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Abstract

It has long been understood that externalities of some kind are responsible for Sen's (1970) theorem on the impossibility of a Paretian liberal. However, Saari and Petron (2006) show that for any social preference cycle generated by combining the weak Pareto principle and individual decisiveness, every decisive individual must suffer at least one strong negative externality. We show that this fundamental result only holds when individual preferences are strict. Building on their contribution, we prove a general theorem for the case of weak preferences.

Keywords: Sen's impossibility theorem; Liberal paradox; Saari-Petron theorem; Externalities; Social preference cycles

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1 Introduction

Sen's (1970) theorem on the impossibility of a Paretian liberal is one of the landmark results of social choice theory.[1] To illustrate the theorem, Sen presents the following example. There are two individuals, Prude and Lewd, and a copy of the risqué *Lady Chatterley's Lover*. There are three options: Prude reads the book (x), Lewd reads the book (y), or no one reads the book (z). Lewd prefers most for Prude to read the book. Sen says that Lewd would "delight" in Prude's discomfort (p. 155). Given a choice between reading the book himself or no one reading it, Lewd would prefer (being a lewd) to read it. His preference ordering is, therefore, $xP_l yP_l z$.[2] Prude prefers most that no one reads it. However, if someone must then he would rather it be him. Prude believes that if Lewd reads it then he could become even more depraved. Prude's ordering is, therefore, $zP_p xP_p y$.

Given that the alternatives are taken to represent social states with "each state being a complete description of society including every individual's position in it",[3] alternatives y and z differ only in ways that are private to Lewd, and x and z differ only in ways that are private to Prude. On foot of this, we might feel that Lewd's and Prude's preferences over these respective pairs should be given special status when determining the social preference. Suppose we make Lewd *decisive* over $\{y, z\}$. This means that $yP_l z$ implies yPz and $zP_l y$ implies zPy. Similarly, make Prude decisive over $\{x, z\}$. If we combine this decisiveness with the *weak Pareto* principle then we obtain a cycle in the social preference relation, $yPzPxPy$.

Many argue that Sen's formal framework is not the correct one for modeling individual rights.[4] However, there are aggregation contexts where Sen's

[1] Suzumura (2011) is a survey. Salles (2008, 2009) and Salles and Zhang (2010) is recent foundational work on Sen's theorem.

[2] Our notation is standard.

[3] Sen (1970, p. 152).

[4] These critics favor a game form approach to rights. See Nozick (1974), Sugden (1985) and Gaertner, Suzumura and Pattanaik (1992).

idea of decisiveness remains relevant.[5] For example, the alternatives could represent candidates for appointment at a university rather than complete social states. In this case, Professor A could be a co-author of candidates y and z and Professor B could be a co-author of candidates x and z. On the basis of the superior information that co-authorship typically provides, the university could assign $\{y, z\}$ decisiveness to A and $\{x, z\}$ decisiveness to B. A similar social preference cycle can arise. We call cycles that are generated by combining both weak Pareto and decisiveness *Sen cycles*.

2 Saari-Petron theorem

Saari and Petron (2006) establish an important necessary condition for Sen cycles. Every individual must suffer at least one *strong negative externality*. This notion of "strong preference" is used in Saari (1995, 2001) and the idea can be traced back to Luce and Raiffa (1957) and Blau (1975). Given Lewd's ordering xP_lyP_lz we can write, following Saari and Petron, $[yz, 0]$ to denote that no alternative separates y and z. Lewd's $\{y, z\}$ ranking is *weak*. For Prude, zP_pxP_py and so $[zy, 1]$. Prude's $\{y, z\}$ ranking is *strong* (and opposed to Lewd's). In this formulation 0 and 1 are meant to indicate the *intensity* of the binary $\{y, z\}$ ranking. This intensity measure can be greater than 1 if there are more alternatives.

Saari and Petron give the following definition (Definition 2, p. 272): For any pair of alternatives $\{x, y\}$, a decisive agent's choice of x creates a strong, negative externality if another agent's sincere ranking is $[yx, \alpha]$ with *positive* α intensity. It is easy to verify that Prude and Lewd both suffer strong negative externalities from each other's choices. However, surprisingly, this property is general. Saari and Petron show that if there is a Sen cycle then *every individual who is decisive over some pair in the cycle* must suffer at

[5] On this point see Risse (2001), Dietrich and List (2008), Li and Saari (2008) and Saari (2011).

least one strong negative externality. We call this the *Saari-Petron theorem*.[6] In this paper, we show that this result only holds when individual preferences are strict, i.e. no individual can be indifferent between alternatives. As the examples below demonstrate, the result does not hold when individual preferences are weak. However, building on their ideas, we can prove a general theorem that covers the case of weak preferences.

We say that individual r suffers a negative externality if there exists $\{a, b\}$ such that aPb and yet for r we have $[ba, \beta]$ with $\beta \geq 0$.

Example 1. There are three individuals i, j and k, and their preferences over the four alternatives are (respectively) $xP_iyP_izP_iw$, $wP_jzP_jxP_jy$ and $zP_kxI_kwP_ky$. Note that individual k is indifferent between x and w. Making i decisive over $\{y, w\}$, j decisive over $\{w, z\}$, and k decisive over $\{z, x\}$ leads to $yPwPzPxPy$ where xPy follows from the weak Pareto principle. For i we have $[xz, 1]$ and for j we have $[wy, 2]$. Both suffer strong negative externalities. However, for k we have *both* $[wy, 0]$ and $[zw, 0]$, and so k suffers only weak negative externalities.

The next example shows that cycles can occur where no individual suffers a strong negative externality.

Example 2. There are four individuals i, j, k and l, and their preferences over the five alternatives are (respectively) $xP_iyI_izP_iwP_iv$, $yP_jvI_jxP_jzP_jw$, $yI_kwP_kvI_kzI_kx$ and $zP_lxI_lyI_lwP_lv$. Making i decisive over $\{x, y\}$, j decisive over $\{v, w\}$, k decisive over $\{w, z\}$ and l decisive over $\{z, x\}$ leads to $xPyPvPwPzPx$ where yPv follows from the weak Pareto principle. It is easy to verify that all individuals suffer only weak negative externalities.

It can be verified in these examples that if individual indifference is broken in some way, then each individual suffers at least one strong negative externality.

[6] Theorem 3 in their paper. The clause that the strong negative externalities are suffered by *decisive* individuals is important. In the Lady Chatterley example, a third individual j could hold the ordering xP_jzP_jy and suffer no strong negative externality.

	$\{x, y\}$	$\{y, z\}$	$\{z, x\}$
r	xP_ry	yP_rz	zP_rx
r	yP_rx	$-$	zP_rx
r	xI_ry	zP_ry	zP_rx

Table 1: Restoring transitivity for r.

3 General theorem

Saari and Petron (p. 269) describe an "information table" (Table 1) and we use this to prove our theorem. We consider two cases: one where the cycle involves three alternatives, and another where the cycle involves four or more alternatives (but a finite number). Let us consider the first case and assume that $xPyPzPx$.

Imagine that we assign to an arbitrary individual r each of these social *pairwise* rankings (Table 1, first row). The result is intransitive preferences for r. In order to restore transitivity (and recover r's original *pairwise* rankings induced by her transitive preferences), we need to change at least one of these pairwise rankings. Assume that zPx follows from the application of the weak Pareto principle. Therefore, we cannot change that pairwise ranking. We have to change at least one pairwise ranking over which the individual is *not* decisive. The individual cannot be decisive over all of the decisive pairs in the Sen cycle, otherwise her preferences would be intransitive. Therefore, there must be some pair (or pairs) of alternatives over which she is not decisive. She could be decisive over no pair.

Without loss of generality, assume that she is not decisive over $\{x, y\}$ and either yP_rx or xI_ry (second and third rows of Table 1 respectively). Note that xI_ry implies zP_ry by transitivity.

This leads immediately to the following result.[7]

Lemma 3. *If there is a Sen cycle involving three alternatives, every individual suffers a negative externality.*

[7]See Bernholz (1982) and Campbell and Kelly (1997).

This statement holds for all individuals (not just decisive ones). If r is decisive over either $\{x, y\}$ or $\{y, z\}$ (but not both) then xP_ry or yP_rz. In the former case, transitivity implies that zP_rxP_ry and in the latter case it implies yP_rzP_rx. Lemma 3 states that r suffers a negative externality, and the preceding argument shows that it must be a strong negative externality. Therefore, we can state the following.

Lemma 4. *If there is a Sen cycle involving three alternatives, then every individual who is decisive over some pair in the cycle suffers a strong negative externality.*

Note that for Examples 1 and 2, the cycles involve more than three alternatives. Lemma 4 says that counter-examples to the Saari-Petron theorem cannot be found with three alternatives.

We now consider the case of cycles involving four or more alternatives. Suppose that $xPyPzPw...vPx$. In the arguments below, and without loss of generality, assume that xPy follows from the application of the weak Pareto principle. The following generalization of Lemma 3 holds.

Lemma 5. *If there is a Sen cycle involving four or more alternatives, every individual suffers a negative externality.*

Proof. Consider the set of pairs in the cycle that follow from the exercise of someone's power of decisiveness. We need to prove that there exists $\{z, w\}$ in this set such that zPw but wP_rz. By contradiction, if this is not the case then zR_rw for all $\{z, w\}$ in the set. Transitivity implies yR_rx. However, this must be false since, by assumption, xPy follows from the weak Pareto principle. $\square$

Like Lemma 3, this statement holds for all individuals. Suppose now that r is decisive over some pair in the cycle. We can establish the following result.

Lemma 6. *Suppose there is a Sen cycle involving four or more alternatives. Take any individual who is decisive over some pair in the cycle. If that* *individual suffers only one negative externality, then that externality must be strong.*

Proof. Lemma 5 implies that r must suffer a negative externality. Recall the information table of Saari and Petron. Without loss of generality assume that r is decisive over $\{z, w\}$ and that the sole externality suffered by r arises from vPx.

	$\{x, y\}$	$\{y, z\}$	$\{z, w\}$	...	$\{v, x\}$
r	xP_ry	$\neg zP_ry$	zP_rw	...	xP_rv
r	xP_ry	yR_rz	zP_rw	...	xP_rv

Table 2: Information table.

The assumption that the only externality suffered arises from vPx implies row 1 of Table 2, which can be written as row 2. Note that $xP_ryR_rzP_rwR_iv$ implies that $[xv, \alpha]$ with $\alpha > 0$. The externality is strong. □

It is now possible to state our main result.

Theorem 7. *If there is a Sen cycle, then every individual who is decisive over some pair in the cycle suffers at least one strong negative externality, or at least two weak negative externalities.*

Proof. If there is a Sen cycle involving three alternatives then Lemma 4 shows that the externality suffered must be strong. If the Sen cycle involves four or more alternatives then Lemma 6 shows that the decisive individual cannot suffer only one weak negative externality. Therefore, the individual must suffer at least one strong negative externality, or at least two weak negative externalities. □

It is easy to verify that individual k suffers two weak negative externalities in Example 1, and that individuals i, j, k and l suffer at least two weak negative externalities in Example 2.

We now state and prove the Saari-Petron theorem. Our proof is different to theirs, and it highlights the role played by the assumption of strict preferences.

Theorem 8. *If there is a Sen cycle, then every individual who is decisive over some pair in the cycle and holds a strict preference ordering suffers at least one strong negative externality.*

Proof. If the Sen cycle involves three alternatives then the result follows from Lemma 4. Suppose the Sen cycle involves four or more alternatives. If there is just one negative externality, then the result follows from Lemma 6. Consider Table 3. Without loss of generality, assume that r is decisive over $\{z, w\}$ and that r suffers negative externalities from sPt and vPx. This gives rise to the first row of the information table. The dots in the columns indicate that r's ranking of that pair in the cycle is identical to the social ranking. By way of contradiction, assume that both the $\{s, t\}$ and $\{v, x\}$ externalities are weak, with s, t, v and x distinct.

	$\{x, y\}$	...	$\{s, t\}$	...	$\{z, w\}$	...	$\{v, x\}$
r	xP_ry	...	tP_rs	...	zP_rw	...	xP_rv
r	xP_ry	...	tP_rs	...	zP_rw	...	vP_rx
r	xP_ry	...	sP_rt	...	zP_rw	...	vP_rx

Table 3: Saari-Petron theorem.

Note that if r holds a strict preference ordering and these two negative externalities are weak, then these pairwise rankings can be reversed (one at a time) without affecting any other pairwise ranking. In row 2, r's $\{v, x\}$ ranking has been reversed, for example, without any other pairwise ranking changing. However, in row 3, r's $\{s, t\}$ ranking has been reversed and this leads to intransitive preferences for r. This is a contradiction and so the $\{s, t\}$ externality must have been strong, not weak.

If $t = v$ then consider Table 4.

	$\{x,y\}$	...	$\{s,t\}$	$\{t,x\}$
r	xP_ry	...	tP_rs	xP_rt

Table 4: When $t = v$.

From row 1, we can see that xP_ryP_rs and also that xP_rtP_rs. Given that the externalities are weak and that r holds a strict preference ordering, we have $xP_rtP_rsP_ryP_rs$. However, this leads to intransitive preferences for r and so one of these externalities must have been strong, not weak. □

When preferences are weak, the argument in Theorem 8 no longer applies. First, if r holds a weak preference ordering and negative externalities are weak, then it does not follow that these pairwise rankings can be reversed (one at a time) without affecting any other pairwise ranking. In Example 1 we have $zP_kxI_kwP_ky$ and k suffers a weak negative externality from yPw. Reversing k's $\{y,w\}$ ranking will affect her $\{x,y\}$ ranking. Second, for the argument in Table 4, the assumption of strict preferences is critical. If preferences are weak, then we could have $xP_ryI_rtP_rs$ and this would not be intransitive.

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