Back to the Sixties: A Note on Multi-Primary-Factor Linear Models with Homogeneous Capital

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Abstract

This paper extends Bruno's (1967) one capital good two-sector growth model with discrete technology by allowing multiple primary factors of production. While the existence of an optimal steady state is established for any positive rate of discount, an example in which three "modified golden rules" exist shows that the optimal steady state is non necessarily unique. The extended model provides a simple exemplification of the more general principle that the presence of multiple primary factors of production into homogeneous capital models can definitively result into the same complications that arise when there is joint production.

Keywords: Homogeneous capital, Multiple primary factors, Linear activity models, Duality.

JEL Classification: C62, O41.

1 Introduction

A recently published unfinished handwritten manuscript by Paul Samuelson (transcripted by Edwin Burmeister) (Samuelson & Burmeister, 2016) outlines a linear activity model with alternative known techniques each involving, along with labour and corn seed, non-reproducible land and then allows the possibility that each category of inputs involve heterogeneous varieties. The model is presented as an example of "the non-Clark Sraffa-Samuelson paradigm" and is contrasted with "the Clark paradigm" in which technology is smooth. Samuelson's plan was to begin with the short period problem of competitive pricing given amounts of the different factors of production and then to consider "the intertemporal phasing of technology" (Samuelson & Burmeister, 2016, p. 160) and, hence, the fact that "the produced inputs currently used [*are*] themselves

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the outputs of earlier periods" (Samuelson & Burmeister, 2016, p. 161). However, the manuscript is highly incomplete and only three conclusions - the last two of which should hold both in the Clark smooth case and in the discrete linear activity case - are clearly stated:

1. the short run competitive factor pricing can be reduced to the solution of a pair of dual linear programming problems;

2. appropriately reinterpreted to take into account that in the discrete case the "production functions" are neither differentiable nor strictly concave, the "neoclassical" comparative statics properties of the Solow-Ramsey model hold if capital is homogeneous and joint production is excluded;

3. only the usual paradoxical behaviors, in particular the existence of a range of the interest rates in which the steady state level of consumption increases with the rate of interest, can appear "as soon as there are joint products and/or multiple heterogeneous capital goods" (Samuelson & Burmeister, 2016, p. 162).

This note has two aims: to recall that conclusion 2 is not robust to the details of the modeling strategy, and to show by means of a specific example that there are indeed very simple multi-primary-factor linear models without proper joint production and with homogeneous capital for which a finite number a multiple steady states exists.[1] In a sense, in these kind of models the multiplicity of primary factors of productions acts as a substitute for joint production in allowing multiple turnpikes as in Liviatan & Samuelson (1969) or Burmeister & Turnovsky (1972). Note that a slight different interpretation has been advanced in Burmeister (1975), where it is suggested that "a kind of joint intrinsic production [...] occurs when the number of primary factors exceeds one" (Burmeister, 1975, p. 500).

Building on the result that multiplicity of the steady states is possible in one capital good Ricardian models with intensive rent (Freni, 1991, 1997), the paper presents an example showing that multiple steady state can also exist in the classical one capital good linear activity model developed in Bruno (1967), provided at least two primary factors (two qualities of labour, for example) are required in the production of the two goods of the system.[2]

The continuous time framework used here precludes a direct comparison with the discrete time case of circulating capital and no attempt is done in this work to establish whether conclusion 2 above stands in discrete time models in which capital is not durable. Strictly speaking indeed, fixed capital cannot be avoided in continuous time. In continuous time, however, joint production occurs if the flow output vector of at least a process contains more than one positive entry and

[1] Note that multiple steady states in the form of a continuum of turnpikes belonging to a convex set occur in all kinds of linear model whenever the stationarity conditions are satisfied at a switch point.

[2] A multi-sector version of the model without primary factors of production and a CRRA utility function has been studied in the endogenous growth literature see e.g. Freni *et al.* (2003, 2006, 2008).

this is not implied by fixed capital as such, as it is instead in discrete time. In particular, since we stick with the usual assumption that the rate of depreciation of capital is a constant not affected by capital utilization, our scenario is one in which there is single production despite the fact that capital is durable. So, what the example shows is that the presence of heterogenous primary factors into the classical one capital two-sector growth model can definitively result into the complications that arise when there is joint production (cfr. Etula, 2008, p. 100).

A multiple-primary-factor extension of Bruno's (1967) two-sector model is briefly reviewed in Section 2. The example is presented in Section 3. Section 4 concludes.

2 A two-sector multiple-primary-factor linear model

Consider the two-sector multiple-technique case of the discrete capital model introduced in Bruno (1967) under the hypothesis that multiple primary factors in fixed supply are used in production. In the system, there are two commodities: a pure capital good and a pure consumption good. The services of s, $s \geq 1$, primary factors of production, different qualities of labour for simplicity, are combined with the services of the stock of capital to produce the two commodities. Technology is of the discrete type without joint production, comprising m, $m \geq 1$, processes for producing the consumption good and n, $n \geq 1$, processes that produce the capital good.

When process j, $j \in \{1, 2, ..., n\}$, is used, a unit of the capital good needs, to be produced, a_{kj} units of the capital good services and $[l_{kj1}, l_{kj2}, ..., l_{kjs}]$ units of the services of the primary factors of production, whereas the production of one unit of the consumption good by means of the i-th, $i \in \{1, 2, ..., m\}$, process requires a_{ci} units of the capital good services and $[l_{ci1}, l_{ci2}, ..., l_{cis}]$ units of the services of the primary factors of production. So the technology is described by a couple of capital coefficients vectors

$$\mathbf{a}_c = \begin{bmatrix} a_{c1} & a_{c2} & ... & a_{cm} \end{bmatrix}^T, \quad \mathbf{a}_k = \begin{bmatrix} a_{k1} & a_{k2} & ... & a_{kn} \end{bmatrix}^T,$$

and a couple of labour coefficients matrices

$$\mathbf{L}_c = [l_{cir}]_{i=1,..,m;r=1,..,s}, \quad \mathbf{L}_k = [l_{kjr}]_{j=1,..,n;r=1,..,s}.$$

Without loss of generality, it is assumed that two processes that produce the same good differ at least for an entry. Moreover, all entries in the above vectors and matrices are assumed to be non negative.

Let $k(t) \geq 0$ represent the stock of capital at a given time $t \geq 0$, and

$$\mathbf{x}_k(t) = \begin{bmatrix} x_{k1}(t) & x_{k2}(t) & ... & x_{kn}(t) \end{bmatrix}^T,$$

$$\mathbf{x}_c(t) = \left[\begin{array}{cccc} x_{c1}(t) & x_{c2}(t) & ... & x_{cm}(t) \end{array}\right]^T,$$

be the intensities of activation of the production processes at that time. Assuming that the flow of new capital is accumulated, that capital decay at a constant rate $\delta > 0$, and that the initial state of the system is $k_0 \geq 0$, then the state equation is given by the differential equation

$$\begin{cases} \dot{k}(t) = \mathbf{x}_k(t)^T \mathbf{e} - \delta k(t), \quad t \geq 0 \\ k(0) = k_0. \end{cases} \tag{1}$$

Assume that the different labour flows available at every t are constant and given by the strictly positive vector $\mathbf{h} = \left[\begin{array}{cccc} h_1 & h_2 & ... & h_s \end{array}\right]^T > \mathbf{0}$, and assume that every unit of capital good instantaneously provides one unit of production services. Under these assumptions the production is subject to the following set of constraints, holding for all $t \geq 0$

$$\mathbf{x}_c(t)^T \mathbf{L}_c + \mathbf{x}_k(t)^T \mathbf{L}_k \leq \mathbf{h}^T, \tag{2}$$

$$\mathbf{x}_c(t)^T \mathbf{a}_c + \mathbf{x}_k(t)^T \mathbf{a}_k \leq k(t), \tag{3}$$

$$\mathbf{x}_c(t) \geq \mathbf{0}, \mathbf{x}_k(t) \geq \mathbf{0}. \tag{4}$$

Let the planner's instantaneous utility be given by the amount of consumption good produced at a given time t and assume that the rate of interest (or discount) is the constant $r \geq 0$, then the planner problem is maximizing

$$J(\mathbf{x}_c(t), \mathbf{x}_k(t)) = \int_0^{+\infty} e^{-rt} \mathbf{x}_c(t)^T \mathbf{e} \, \mathrm{d}t \tag{5}$$

over the set of admissible controls

$$\mathcal{U}(k_0) = \{(\mathbf{x}_c(t), \mathbf{x}_k(t)) \in L^1_{loc}(0, +\infty; \mathbb{R}^{n+m}_+) \ : \ (1) - (4) \text{ hold at all } t \geq 0\}.$$

To simplify the analysis of the special features of the Hamiltonians of the problem at hand, let us make three more specific assumptions about the technology:

(H1) $\mathbf{a}_c > \mathbf{0}$ and $\mathbf{a}_k > \mathbf{0}$;

(H2) the set $\mathcal{A} = \{(\mathbf{x}_c, \mathbf{x}_k) \in \mathbb{R}^{n+m}_+ : \text{ (2) holds}\}$ is bounded;

(H3) $\exists j \in \{1, 2, ..., n\} : \delta \mathbf{e}_j^T \mathbf{a}_k < 1$.

The current value pre-Hamiltonian associated to the problem is

$$h(k, v_k, \mathbf{x}_c, \mathbf{x}_k) = \mathbf{x}_c \mathbf{e} + (\mathbf{x}_k \mathbf{e} - \delta k) v_k,$$

where v_k is the price of the capital good, while the current value Hamiltonian is

$$H(k, v_k) = -\delta k v_k$$
$$+ \sup\{\mathbf{x}_c \mathbf{e} + \mathbf{x}_k \mathbf{e} v_k : (\mathbf{x}_c, \mathbf{x}_k) \in \mathbb{R}^{n+m}_+, \, \mathbf{x}_c^T \mathbf{L}_c + \mathbf{x}_k^T \mathbf{L}_k \leq \mathbf{h}^T, \, \mathbf{x}_c^T \mathbf{a}_c + \mathbf{x}_k^T \mathbf{a}_k \leq k\}.$$

The maximization process through which the Hamiltonian is computed is therefore equivalent to the following Linear Programming problem

$$\max[\mathbf{x}_c^T \mathbf{e} + \mathbf{x}_k^T \mathbf{e} v_k] \tag{6}$$

subject to

$$\mathbf{x}_c^T \mathbf{L}_c + \mathbf{x}_k^T \mathbf{L}_k \leq \mathbf{h}^T, \tag{7}$$

$$\mathbf{x}_c^T \mathbf{a}_c + \mathbf{x}_k^T \mathbf{a}_k \leq k, \tag{8}$$

$$\mathbf{x}_c \geq \mathbf{0}, \mathbf{x}_k \geq \mathbf{0}. \tag{9}$$

Under assumption (H1) (or (H2)) the feasible region is bounded and the maximum exists. The corresponding dual problem is

$$\min[kq + \mathbf{h}^T \mathbf{w}] \tag{10}$$

subject to

$$\mathbf{e} \leq \mathbf{a}_c q + \mathbf{L}_c \mathbf{w} \tag{11}$$

$$\mathbf{e} v_k \leq \mathbf{a}_k q + \mathbf{L}_k \mathbf{w} \tag{12}$$

$$q \geq 0, \mathbf{w} \geq \mathbf{0}, \tag{13}$$

where $q \in \mathbb{R}_+$ and $\mathbf{w} \in \mathbb{R}_+^s$ are dual control variables having the economic meaning, respectively, of the rental rate of the capital good and the wage rates. Since the primal has an optimal solution, the dual has an optimal solution too (see e.g. Franklin, 1980). For any given stock of capital at time t, $k(t)$, the short run competitive factor prices are the solutions of this dual problem, thus confirming Samuelson's conclusion 1 in Section 1.

A *modified golden rule* (or simply a *golden rule* if the rate of interest is zero) is a solution of the above primal and dual linear programs that satisfies the additional stationary conditions:

$$q = (\delta + r) v_k, \tag{14}$$

$$\mathbf{x}_k^T \mathbf{e} = \delta k. \tag{15}$$

It is known that a primal component of a modified golden rule is a stationary solution of the planner problem (5), and that, vice versa, any stationary optimal solution of the planner problem (5) can be supported by a stationary price system. Moreover, it is also known that if the golden rule capital is unique, then the primal component of a golden rule is an optimal overtaking solution of the planner problem (5) (see Lemma 3.3 and Example 4.1 of Leizarowitz, 1985, see also Example 4.4 of Carlson *et al.* , 1991). We can now use equation (15) to substitute for k into inequality (8) and equation (14) to substitute for v_k into inequality (12). This reduces the problem of identifying the modified golden rules (and the golden rules) to finding the solutions of the following *linear complementarity problem*:

$$\begin{bmatrix} \mathbf{x}_c^T & \mathbf{x}_k^T \end{bmatrix} \begin{bmatrix} \mathbf{L}_c & \mathbf{a}_c \\ \mathbf{L}_k & \mathbf{a}_k - \delta^{-1}\mathbf{e} \end{bmatrix} \leq \begin{bmatrix} \mathbf{h}^T & 0 \end{bmatrix} \tag{16}$$

$$\begin{bmatrix}\mathbf{x}_c^T & \mathbf{x}_k^T\end{bmatrix}\begin{bmatrix}\mathbf{L}_c & \mathbf{a}_c \\ \mathbf{L}_k & \mathbf{a}_k - \delta^{-1}\mathbf{e}\end{bmatrix}\begin{bmatrix}\mathbf{w} \\ q\end{bmatrix} = \mathbf{h}^T\mathbf{w} \tag{17}$$

$$\begin{bmatrix}\mathbf{e} \\ \mathbf{0}\end{bmatrix} \leq \begin{bmatrix}\mathbf{L}_c & \mathbf{a}_c \\ \mathbf{L}_k & \mathbf{a}_k - (\delta + r)^{-1}\mathbf{e}\end{bmatrix}\begin{bmatrix}\mathbf{w} \\ q\end{bmatrix} \tag{18}$$

$$\mathbf{x}_c^T\mathbf{e} = \begin{bmatrix}\mathbf{x}_c^T & \mathbf{x}_k^T\end{bmatrix}\begin{bmatrix}\mathbf{L}_c & \mathbf{a}_c \\ \mathbf{L}_k & \mathbf{a}_k - (\delta + r)^{-1}\mathbf{e}\end{bmatrix}\begin{bmatrix}\mathbf{w} \\ q\end{bmatrix} \tag{19}$$

$$\begin{bmatrix}\mathbf{x}_c^T & \mathbf{x}_k^T\end{bmatrix} \geq \begin{bmatrix}\mathbf{0}^T & \mathbf{0}^T\end{bmatrix}, \quad \begin{bmatrix}\mathbf{w} \\ q\end{bmatrix} \geq \begin{bmatrix}\mathbf{0} \\ 0\end{bmatrix}. \tag{20}$$

Define $R = inf\{r \in \mathbb{R} \ : \ \mathbf{a}_k - (\delta + r)^{-1}\mathbf{e} > \mathbf{0}\}$ (observe $R > 0$ by assumption (H3)). If $r > R$, then in any solution of the complementary problem (16)-(20) $\mathbf{x}_k = \mathbf{0}$ and hence, using inequality $\mathbf{x}_c^T\mathbf{a}_c + \mathbf{x}_k^T(\mathbf{a}_k - \delta^{-1}\mathbf{e}) \leq 0$ in (16), $\mathbf{x}_c = \mathbf{0}$. Then noticing that a null vector of wage rates and any rental rate q such that $\mathbf{e} \leq \mathbf{a}_c q$ constitute a set of supporting factor prices for the quantity vector $\mathbf{x}_k = \mathbf{0}$, $\mathbf{x}_c = \mathbf{0}$ proves that the complementary problem (16)-(20) has a solution. For $r = R$, although non trivial steady states can be proved to exist, $\mathbf{x}_k = \mathbf{0}$ and $\mathbf{x}_c = \mathbf{0}$ with the above prices is still a solution for the system. Finally, when $R > r \geq 0$ our assumptions imply that the key hypothesis of the Complementary Construction Theorem in Dantzig & Manne (1974) holds.[3] So a solution to the

[3]Define $\mathbf{C}^T = \begin{bmatrix}\mathbf{L}_c & \mathbf{a}_c \\ \mathbf{L}_k & \mathbf{a}_k - \delta^{-1}\mathbf{e}\end{bmatrix}$, $\mathbf{D}^T = \begin{bmatrix}\mathbf{0} & \mathbf{0} \\ \mathbf{0} & r\delta^{-1}(\delta + r)^{-1}\mathbf{e}\end{bmatrix}$, $\mathbf{x} = \begin{bmatrix}\mathbf{x}_c \\ \mathbf{x}_k\end{bmatrix}$, $\mathbf{v} = \begin{bmatrix}w \\ \mathbf{q}\end{bmatrix}$, $\mathbf{l} = \begin{bmatrix}\mathbf{h} \\ 0\end{bmatrix}$ and $-\mathbf{f} = \begin{bmatrix}-\mathbf{e} \\ \mathbf{0}\end{bmatrix}$ and rewrite the linear complementary problem (16)-(20) as follows:

$$\begin{bmatrix}\mathbf{u}_1 \\ \mathbf{u}_2\end{bmatrix} = \begin{bmatrix}\mathbf{0} & -\mathbf{C} \\ \mathbf{C}^T + \mathbf{D}^T & \mathbf{0}\end{bmatrix}\begin{bmatrix}\mathbf{v} \\ \mathbf{x}\end{bmatrix} + \begin{bmatrix}\mathbf{l} \\ -\mathbf{f}\end{bmatrix} \geq \mathbf{0}$$

$$\begin{bmatrix}\mathbf{v}^T & \mathbf{x}^T\end{bmatrix}\begin{bmatrix}\mathbf{u}_1 \\ \mathbf{u}_2\end{bmatrix} = 0.$$

Dantzig & Manne (1974) proved that the Lemke algorithm leads to a solution of this problem, provided the sets of optimal solutions of the two following linear programming problems are both nonempty and bounded:

$$\max \begin{bmatrix}\mathbf{e}^T & \mathbf{0}^T\end{bmatrix}\begin{bmatrix}\mathbf{x}_c \\ \mathbf{x}_k\end{bmatrix} \tag{21}$$

subject to

$$\begin{bmatrix}\mathbf{C}\end{bmatrix}\begin{bmatrix}\mathbf{x}_c \\ \mathbf{x}_k\end{bmatrix} \leq \begin{bmatrix}\mathbf{h} \\ 0\end{bmatrix} \tag{22}$$

$$\begin{bmatrix}\mathbf{x}_c \\ \mathbf{x}_k\end{bmatrix} \geq \begin{bmatrix}\mathbf{0} \\ \mathbf{0}\end{bmatrix}; \tag{23}$$

and

$$\min \begin{bmatrix}\mathbf{h}^T & 0\end{bmatrix}\begin{bmatrix}\mathbf{w} \\ q\end{bmatrix} \tag{24}$$

subject to

$$\begin{bmatrix}\mathbf{C}^T + \mathbf{D}^T\end{bmatrix}\begin{bmatrix}\mathbf{w} \\ q\end{bmatrix} \geq \begin{bmatrix}\mathbf{e} \\ \mathbf{0}\end{bmatrix} \tag{25}$$

$$\begin{bmatrix}\mathbf{w} \\ q\end{bmatrix} \geq \begin{bmatrix}\mathbf{0} \\ 0\end{bmatrix}. \tag{26}$$

The null vector is feasible both for problem (21)-(23) and for the dual of problem (24)-(26). Moreover, choosing a sufficiently large $m > 0$, the vector $\begin{bmatrix}m\mathbf{e} \\ 0\end{bmatrix}$ is feasible both for problem linear complementary problem (16)-(20) exists at any given non negative rate of interest. When $r = 0$, the gap between the matrix in (16)/(17) and the matrix in (18)/(19) vanishes, and the linear complementarity problem collapses to the standard pair of dual linear programs that characterize the golden rule: the primal maximizes steady state consumption, the dual minimizes steady state rents for the primary factors. Note that under the expansibility assumption (H3) the golden rule consumption and capital are both strictly positive.

Let now $\mathbf{X}_c(r)$ and $\mathbf{X}_k(r)$ be the sets of the intensity levels vectors $\mathbf{x}_c$ and $\mathbf{x}_k$ belonging to a solution of (16)-(20) at a given $r \geq 0$. Define the two following correspondences: the steady state capital stock $K^s(r) = \{k : k = \delta^{-1}\mathbf{x}_k^T\mathbf{e},\ \mathbf{x}_k \in \mathbf{X}_k(r)\}$, and the steady state consumption flow $C^s(r) = \{c : c = \mathbf{x}_c^T\mathbf{e},\ \mathbf{x}_c \in \mathbf{X}_c(r)\}$, and the following four step functions: $K_+(r) = \max_{k\in K^s(r)} k$, $K_-(r) = \min_{k\in K^s(r)} k$, $C_+(r) = \max_{c\in C^s(r)} c$ and $C_-(r) = \min_{c\in C^s(r)} c$. The following result can be proved:

Proposition 2.1 *Assume at least one of the following conditions holds: (i) $\mathbf{L}_k = \mathbf{L}_c \equiv \mathbf{L}$ and $\mathbf{a}_k = \mathbf{a}_c \equiv \mathbf{a}$, (ii) $s = 1$. Then (a) $K_+(r)$, $K_-(r)$, $C_+(r)$, $C_-(r)$ are decreasing step functions, (b) $K_+(r) \neq K_-(r)$ and $C_+(r) \neq C_-(r)$ only for a finite set of values of the interest rate r, (c) $K_+(r) \neq K_-(r)$ and $C_+(r) \neq C_-(r)$ imply $K^s(r) = [K_-(r), K_+(r)]$ and $C^s(r) = [C_-(r), C_+(r)]$.*

Proof. We give a self contained proof of these classical results.

Proof if (i) holds. After reducing the model to an aggregate model, the argument procedes along the standard lines for the one sector Ramsey-Solow model. Note that the marginal rate of transformation between the consumption good and the capital good is 1. So across all the steady states (where both goods are produced), $v_k = 1$ holds. Then linear parametric programming can be used to construct the aggregate "Ricardian" production function

$$y(k) = \max[\mathbf{y}^T\mathbf{e}] \tag{27}$$

subject to

$$\mathbf{y}^T\mathbf{L} \leq \mathbf{h}^T, \tag{28}$$

$$\mathbf{y}^T\mathbf{a} \leq k, \tag{29}$$

$$\mathbf{y} \geq \mathbf{0}, \tag{30}$$

where $\mathbf{y} = \mathbf{x}_c + \mathbf{x}_k$. By the linear parametric programming theory, the function $y(k)$ is piecewise linear and concave, by assumption (H1), $y(0) = 0$ and by assumption (H3), $y'_+(0) > \delta$. Since duality implies $r + \delta = q \in \partial y(k)$, then $K^s(r)$ is a convex correspondence nonincreasing in r. The result for the steady state consumption follows by noticing that $k \in K^s(r) \iff y(k) - \delta k \in C^s(r)$.

(24)-(26) and for the dual of problem (21)-(23). So both the above liner programming problems have optimal solutions (see Franklin, 1980). The set of optimal solutions of the linear program (21)-(23) is bounded because of assumption (H2). For problem (24)-(26), note that since the vector of optimal wage rates $\mathbf{w}$ is obviously bounded, then also the optimal rental rate q is bounded. Note indeed that $\left[(\delta + r)^{-1}\mathbf{e} - \mathbf{a}_k\right] q \leq \mathbf{L}_k\mathbf{w}$ from inequality (18) and that the vector $\left[(\delta + r)^{-1}\mathbf{e} - \mathbf{a}_k\right]$ has at least a positive entry because $r < R$.

Proof if (ii) holds. Let $\mathbf{l}_c$ and $\mathbf{l}_k$ denote in this case the $m \times 1$ and $n \times 1$ homogeneous labour input matrices. Note that assumption (H2) implies that $\mathbf{l}_c$ and $\mathbf{l}_k$ are both strictly positive. Normalize to 1 the constant flow of labour, and define a technique as a 2×2 matrix of the following form:

$$\mathbf{T}_{ij} = \begin{bmatrix} l_{ci} & a_{ci} \\ l_{kj} & a_{kj} \end{bmatrix}.$$

First, we argue that if a couple (w, q) belongs to a solution of the linear complementary problem (16)-(20) at a given rate of interest, then it solves the linear programming problem

$$w^*(r) = \min w \tag{31}$$

subject to

$$\begin{bmatrix} \mathbf{e} \\ \mathbf{0} \end{bmatrix} \leq \begin{bmatrix} \mathbf{l}_c & \mathbf{a}_c \\ \mathbf{l}_k & \mathbf{a}_k - (\delta + r)^{-1}\mathbf{e} \end{bmatrix} \begin{bmatrix} w \\ q \end{bmatrix} \tag{32}$$

$$\begin{bmatrix} w \\ q \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{33}$$

Consider the three cases $0 \leq r < R$, $r = R$, and $r > R$. Figures 1 and 2 show how the non extra-profits region defined by (32) and (33) typically looks like in the extreme cases $r = 0$ and $r > R$. Since $(\mathbf{e}_j^T \mathbf{l}_k)^{-1}[(\delta + r)^{-1} - \mathbf{e}_j^T \mathbf{a}_k]$ gives the slope of the zero-profit line for the capital producing j-th method, increasing the interest rate with $r \geq 0$ generates a clockwise rotation of all the n capital related lines. At $r = R$ the slope of the zero-profit line of any activable capital producing process is 0. Note also that the set of the admissible wage rates is bounded away from zero for each $0 \leq r < R$.

If $r > R$, we already know that $\mathbf{x}_k = \mathbf{0}$ and $\mathbf{x}_c = \mathbf{0}$ in any solution of the complementary problem (16)-(20), so labour is unemployed and then $w = 0$. If $r = R$ and something is produced, then some capital is produced and hence again $w = 0$. Moreover, since $\mathbf{e} \leq \mathbf{a}_c q$ implies $\mathbf{x}_c^T \mathbf{a}_c + \mathbf{x}_k^T(\mathbf{a}_k - \delta^{-1}\mathbf{e}) = 0$ and we have $\mathbf{x}_k^T(\mathbf{a}_k - \delta^{-1}\mathbf{e}) < 0$, then a process producing the consumption good is activated. So $q \min_i(a_{ci}) = 1$. Finally, if $0 \leq r < R$, then the wage rate is positive, so there is full employment of labour and, hence, something is necessary produced. Then, some capital is produced and so $q > 0$ and, thus, $\mathbf{x}_c^T \mathbf{a}_c + \mathbf{x}_k^T(\mathbf{a}_k - \delta^{-1}\mathbf{e}) = 0$. Once again, the fact that $\mathbf{x}_k^T(\mathbf{a}_k - \delta^{-1}\mathbf{e}) < 0$ implies that also the consumption good is produced. If both goods are produced, then clearly (w, q) solves (31)-(33) (see Figure 1). Note that byproducts of our argument are: the existence of a solution for the linear program (31)-(33) and the facts that $K^s(r) = \{0\}$ and $C^s(r) = \{0\}$ for $r > R$. So only the case $0 \leq r \leq R$ needs to be considered.

To construct the solutions of the linear complementary problem (16)-(20) in this last case, consider now the dual of the linear programming problem (31)-(33)

$$\max[\mathbf{x}_c^T \mathbf{e}] \tag{34}$$

subject to

$$\begin{bmatrix} \mathbf{x}_c^T & \mathbf{x}_k^T \end{bmatrix} \begin{bmatrix} \mathbf{l}_c & \mathbf{a}_c \\ \mathbf{l}_k & \mathbf{a}_k - (\delta + r)^{-1}\mathbf{e} \end{bmatrix} \leq \begin{bmatrix} 1 & 0 \end{bmatrix} \tag{35}$$

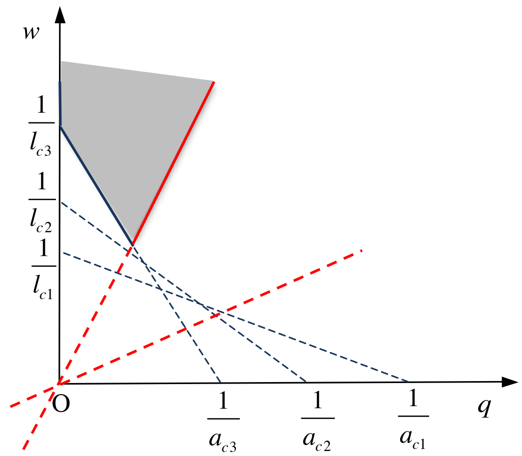

Figure 1: Case $r = 0$. In this example, there are three processes for the production of the consumption good (blue) and (at least) two processes for producing the capital good (red). The shadowed region indicates where profits are non positive and the non negativity conditions are satisfied.

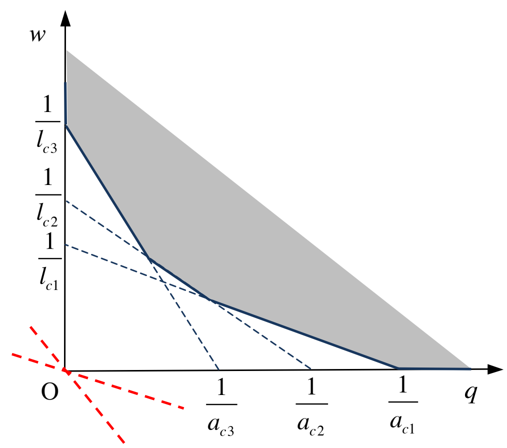

Figure 2: Case $r > R$. No process producing the capital good appears on the frontier of the feasible region. So the capital good cannot be produced without incurring losses.

$$\begin{bmatrix}\mathbf{x}_c^T & \mathbf{x}_k^T\end{bmatrix} \geq \begin{bmatrix}\mathbf{0}^T & \mathbf{0}^T\end{bmatrix}. \tag{36}$$

At a given $0 \leq r \leq R$, any optimal basic solution of this problem identifies a technique

$$\mathbf{T}_{i^*(r)j^*(r)} = \begin{bmatrix} l_{ci^*(r)} & a_{ci^*(r)} \\ l_{kj^*(r)} & a_{kj^*(r)} \end{bmatrix}$$

known as a cost-minimizing technique. Since $a_{kj^*(r)} - \delta^{-1} < 0$, the following inequality clearly holds:

$$\begin{bmatrix} x_{ci^*(r)j^*(r)} & x_{ki^*(r)j^*(r)} \end{bmatrix} \equiv \begin{bmatrix} 1 & 0 \end{bmatrix} \left[\mathbf{T}_{i^*(r)j^*(r)} - \mathbf{D}_1\right]^{-1} > \begin{bmatrix} 0 & 0 \end{bmatrix}, \tag{37}$$

where $\mathbf{D}_1 = \begin{bmatrix} 0 & 0 \\ 0 & \delta^{-1} \end{bmatrix}$, and thus $x_{ci^*(r)j^*(r)} \in C^s(r)$ and $\delta^{-1} x_{ki^*(r)j^*(r)} \in K^s(r)$. More in general, there is a one to one correspondence between the basic solutions of the above linear program and the basic solutions of the linear complementary problem (16)-(20), meaning that if $I^*(r) \times J^*(r) \subseteq \{1, 2, ..., m\} \times \{1, 2, ..., n\}$ denotes the set of cost-minimizing techniques at a given r, $0 \leq r \leq R$, then, for $r < R$, all solutions of (16)-(20) are obtained by solving the linear program (31)-(33) and by taking the convex hull of the "derived" basic solutions identified by means of the basic solutions of its dual. At $r = R$, labour unemployment can occur, implying that in combining the "derived" basic solutions to obtain a solution of (16)-(20) the sum of weights can be equal or less than 1. Note, however, that $C^s(r)$ and $K^s(r)$ are almost everywhere singleton, as generically only one technique is cost-minimizing. For $0 \leq r < R$, there is only a finite subset of values of the rate of interest where two (or more) techniques can be used to produce the two goods and where, therefore, $C^s(r)$ and $K^s(r)$ are two closed intervals. We will call an element in this set a switch point.

To complete the proof, let $\mathbf{T}_{i^M(r)j^M(r)}$ and $\mathbf{T}_{i^m(r)j^m(r)}$ be, respectively, the most capital intensive and the most labour intensive cost minimizing technique at the rate of interest r and define

$$\begin{bmatrix} x_{ci^M(r)j^M(r)} & x_{ki^M(r)j^M(r)} \end{bmatrix} \equiv \begin{bmatrix} 1 & 0 \end{bmatrix} \left[\mathbf{T}_{i^M(r)j^M(r)} - \mathbf{D}_1\right]^{-1} > \begin{bmatrix} 0 & 0 \end{bmatrix}, \tag{38}$$

$$\begin{bmatrix} x_{ci^m(r)j^m(r)} & x_{ki^m(r)j^m(r)} \end{bmatrix} \equiv \begin{bmatrix} 1 & 0 \end{bmatrix} \left[\mathbf{T}_{i^m(r)j^m(r)} - \mathbf{D}_1\right]^{-1} > \begin{bmatrix} 0 & 0 \end{bmatrix}. \tag{39}$$

We claim that:

1. $$C^s(r) = [x_{ci^m(r)j^m(r)} \quad x_{ci^M(r)j^M(r)}]$$

   and

   $$K^s(r) = [\delta^{-1} x_{ki^m(r)j^m(r)} \quad \delta^{-1} x_{ki^M(r)j^M(r)}]$$

   if $0 \leq r < R$, while

   $$C^s(R) = [0 \quad x_{ci^M(R)j^M(R)}]$$

   and

   $$K^s(R) = [0 \quad \delta^{-1} x_{ki^M(R)j^M(R)}];$$

2. for any r in the interval $[0 \quad R)$, there exists a right neighbourhood where $C^s(r) = \{x_{ci^m(r)j^m(r)}\}$ and $K^s(r) = \{\delta^{-1}x_{ki^m(r)j^m(r)}\}$, and for any r in the interval $(0 \quad R]$ there exists a left neighbourhood where $C^s(r) = \{x_{ci^M(r)j^M(r)}\}$ and $K^s(r) = \{\delta^{-1}x_{ki^M(r)j^M(r)}\}$. This holds in particular at every switch point, where the techniques in the right and the left neighbourhood differ.

Point 2 means that both sectors tend to become more labour intensive when the rate of interest rises. For the capital good sector, the result follows from the fact that if $(w^*(r), q^*(r))$ solves the linear program (31)-(33), then $w^*(r)/q^*(r) = (w^*(r)/v_k)/(q^*(r)/v_k) = \max_j((\mathbf{e}_j^T\mathbf{l}_k)^{-1}[(\delta + r)^{-1} - \mathbf{e}_j^T\mathbf{a}_k])$. So $(w^*(r)/v_k) = \max_i((\mathbf{e}_i^T\mathbf{l}_k)^{-1}[1 - (\delta + r)\mathbf{e}_i^T\mathbf{a}_k])$ is the upper envelope of n linear functions with slope $-(\mathbf{e}_j^T\mathbf{l}_k)^{-1}\mathbf{e}_j^T\mathbf{a}_k$, and hence more labour intensive methods are chosen at higher levels of the rate of interest. As this implies that the ratio $w^*(r)/q^*(r)$ decreases when r rises, inspection of Figure 1 shows that the same result holds for the consumption sector. For point 1, consider any basic solution $\begin{bmatrix} x_{ci^*(r)j^*(r)} & x_{ki^*(r)j^*(r)} \end{bmatrix}$ and the associated solution of the the linear program (31)-(33) $(w^*(r), q^*(r))$. Observe that adding $\begin{bmatrix} x_{ci^*(r)j^*(r)} & x_{ki^*(r)j^*(r)} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & r\delta^{-1}(\delta + r)^{-1} \end{bmatrix}$ to both sides of the equation

$$\begin{bmatrix} x_{ci^*(r)j^*(r)} & x_{ki^*(r)j^*(r)} \end{bmatrix} [\mathbf{T}_{i^*(r)j^*(r)} - \mathbf{D}_1] = \begin{bmatrix} 1 & 0 \end{bmatrix} \tag{40}$$

and then multiplying by $\begin{bmatrix} w^*(r) \\ q^*(r) \end{bmatrix}$ we have

$$x_{ci^*(r)j^*(r)} = w^*(r) + r\delta^{-1}(\delta + r)^{-1}q^*(r)x_{ki^*(r)j^*(r)}. \tag{41}$$

Since the basic solution satisfies also the full occupation constraint

$$\begin{bmatrix} x_{ci^*(r)j^*(r)} & x_{ki^*(r)j^*(r)} \end{bmatrix} [T_{i^*(r)j^*(r)}] \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 1, \tag{42}$$

it is clear, as shown in Figure 3, that for $r < R$ $\begin{bmatrix} x_{ci^M(r)j^M(r)} & x_{ki^M(r)j^M(r)} \end{bmatrix}$ and $\begin{bmatrix} x_{ci^m(r)j^m(r)} & x_{ki^m(r)j^m(r)} \end{bmatrix}$ are the extreme basic solutions.[4] For $r = R$, equation (41) defines the ray from the origin

$$x_{ci^*(R)j^*(R)} = R\delta^{-1}(\delta + R)^{-1}q^*(R)x_{ki^*(R)j^*(R)}, \tag{43}$$

and $C^s(R)$ and $K^s(R)$ are then given by the subset of non negative solutions of the above equation for which the following inequality holds:

$$\begin{bmatrix} x_{ci^*(R)j^*(R)} & x_{ki^*(R)j^*(R)} \end{bmatrix} [T_{i^M(R)j^M(R)}] \begin{bmatrix} 1 \\ 0 \end{bmatrix} \leq 1. \tag{44}$$

From 1 and 2 above, points (a), (b) and (c) clearly follow. □

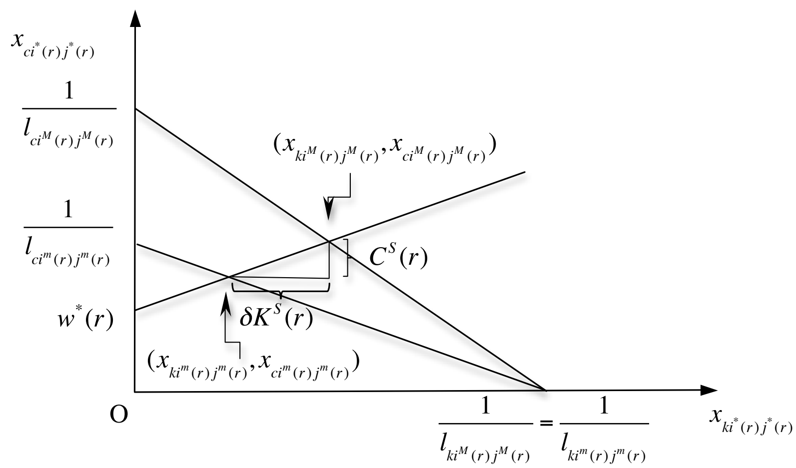

Figure 3: The sets $C^s(r)$ and $K^s(r)$ at a switch point where two (or more) processes that produce the consumption good are cost minimizing.

The next section shows that the ”neoclassical” properties in Proposition 2.1 do not carry over to the general two-sector one capital good multi-primary-factor model even if joint production is excluded.

3 An example

A minimal example to show that the linear complementary problem (16)-(20) can have multiple solutions is provided in this section. In the example, only two primary factors of production (two qualities of labour) are used in both sectors and only a single process for the production of the capital good and two processes for the production of the consumption good are available. The labour matrices and the capital vectors of the coefficients are the following ones:

$$\mathbf{L}_c = \begin{bmatrix} \frac{95}{100} & 1 \\ 1 & 2 \end{bmatrix}$$

$$\mathbf{l}_k = \begin{bmatrix} \frac{1}{100} & \frac{11}{10} \end{bmatrix}$$

$$\mathbf{a}_c = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} \end{bmatrix}^T$$

[4] Although in this paper we will not dig into the question, there is a close link between the procedure used here to construct the basic solutions of the linear complementarity problem (16)-(20) and the construction of the so called master function in Samuelson & Etula (2006) and Etula (2008).

$$a_k = \frac{1}{4},$$

and the rate of decay of capital δ is assumed to be 1. Moreover, the system is endowed with 1 unit of labour of type 1 and 2 units of labour of type 2. So we have $\mathbf{h} = \begin{bmatrix} 1 & 2 \end{bmatrix}^T$, and thus the linear complementary problem (16)-(20) in this specific example takes the form:

$$\begin{bmatrix} x_{c1} & x_{c2} & x_k \end{bmatrix} \begin{bmatrix} \frac{95}{100} & 1 & \frac{1}{2} \\ 1 & 2 & \frac{1}{4} \\ \frac{1}{100} & \frac{11}{10} & \frac{1}{4} - 1 \end{bmatrix} \leq \begin{bmatrix} 1 & 2 & 0 \end{bmatrix} \tag{45}$$

$$\begin{bmatrix} x_{c1} & x_{c2} & x_k \end{bmatrix} \begin{bmatrix} \frac{95}{100} & 1 & \frac{1}{2} \\ 1 & 2 & \frac{1}{4} \\ \frac{1}{100} & \frac{11}{10} & \frac{1}{4} - 1 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ q \end{bmatrix} = w_1 + 2w_2 \tag{46}$$

$$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \leq \begin{bmatrix} \frac{95}{100} & 1 & \frac{1}{2} \\ 1 & 2 & \frac{1}{4} \\ \frac{1}{100} & \frac{11}{10} & \frac{1}{4} - \frac{1}{1+r} \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ q \end{bmatrix} \tag{47}$$

$$x_{c1} + x_{c2} = \begin{bmatrix} x_{c1} & x_{c2} & x_k \end{bmatrix} \begin{bmatrix} \frac{95}{100} & 1 & \frac{1}{2} \\ 1 & 2 & \frac{1}{4} \\ \frac{1}{100} & \frac{11}{10} & \frac{1}{4} - \frac{1}{1+r} \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ q \end{bmatrix} \tag{48}$$

$$\begin{bmatrix} x_{c1} & x_{c2} & x_k \end{bmatrix} \geq \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} w_1 \\ w_2 \\ q \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \tag{49}$$

Note that $R = 3$. For $0 \leq r < 3$ both goods are produced, thus either one of the processes producing the consumption good is not operated or all three processes are activated. In the latter case, system (47) must hold with equality, which means, after substituting for q using the last equation in (47) into the first two equations, that the system

$$[\frac{95}{100} + \frac{1+r}{50(3-r)}]w_1 + [1 + \frac{11(1+r)}{5(3-r)}]w_2 = 1 \tag{50}$$

$$[1 + \frac{1+r}{100(3-r)}]w_1 + [2 + \frac{11(1+r)}{10(3-r)}]w_2 = 1 \tag{51}$$

must hold. A straightforward calculation shows that a non negative solution for this system exists if and only if $\frac{19}{21} \leq r \leq \frac{7}{3}$. At $r = \frac{19}{21}$, $w_1 = 0$ and $w_2 = \frac{1}{3}$, while at $r = \frac{7}{3}$, $w_1 = \frac{20}{21}$ and $w_2 = 0$. In between, both wage rates are positive. Since solving system (45) with equality gives

$$\begin{bmatrix} x_{c1} & x_{c2} & x_k \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} \frac{95}{100} & 1 & \frac{1}{2} \\ 1 & 2 & \frac{1}{4} \\ \frac{1}{100} & \frac{11}{10} & \frac{1}{4} - 1 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{24}{35} & \frac{12}{35} & \frac{20}{35} \end{bmatrix}, \qquad (52)$$

then $\frac{36}{35} \in C^s(r)$ and $\frac{20}{35} \in K^s(r)$ for $\frac{19}{21} \leq r \leq \frac{7}{3}$. Moreover, no other solution in which all the three process are operated exists for $\frac{19}{21} < r < \frac{7}{3}$.

Consider next the case in which a single process is operated in the consumption sector. In this case, system (45) cannot hold with equality, implying that unemployment of one type of labour arises. Since the first process uses type 1 labour more intensively than the second process, operating the first process induces unemployment of type 2 labour, while activating the second process leads to less than full employment for the labour of kind 1. In particular, setting $x_{c2} = 0$ and assuming that the first and the last inequality in system (45) hold with equality leads to the system

$$\frac{95}{100} x_{c1} + \frac{1}{100} x_k = 1 \qquad (53)$$

$$x_{c1} + \frac{11}{10} x_k \leq 2 \qquad (54)$$

$$\frac{1}{2} x_{c1} - \frac{3}{4} x_k = 0, \qquad (55)$$

whose solution is given by $x_{c1} = \frac{300}{287}$, $x_k = \frac{200}{287}$. Setting instead $x_{c1} = 0$ and assuming that the second and the last inequality in system (45) hold with equality leads to the system

$$x_{c2} + \frac{1}{100} x_k \leq 1 \qquad (56)$$

$$2x_{c2} + \frac{11}{10} x_k = 2 \qquad (57)$$

$$\frac{1}{4} x_{c2} - \frac{3}{4} x_k = 0, \qquad (58)$$

that has the solution $x_{c2} = \frac{60}{71}$, $x_k = \frac{20}{71}$.

Note that the $\frac{300}{287}$ is the golden rule consumption flow (i. e. $\frac{300}{287}$ is the only element in $C^s(0)$). This implies that there is a price support for $\begin{bmatrix} x_{c1} & x_{c2} & x_k \end{bmatrix} = \begin{bmatrix} \frac{300}{287} & 0 & \frac{200}{287} \end{bmatrix}$ if the rate of interest is literally zero or close to zero. To find explicitly the supporting prices and the interval of existence of these prices, set $w_2 = 0$ in system (47) and assume the first and the last inequality hold with equality. This leads to

$$\frac{95}{100} w_1 + \frac{1}{2} q = 1 \qquad (59)$$

$$w_1 + \frac{1}{4}q \geq 1 \tag{60}$$

$$\frac{1}{100}w_1 + [\frac{1}{4} - \frac{1}{1+r}]q = 0, \tag{61}$$

that for $0 \leq r \leq \frac{7}{3}$ has the solution $w_1 = [\frac{95}{100} + \frac{1+r}{50(3-r)}]^{-1} > 0$, $q = \frac{1}{100}[\frac{95}{100} + \frac{1+r}{50(3-r)}]^{-1}[\frac{1}{1+r} - \frac{1}{4}]^{-1} > 0$. Note that at $r = \frac{7}{3}$ inequality (60) holds with equality. In the same way, setting $w_1 = 0$ and verifying that for $\frac{19}{21} \leq r \leq 3$ the system

$$w_2 + \frac{1}{2}q \geq 1 \tag{62}$$

$$2w_2 + \frac{1}{4}q = 1 \tag{63}$$

$$\frac{11}{10}w_2 + [\frac{1}{4} - \frac{1}{1+r}]q = 0, \tag{64}$$

has the solution $w_2 = \frac{10(3-r)}{71-9r} \geq 0$, $q = 4 - \frac{80(3-r)}{71-9r} > 0$ proves that $\begin{bmatrix} x_{c1} & x_{c2} & x_k \end{bmatrix} = \begin{bmatrix} 0 & \frac{60}{71} & \frac{20}{71} \end{bmatrix}$ has a support for $\frac{19}{21} \leq r \leq 3$. For $r = 3$ both wage rates are zero, so that both kinds of labour can be unemployed. This implies that at $r = 3$ there is a supporting price for any vector $\begin{bmatrix} x_{c1} & x_{c2} & x_k \end{bmatrix} = \begin{bmatrix} 0 & \theta\frac{60}{71} & \theta\frac{20}{71} \end{bmatrix}$, where $0 \leq \theta \leq 1$. The above argument, therefore, establishes three further facts. First, $\frac{300}{287} \in C^s(r)$ and $\frac{200}{287} \in K^s(r)$ for $0 \leq r \leq \frac{7}{3}$, and $\frac{60}{71} \in C^s(r)$ and $\frac{20}{71} \in K^s(r)$ for $\frac{19}{21} \leq r \leq 3$. Second, no other solution in which two process are operated exists for $0 \leq r < 3$. Third, $[0 \quad \frac{60}{71}] \subseteq C^s(3)$ and $[0 \quad \frac{20}{71}] \subseteq K^s(3)$.

Finally, consider the critical cases $r = \frac{19}{21}$, where the first kind of labour needs not be fully employed, and $r = \frac{7}{3}$, where instead some labour of type 2 can be unemployed. Since the price system at $r = \frac{19}{21}$ supports any convex combinations of the two intensity vectors $\begin{bmatrix} \frac{24}{35} & \frac{12}{35} & \frac{20}{35} \end{bmatrix}$ and $\begin{bmatrix} 0 & \frac{60}{71} & \frac{20}{71} \end{bmatrix}$, then $[\frac{60}{71} \quad \frac{36}{35}] \subseteq C^s(\frac{19}{21})$ and $[\frac{20}{71} \quad \frac{20}{35}] \subseteq K^s(\frac{19}{21})$. Analogously, the price system at $r = \frac{7}{3}$ supports any convex combinations of the two intensity vectors $\begin{bmatrix} \frac{24}{35} & \frac{12}{35} & \frac{20}{35} \end{bmatrix}$ and $\begin{bmatrix} \frac{300}{287} & 0 & \frac{200}{287} \end{bmatrix}$. Thus, $[\frac{36}{35} \quad \frac{300}{287}] \subseteq C^s(\frac{7}{3})$ and $[\frac{20}{35} \quad \frac{200}{287}] \subseteq K^s(\frac{7}{3})$. This concludes the construction of the $C^s(r)$ and $K^s(r)$ correspondences for the current example.

The results for $K^s(r)$ are plotted in Figure 4. The graph of the $C^s(r)$ correspondence is similar. For $\frac{19}{21} < r < \frac{7}{3}$ there are three steady states. Moreover, by a straightforward analysis of the Hamiltonian dynamics, that is not developed here, it can be proved that the equilibria are alternately stable and unstable (see e.g., Liviatan & Samuelson, 1969; Freni, 1997). Then, the instability of the intermediate equilibrium (where the $K^s(r)$ correspondence is increasing) precludes comparative statics perverse effects, but nevertheless the long run wages distribution is history dependent in this example.[5] Interestingly, the results also show that the equilibria with a wide wage gap are the stable ones.

[5]For the working of Samuelson's Correspondence Principle with a scalar state variable see Burmeister & Long (1977).

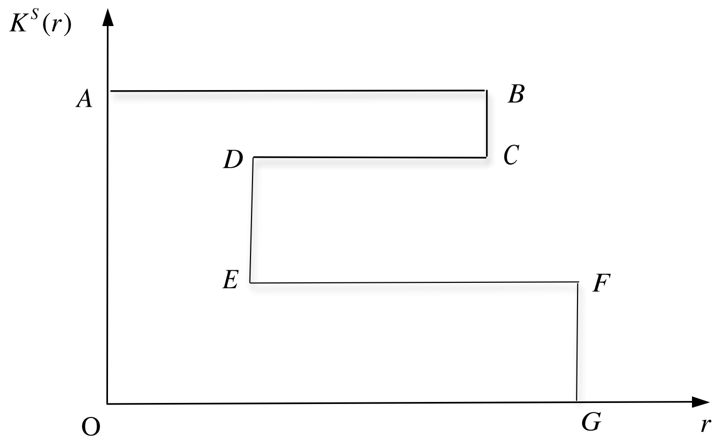

Figure 4: The graph of the steady state $K^s(r)$ correspondence. $OG = 3$ gives the maximum rate of interest and $OA = \frac{200}{287}$ is the golden rule stock level. $AB = \frac{7}{3}$ and $DC = \frac{7}{3} - \frac{19}{21}$. Two relevant modified golden rules stock levels are given by $FG = \frac{20}{71}$ and $FG + DE = \frac{20}{35}$.

4 Concluding remarks

This paper shows non uniqueness of the steady state for the Bruno's (1967) one capital good two-sector growth model with a discrete technology when there are many primary factors of production. Of course, it is natural to expect that the same problem arises in the smooth neoclassical case. The exploration of this issue is left for future investigation.

The model can be reinterpreted as a two-agricultural-good Ricardian model with multiple qualities of lands (Samuelson, 1959; Pasinetti, 1960). In this case, if capitalists require a positive rate of profits to carry on a stationary stock, then our results prove that the uniqueness of the stationary state is not guaranteed. The smooth Clark's version of the model, often slightly modified to include a non-linear utility function, has been also used as a dynamic Heckscher-Ohlin model for the analysis of convergence among open economies (see Atkeson & Kehoe, 2000). At times, multiple primary factors have been included in the model (e.g. Nishimura *et al.* 2006, Guilló & Perez-Sebastian, 2015), but the full extended model has not yet been worked out.

While the problem of uniqueness of the steady state in growth models has been explored in some generality in the case of a single primary factor (see e. g., Brock, 1973; Brock & Burmeister, 1976; Burmeister, 1981),[6] not much is known

[6] All these results apply to smooth economies.

about what kind of economically relevant conditions lead to uniqueness in multi primary factor models. This too seems an interesting question for future work.

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