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A Bayesian implementable social choice function cannot be implemented by a direct mechanism

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Abstract

The revelation principle is a fundamental theorem in many economics fields such as game theory, mechanism design and auction theory etc. In this paper, I construct an example to show that a social choice function which can be implemented in Bayesian Nash equilibrium by an indirect mechanism cannot be implemented by a direct mechanism. The key point is that agents pay cost in the indirect mechanism, but pay nothing in the direct mechanism. As a result, the revelation principle does not hold at all.

JEL codes: D70

Key words: Revelation principle; Game theory; Mechanism design; Auction theory.

1 Introduction

The revelation principle plays an important role in microeconomics theory and has been applied to many other fields such as auction theory, game theory etc. According to the wide-spread textbook given by Mas-Colell, Whinston and Green (Page 884, Line 24 [1]): "*The implication of the revelation principle is ... to identify the set of implementable social choice functions, we need only identify those that are truthfully implementable.*" Related definitions about the revelation principle can be seen in Appendix, which are cited from Section 23.B and 23.D of MWG's textbook[1].

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However, in this paper, I will construct a simple labor model to show the revelation principle does not hold at all. Section 2 is the main part of this paper, and Section 3 draws conclusions.

2 A labor model

Here we consider a simple labor model which uses some ideas from the first-price sealed auction model in Example 23.B.5 [1] and the signaling model in Section 13.C [1]. There are one firm and two workers. The firm wants to hire a worker, and two workers compete for this job offer. Worker 1 and Worker 2 differ in the number of units of output they produce if hired by the firm, which is denoted by productivity type.

For simplicity, we make the following assumptions:
1) The possible productivity types of two workers are: $\theta_1, \theta_2 \in \{\theta_L, \theta_H\}$, where $\theta_H > \theta_L > 0$. Each worker's productivity is a random variable chosen independently, and is private information for each worker.
2) Before confronting the firm, each worker $i = 1, 2$ can get some education. The possible levels of education are: e_L and e_H, where $e_H > e_L > 0$. Each worker's education is observable to the firm. Education does nothing for a worker's productivity.
3) The cost of obtaining education level e for a worker of some type θ is given by a function $c(e, \theta) = e/\theta$. That is, the cost of education is lower for a high-productivity worker.

The model's outcome can be represented by a vector (y_1, y_2), where y_i denotes the probability that worker i gets the job offer. Recall that the firm does not know the exact productivity types of two workers, but its aim is to hire a worker with productivity as high as possible. This aim can be represented by a social choice function $f(\vec{\theta}) = (y_1(\vec{\theta}), y_2(\vec{\theta}))$, in which $\vec{\theta} = (\theta_1, \theta_2)$,

$$y_1(\vec{\theta}) = \begin{cases} 1, & if \theta_1 > \theta_2 \\ 0.5, & if \theta_1 = \theta_2 \\ 0, & if \theta_1 < \theta_2 \end{cases}, y_2(\vec{\theta}) = \begin{cases} 1, & if \theta_1 < \theta_2 \\ 0.5, & if \theta_1 = \theta_2 \\ 0, & if \theta_1 > \theta_2 \end{cases} \quad (1)$$

In order to implement the above $f(\vec{\theta})$, the firm designs an indirect mechanism $\Gamma = (S_1, S_2, g)$ as follows:
1) A random move of nature determines the productivity of workers: $\theta_1, \theta_2 \in \{\theta_L, \theta_H\}$.
2) Conditional on his type θ_i, each worker $i = 1, 2$ chooses his education level as a bid $b_i : \{\theta_L, \theta_H\} \to \{e_L, e_H\}$. The strategy set S_i is the set of all possible

bids $b_i(\theta_i)$, and the outcome function g is defined as:

$$g(b_1, b_2) = (p_1, p_2) = \begin{cases} (1,0), & if b_1 > b_2 \\ (0.5, 0.5), & if b_1 = b_2 \\ (0,1), & if b_1 < b_2 \end{cases} \quad (2)$$

where $(1,0)$ means worker 1 gets the offer, $(0,1)$ means worker 2 gets the offer; $(0.5, 0.5)$ means that each worker has the same probability 0.5 to be the winner.

Let u_0 be the utility of the firm, and u_1, u_2 be the utilities of worker $1, 2$ respectively, then $u_0(b_1, b_2) = p_1\theta_1 + p_2\theta_2 - w$, and for $i, j = 1, 2$, $i \neq j$,

$$u_i(b_i, b_j; \theta_i) = \begin{cases} w - b_i/\theta_i, & if b_i > b_j \\ 0.5w - b_i/\theta_i, & if b_i = b_j \\ -b_i/\theta_i, & if b_i < b_j \end{cases} \quad (3)$$

Proposition 1: The social choice function $f(\vec{\theta})$ can be implemented by the indirect mechanism Γ in Bayesian Nash equilibrium.
Proof: Consider a separating strategy, *i.e.*, workers with different productivity types choose different education levels,

$$b_1(\theta_1) = \begin{cases} e_H, & if \theta_1 = \theta_H \\ e_L, & if \theta_1 = \theta_L \end{cases}, b_2(\theta_2) = \begin{cases} e_H, & if \theta_2 = \theta_H \\ e_L, & if \theta_2 = \theta_L \end{cases}. \quad (4)$$

Now let us check whether this separating strategy yields a Bayesian Nash equilibrium. Assume $b_j^*(\theta_j)$ takes this form, *i.e.*,

$$b_j^*(\theta_j) = \begin{cases} e_H, & if \theta_j = \theta_H \\ e_L, & if \theta_j = \theta_L \end{cases}, \quad (5)$$

then consider worker i's problem ($i \neq j$). For each $\theta_i \in \{\theta_L, \theta_H\}$, worker i solves the maximization problem $\max_{b_i} h(b_i, \theta_i)$, where by Eq (3) the object function is

$$h(b_i, \theta_i) = (w - b_i/\theta_i)P(b_i > b_j^*(\theta_j)) + (0.5w - b_i/\theta_i)P(b_i = b_j^*(\theta_j)) - (b_i/\theta_i)P(b_i < b_j^*(\theta_j)) \quad (6)$$

We discuss this maximization problem in four different cases:
1) Suppose $\theta_i = \theta_j = \theta_L$, then $b_j^*(\theta_j) = e_L$ by Eq (5).

$$\begin{aligned} h(b_i, \theta_i) &= (w - b_i/\theta_L)P(b_i > e_L) + (0.5w - b_i/\theta_L)P(b_i = e_L) - (b_i/\theta_L)P(b_i < e_L) \\ &= \begin{cases} w - e_H/\theta_L, & if b_i = e_H \\ 0.5w - e_L/\theta_L, & if b_i = e_L \end{cases} \end{aligned}$$

Thus, if $w < 2(e_H - e_L)/\theta_L$, then $h(e_H, \theta_i) < h(e_L, \theta_i)$, which means the optimal value of $b_i(\theta_i)$ is e_L. In this case, $b_i^*(\theta_L) = e_L$.

2) Suppose $\theta_i = \theta_L$, $\theta_j = \theta_H$, then $b_j^*(\theta_j) = e_H$ by Eq (5).

$$\begin{aligned} h(b_i, \theta_i) &= (w - b_i/\theta_L)P(b_i > e_H) + (0.5w - b_i/\theta_L)P(b_i = e_H) - (b_i/\theta_L)P(b_i < e_H) \\ &= \begin{cases} 0.5w - e_H/\theta_L, & if b_i = e_H \\ -e_L/\theta_L, & if b_i = e_L \end{cases} \end{aligned}$$

Thus, if $w < 2(e_H - e_L)/\theta_L$, then $h(e_H, \theta_i) < h(e_L, \theta_i)$, which means the optimal value of $b_i(\theta_i)$ is e_L. In this case, $b_i^*(\theta_L) = e_L$.

3) Suppose $\theta_i = \theta_H$, $\theta_j = \theta_L$, then $b_j^*(\theta_j) = e_L$ by Eq (5).

$$\begin{aligned} h(b_i, \theta_i) &= (w - b_i/\theta_H)P(b_i > e_L) + (0.5w - b_i/\theta_H)P(b_i = e_L) - (b_i/\theta_H)P(b_i < e_L) \\ &= \begin{cases} w - e_H/\theta_H, & if b_i = e_H \\ 0.5w - e_L/\theta_H, & if b_i = e_L \end{cases} \end{aligned}$$

Thus, if $w > 2(e_H - e_L)/\theta_H$, then $h(e_H, \theta_i) > h(e_L, \theta_i)$, which means the optimal value of $b_i(\theta_i)$ is e_H. In this case, $b_i^*(\theta_H) = e_H$.

4) Suppose $\theta_i = \theta_j = \theta_H$, then $b_j^*(\theta_j) = e_H$ by Eq (5).

$$\begin{aligned} h(b_i, \theta_i) &= (w - b_i/\theta_H)P(b_i > e_H) + (0.5w - b_i/\theta_H)P(b_i = e_H) - (b_i/\theta_H)P(b_i < e_H) \\ &= \begin{cases} 0.5w - e_H/\theta_H, & if b_i = e_H \\ -e_L/\theta_H, & if b_i = e_L \end{cases} \end{aligned}$$

Thus, if $w > 2(e_H - e_L)/\theta_H$, then $h(e_H, \theta_i) > h(e_L, \theta_i)$, which means the optimal value of $b_i(\theta_i)$ is e_H. In this case, $b_i^*(\theta_H) = e_H$.

In summary, if $w \in [2(e_H - e_L)/\theta_H, 2(e_H - e_L)/\theta_L]$, then the the strategy $b_i^*(\theta_i)$ of worker i

$$b_i^*(\theta_i) = \begin{cases} e_H, & if \theta_i = \theta_H \\ e_L, & if \theta_i = \theta_L \end{cases} \tag{7}$$

is the optimal response to the strategy $b_j^*(\theta_j)$ of worker j ($j \neq i$) given in Eq (5). Therefore, the strategy combination $(b_1^*(\theta_1), b_2^*(\theta_2))$ is a Bayesian Nash equilibrium of the game induced by Γ.

By Eq(2) and Eq(7), for any $\vec{\theta} = (\theta_1, \theta_2)$, $\theta_1, \theta_2 \in \{\theta_L, \theta_H\}$,

$$g(b_1^*(\theta_1), b_2^*(\theta_2)) = \begin{cases} (1, 0), & if \theta_1 > \theta_2 \\ (0.5, 0.5), & if \theta_1 = \theta_2 \\ (0, 1), & if \theta_1 < \theta_2 \end{cases} \tag{8}$$

which implements the social choice function $f(\vec{\theta})$. **Q.E.D**

Proposition 2: The social choice function $f(\vec{\theta})$ cannot be implemented truthfully in a direct mechanism.
Proof: Consider a direct mechanism Γ_{direct}, the timing steps of which is as follows:
1) Each worker $i = 1, 2$ announces his productivity type $\hat{\theta}_i \in \{\theta_L, \theta_H\}$ to a virtual mediator.
2) The mediator submits $b_i^*(\hat{\theta}_i)$ $(i = 1, 2)$ to the firm:

$$b_i^*(\hat{\theta}_i) = \begin{cases} e_H, & if \hat{\theta}_i = \theta_H \\ e_L, & if \hat{\theta}_i = \theta_L \end{cases}$$

3) The firm performs the outcome function $g(b_1, b_2)$, and hires the winner.

Since each worker i does not need to pay the cost b_i/θ_i when playing in the direct mechanism, the utility function of each worker $i = 1, 2$ is changed from Eq (3) to the follows:

$$u_i(\hat{\theta}_i, \hat{\theta}_j; \theta_i) = \begin{cases} w, & if \hat{\theta}_i > \hat{\theta}_j \\ 0.5w, & if \hat{\theta}_i = \hat{\theta}_j \\ 0, & if \hat{\theta}_i < \hat{\theta}_j \end{cases} \tag{9}$$

The utility matrix is as follows.

$\hat{\theta}_i$ \ $\hat{\theta}_j$	θ_L	θ_H
θ_L	$[0.5w, 0.5w]$	$[0, w]$
θ_H	$[w, 0]$	$[0.5w, 0.5w]$

Obviously, the dominant strategy for each worker $i = 1, 2$ is to definitely announce $\hat{\theta}_i = \theta_H$, no matter what his true productivity type is. Consequently, the social choice function $f(\vec{\theta})$ cannot be implemented truthfully in a direct mechanism. **Q.E.D**

3 Conclusions

From Proposition 1 and 2, we can see that:
1) In the indirect mechanism Γ, the utility function of each worker $i = 1, 2$ is given by Eq (3). It is the cost b_i/θ_i of a type θ_i worker choosing the bid b_i that makes the strategy combination $(b_1^*(\theta_1), b_2^*(\theta_2))$ become a Bayesian Nash equilibrium. The cost of a worker is irrevocable even if he loses the job competition.

2) In the direct mechanism Γ_{direct}, the utility function of each worker $i = 1, 2$ is given by Eq (9), where the cost disappears. Thus, each worker is free to pretend to be a high-productivity worker without any suffer. There is no way for the direct mechanism to discriminate this counterfeit.
In summary, the revelation principle does not hold at all.

Appendix: Definitions in Section 23.B and 23.D [1]

Definition 23.B.1: A social choice function is a function $f : \Theta_1 \times \cdots \times \Theta_I \to X$ that, for each possible profile of the agents' types $(\theta_1, \cdots, \theta_I)$, assigns a collective choice $f(\theta_1, \cdots, \theta_I) \in X$.

Definition 23.B.3: A mechanism $\Gamma = (S_1, \cdots, S_I, g(\cdot))$ is a collection of I strategy sets $S_1, \cdots, S_I$ and an outcome function $g : S_1 \times \cdots \times S_I \to X$.

Definition 23.B.5: A direct revelation mechanism is a mechanism in which $S_i = \Theta_i$ for all i and $g(\theta) = f(\theta)$ for all $\theta \in \Theta_1 \times \cdots \times \Theta_I$.

Definition 23.D.1: The strategy profile $s^*(\cdot) = (s_1^*(\cdot), \cdots, s_I^*(\cdot))$ is a *Bayesian Nash equilibrium* of mechanism $\Gamma = (S_1, \cdots, S_I, g(\cdot))$ if, for all i and all $\theta_i \in \Theta_i$,

$$E_{\theta_{-i}}[u_i(g(s_i^*(\theta_i), s_{-i}^*(\theta_{-i})), \theta_i)|\theta_i] \geq E_{\theta_{-i}}[u_i(g(\hat{s}_i, s_{-i}^*(\theta_{-i})), \theta_i)|\theta_i]$$

for all $\hat{s}_i \in S_i$.

Definition 23.D.2: The mechanism $\Gamma = (S_1, \cdots, S_I, g(\cdot))$ implements the social choice function $f(\cdot)$ in Bayesian Nash equilibrium if there is a Bayesian Nash equilibrium of Γ, $s^*(\cdot) = (s_1^*(\cdot), \cdots, s_I^*(\cdot))$, such that $g(s^*(\theta)) = f(\theta)$ for all $\theta \in \Theta$.

Definition 23.D.3: The social choice function $f(\cdot)$ is truthfully implementable in Bayesian Nash equilibrium if $s_i^*(\theta_i) = \theta_i$ (for all $\theta_i \in \Theta_i$ and $i = 1, \cdots, I$) is a Bayesian Nash equilibrium of the direct revelation mechanism $\Gamma = (\Theta_1, \cdots, \Theta_I, f(\cdot))$. That is, if for all $i = 1, \cdots, I$ and all $\theta_i \in \Theta_i$,

$$E_{\theta_{-i}}[u_i(f(\theta_i, \theta_{-i})), \theta_i)|\theta_i] \geq E_{\theta_{-i}}[u_i(f(\hat{\theta}_i, \theta_{-i}), \theta_i)|\theta_i], \qquad (23.D.1)$$

for all $\hat{\theta}_i \in \Theta_i$.

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References

[1] A. Mas-Colell, M.D. Whinston and J.R. Green, Microeconomic Theory, Oxford University Press, 1995.