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Abstract

The revelation principle is a fundamental theorem in many economics fields such as game theory, mechanism design and auction theory etc. In this paper, I construct an example to show that a social choice function which can be implemented in Bayesian Nash equilibrium is not truthfully implementable. The key point is that agents pay cost in the indirect mechanism, but pay nothing in the direct mechanism. As a result, the revelation principle may not hold when agent's cost cannot be neglected in the indirect mechanism.

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Key words: Revelation principle; Game theory; Mechanism design; Auction theory.

1 Introduction

The revelation principle plays an important role in microeconomics theory and has been applied to many other fields such as auction theory, game theory etc. According to the wide-spread textbook given by Mas-Colell, Whinston and Green (Page 884, Line 24 [1]): "*The implication of the revelation principle is ... to identify the set of implementable social choice functions, we need only identify those that are truthfully implementable.*" Related definitions about the revelation principle can be seen in Appendix, which are cited from Section 23.B and 23.D of MWG's textbook[1].

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However, in this paper, I will construct a simple labor model to show that the revelation principle may not hold when agent's cost cannot be neglected in the indirect mechanism. Section 2 is the main part of this paper, and Section 3 draws conclusions.

2 A labor model

Here we consider a simple labor model which uses some ideas from the first-price sealed auction model in Example 23.B.5 [1] and the signaling model in Section 13.C [1]. There are one firm and two workers. The firm wants to hire a worker, and two workers compete for this job offer. Worker 1 and Worker 2 differ in the number of units of output they produce if hired by the firm, which is denoted by productivity type.

For simplicity, we make the following assumptions:
1) The possible productivity types of two workers are: $\theta_1, \theta_2 \in \{\theta_L, \theta_H\}$, where $\theta_H > \theta_L > 0$. Each worker's productivity is a random variable chosen independently, and is private information for each worker.
2) Before confronting the firm, each worker $i = 1, 2$ can get some education. The possible levels of education are: e_L and e_H, where $e_H > e_L \geq 0$. Each worker's education is observable to the firm. Education does nothing for a worker's productivity.
3) The cost of obtaining education level e for a worker of some type θ is given by a function $c(e, \theta) = e/\theta$. That is, the cost of education is lower for a high-productivity worker.

The model's outcome can be represented by a vector (y_1, y_2), where y_i denotes the probability that worker i gets the job offer with wage $w > 0$. Recall that the firm does not know the exact productivity types of two workers, but its aim is to hire a worker with productivity as high as possible. This aim can be represented by a social choice function $f(\vec{\theta}) = (y_1(\vec{\theta}), y_2(\vec{\theta}))$, in which $\vec{\theta} = (\theta_1, \theta_2)$,

$$y_1(\vec{\theta}) = \begin{cases} 1, & if \theta_1 > \theta_2 \\ 0.5, & if \theta_1 = \theta_2 \\ 0, & if \theta_1 < \theta_2 \end{cases}, y_2(\vec{\theta}) = \begin{cases} 1, & if \theta_1 < \theta_2 \\ 0.5, & if \theta_1 = \theta_2 \\ 0, & if \theta_1 > \theta_2 \end{cases} \quad (1)$$

In order to implement the above $f(\vec{\theta})$, the firm designs an indirect mechanism $\Gamma = (S_1, S_2, g)$ as follows:
1) A random move of nature determines the productivity of workers: $\theta_1, \theta_2 \in \{\theta_L, \theta_H\}$.
2) Conditional on his type θ_i, each worker $i = 1, 2$ chooses his education level

as a bid $b_i : \{\theta_L, \theta_H\} \to \{e_L, e_H\}$. The strategy set S_i is the set of all possible bids $b_i(\theta_i)$, and the outcome function g is defined as:

$$g(b_1, b_2) = (p_1, p_2) = \begin{cases} (1,0), & if b_1 > b_2 \\ (0.5, 0.5), & if b_1 = b_2 \\ (0,1), & if b_1 < b_2 \end{cases} \tag{2}$$

where p_i $(i = 1, 2)$ is the probability that worker i gets the offer.

Let u_0 be the utility of the firm, and u_1, u_2 be the utilities of worker $1, 2$ respectively, then $u_0(b_1, b_2) = p_1\theta_1 + p_2\theta_2 - w$, and for $i, j = 1, 2$, $i \neq j$,

$$u_i(b_i, b_j; \theta_i) = \begin{cases} w - b_i/\theta_i, & if b_i > b_j \\ 0.5w - b_i/\theta_i, & if b_i = b_j \\ -b_i/\theta_i, & if b_i < b_j \end{cases} \tag{3}$$

Proposition 1: If $w \in (2(e_H - e_L)/\theta_H, 2(e_H - e_L)/\theta_L)$, the social choice function $f(\vec{\theta})$ can be implemented by the indirect mechanism Γ in Bayesian Nash equilibrium.
Proof: Consider a separating strategy, *i.e.*, workers with different productivity types choose different education levels,

$$b_1(\theta_1) = \begin{cases} e_H, & if \theta_1 = \theta_H \\ e_L, & if \theta_1 = \theta_L \end{cases}, b_2(\theta_2) = \begin{cases} e_H, & if \theta_2 = \theta_H \\ e_L, & if \theta_2 = \theta_L \end{cases}. \tag{4}$$

Now let us check whether this separating strategy yields a Bayesian Nash equilibrium. Assume $b_j^*(\theta_j)$ takes this form, *i.e.*,

$$b_j^*(\theta_j) = \begin{cases} e_H, & if \theta_j = \theta_H \\ e_L, & if \theta_j = \theta_L \end{cases}, \tag{5}$$

then consider worker i's problem $(i \neq j)$. For each $\theta_i \in \{\theta_L, \theta_H\}$, worker i solves the maximization problem $\max_{b_i} h(b_i, \theta_i)$, where by Eq (3) the object function is

$$h(b_i, \theta_i) = (w - b_i/\theta_i)P(b_i > b_j^*(\theta_j)) + (0.5w - b_i/\theta_i)P(b_i = b_j^*(\theta_j)) - (b_i/\theta_i)P(b_i < b_j^*(\theta_j)) \tag{6}$$

We discuss this maximization problem in four different cases:
1) Suppose $\theta_i = \theta_j = \theta_L$, then $b_j^*(\theta_j) = e_L$ by Eq (5).

$$\begin{aligned} h(b_i, \theta_i) &= (w - b_i/\theta_L)P(b_i > e_L) + (0.5w - b_i/\theta_L)P(b_i = e_L) - (b_i/\theta_L)P(b_i < e_L) \\ &= \begin{cases} w - e_H/\theta_L, & if b_i = e_H \\ 0.5w - e_L/\theta_L, & if b_i = e_L \end{cases} \end{aligned}$$

Thus, if $w < 2(e_H - e_L)/\theta_L$, then $h(e_H, \theta_i) < h(e_L, \theta_i)$, which means the optimal value of $b_i(\theta_i)$ is e_L. In this case, $b_i^*(\theta_L) = e_L$.

2) Suppose $\theta_i = \theta_L$, $\theta_j = \theta_H$, then $b_j^*(\theta_j) = e_H$ by Eq (5).

$$\begin{aligned} h(b_i, \theta_i) &= (w - b_i/\theta_L)P(b_i > e_H) + (0.5w - b_i/\theta_L)P(b_i = e_H) - (b_i/\theta_L)P(b_i < e_H) \\ &= \begin{cases} 0.5w - e_H/\theta_L, & if b_i = e_H \\ -e_L/\theta_L, & if b_i = e_L \end{cases} \end{aligned}$$

Thus, if $w < 2(e_H - e_L)/\theta_L$, then $h(e_H, \theta_i) < h(e_L, \theta_i)$, which means the optimal value of $b_i(\theta_i)$ is e_L. In this case, $b_i^*(\theta_L) = e_L$.

3) Suppose $\theta_i = \theta_H$, $\theta_j = \theta_L$, then $b_j^*(\theta_j) = e_L$ by Eq (5).

$$\begin{aligned} h(b_i, \theta_i) &= (w - b_i/\theta_H)P(b_i > e_L) + (0.5w - b_i/\theta_H)P(b_i = e_L) - (b_i/\theta_H)P(b_i < e_L) \\ &= \begin{cases} w - e_H/\theta_H, & if b_i = e_H \\ 0.5w - e_L/\theta_H, & if b_i = e_L \end{cases} \end{aligned}$$

Thus, if $w > 2(e_H - e_L)/\theta_H$, then $h(e_H, \theta_i) > h(e_L, \theta_i)$, which means the optimal value of $b_i(\theta_i)$ is e_H. In this case, $b_i^*(\theta_H) = e_H$.

4) Suppose $\theta_i = \theta_j = \theta_H$, then $b_j^*(\theta_j) = e_H$ by Eq (5).

$$\begin{aligned} h(b_i, \theta_i) &= (w - b_i/\theta_H)P(b_i > e_H) + (0.5w - b_i/\theta_H)P(b_i = e_H) - (b_i/\theta_H)P(b_i < e_H) \\ &= \begin{cases} 0.5w - e_H/\theta_H, & if b_i = e_H \\ -e_L/\theta_H, & if b_i = e_L \end{cases} \end{aligned}$$

Thus, if $w > 2(e_H - e_L)/\theta_H$, then $h(e_H, \theta_i) > h(e_L, \theta_i)$, which means the optimal value of $b_i(\theta_i)$ is e_H. In this case, $b_i^*(\theta_H) = e_H$.

In summary, when $w \in (2(e_H - e_L)/\theta_H, 2(e_H - e_L)/\theta_L)$, the strategy $b_i^*(\theta_i)$ of worker i

$$b_i^*(\theta_i) = \begin{cases} e_H, & if \theta_i = \theta_H \\ e_L, & if \theta_i = \theta_L \end{cases} \tag{7}$$

is the optimal response to the strategy $b_j^*(\theta_j)$ of worker j ($j \neq i$) given in Eq (5). Therefore, the strategy profile $(b_1^*(\theta_1), b_2^*(\theta_2))$ is a Bayesian Nash equilibrium of the game induced by Γ.

By Eq(2) and Eq(7), for any $\vec{\theta} = (\theta_1, \theta_2)$, $\theta_1, \theta_2 \in \{\theta_L, \theta_H\}$,

$$g(b_1^*(\theta_1), b_2^*(\theta_2)) = \begin{cases} (1, 0), & if \theta_1 > \theta_2 \\ (0.5, 0.5), & if \theta_1 = \theta_2 \\ (0, 1), & if \theta_1 < \theta_2 \end{cases} \tag{8}$$

which implements the social choice function $f(\vec{\theta})$. **Q.E.D**

Proposition 2: The social choice function $f(\vec{\theta})$ is not truthfully implementable in Bayesian Nash equilibrium.
Proof: Consider the direct revelation mechanism $\Gamma_{direct} = (\Theta_1, \cdots, \Theta_I, f(\vec{\theta}))$, in which $\Theta_i = \{\theta_L, \theta_H\}$, $\vec{\theta} \in \Theta_1 \times \Theta_2$, $i = 1, 2$. The timing steps of Γ_{direct} are as follows:
1) Each worker i announces his productivity type $\hat{\theta}_i \in \{\theta_L, \theta_H\}$ to a virtual mediator. Note that $\hat{\theta}_i$ may not be his true type θ_i.
2) The mediator submits $b_i^*(\hat{\theta}_i)$ $(i = 1, 2)$ to the firm:

$$b_i^*(\hat{\theta}_i) = \begin{cases} e_H, & if \hat{\theta}_i = \theta_H \\ e_L, & if \hat{\theta}_i = \theta_L \end{cases}$$

3) The firm performs the outcome function $g(b_1, b_2)$, and hires the winner.

Since each worker i does not need to pay the cost b_i/θ_i when playing in the direct mechanism, the utility function of each worker $i = 1, 2$ is changed from Eq (3) to the follows:

$$u_i(\hat{\theta}_i, \hat{\theta}_j; \theta_i) = \begin{cases} w, & if \hat{\theta}_i > \hat{\theta}_j \\ 0.5w, & if \hat{\theta}_i = \hat{\theta}_j \\ 0, & if \hat{\theta}_i < \hat{\theta}_j \end{cases} \tag{9}$$

The utility matrix can be expressed as follows.

$\hat{\theta}_i$ \ $\hat{\theta}_j$	θ_L	θ_H
θ_L	$[0.5w, 0.5w]$	$[0, w]$
θ_H	$[w, 0]$	$[0.5w, 0.5w]$

Obviously, the dominant strategy for each worker $i = 1, 2$ is to definitely announce $\hat{\theta}_i = \theta_H$, no matter what his true productivity type is. Consequently, the social choice function $f(\vec{\theta})$ is not truthfully implementable in Bayesian Nash equilibrium. **Q.E.D**

3 Conclusions

From Proposition 1 and 2, it can be seen that:
1) In the indirect mechanism Γ, the utility function of each worker $i = 1, 2$ is given by Eq (3). The cost b_i/θ_i is the key item that makes the separating strategy profile $(b_1^*(\theta_1), b_2^*(\theta_2))$ be a Bayesian Nash equilibrium, when the wage

$w \in (2(e_H - e_L)/\theta_H, 2(e_H - e_L)/\theta_L)$.
2) In the direct mechanism Γ_{direct}, the utility function of each worker $i = 1, 2$ is given by Eq (9), where the cost disappears. Thus, a low-productivity worker is free to pretend to be a high-productivity worker without any suffer. There is no way for the direct revelation mechanism to avoid this counterfeit.
In summary, the revelation principle may not hold when agent's cost cannot be neglected in the indirect mechanism.

Appendix: Definitions in Section 23.B and 23.D [1]

Definition 23.B.1: A *social choice function* is a function $f : \Theta_1 \times \cdots \times \Theta_I \to X$ that, for each possible profile of the agents' types $(\theta_1, \cdots, \theta_I)$, assigns a collective choice $f(\theta_1, \cdots, \theta_I) \in X$.

Definition 23.B.3: A *mechanism* $\Gamma = (S_1, \cdots, S_I, g(\cdot))$ is a collection of I strategy sets $S_1, \cdots, S_I$ and an outcome function $g : S_1 \times \cdots \times S_I \to X$.

Definition 23.B.5: A *direct revelation mechanism* is a mechanism in which $S_i = \Theta_i$ for all i and $g(\theta) = f(\theta)$ for all $\theta \in \Theta_1 \times \cdots \times \Theta_I$.

Definition 23.D.1: The strategy profile $s^*(\cdot) = (s_1^*(\cdot), \cdots, s_I^*(\cdot))$ is a *Bayesian Nash equilibrium* of mechanism $\Gamma = (S_1, \cdots, S_I, g(\cdot))$ if, for all i and all $\theta_i \in \Theta_i$,

$$E_{\theta_{-i}}[u_i(g(s_i^*(\theta_i), s_{-i}^*(\theta_{-i})), \theta_i)|\theta_i] \geq E_{\theta_{-i}}[u_i(g(\hat{s}_i, s_{-i}^*(\theta_{-i})), \theta_i)|\theta_i]$$

for all $\hat{s}_i \in S_i$.

Definition 23.D.2: The mechanism $\Gamma = (S_1, \cdots, S_I, g(\cdot))$ *implements the social choice function* $f(\cdot)$ *in Bayesian Nash equilibrium* if there is a Bayesian Nash equilibrium of Γ, $s^*(\cdot) = (s_1^*(\cdot), \cdots, s_I^*(\cdot))$, such that $g(s^*(\theta)) = f(\theta)$ for all $\theta \in \Theta$.

Definition 23.D.3: The social choice function $f(\cdot)$ is *truthfully implementable in Bayesian Nash equilibrium* if $s_i^*(\theta_i) = \theta_i$ (for all $\theta_i \in \Theta_i$ and $i = 1, \cdots, I$) is a Bayesian Nash equilibrium of the direct revelation mechanism $\Gamma = (\Theta_1, \cdots, \Theta_I, f(\cdot))$. That is, if for all $i = 1, \cdots, I$ and all $\theta_i \in \Theta_i$,

$$E_{\theta_{-i}}[u_i(f(\theta_i, \theta_{-i})), \theta_i)|\theta_i] \geq E_{\theta_{-i}}[u_i(f(\hat{\theta}_i, \theta_{-i}), \theta_i)|\theta_i], \qquad (23.D.1)$$

for all $\hat{\theta}_i \in \Theta_i$.

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References

[1] A. Mas-Colell, M.D. Whinston and J.R. Green, Microeconomic Theory, Oxford University Press, 1995.