Two person zero-sum game with two sets of strategic variables

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20 August 2016

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Abstract

We consider a two-person zero-sum game with two sets of strategic variables which are related by invertible functions. They are denoted by $(s_A, s_B) \in (S_A, S_B)$ and $(t_A, t_B) \in (T_A, T_B)$ for players A and B. The payoff function of Player A is u_A. Then, the payoff function of Player B is $-u_A$. u_A is upper semi-continuous and quasi-concave on S_A for each $s_B \in S_B$ (or each $t_B \in T_B$), upper semi-continuous and quasi-concave on T_A for each $t_B \in T_B$ (or each $s_B \in S_B$), and lower semi-continuous and quasi-convex on S_B for each $s_A \in S_A$ (or each $t_A \in T_A$), lower semi-continuous and quasi-convex on T_B for each $t_A \in T_A$ (or each $s_A \in S_A$). We do not postulate differentiability of payoff functions.

We will show that the following four patterns of competition are equivalent, that is, they yield the same outcome.

1. Player A and B choose s_A and s_B (competition by (s_A, s_B)).
2. Player A and B choose t_A and t_B (competition by (t_A, t_B)).
3. Player A and B choose t_A and s_B (competition by (t_A, s_B)).
4. Player A and B choose s_A and t_B (competition by (s_A, t_B)).

Keywords. zero-sum game, two strategic variables, equivalence of outcome.
JEL Classification code. C72, L13.

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1 Introduction

We consider a two-person zero-sum game with two sets of strategic variables which are related by invertible functions. They are denoted by $(s_A, s_B) \in (S_A, S_B)$ and $(t_A, t_B) \in (T_A, T_B)$ for players A and B. The payoff function of Player A is u_A. Then, the payoff function of Player B is $-u_A$. u_A is upper semi-continuous and quasi-concave on S_A for each $s_B \in S_B$ (or each $t_B \in T_B$), upper semi-continuous and quasi-concave on T_A for each $t_B \in T_B$ (or each $s_B \in S_B$), and lower semi-continuous and quasi-convex on S_B for each $s_A \in S_A$ (or each $t_A \in T_A$), lower semi-continuous and quasi-convex on T_B for each $t_A \in T_A$ (or each $s_A \in S_A$). We do not postulate differentiability of payoff functions.

We will show that the following four patterns of competition are equivalent, that is, they yield the same outcome.

(1) Player A and B choose s_A and s_B (competition by (s_A, s_B)).

(2) Player A and B choose t_A and t_B (competition by (t_A, t_B)).

(3) Player A and B choose t_A and s_B (competition by (t_A, s_B)).

(4) Player A and B choose s_A and t_B (competition by (s_A, t_B)).

Relative profit maximization in duopoly with differentiated goods is an example of zero-sum game with two alternative strategic variables[1]. Each firm chooses its output or price. The results of this paper imply that when firms in duopoly maximize their relative profits, Cournot and Bertrand equilibria are equivalent, and price-setting behavior and output-setting behavior are equivalent[2].

The key to our results is Lemma 4 in Section 6. This lemma implies that the maximin strategies in four patterns of competition are equivalent, and the minimax strategies in four patterns of competition are equivalent.

2 The model

Consider a two-person zero-sum game as follows. There are two players, A and B. They have two sets of alternative strategic variables, $(s_A, s_B) \in S_A \times S_B$ and $(t_A, t_B) \in T_A \times T_B$. S_A, S_B, T_A and T_B are compact sets in metric spaces. The relations of them are represented by

$$s_A = f_A(t_A, t_B), \text{ and } s_B = f_B(t_A, t_B).$$

(f_A, f_B) is a continuous invertible function, and so it is a one-to-one and onto function. We denote

$$t_A = g_A(s_A, s_B), \text{ and } t_B = g_B(s_A, s_B).$$

[1]A game of relative profit maximization in duopoly is a zero-sum game because the sum of the relative profits of firms is zero.

[2]About relative profit maximization under imperfect competition please see Matsumura, Matsushima and Cato (2013), Satoh and Tanaka (2013), Satoh and Tanaka (2014a), Satoh and Tanaka (2014b), Tanaka (2013a), Tanaka (2013b) and Vega-Redondo (1997).

(g_A, g_B) is also a continuous invertible function. The payoff function of Player A is $u_A(s_A, s_B)$ and the payoff function of Player B is $u_B(s_A, s_B)$. Since the game is zero-sum, we have $u_B(s_A, s_B) = -u_A(s_A, s_B)$. u_A is upper semi-continuous and quasi-concave on S_A for each $s_B \in S_B$ (or each $t_B \in T_B$), upper semi-continuous and quasi-concave on T_A for each $t_B \in T_B$ (or each $s_B \in S_B$), and lower semi-continuous and quasi-convex on S_B for each $s_A \in S_A$ (or each $t_A \in T_A$), lower semi-continuous and quasi-convex on T_B for each $t_A \in T_A$ (or each $s_A \in S_A$). We do not postulate differentiability of payoff functions[3].

3 Competition by (s_A, s_B)

First consider competition by (s_A, s_B). Let s_A^* and s_B^* be the values of s_A and s_B which, respectively, (locally) maximizes $u_A(s_A, s_B)$ given s_B^* and (locally) maximizes $u_B(s_A, s_B)$ given s_A^* in a neighborhood around (s_A^*, s_B^*) in $S_A \times S_B$. Then,

$$u_A(s_A^*, s_B^*) \geq u_A(s_A, s_B^*) \text{ for all } s_A \neq s_A^*,$$

and

$$u_B(s_A^*, s_B^*) \geq u_B(s_A^*, s_B) \text{ for all } s_B \neq s_B^*.$$

Since $u_B = -u_A$, this is rewritten as

$$u_A(s_A^*, s_B) \geq u_A(s_A^*, s_B^*), \text{ for all } s_B \neq s_B^*.$$

Thus, we obtain

$$u_A(s_A^*, s_B) \geq u_A(s_A^*, s_B^*) \geq u_A(s_A, s_B^*) \text{ for all } s_A \neq s_A^*, \text{ and all } s_B \neq s_B^*.$$

This is equivalent to

$$u_A(s_A^*, s_B^*) = \max_{s_A} u_A(s_A, s_B^*) = \min_{s_B} u_A(s_A^*, s_B).$$

(s_A^*, s_B^*) is a Nash equilibrium of competition by (s_A, s_B) game.

On the other hand, by the Sion's minimax theorem (Sion (1958), Komiya (1988), Kindler (2005)) we have

$$v_A^s \equiv \max_{s_A} \min_{s_B} u_A(s_A, s_B) = \min_{s_B} \max_{s_A} u_A(s_A, s_B) \equiv v_B^s.$$

We can show the following lemma.

Lemma 1. *The following three statements are equivalent.*

(1) *There exists a Nash equilibrium in competition by (s_A, s_B) game.*

[3] In Satoh and Tanaka (2016) we analyze maximin and minimax strategies in duopoly when payoff functions of firms are differentiable.

(2) The following relation holds.

$$v_A^s \equiv \max_{s_A} \min_{s_B} u_A(s_A, s_B) \equiv \min_{s_B} \max_{s_A} u_A(s_A, s_B) = v_B^s,$$

in a neighborhood around $(\arg\max_{s_A} \min_{s_B} u_A(s_A, s_B), \arg\min_{s_B} \max_{s_A} u_A(s_A, s_B))$ *in* $S_A \times S_B$.

(3) There exists a real number $\mathbf{v}_s$, s_A^m *and* s_B^m *such that*

$$u_A(s_A^m, s_B) \geq \mathbf{v}_s \text{ for any } s_B, \text{ and } u_A(s_A, s_B^m) \leq \mathbf{v}_s \text{ for any } s_A \quad (1)$$
$$\text{in a neighborhood around } (s_A^m, s_B^m) \text{ in } S_A \times S_B.$$

Proof. $(1 \to 2)$

Let s_A^* and s_B^* be the equilibrium strategies. Then,

$$\begin{aligned} v_B^s &= \min_{s_B} \max_{s_A} u_A(s_A, s_B) \leq \max_{s_A} u_A(s_A, s_B^*) = u_A(s_A^*, s_B^*) \\ &= \min_{s_B} u_A(s_A^*, s_B) \leq \max_{s_A} \min_{s_B} u_A(s_A, s_B) = v_A^s. \end{aligned}$$

On the other hand, $\min_{s_B} u_A(s_A, s_B) \leq u_A(s_A, s_B)$, then $\max_{s_A} \min_{s_B} u_A(s_A, s_B) \leq \max_{s_A} u_A(s_A, s_B)$, and so $\max_{s_A} \min_{s_B} u_A(s_A, s_B) \leq \min_{s_B} \max_{s_A} u_A(s_A, s_B)$. Thus, $v_A^s \leq v_B^s$, and we have $v_A^s = v_B^s$.

$(2 \to 3)$

Let $s_A^m = \arg\max_{s_A} \min_{s_B} u_A(s_A, s_B)$ (the maximin strategy), $s_B^m = \arg\min_{s_B} \max_{s_A} u_A(s_A, s_B)$ (the minimax strategy), and let $\mathbf{v}_s = v_A^s = v_B^s$. Then, we have

$$\begin{aligned} u_A(s_A^m, s_B) &\geq \min_{s_B} u_A(s_A^m, s_B) = \max_{s_A} \min_{s_B} u_A(s_A, s_B) = \mathbf{v}_s \\ &= \min_{s_B} \max_{s_A} u_A(s_A, s_B) = \max_{s_A} u_A(s_A, s_B^m) \geq u_A(s_A, s_B^m). \end{aligned}$$

$(3 \to 1)$

From (1)

$$u_A(s_A^m, s_B) \geq \mathbf{v}_s \geq u_A(s_A, s_B^m) \text{ for all } s_A \in S_A,\ s_B \in S_B.$$

Putting $s_A = s_A^m$ and $s_B = s_B^m$, we see $\mathbf{v}_s = u_A(s_A^m, s_B^m)$ and (s_A^m, s_B^m) is an equilibrium.

□

We write $(s_A^m, s_B^m) = (s_A^*, s_B^*)$. Denote the value of t_A which is derived from $t_A = g_A(s_A^*, s_B^*)$ by t_A^*, and denote the value of t_B which is derived from $t_B = g_B(s_A^*, s_B^*)$ by t_B^*.

4 Competition by (t_A, t_B)

Next consider competition by (t_A, t_B). Substituting f_A and f_B into u_A and u_B yields

$$u_A = u_A(f_A(t_A, t_B), f_B(t_A, t_B)),\ u_B = u_B(f_A(t_A, t_B), f_B(t_A, t_B)).$$

Let $\tilde{t}_A$ and $\tilde{t}_B$ be the values of t_A and t_B which, respectively, (locally) maximizes $u_A(f_A(t_A, t_B), f_B(t_A, t_B))$ given $\tilde{t}_B$ and (locally) maximizes $u_B(f_A(t_A, t_B), f_B(t_A, t_B))$ given $\tilde{t}_A$ in a neighborhood around $(\tilde{t}_A, \tilde{t}_B)$ in $T_A \times T_B$. Then,

$$u_A(f_A(\tilde{t}_A, \tilde{t}_B), f_B(\tilde{t}_A, \tilde{t}_B)) \geq u_A(f_A(t_A, \tilde{t}_B), f_B(t_A, \tilde{t}_B)) \text{ for all } t_A \neq \tilde{t}_A,$$

and

$$u_B(f_A(\tilde{t}_A, \tilde{t}_B), f_B(\tilde{t}_A, \tilde{t}_B)) \geq u_B(f_A(\tilde{t}_A, t_B), f_B(\tilde{t}_A, t_B)) \text{ for all } t_B \neq \tilde{t}_B.$$

Since $u_B = -u_A$, this is rewritten as

$$u_A(f_A(\tilde{t}_A, t_B), f_B(\tilde{t}_A, t_B)) \geq u_A(f_A(\tilde{t}_A, \tilde{t}_B), f_B(\tilde{t}_A, \tilde{t}_B)) \text{ for all } t_B \neq \tilde{t}_B.$$

Thus, we obtain

$$u_A(f_A(\tilde{t}_A, t_B), f_B(\tilde{t}_A, t_B)) \geq u_A(f_A(\tilde{t}_A, \tilde{t}_B), f_B(\tilde{t}_A, \tilde{t}_B)) \geq u_A(f_A(t_A, \tilde{t}_B), f_B(t_A, \tilde{t}_B))$$
$$\text{for all } t_A \neq \tilde{t}_A, \text{ and all } t_B \neq \tilde{t}_B.$$

This is equivalent to

$$u_A(f_A(\tilde{t}_A, \tilde{t}_B), f_A(\tilde{t}_A, \tilde{t}_B)) = \max_{t_A} u_A(f_A(t_A, \tilde{t}_B), f_B(t_A, \tilde{t}_B))$$
$$= \min_{t_B} u_A(f_A(\tilde{t}_A, t_B), f_B(\tilde{t}_A, t_B)).$$

Similarly to Lemma 1 we can show.

Lemma 2. *The following three statements are equivalent.*

(1) *There exists a Nash equilibrium in competition by* (t_A, t_B) *game.*

(2) *The following relation holds.*

$$v_A^t \equiv \max_{t_A} \min_{t_B} u_A(f_A(t_A, t_B), f_B(t_A, t_B)) = \min_{t_B} \max_{t_A} u_A(f_A(t_A, t_B), f_B(t_A, t_B)) \equiv v_B^t,$$

in a neighborhood around

$$(\arg\max_{t_A} \min_{t_B} u_A(f_A(t_A, t_B), f_B(t_A, t_B)), \arg\min_{t_B} \max_{t_A} u_A(f_A(t_A, t_B), f_B(t_A, t_B)))$$

in $T_A \times T_B$.

(3) *There exists a real number* v_t, $t_A^m \in T_A$ *and* $t_B^m \in T_B$ *such that*

$u_A(f_A(t_A^m, t_B), f_B(t_A^m, t_B)) \geq \mathrm{v}_t$ for any $t_B \in T_B$, and $u_A(f_A(t_A, t_B^m), f_B(t_A, t_B^m)) \leq \mathrm{v}_t$ for any $t_A \in T_A$ in a neighborhood around (t_A^m, t_B^m) in $T_A \times T_B$.

We write $(t_A^m, t_B^m) = (\tilde{t}_A, \tilde{t}_B)$. Denote the value of s_A which is derived from $s_A = f_A(\tilde{t}_A, \tilde{t}_B)$ by $\tilde{s}_A$, and denote the value of s_B which is derived from $s_B = f_B(\tilde{t}_A, \tilde{t}_B)$ by $\tilde{s}_B$.

5 Competition by (t_A, s_B)

Next consider competition by (t_A, s_B). we have

$$s_A = f_A(t_A, g_B(s_A, s_B)),\ t_B = g_B(f_A(t_A, t_B), s_B).$$

The payoffs of Player A and B are written as

$$u_A(s_A, s_B) = u_A(f_A(t_A, t_B), s_B),\ u_B(s_A, s_B) = u_B(f_A(t_A, t_B), s_B).$$

Let $\bar{t}_A$ and $\bar{s}_B$ be the values of t_A and s_B which, respectively, (locally) maximizes u_A given $\bar{s}_B$ and (locally) maximizes u_B given $\bar{t}_A$ in a neighborhood around $(\bar{t}_A, \bar{s}_B)$ in $T_A \times S_B$. Then,

$$u_A(f_A(\bar{t}_A, t_B), \bar{s}_B) \geq u_A(f_A(t_A, t_B), \bar{s}_B) \text{ for all } t_A \neq \bar{t}_A,$$

and

$$u_B(f_A(\bar{t}_A, t_B), \bar{s}_B)) \geq u_B(f_A(\bar{t}_A, t_B), s_B)) \text{ for all } s_B \neq \bar{s}_B.$$

Since $u_B = -u_A$, this is rewritten as

$$u_A(f_A(\bar{t}_A, t_B), s_B)) \geq u_A(f_A(\bar{t}_A, t_B), \bar{s}_B)) \text{ for all } s_B \neq \bar{s}_B.$$

Thus, we obtain

$$\begin{aligned} &u_A(f_A(\bar{t}_A, t_B), s_B)) \geq u_A(f_A(\bar{t}_A, t_B), \bar{s}_B)) \geq u_A(f_A(t_A, t_B), \bar{s}_B)) \\ &\text{for all } t_A \neq \bar{t}_A, \text{ and all } s_B \neq \bar{s}_B. \end{aligned}$$

This is equivalent to

$$u_A(f_A(\bar{t}_A, t_B), \bar{s}_B) = \max_{t_A} u_A(f_A(t_A, t_B), \bar{s}_B) = \min_{s_B} u_A(f_A(\bar{t}_A, t_B), s_B).$$

Similarly to Lemma 1 we can show.

Lemma 3. *The following three statements are equivalent.*

(1) *There exists a Nash equilibrium in competition by (t_A, s_B) game.*

(2) *The following relation holds.*

$$v_A^{ts} \equiv \max_{t_A} \min_{s_B} u_A(f_A(t_A, t_B), s_B) = \min_{s_B} \max_{t_A} u_A(f_A(t_A, t_B), s_B) \equiv v_B^{ts},$$

in a neighborhood around

$$(\arg\max_{t_A} \min_{s_B} u_A(f_A(t_A, t_B), s_B), \arg\min_{s_B} \max_{t_A} u_A(f_A(t_A, t_B), s_B))$$

in $T_A \times S_B$.

(3) *There exists a real number* $\mathbf{v}_{ts}$, $t_A^{ts} \in T_A$ *and* $s_B^{ts} \in S_B$ *such that*

$u_A(f_A(t_A^{ts}, t_B), s_B) \geq \mathbf{v}_{ts}$ for any $s_B \in S_B$, and $u_A(f_A(t_A, t_B), s_B^{ts}) \leq \mathbf{v}_{ts}$ for any $t_A \in T_A$ in a neighborhood around (t_A^{ts}, s_B^{ts}) in $T_A \times S_B$.

We write $(t_A^{ts}, s_B^{ts}) = (\bar{t}_A, \bar{s}_B)$. Denote the value of s_A which is derived from $s_A = f_A(\bar{t}_A, g_B(s_A, \bar{s}_B))$ by $\bar{s}_A$, and denote the value of t_B which is derived from $t_B = g_B(f_A(\bar{t}_A, t_B)\bar{s}_B)$, by $\bar{t}_B$. Then, $\bar{t}_A$ and $\bar{s}_B$ are written as

$$\bar{t}_A = g_A(\bar{s}_A, \bar{s}_B), \text{ and } \bar{s}_B = f_B(\bar{t}_A, \bar{t}_B).$$

6 Equivalence of four patterns of competition

In this section we show the equivalence of four patterns of competition. First we show the following lemma which is key to our results.

Lemma 4. *The following relations hold.*

(1) $\max_{t_A} \min_{s_B} u_A(f_A(t_A, t_B), s_B) = \max_{s_A} \min_{s_B} u_A(s_A, s_B)$.

(2) $\min_{s_B} \max_{t_A} u_A(f_A(t_A, t_B), s_B) = \min_{t_B} \max_{t_A} u_A(f_A(t_A, t_B), f_B(t_A, t_B))$.

Proof. (1) $\min_{s_B} u_A(f_A(t_A, t_B), s_B)$ is the minimum of u_A with respect to s_B given t_A. Let $s_B(t_A) = \arg\min_{s_B} u_A(f_A(t_A, t_B), s_B)$, and fix the value of s_A at $f_A(t_A, g_B(s_A, s_B(t_A)))$. Then, we have

$$\min_{s_B} u_A(f_A(t_A, g_B(s_A, s_B(t_A))), s_B)$$
$$\leq u_A(f_A(t_A, g_B(s_A, s_B(t_A))), s_B(t_A)) = \min_{s_B} u_A(f_A(t_A, t_B), s_B),$$

where $\min_{s_B} u_A(f_A(t_A, g_B(s_A, s_B(t_A))), s_B)$ is the minimum of u_A with respect to s_B given the value of s_A at $f_A(t_A, g_B(s_A, s_B(t_A)))$. This holds for any t_A. Thus,

$$\max_{f_A(t_A, g_B(s_A, s_B(t_A)))} \min_{s_B} u_A(f_A(t_A, g_B(s_A, s_B(t_A))), s_B) \leq \max_{t_A} \min_{s_B} u_A(f_A(t_A, t_B), s_B).$$

We assume $s_B(t_A)$ is single-valued. By the maximum theorem and continuity of the functions, u_A and f_A, $s_B(t_A)$ is continuous. The values of s_A in some neighborhood around $(\bar{s}_A, \bar{s}_B)$ can be realized by appropriately choosing t_A given s_B as $s_A = f_A(t_A, g_B(s_A, s_B(t_A)))$. Therefore, this can be rewritten as

$$\max_{s_A} \min_{s_B} u_A(s_A, s_B) \leq \max_{t_A} \min_{s_B} u_A(f_A(t_A, t_B), s_B). \tag{2}$$

On the other hand, $\min_{s_B} u_A(s_A, s_B)$ is the minimum of u_A with respect to s_B given s_A. Let $s_B(s_A) = \arg\min_{s_B} u_A(s_A, s_B)$, and fix the value of t_A at $g_A(s_A, s_B(s_A))$. Then, we have

$$\begin{aligned}&\min_{s_B} u_A(f_A(g_A(s_A, s_B(s_A)), g_B(s_A, s_B(s_A))), s_B)\\&\leq u_A(f_A(g_A(s_A, s_B(s_A)), g_B(s_A, s_B(s_A))), s_B(s_A)) = u_A(s_A, s_B(s_A)) = \min_{s_B} u_A(s_A, s_B),\end{aligned}$$

where $\min_{s_B} u_A(f_A(g_A(s_A, s_B(s_A)), g_B(s_A, s_B(s_A))), s_B)$ is the minimum of u_A with respect to s_B given the value of t_A at $g_A(s_A, s_B(s_A))$. This holds for any s_A. Thus,

$$\max_{g_A(s_A, s_B(s_A))} \min_{s_B} u_A(f_A(g_A(s_A, s_B(s_A)), g_B(s_A, s_B(s_A))), s_B) \leq \max_{s_A} \min_{s_B} u_A(s_A, s_B)$$

We assume $s_B(s_A)$ is single-valued. By the maximum theorem and continuity of u_A, $s_B(s_A)$ is continuous. The values of t_A in some neighborhood around $(\bar{t}_A, \bar{s}_B)$ can be realized by appropriately choosing s_A given s_B as $t_A = g_A(s_A, s_B(s_A))$. Therefore, this can be rewritten as

$$\max_{t_A} \min_{s_B} u_A(f_A(t_A, t_B), s_B) \leq \max_{s_A} \min_{s_B} u_A(s_A, s_B). \tag{3}$$

Combining (2) and (3), we get

$$\max_{s_A} \min_{s_B} u_A(s_A, s_B) = \max_{t_A} \min_{s_B} u_A(f_A(t_A, t_B), s_B).$$

(2) $\max_{t_A} u_A(f_A(t_A, t_B), s_B)$ is the maximum of u_A with respect to t_A given s_B. Let $t_A(s_B) = \arg\max_{t_A} u_A(f_A(t_A, t_B), s_B)$, and fix the value of t_B at $g_B(f_A(t_A(s_B), t_B), s_B)$. Then, we have

$$\begin{aligned}&\max_{t_A} u_A(f_A(t_A, g_B(f_A(t_A(s_B), t_B), s_B)), s_B)\\&= \max_{t_A} u_A(f_A(t_A, g_B(f_A(t_A(s_B), t_B), s_B)), f_B(t_A, g_B(f_A(t_A(s_B), t_B), s_B)))\\&\geq u_A(f_A(t_A(s_B), g_B(f_A(t_A(s_B), t_B), s_B)), s_B) = \max_{t_A} u_A(f_A(t_A, t_B), s_B),\end{aligned}$$

where $\max_{t_A} u_A(f_A(t_A, g_B(f_A(t_A(s_B), t_B), s_B)), s_B)$ is the maximum of u_A with respect to t_A given the value of t_B at $g_B(f_A(t_A(s_B), t_B), s_B))$. This holds for any s_B. Thus,

$$\begin{aligned}\min_{g_B(f_A(t_A(s_B), t_B), s_B))} \max_{t_A} &u_A(f_A(t_A, g_B(f_A(t_A(s_B), t_B), s_B)), f_B(t_A, g_B(f_A(t_A(s_B), t_B), s_B)))\\&\geq \min_{s_B} \max_{t_A} u_A(f_A(t_A, t_B), s_B).\end{aligned}$$

We assume $t_A(s_B)$ is single-valued. By the maximum theorem and continuity of the functions, u_A and f_A, $t_A(s_B)$ is continuous. The values of t_B in some neighborhood around $(\bar{t}_A, \bar{t}_B)$ can be realized by appropriately choosing s_B given t_A as $t_B = g_B(f_A(t_A(s_B), t_B), s_B)$. Therefore, this can be rewritten as

$$\min_{t_B} \max_{t_A} u_A(f_A(t_A, t_B), f_B(t_A, t_B)) \geq \min_{s_B} \max_{t_A} u_A(f_A(t_A, t_B), s_B). \tag{4}$$

On the other hand, $\max_{t_A} u_A(f_A(t_A,t_B), f_B(t_A,t_B))$ is the maximum of u_A with respect to t_A given t_B. Let $t_A(t_B) = \arg\max_{t_A} u_A(f_A(t_A,t_B), f_B(t_A,t_B))$, and fix the value of s_B at $f_B(t_A(t_B),t_B)$. Then, we have

$$\begin{aligned}&\max_{t_A} u_A(f_A(t_A,t_B), f_B(t_A(t_B),t_B))\\ &\geq u_A(f_A(t_A(t_B),t_B), f_B(t_A(t_B),t_B)) = \max_{t_A} u_A(f_A(t_A,t_B), f_B(t_A,t_B)),\end{aligned}$$

where $\max_{t_A} u_A(f_A(t_A,t_B), f_B(t_A(t_B),t_B))$ is the maximum of u_A with respect to t_A given the value of s_B at $f_B(t_A(t_B),t_B)$. This holds for any t_B. Thus,

$$\min_{f_B(t_A(t_B),t_B)} \max_{t_A} u_A(f_A(t_A,t_B), f_B(t_A(t_B),t_B)) \geq \min_{t_B}\max_{t_A} u_A(f_A(t_A,t_B), f_B(t_A,t_B)).$$

We assume $t_A(t_B)$ is single-valued. By the maximum theorem and continuity of the functions, u_A, f_A and f_B, $t_A(t_B)$ is continuous. The values of s_B in some neighborhood around $(\bar{t}_A, \bar{s}_B)$ can be realized by appropriately choosing t_B given t_A as $s_B = f_B(t_A(t_B),t_B)$. Therefore, this can be rewritten as

$$\min_{s_B}\max_{t_A} u_A(f_A(t_A,t_B), s_B) \geq \min_{t_B}\max_{t_A} u_A(f_A(t_A,t_B), f_B(t_A,t_B)). \tag{5}$$

Combining (4) and (5), we get

$$\min_{t_B}\max_{t_A} u_A(f_A(t_A,t_B), f_B(t_A,t_B)) = \min_{s_B}\max_{t_A} u_A(f_A(t_A,t_B), s_B).$$

□

Now we show the following propositions.

Proposition 1. (1) *Competition by (s_A, s_B) and competition by (t_A, s_B) are equivalent.*

(2) *Competition by (t_A, s_B) and competition by (t_A, t_B) are equivalent.*

Proof. (1) We show that the condition for $(\bar{s}_A, \bar{s}_B)$ and the condition for (s_A^*, s_B^*) are the same. From Lemma (3)

$$\max_{t_A}\min_{s_B} u_A(f_A(t_A,t_B), s_B) = \min_{s_B}\max_{t_A} u_A(f_A(t_A,t_B), s_B) = u_A(\bar{s}_A, \bar{s}_B).$$

Since any value of s_A can be realized by appropriately choosing t_A given s_B, we have $\max_{t_A} u_A(f_A(t_A,t_B), s_B) = \max_{s_A} u_A(s_A, s_B)$ for any s_B. Thus,

$$\min_{s_B}\max_{s_A} u_A(s_A, s_B) = \min_{s_B}\max_{t_A} u_A(f_A(t_A,t_B), s_B) = u_A(\bar{s}_A, \bar{s}_B).$$

From Lemma 4 we have $\max_{t_A}\min_{s_B} u_A(f_A(t_A,t_B), s_B) = \max_{s_A}\min_{s_B} u_A(s_A, s_B)$. Therefore, we obtain

$$\max_{s_A}\min_{s_B} u_A(s_A, s_B) = \min_{s_B}\max_{s_A} u_A(s_A, s_B) = u_A(\bar{s}_A, \bar{s}_B).$$

This is 2 of Lemma 1.

(2) We show that the condition for $(\bar{t}_A, \bar{t}_B)$ and the condition for $(\tilde{t}_A, \tilde{t}_B)$ are the same. From Lemma (3)

$$\max_{t_A} \min_{s_B} u_A(f_A(t_A, t_B), s_B) = \min_{s_B} \max_{t_A} u_A(f_A(t_A, t_B), s_B) = u_A(f_A(\bar{t}_A, \bar{t}_B), f_B(\bar{t}_A, \bar{t}_B)).$$

Since any value of t_B can be realized by appropriately choosing s_B given t_A, we have $\min_{s_B} u_A(f_A(t_A, t_B), s_B) = \min_{t_B} u_A(f_A(t_A, t_B), f_B(t_A, t_B))$ for any t_A. Thus,

$$\max_{t_A} \min_{t_B} u_A(f_A(t_A, t_B), f_B(t_A, t_B)) = \max_{t_A} \min_{s_B} u_A(f_A(t_A, t_B), s_B) = u_A(f_A(\bar{t}_A, \bar{t}_B), f_B(\bar{t}_A, \bar{t}_B)).$$

From Lemma 4 we have $\min_{s_B} \max_{t_A} u_A(f_A(t_A, t_B), s_B) = \min_{t_B} \max_{t_A} u_A(f_A(t_A, t_B), f_B(t_A, t_B))$. Therefore, we obtain

$$\begin{aligned}
&\max_{t_A} \min_{t_B} u_A(f_A(t_A, t_B), f_B(t_A, t_B)) = \min_{t_B} \max_{t_A} u_A(f_A(t_A, t_B), f_B(t_A, t_B)) \\
&= u_A(f_A(\bar{t}_A, \bar{t}_B), f_B(\bar{t}_A, \bar{t}_B)).
\end{aligned}$$

This is 2 of Lemma 2.

□

Exchanging A with B we can show the following proposition.

Proposition 2. (1) *Competition by (s_A, s_B) and competition by (s_A, t_B) are equivalent.*

(2) *Competition by (s_A, t_B) and competition by (t_A, t_B) are equivalent.*

Finally, from these results we get

Proposition 3. *Competition by (s_A, s_B) and competition by (t_A, t_B) are equivalent.*

Therefore, all of four patterns of competition are equivalent.

7 Concluding Remark

We have shown that in a two-person zero-sum game with two sets of alternative strategic variables, any pattern of competition is equivalent, and any selection of strategic variables is equivalent. We want to extend the results of this paper to a symmetric n-person zero-sum game[4].

[4]In an asymmetric situation the equivalence does not hold with more than two players.

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