this equation t more times yields $F(S^0) = (\sum_{j=0}^{t} C(S^j)) + F(S^{t+1})$, implying $F(S^{t+1}) = F(S^0) - \sum_{j=0}^{t} C(S^j)$ for every integer $t \geq 0$. Recall that given any $S \in \Sigma_0^2$, the IKSNC solution selects the allocation $\lim_{t\to\infty} c^t(S^0)$ in S. Now suppose that $F(S^0) \neq \lim_{t\to\infty} c^t(S^0)$. Then, one can easily show by the geometry of the rule F that there exists $k \geq 0$ such that $F(S^{k+1}) \notin S^{k+1}$, a contradiction. Therefore, $F(S^0) = \lim_{t\to\infty} c^t(S^0)$. Since $S \in \Sigma_0^2$ was arbitrarily picked, F must coincide with the IKSNC solution. □

4 Conclusion

In this paper, we have attempted to offer, for two-person games, an alternative characterization of Iterated Kalai-Smorodinsky-Nash Compromise, a new bargaining solution introduced by Saglam (2016) for n-person games. To that end, we have introduced an axiom called Γ-Decomposability, which requires that given a reference solution Γ the outcome of the solution F on any bargaining problem S can be obtained by adding the solution point $\Gamma(S)$ to the solution point chosen by F on the subproblem of S admitting $\Gamma(S)$ as its starting point. We have showed that the axiom of Γ-Decomposability uniquely characterizes the IKSNC solution whenever the solution Γ associated with the axiom is requested to satisfy the standard axioms of IEUR and SY, along with three additional axioms we have introduced in this paper, namely RMIBE, WIIA, and WPO-S. These five axioms in fact characterize the KSNC solution, which -if indefinitely repeated- yields the IKSNC solution. Of these five axioms, IEUR, SY, and WPO-S account for the common attributes of the Kalai-Smorodinsky and Nash solutions over which a one-shot compromise yields the KSNC solution. On the other hand, RMIBE and WIIA are needed to axiomatize the uncommon attributes of the Kalai-Smorodinsky and the Nash solutions, respectively.

The future research might extend our work to n-person games. We should recall here that the characterization of the IKSNC solution critically depends on the characterization of the KSNC solution that chooses for each player the minimum of the utility payoffs he or she would have received under the Kalai-Smorodinsky and Nash solutions. As already known, the axiomatization result by Nash (1950) for two-person games straightforwardly extends to n-person games. On the other hand, for the Kalai-Smorodinsky solution axiomatization results are nontrivially different for the two-person and n-person games (when

$n \geq 3$) as shown by the work of Thomson (1983).[5] Thus, one may conjecture that the axiomatization of the KSNC solution could also be different for the two types of games.

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[5] Thomson (1983) showed that in the n-person case the Kalai-Smorodinsky solution turns out to be the only solution that possesses Anonymity (AN), Continuity (CONT), IEUR, Monotonicity With Respect to Changes in the Number of Agents (MON), and WPO. Of these axioms, IEUR and WPO are generalizations of the former conditions used by Kalai and Smorodinsky (1975) for the two-person case. The axiom AN is a strengthening of SY and it requires that not only the names of the agents in a given group do not matter but also that any other group of agents of the same size would reach the same bargaining outcome. CONT implores that a small change in the bargaining set causes only a small change in the bargaining outcome. Finally, MON says that if the expansion of a group of agents requires a sacrifice to support the entrants, then every incumbent must contribute. The Nash solution satisfies AN, CONT, IEUR, and WPO; but it does not satisfy MON as directly shown by Thomson and Lensberg (1989, pp. 41-42).

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