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Abstract: The study analyses quantitative models for financial markets by starting from geometric Brown process and Wiener process by analyzing Ito's lemma and first passage model. Furthermore, it is analyzed the prices of the options, Vanilla & Exotic, by using the expected value and numerical model with geometric applications. From contingent claim approach ALM strategies are also analyzed so to get the effective duration measure of liabilities by assuming that clients buy options for protection and liquidity by assuming defaults protection barrier as well. Furthermore, the study analyses interest rate models by showing that the yields curve is given by the average of the expected short rates & variation of GDP with the liquidity risk, but in the case we have crisis it is possible to have risk premium as well, the study is based on simulated modelisation by using the drift condition in combination with the inflation models as expectation of the markets. Moreover, the CIR process is considered as well by getting with modification of the diffusion process the same result of the simulated modelisation but we have to consider that the CIR process is considered in the simulated environment as well. The credit risk model is considered as well in intensity model & structural model by getting the liquidity and risk premium and the PD probability from the Rating Matrix as well by using the diagonal. Furthermore, the systemic risk is considered as well by using a deco relation concept by copula approaches. Moreover, along the equilibrium condition between financial markets is achieved the equity pricing with implications for the portfolio construction in simulated environment with Bayesian applications for smart beta. Finally, Value at Risk is also analyzed both static and dynamic with implications for the percentile of daily return and the tails risks by using a simulated approach.

Stochastic Differential Equations

In finance is made a large use of Wiener process and geometric Brown process, the name came from George Brown in the 1827 that noted that the volatility of a small particle suspended in a liquid increases with the time, Wiener gave a mathematical formal assumption on the phenomena from this the term Wiener process. The geometric Brown process is used in finance to indicate a formal assumption for the dynamic of the prices that does not permit to assume negative value, formally we have:

$$\frac{dS(t)}{S(t)} = \mu(t)dt + \sigma dWs$$

Where $\mu(t)$denotes the drift of the distribution and it is the average in the *dt* , σ denotes the volatility of the distribution and dWs denotes a Wiener process such that it may be decomposed by the following:

$$dWs = N[0,1]\sqrt{dt}$$

We may assume the following for the Wiener process:

$$E\left[\oint_t^T dWs^2\right] = T - t \implies dWs^2 \sim dt$$

This means that a Wiener process is a forward process, the uncertainty is to the end of the process in *T + dt* . From this we may obtain explanation of Ito's lemma by using Taylor series, if we take a function of S as F(S) we may write Ito's lemma in the following way:

$$dF(S) = \frac{\partial F}{\partial t}dt + \frac{\partial F}{\partial S}dS + \frac{1}{2}\left(\frac{\partial^2 F}{\partial^2 S}\, dS^2 + \frac{\partial^2 F}{\partial^2 t}dt^2 + 2\,\frac{\partial^2 F}{\partial S \partial t}\, dt\, dS\right) + Q$$

We may note that:

$$dS^2 = \mu^2\, S^2 dt^2 + 2\,\mu\,\sigma\, S^2 dt^{3/2} + \sigma^2 S^2\, dt$$

$$dS\, dt = \mu\, S\, dt^2 + \sigma\, S\, dt^{3/2}$$

From this we obtain as *dt* tends to zero:

$$dF(S) = \frac{\partial F}{\partial t}dt + \frac{\partial F}{\partial S}dS + \frac{1}{2}\left(\frac{\partial^2 F}{\partial^2 S}\; S^2\sigma^2 dt\right)$$

By substituting *d*S we obtain Ito's lemma:

$$dF(S) = \left(\frac{\partial F}{\partial t} + \mu\frac{\partial F}{\partial S}S + \frac{1}{2}\frac{\partial^2 F}{\partial^2 S}\; S^2\sigma^2\right)dt + \sigma\frac{\partial F}{\partial S}\; SdZ$$

We may see now the solution of geometric Brown process:

$$F(S) = \ln S(t) \quad dF(S) = \ln\big(dS(t)\big)$$

$$dF(S) = \frac{1}{S}\, dS + \frac{1}{2}\left(-\frac{1}{S^2}\right)\, dS^2$$

Because:

$$\frac{\partial F}{\partial t} = 0$$

As such we have:

$$dF(S) = \left(\mu(t) - \frac{1}{2}\,\sigma^2\right) dt + \sigma dWs$$

$$F(S(T)) = F\big(S(t)\big) + \ dF(S)$$

$$\ln S(T) = \ \ln S(t) + \ \left(\mu(t) - \frac{1}{2}\,\sigma^2\right) dt + \sigma dWs$$

$$S(T) = \ S(t)e^{\left(\mu(t) - \frac{1}{2}\sigma^2\right)dt + \ \sigma dWs}$$

Now we may see the solution of the geometric process, as we will see the process cannot assume negative value, in fact we have the following solution:

$$E[S(T)] = \ S(t)e^{\mu T + \frac{1}{2}\sigma^2 T}$$

As we may note the expected value of σdWs is $\frac{1}{2}\,\sigma^2 dt$ as result we may assume that a geometric process is given by the following process:

$$\frac{dS(t)}{S(t)} = (\mu(t) - \frac{1}{2}\,\sigma^2)dt \ + \sigma dWs$$

As such we have that the expected value is given by:

$$E[S(T)] = \ S(t)e^{\mu T}$$

From this we may note that to keep the average of the distribution the geometric process may be characterized by the following distribution:

$$S(T) = \ S(t)e^{(\mu(t) - \frac{1}{2}\sigma^2)dt + \ \sigma dWs}$$

This result may be obtained by using Ito's lemma, so we refer for the rest of book to this result as Ito's lemma. We may see as to obtain the expected value of a normal distribution as such we have the following:

$$\int f(z) = \int z\, f(z)$$

As such we have the following:

$$\int \frac{1}{\sigma\sqrt{2\pi}}\ z\, e^{-\frac{1}{2}z^2}$$

This may be rewritten by:

$$\int \frac{1}{\sigma}\ z\frac{1}{2}\ z^2 \sim \frac{1}{2}\ \sigma^2$$

From this we may obtain explanation for Ito's lemma, if we take a function of S as F(S) we may write Ito's lemma in the following way:

$$dF(S) = \frac{\partial F}{\partial t} + \ \mu\frac{\partial F}{\partial S}S + \frac{1}{2}\frac{\partial^2 F}{\partial^2 S}\ S^2\sigma^2 dt + \ \sigma\frac{\partial F}{\partial S}\ SdZ$$

Where:

$$\sigma dZ = N[\frac{z-\mu}{\sigma\sqrt{dt}}\ ; \sigma^2 dt]$$

As result:

$$E\ [\sigma dZ] = 0$$

Because:

$$\int f(z) = \int \frac{z-\mu}{\sigma\sqrt{dt}}\ f(z) = 0$$

Now we may analyze the following parabolic problem:

$$\frac{\partial F}{\partial t} + \ \mu\frac{\partial F}{\partial S}S + \frac{1}{2}\frac{\partial^2 F}{\partial^2 S}\ S^2\sigma^2 dt = 0$$

Subject to the following constraint:

$$F\big(S(T)\big) = F(S)$$

The solution it is easy to solve, because if we take ito's lemma and we take the expectation we obtain that the solution to the parabolic problem is given by:

$$F\big(S(T)\big) = E\ [F(S)]$$

As result we may rewrite Ito's lemma in the following form:

$$dF(S) = \frac{\partial F}{\partial t} + \ \mu\frac{\partial F}{\partial S}S - \frac{1}{2}\frac{\partial^2 F}{\partial^2 S}\ S^2\sigma^2 dt - \ \sigma\frac{\partial F}{\partial S}\ SdZ$$

Now it is interesting to analyses first passage model:

$$f(z) - \ f(z-H)$$

Where:

$$f(z-H)\ \exists\ \forall\ z>H$$

As result we have:

$$\int_{-\infty}^{+\infty} f(z)-\ f(z-H)=\int_{-\infty}^{+\infty} f(z)-\int_{H}^{+\infty} z\,f(z)=1-f(H)^{-1}\,N[-H]$$

Interest Rate Models

The price of a zero coupon bond P(T) is given by the following:

$$P(T) = E[\, e^{-\int_t^T r(t)dt}]$$

Where *r(t)* denotes the short rate that is given by the following stochastic differential equation:

$$dr(t) = \mu(t)dt + \sigma_r \, dWr$$

$$r(t) = R(t)$$

As result by applying Ito's lemma we have the following for the price of a zero coupon bond:

$$P(T) = e^{-(r(t)T+\mu(t)T^2)}$$

From this we may see that the internal compounded *R(T)* interest rate is given by the following:

$$R(T) = r(t) + \mu(t)T$$

We may investigate the drift by using the forward process as result we have the following:

$$Forward = -\frac{\partial LogP(T)}{\partial T}$$

By applying Ito's lemma we have the following process:

$$dLogP(T) = \left(r(t) - \frac{1}{2}\,\sigma_r^2 T\right)dt - \sigma_r T dWr$$

As such we have:

$$-\frac{\partial dLogP(T)}{\partial T} = r(t) + \sigma_r^2 dt + \sigma_r dWr$$

As result we have:

$$R(T) = r(t) + \sigma_r^2 T$$

That is the drift condition. Now if we build a portfolio of default free bonds by shorting the bonds overvalued and acquiring the bonds undervalued we obtain a relation rule that the yields curve must respect given by the following:

$$\gamma = \frac{R(T) - r(t)}{\sigma_r T}$$

From this we may derive that in absence of arbitrage opportunities we have by assumption the following:

$$\gamma = \sigma_r$$

This is the risk premium requested by the markets, and it is a function of the risk associated with the volatility of the short rate. In absence of arbitrage opportunities the stochastic differential equation that a default free bond must satisfy is given by the following for P(T) = F(r,t):

$$\frac{\partial F}{\partial t} + (\mu - \gamma\sigma_r)\frac{\partial F}{\partial r} + \frac{1}{2}\frac{\partial^2 F}{\partial^2 r}\ \sigma^2 - rF(r,t) = 0$$

The solution to this parabolic problem is given by the integrant factor, as such we have the following:

$$F(r,t) = E(e^{-\int_t^T r(t)\,dt}\ x\ \ 1)$$

Where:

$$dR(T) = \ (\mu - \gamma\sigma_r)dt + \sigma_r\ dWr$$

$$r(t)\ \ = R(t)$$

As result F(r,t) is the risk free as such we have the following solution:

$$R(T) = r(t) + \sigma_r^2 T$$

This is the future value of the short rate in fact if take the average of R(T) for each maturities we have that the risk free rate is given by:

$$r(T) = \ r(t) + \frac{1}{2}\sigma_r^2 T$$

As such we have that $\frac{1}{2}\sigma_r^2 T$ represents the liquidity risk. Because:

$$dR(t) = \ (\mu + \gamma\sigma_r)dt + \sigma_r\ dWr$$

As result we have without arbitrage conditions the following:

$$R(T) = r(t) + \sigma_r^2 T + \sigma_r^2 T$$

Where $\sigma_r^2 T$ denotes the risk premium, so in the yield curve it is possible to have liquidity risk and risk premium. If we take in consideration the inflation we have the following;

$$\sigma_I^2 T = \sigma_r^2 T + \sigma_r^2 T$$

Where $\sigma_I^2 T$ is the expected inflation for each maturities, where σ_I^2 is the variance of the inflation, that is the drift condition such that we have:

$$r(T) = \ r(t) + \sigma_I^2 T$$

As result we have that:

$$Average\ (\sigma_I^2 T) = \sigma_r^2 T$$

The compounded inflation is equal to $\sigma_r^2 T$, so there is an equilibrium relation between the volatility of inflation and the volatility of the short rate. Now it is interesting to analyse the inflation, as result we have the following for the inflation swap as expectation;

$$Inflation\ Swap = \sigma_I^2 T + \frac{1}{2}\sigma_I^2 T$$

and the following for the inflation simulated:

$$Inflation\ Simulated = \sigma_I^2 T + \sigma_I^2 T$$

We may have an inflation, decreasing, or stable, i.e. there are different kind of equilibrium with the short rate, and it is possible to have just liquidity risk or risk premium as well. Now we may assume the following affine form that F(r,t) must satisfy:

$$F(r,t) = A(t,T)e^{-B(t,T)r(t)}$$

Where:

$$dr(t) = (b - ar(t))dt + \sigma_r\, dWr$$

$$r(t) = R(t)$$

We may note that:

$$\mu(t) = b - ar(t)$$

At this point we may solve the stochastic differential equation:

$$A_t - rAB_t - ABb + ABar + \frac{1}{2}AB^2\sigma^2 - rA = 0$$

Where:

$$A(T,T) = 1$$

$$B(T,T) = 0$$

$$A_t - ABb + \frac{1}{2}AB^2\sigma^2 = 0$$

$$B_t + Ba - 1 = 0$$

From this we obtain:

$$B(t,T) = \frac{1 - e^{-a\,(T-t)}}{a}$$

$$A(t,T) = exp\int_t^T Bb - \frac{1}{2}B^2\sigma^2 = \exp\frac{(B(T,t) - T + t)(ab - \frac{1}{2}\sigma^2)}{a^2} - \sigma^2\frac{B(t,T)^2}{4a}$$

We may now assume the following distribution for the short rate:

$$dr(t) = (b - ar(t))dt + \sqrt{r(t)}\sigma_r \, dWr$$

$$r(t) = R(t)$$

At this point we may solve the stochastic differential equation:

$$A_t - rAB_t - ABb + ABar + \frac{1}{2} AB^2 r\sigma^2 - rA = 0$$

As such we have the following system:

$$A_t - ABb = 0$$

$$B_t - Ba - \frac{1}{2} B^2\sigma^2 + 1 = 0$$

From this we may obtain a simplified solution:

$$B(t,T) = \frac{2\,(e^{\gamma(T-t)} - 1)}{(\gamma + a)\,(e^{\gamma(T-t)} - 1) + 2\gamma}$$

$$A(t,T) = (\frac{2\gamma\, e^{\frac{(\gamma+a)(T-t)}{2}}}{(\gamma + a)\,(e^{\gamma(T-t)} - 1) + 2\gamma})^{\frac{2ab}{\sigma^2}}$$

$$\gamma = a + 2\sigma$$

We may note that:

$$R(T) = -\frac{lnA(t,T) - B(t,T)r(t)}{T - t}$$

We may note that if we take the expected value of the distribution we obtain directly the risk free rate, as such we have the following pricing formula:

$$B(t,T) = \frac{2\,(e^{\gamma(T-t)} - 1)}{(\gamma + a)\,(e^{\gamma(T-t)} - 1) + 2\gamma}$$

$$A(t,T) = (\frac{2\gamma\, e^{\frac{(\gamma+a)(T-t)}{2}}}{(\gamma + a)\,(e^{\gamma(T-t)} - 1) + 2\gamma})^{\frac{2b}{\sigma^2}}$$

$$\gamma = \sqrt{a^2 + 2\sigma^2}$$

On this we have to add liquidity and risk premium to obtain the yield curve without arbitrage condition, the simplified solution is the forward of this solution. Indeed, we do not know how much risk premium there is embedded in the yield curve, for instance if there is just liquidity premium or risk premium as well, or if the yield curve is the risk free such that the forward rate is the forward. Indeed, the expected value of the solution does not calibrate the yields curve if there is risk premium or a different forward structure with respect the assumptions of the model, instead the simplified solution permits to calibrate the yields curve where the forward rate is the forward such that does not consider the liquidity and risk premium, it is just the case of risk free rate. The solution to the problem was proposed by Brigo, Mercurio (2006) with the CIR++ as such we have the following:

$$\varphi^{CIR}(t,a) = f^{MKT}(0,t) - f^{CIR}(0,t,a)$$

$$f^{CIR}(0,t,a) = \frac{2b(e^{t\gamma}-1)}{2\gamma+(a+\gamma)(e^{t\gamma}-1)} + x_0 \frac{4\gamma^2 e^{t\gamma}}{[2\gamma+(a+\gamma)(e^{t\gamma}-1)]^2}$$

$$\gamma = \sqrt{a^2+2\sigma^2}$$

The price of a zero coupon bond maturing at time T is given by:

$$P(t,T) = C(t,T)e^{-B(t,T)r(t)}$$

Where:

$$C(t,T) = \frac{P^{MKT}(0,T)A(0,t)e^{-B(0,t)x_0}}{P^{MKT}(0,t)A(0,T)e^{-B(0,T)x_0}} A(t,T)e^{B(t,T)\varphi^{CIR}(t,a)}$$

$$B(t,T) = \frac{2\,(e^{\gamma(T-t)}-1)}{(\gamma+a)\,(e^{\gamma(T-t)}-1)+2\gamma}$$

$$A(t,T) = \left(\frac{2\gamma\, e^{\frac{(\gamma+a)(T-t)}{2}}}{(\gamma+a)\,(e^{\gamma(T-t)}-1)+2\gamma}\right)^{\frac{2b}{\sigma^2}}$$

Option Pricing Models

The option pricing model is based on the arbitrage setting, the main idea is that the pay off and the price of the option may be replicated so its value is directly determinate to avoid arbitrage opportunity called hedging relation. Further application was about the dividend because when the stock pays the dividend its prices will decrease for the same amount but we have to observe that speculators in the hedged portfolio will income the dividend such that they have the same final payoff. We use a different approach in this text book we will price option as expected value of its final pay off along the equilibrium relation between financial markets as we will see in the rest of the book. The final pay off of a Call and Put option is given by the following:

$$Call = Max[S(T) - K\,; 0]$$

$$Put = Max[K - S(T)\,; 0]$$

The prices of the options are given by the expectation of the final pay off discounted:

$$C(S,T,K) = P(T)E[Max(S(T) - K\,; 0)]$$

$$P(S,T,K) = P(T)E[Max(K - S(T)\,; 0)]$$

Now we assume the following distribution for the stock prices;

$$\frac{dS(t)}{S(t)} = (r(t) + \delta - q)dt + \sigma_S\, dWs$$

δ may be the risk premium or the liquidity premium, depends from the equilibrium in the treasury market, instead q denotes the dividend yield, instead we assume the following process for the default free zero coupon bond:

$$\frac{dP(T)}{P(T)} = r(t)dt + \sigma_P dWp$$

$$\sigma_P = \sigma_r T$$

Now to compute the value of the option is a problem because we have stochastic interest rate so the solution is to take the default free zero coupon bond as forward measure, so by using it as numeraire we have the following process in equilibrium between financial markets:

$$\frac{dN(t)}{N(t)} = \sigma_N dWn$$

Where:

$$N(t) = \frac{S(t)}{P(T)}$$

$$\sigma_N^2 = \frac{\int_t^T \sigma_S^2 + \sigma_P^2 - \ 2\rho\sigma_S\sigma_P \ dt}{T - t}$$

Now we derive the price of a Call option, as such we have the following:

$$\frac{C(S,T,K)}{P(T)} = \int_{-\infty}^{+\infty} Max \ [N(t)e^{-\frac{1}{2}\sigma_N^2 T + \sigma_N z\sqrt{T}} - K; 0 \ ] \ f(z)dz$$

The integral vanishes when N(T) < K, thus by solving for z we have:

$$z^{\circ} = \frac{\ln\left(\frac{KP(T)}{S}\right) + \frac{1}{2}\sigma_N^2 T}{\sigma_N\sqrt{T}}$$

As result we may rewrite the integral in the following form:

$$\int_{z^{\circ}}^{+\infty} [N(t)e^{-\frac{1}{2}\sigma_N^2 T + \sigma_N z\sqrt{T}}] \ f(z)dz \int_{z^{\circ}}^{+\infty} K \ f(z)dz$$

By using the symmetry property of a normal distribution we obtain the following pricing formula:

$$\frac{C(S,T,K)}{P(T)} = N(t)N\left[-z^{\circ} + \sigma_N\sqrt{T}\right] - \ KN[-z^{\circ}]$$

As result the price of a Call option is given by:

$$C(S,T,K) = S(t)N[d1] - \ P(T)KN[d2]$$

Where:

$$d1 = \frac{\ln\left(\frac{S}{KP(T)}\right) + \frac{1}{2}\sigma_N^2 T}{\sigma_N\sqrt{T}}$$

$$d2 = \frac{\ln\left(\frac{S}{KP(T)}\right) - \frac{1}{2}\sigma_N^2 T}{\sigma_N\sqrt{T}}$$

As result the value of a Put option is given by:

$$P(S,T,K) = P(T)KN[-d2] - \ S(t)N[-d1]$$

Where:

$$-d2 = \frac{\ln\left(\frac{KP(T)}{S}\right) + \frac{1}{2}\sigma_N^2 T}{\sigma_N\sqrt{T}}$$

$$-d1 = \frac{\ln\left(\frac{KP(T)}{S}\right) - \frac{1}{2}\sigma_N^2 T}{\sigma_N \sqrt{T}}$$

We may note that between the two formulations there is a parity relation such that we have:

$$P(S,T,K) - C(S,T,K) + S(t) = P(T)K$$

Now we may note that the Put option formula may be less than its payoff, this not a good news because we may have European and American options, the European options may be exercised just at maturity, instead the American options may be exercised before of maturity as such if the value of the options is greater or equal to the pays off they will not be exercised before of maturity, furthermore, the early exercised opportunity may bring in the Put Call parity to have a greater earning with respect the risk free rate so we may value the American Put option such that there is parity relation in the world of numeraire i.e. with interest rate nil. As such we have the following pricing formula for the American Put option that is greater or equal to the pay off:

$$P(S,T,K) = KN[h1] - S(t)N[h2]$$

Where:

$$h1 = \frac{\ln\left(\frac{K}{S}\right) + \frac{1}{2}\sigma_N^2 T}{\sigma_N \sqrt{T}}$$

$$h2 = \frac{\ln\left(\frac{K}{S}\right) - \frac{1}{2}\sigma_N^2 T}{\sigma_N \sqrt{T}}$$

Indeed, we have got the same formulation of Black, Scholes (1973) with the changes of measure and by considering that the dividend is income so to have the same final pay off in the hedge portfolio. A different approach that gives the same result of finite difference methods is the lattice methods that is a kind of discretizetion. The lattice methods has the appealing feature to permit to simulate in a binomial trees the price of the option by backward iterations. As such, in the discounting process is chose the greater between the prices and the pays off, this does not permit to the final value of the option to be less than its pay off, this feature is very appealing for American option if we consider the dividend or for the case of American Put option because the European Put option may be less than its pay off. Now It is interesting to introduce the lattice methods in binomial model as such we assume the following:

$$u = e^{\sigma\sqrt{\Delta t}} \quad d = e^{-\sigma\sqrt{\Delta t}} \quad a = e^{r\Delta t}$$

The risk neutral probability is given by the following for up and down respectively:

$$p = \frac{a-d}{u-d} \quad 1-p$$

The pays off are given by:

$$C_{N,i} = Max\left[\, Su^i d^{N-i} - K\,, 0\,\right]$$

$$P_{N,i} = Max\left[\, K - Su^i d^{N-i}, 0\,\right]$$

The prices are given for European options by:

$$C_{j,i} = e^{-r\Delta t}\,[p\, C_{j+1,i+1} + (1-p)C_{J+1,i}]$$

$$P_{j,i} = e^{-r\Delta t}\,[p\, P_{j+1,i+1} + (1-p)P_{J+1,i}]$$

Instead, for American options by:

$$C_{j,i} = Max\,[Su^i d^{j-i} - K\,, e^{-r\Delta t}\,(p\, C_{j+1,i+1} + (1-p)C_{J+1,i})\,]$$

$$P_{j,i} = Max\,[K - Su^i d^{j-i}\,, e^{-r\Delta t}\,(p\, P_{j+1,i+1} + (1-p)P_{J+1,i})]$$

As such we may compare the lattice methods with the expected value. So we have the following prospect for the American Call options:

σ_S	K	r	S_t	T	Lattice 150 nodes	Expected
0,08	1	0,03	1,5	1	0,52955	0,52955
0,08	1	0,03	1,25	1	0,27957	0,27957
0,08	1	0,03	1	1	0,04834	0,04840
0,08	1	0,03	0,75	1	0,00001	0,00001
0,08	1	0,03	0,5	1	0,00000	0,00000
0,08	1	0,03	1,5	2	0,55824	0,55824
0,08	1	0,03	1,25	2	0,30847	0,30848
0,08	1	0,03	1	2	0,07884	0,07893
0,08	1	0,03	0,75	2	0,00076	0,00078
0,08	1	0,03	0,5	2	0,00000	0,00000
0,08	1	0,03	1,5	5	0,63934	0,63935
0,08	1	0,03	1,25	5	0,39054	0,39054
0,08	1	0,03	1	5	0,15775	0,15788
0,08	1	0,03	0,75	5	0,01801	0,01818
0,08	1	0,03	0,5	5	0,00003	0,00004

And the following prospect for the European Call options:

σ_S	K	r	S_t	T	Lattice 150 nodes	Expected
0,08	1	0,03	1,5	1	0,52955	0,52955
0,08	1	0,03	1,25	1	0,27957	0,27957
0,08	1	0,03	1	1	0,04834	0,04840
0,08	1	0,03	0,75	1	0,00001	0,00001
0,08	1	0,03	0,5	1	0,00000	0,00000
0,08	1	0,03	1,5	2	0,55824	0,55824
0,08	1	0,03	1,25	2	0,30847	0,30848
0,08	1	0,03	1	2	0,07884	0,07893

0,08	1	0,03	0,75	2	0,00076	0,00078
0,08	1	0,03	0,5	2	0,00000	0,00000
0,08	1	0,03	1,5	5	0,63934	0,63935
0,08	1	0,03	1,25	5	0,39054	0,39054
0,08	1	0,03	1	5	0,15775	0,15788
0,08	1	0,03	0,75	5	0,01801	0,01818
0,08	1	0,03	0,5	5	0,00003	0,00004

As we may see the two formulations converge, this is due to the fact that binomial distribution converges in the limit to the normal distribution. Now we may see the case of American Put options:

σ_S	K	r	S_t	T	Lattice 150 nodes	Expected
0,08	1	0,03	1,5	1	0,00000	0,00000
0,08	1	0,03	1,25	1	0,00002	0,00007
0,08	1	0,03	1	1	0,02184	0,03191
0,08	1	0,03	0,75	1	0,25000	0,25000
0,08	1	0,03	0,5	1	0,50000	0,50000
0,08	1	0,03	1,5	2	0,00000	0,00001
0,08	1	0,03	1,25	2	0,00026	0,00115
0,08	1	0,03	1	2	0,02677	0,04511
0,08	1	0,03	0,75	2	0,25000	0,25017
0,08	1	0,03	0,5	2	0,50000	0,50000
0,08	1	0,03	1,5	5	0,00006	0,00088
0,08	1	0,03	1,25	5	0,00165	0,01014
0,08	1	0,03	1	5	0,03254	0,07127
0,08	1	0,03	0,75	5	0,25000	0,25352
0,08	1	0,03	0,5	5	0,50000	0,50000

We may note that the lattice method undervalues the option at the money, but we may see that the two formulations converge if we assume interest rate nil:

σ_S	K	r	S_t	T	Lattice 150 nodes	Expected
0,08	1	0	1,5	1	0,00000	0,00000
0,08	1	0	1,25	1	0,00007	0,00007
0,08	1	0	1	1	0,03185	0,03191
0,08	1	0	0,75	1	0,25000	0,25000
0,08	1	0	0,5	1	0,50000	0,50000
0,08	1	0	1,5	2	0,00001	0,00001
0,08	1	0	1,25	2	0,00113	0,00115
0,08	1	0	1	2	0,04504	0,04511
0,08	1	0	0,75	2	0,25017	0,25017
0,08	1	0	0,5	2	0,50000	0,50000
0,08	1	0	1,5	5	0,00086	0,00088
0,08	1	0	1,25	5	0,01017	0,01014
0,08	1	0	1	5	0,07115	0,07127
0,08	1	0	0,75	5	0,25350	0,25352
0,08	1	0	0,5	5	0,50000	0,50000

As such we have the following prospect for European Put options:

σ_S	K	r	S_t	T	Lattice 150 nodes	Expected
0,08	1	0,03	1,5	1	0,00000	0,00000
0,08	1	0,03	1,25	1	0,00002	0,00002
0,08	1	0,03	1	1	0,01879	0,01884
0,08	1	0,03	0,75	1	0,22046	0,22046
0,08	1	0,03	0,5	1	0,47045	0,47045
0,08	1	0,03	1,5	2	0,00000	0,00000
0,08	1	0,03	1,25	2	0,00024	0,00024
0,08	1	0,03	1	2	0,02061	0,02069
0,08	1	0,03	0,75	2	0,19252	0,19254
0,08	1	0,03	0,5	2	0,44176	0,44176
0,08	1	0,03	1,5	5	0,00005	0,00005
0,08	1	0,03	1,25	5	0,00125	0,00124
0,08	1	0,03	1	5	0,01845	0,01859
0,08	1	0,03	0,75	5	0,12871	0,12889
0,08	1	0,03	0,5	5	0,36074	0,36075

As we may see the two formulations converge, this is due to the fact that binomial distribution converges in the limit to the normal distribution. As we have seen the lattice methods may be an alternative valuation model in the pricing of options. Now if we take the three month volatility we will not get the effective market prices of the options, this is due to the fact that the market prices are not continuous processes, so we have to model the dynamic of the jump to obtain the effective market prices of the options. We start with the presentation of a jump diffusion process:

$$\frac{dS(t)}{S(t)} = \mu(t)dt \ + \sigma(t)dWs + P(dt)N[a, \sigma]$$

P(dt) denotes a Poisson distribution and counts the number of jumps that are measured by the Normal distribution that is perfectly correlated with the Wiener process, the jump has the same direction. The problem it is easy to solve because in real markets the jump happens in every instant because the markets prices are not continuous as result we may solve the equation in the following way:

$$\frac{dS(t)}{S(t)} = (\mu - a)dt \ + \sigma(J)dWj$$

Where:

$$\sigma(J) = \sigma(t) + \sigma$$

So the effective volatility may be decomposed in two parts, a continuous part and a jump part so by taking the instantaneous volatility we may obtain the effective market prices of the options that may be approximate by sharing for two the three month volatility.

Now it is interesting to introduce barrier option, we may note that a knock in barrier option Call exists is the strike price K is less than the barrier H as result we have the following pricing formula:

$$Call\ in\ (S, K < H, T) = S\left(N[d1] - \frac{S}{HP(T)}\ N[h1]\right)$$

$$- P(T)K\left(N[d2] - \frac{HP(T)}{S}\ N[h2]\right)$$

Where:

$$d1 = \frac{\ln\left(\frac{S}{KP(T)}\right) + \frac{1}{2}\sigma_N^2 T}{\sigma_N\sqrt{T}}$$

$$h1 = \frac{\ln\left(\frac{S}{HP(T)}\right) + \frac{1}{2}\sigma_N^2 T}{\sigma_N\sqrt{T}}$$

$$d2 = \frac{\ln\left(\frac{S}{KP(T)}\right) - \frac{1}{2}\sigma_N^2 T}{\sigma_N\sqrt{T}}$$

$$h2 = \frac{\ln\left(\frac{S}{HP(T)}\right) - \frac{1}{2}\sigma_N^2 T}{\sigma_N\sqrt{T}}$$

At the same way a knock in barrier Put exists if the strike price K is greater than the barrier H, as result we have the following pricing formula:

$$Put\ in(S, K > H, T) = -S\left(N[-d1] - \frac{S}{HP(T)}\ N[-h1]\right) + P(T)K\left(N[-d2] - \frac{HP(T)}{S}\ N[-h2]\right)$$

As such we may obtain the value of a knock out barrier option as follows:

$$Call\ out(S, K < H, T) = S\left(1 - \frac{S}{HP(T)}\ N[h1]\right)$$

$$- P(T)K\left(1 - \frac{HP(T)}{S}\ N[h2]\right)$$

$$Put\ out(S, K > H, T) = -S\left(1 - \frac{S}{HP(T)}\ N[-h1]\right)$$

$$+ P(T)K\left(1 - \frac{HP(T)}{S}\ N[-h2]\right)$$

Otherwise, we may use the following equalities:

$$C(S, K, T) - C(S, H, T) = Call\ out(S, K < H, T) + Call\ in(S, K < H, T)$$

$$P(S, K, T) - P(S, H, T) = Put\ out(S, K > H, T) + Put\ in(S, K > H, T)$$

As such we may obtain the value of in barrier alive from the following equalities:

$$C(S,K,T) = Call\ out(S,K < H,T) + \ Call\ in\ alive(S,K < H,T)$$

$$P(S,K,T) = Put\ out(S,K > H,T) + \ Put\ in\ alive(S,K > H,T)$$

As such we may obtain the survival probabilities at first passage model with respect the barrier H:

$$\left(N[d1] - \frac{S}{HP(T)}\ N[h1]\right)$$

Where:

$$d1 = \frac{\ln\left(\frac{S}{HP(T)}\right) + \frac{1}{2}\sigma_N^2 T}{\sigma_N \sqrt{T}}$$

$$h1 = \frac{\ln\left(\frac{HP(T)}{S}\right) - \frac{1}{2}\sigma_N^2 T}{\sigma_N \sqrt{T}}$$

As such we may obtain the PD probabilities with the following:

$$\left(N[-d1] + \frac{S}{HP(T)}\ N[h1]\right)$$

We may note that the pay off of in option is deterministic and depends from the probability that the underlying will touch the barrier, indeed is a binary option, as such we have the following:

$$Call\ in(S,K < H,T) = N[\dots]\ P(T)\ (\ H - K)$$

$$N[\dots] = \left(N[d1] - \frac{S}{HP(T)}\ N[h1]\right)$$

Where:

$$d1 = \frac{\ln\left(\frac{S}{HP(T)}\right) + \frac{1}{2}\sigma_N^2 T}{\sigma_N \sqrt{T}}$$

$$h1 = \frac{\ln\left(\frac{HP(T)}{S}\right) - \frac{1}{2}\sigma_N^2 T}{\sigma_N \sqrt{T}}$$

$$Put\ in(S,K > H,T) = N[\dots]\ P(T)\ (K - H)$$

$$N[\dots] = \left(N[d1] - \frac{HP(T)}{S}\ N[h1]\right)$$

Where:

$$d1 = \frac{\ln\left(\frac{HP(T)}{S}\right) + \frac{1}{2}\sigma_N^2 T}{\sigma_N \sqrt{T}}$$

$$h1 = \frac{\ln\left(\frac{S}{HP(T)}\right) - \frac{1}{2}\sigma_N^2 T}{\sigma_N \sqrt{T}}$$

As such we have:

$$Call\ out(S, K < H, T) = C(S, K, T) - C(S, H, T) - Call\ in(S, K < H, T)$$

$$Put\ out(S, K > H, T) = P(S, K, T) - P(S, H, T) - Put\ in(S, K > H, T)$$

As such we may obtain the value of in barrier alive from the following equalities:

$$C(S, K, T) = Call\ out(S, K < H, T) + Call\ in\ alive(S, K < H, T)$$

$$P(S, K, T) = Put\ out(S, K > H, T) + Put\ in\ alive(S, K > H, T)$$

From this we have:

$$Call\ in\ alive(S, K < H, T) = C(S, H, T) + Call\ in(S, K < H, T)$$

$$Put\ in\ alive(S, K > H, T) = P(S, H, T) + Put\ in(S, K > H, T)$$

Now it is interesting to analyze look back options, the pays off may be given by the maximum less the minimum of prices realizations, the formulations for the market prices may be obtained by computing the expected value on the following pays off for Call options and Put options respectively:

$$Call = P(T)E[Max(S - Minimum, 0)]$$

$$Put = P(T)E[Max(Maximum - S, 0)]$$

Another kind of look back options may be given for Call options by the maximum of prices realizations less the strike price and for Put options by the strike price less the minimum of prices realizations, the formulations for the market prices may be obtained by computing the expected value on the following pays off for Call options and Put options respectively:

$$Call = P(T)(Maximum - K) + P(T)E[Max(S - Maximum, 0)]$$

$$Put = P(T)(K - Minimum) + P(T)E[Max(Mimimum - S, 0)]$$

Now it is interesting to analyze chooser option, that gives the option to choose between a Put option and a Call option, its pay off is given by the following:

$$Chooser\ Option = Max\ [\ P(S, K, T); C(S, K, T)]$$

By using the fair Put Call parity we have the following:

$$Chooser\ Option = Max\ [\ P(S, K, T); S - P(T)K + P(S, K, T]$$

$$Chooser\ Option = Max[\,-S + P(T)K; 0]$$

$$Chooser\ Option = Max[\,P(T)K - S; 0]$$

It is easy to note that we may obtain the solution by using $P(T)K$ as numeraire, along the same line we may explore exchange option, the pay off is given by the following:

$$Exchange\ Option = Max[\,S_1; S_2]$$

That may be rewritten as:

$$Exchange\ Option = Max[\frac{S_1}{S_2}\ ; 1]$$

As result we may obtain the solution by using S_2 as numeraire, along the same line we may analyze spread option, the pay off is given by the following:

$$Spread\ Option = Max[(S_1 - S_2) -\ K; 0]$$

We may rewrite the pay off in the following way:

$$Spread\ Option = Max[\frac{S_1}{S_2} -\ 1 - \frac{K}{S_2}, 0]$$

$$Spread\ Option = Max[\frac{S_1}{S_2} - \left(\frac{K}{S_2} +\ 1\right), 0]$$

As such by using S_2 as numeraire we may obtain the price of a spread option by approximating although the strike price is stochastic. Now it is interesting to introduce the concept of stochastic volatility, in fact if the volatility is stochastic the solution is not unique, but anyway we may approximate the value of the option by using taylor series where the first moment is that of volatility, this gives real good approximation of real prices, in fact, the second and third order series converge faster to zero. Alternative solution may be to estimate the partial differential equation that the option prices must satisfy by assuming stochastic volatility as solving it by using numerical procedure. This let us to introduce arbitrage theory, in practice if we built the following portfolio we have:

$$V_t = S\ \frac{\partial F(S)}{\partial S}\ \pm\ \ F(S)$$

The portfolio is risk free, as such by using Ito's lemma we obtain the following stochastic differential equation:

$$\frac{\partial F}{\partial t} +\ r\frac{\partial F}{\partial S}S + \frac{1}{2}\frac{\partial^2 F}{\partial^2 S}\ S^2\sigma^2 \pm rF(S) = 0$$

We may solve the stochastic differential equation by using the integrant factor:

$$F(S) = Z(S)e^{\pm\, r\, T}$$

By solving the stochastic differential equation for Z(S) we obtain that the solution is given by:

$$Z(S) = E(S(T))$$

By solving we obtain:

$$F(S) = E(S(T)e^{\pm r\,T})$$

This means that if we replicate the prices of options by using delta hedging the value of options are given again by the expected value of the pay off discounted where the drift of the process is given again by the short rate, this is what it is called risk neutral world. Instead, for the Put option if we are long on the stock we pay the insurance on the portfolio, but we may assume that the return on the portfolio is zero, i.e. r = 0 because we have a zero variation and the insurance is already represented by the premium of the options as result we obtain again the formulation for the American Put option such that there is parity relation Put Call in the world of numeraire. We have already seen the solution by using the expected value, but now it is interesting to introduce the explicit numerical model, in fact, the value of the options may be obtained by solving numerically the stochastic differential equation, As we know the value of the stock simulated is given by:

$$S(T) = S(t)e^{(r(t) + \sigma^2)T}$$

As such we may solve the stochastic differential equation in the following way:

$$\frac{\partial F}{\partial S} = \frac{(k-1)}{kS}$$

Where:

$$k = \frac{Se^{(r(t) + \sigma^2)T} - S}{S^2} \sim e^{(r(t)+ \sigma^2)T}$$

$$\frac{\partial^2 F}{\partial^2 S} = -\frac{(k-1)}{kS^2}$$

$$\frac{\partial F}{\partial t} \sim \sigma^2 S \left(\frac{T}{\frac{Grids}{T}\ Grids} \right)$$

By substitute these values in the stochastic differential equation we obtain the price of the options, the problem of this procedure is that the price of the option is unique but we may assume the following:

$$F(0) = Max\ (S-K;0) for\ Call = Max\ (K-S;0) for\ Put\ as\ such\ we\ have\ F\big(S(T)\big) = F(0) + F(S)$$

The interesting fact is to analyze hedging strategies as such we have the following:

$$\frac{\partial F}{\partial S}\,\partial S = \partial F$$

As such we may write the prices of the options by assuming the following for Call options and Put options, respectively:

$$Max\left(\frac{(k-1)}{k}\,S + S - K;0\right)$$

$$Max\left(\frac{(k-1)}{k}S+P(T)K-S;0\right)$$

We may note that the options positions are immunized with respect a variation of 100% of the underlying, the assumptions may be used for pricing purpose as well so we may compare the model with the expected approach for European Call options, as such we have the following:

σ_S	K	r	S_t	T	Hedging	Expected
0,08	1	0,03	1,5	1	0,55362	0,52955
0,08	1	0,03	1,25	1	0,29468	0,27957
0,08	1	0,03	1	1	0,03575	0,04840
0,08	1	0,03	0,75	1	0,00000	0,00001
0,08	1	0,03	0,5	1	0,00000	0,00000
0,08	1	0,03	1,5	2	0,60853	0,55824
0,08	1	0,03	1,25	2	0,34044	0,30848
0,08	1	0,03	1	2	0,07235	0,07893
0,08	1	0,03	0,75	2	0,00000	0,00078
0,08	1	0,03	0,5	2	0,00000	0,00000
0,08	1	0,03	1,5	5	0,78142	0,63935
0,08	1	0,03	1,25	5	0,48452	0,39054
0,08	1	0,03	1	5	0,18761	0,15788
0,08	1	0,03	0,75	5	0,00000	0,01818
0,08	1	0,03	0,5	5	0,00000	0,00004

Instead, we may compare the model for European Put options:

σ_S	K	r	S_t	T	Hedging	Expected
0,08	1	0,03	1,5	1	0,00000	0,00000
0,08	1	0,03	1,25	1	0,00000	0,00002
0,08	1	0,03	1	1	0,00619	0,01884
0,08	1	0,03	0,75	1	0,24725	0,22046
0,08	1	0,03	0,5	1	0,48832	0,47045
0,08	1	0,03	1,5	2	0,00000	0,00000
0,08	1	0,03	1,25	2	0,00000	0,00024
0,08	1	0,03	1	2	0,01412	0,02069
0,08	1	0,03	0,75	2	0,24603	0,19254
0,08	1	0,03	0,5	2	0,47794	0,44176
0,08	1	0,03	1,5	5	0,00000	0,00005
0,08	1	0,03	1,25	5	0,00000	0,00124
0,08	1	0,03	1	5	0,04832	0,01859
0,08	1	0,03	0,75	5	0,25142	0,12889
0,08	1	0,03	0,5	5	0,45451	0,36075

We may note that the numerical methods capture the skew in the implied volatility embedded in the value of the options without varying the volatility. As such the hedging strategies permit to immunize the positions without varying continually the underlying. We may further develop the numerical methods by assuming the following:

$$S_i = ih \; T_j = jdt$$

$$\frac{\partial F}{\partial t} = \frac{C(i,j) - C(i,j-1)}{dt}$$

$$\frac{\partial F}{\partial S} = \frac{\big(C(i+1,j) - C(i,j)\big) + \big(C(i,j) - C(i-1,j)\big)}{h}$$

$$\frac{\partial^2 F}{\partial S^2} = \frac{\big(C(i+1,j) - C(i,j)\big) - \big(C(i,j) - C(i-1,j)\big)}{h^2}$$

By substituting these values in the stochastic differential equation we obtain the following:

$$aC(i-1,j) + bC(i,j) + cC(i+1,j) = C(i,j-1)$$

$$cP(i-1,j) + bP(i,j) + aP(i+1,j) = P(i,j-1)$$

Where:

$$a = \frac{1}{2}\sigma^2 i^2 dt - dt\, r\, i$$

$$b = 1 - \sigma^2 i^2 dt + r\, dt$$

$$c = dt\, r\, i + \frac{1}{2}\sigma^2 i^2 dt$$

We assume the following:

$$u = e^{\sigma\sqrt{\Delta t}} \qquad d = e^{-\sigma\sqrt{\Delta t}}$$

As result the pays off are given by:

$$C_{N,i} = Max\left[\, Su^i d^{N-i} - K\, , 0\,\right]$$

$$P_{N,i} = Max\left[\, K - Su^i d^{N-i}, 0\,\right]$$

As result the prices for European options are given by:

$$C_{j,i} = a\, C_{j+1,i-1} + b\, C_{j+1,i} + c\, C_{j+1,i+1}$$

$$P_{j,i} = a\, P_{j+1,i+1} + b\, P_{j+1,i} + c\, P_{j+1,i-1}$$

We may compare the result with European Call options:

σ_S	K	r	S_t	T	Numerical 3 Grids	Expected
0,08	1	0,03	1,5	1	0,57634	0,52955
0,08	1	0,03	1,25	1	0,31435	0,27957
0,08	1	0,03	1	1	0,05099	0,04840
0,08	1	0,03	0,75	1	0,00000	0,00001
0,08	1	0,03	0,5	1	0,00000	0,00000
0,08	1	0,03	1,5	2	0,61843	0,55824
0,08	1	0,03	1,25	2	0,35021	0,30848

0,08	1	0,03	1	2	0,07827	0,07893
0,08	1	0,03	0,75	2	0,00000	0,00078
0,08	1	0,03	0,5	2	0,00000	0,00000
0,08	1	0,03	1,5	5	0,73670	0,63935
0,08	1	0,03	1,25	5	0,45207	0,39054
0,08	1	0,03	1	5	0,15310	0,15788
0,08	1	0,03	0,75	5	0,00274	0,01818
0,08	1	0,03	0,5	5	0,00000	0,00004

We may compare the result with European Put options:

σ_S	*K*	*r*	S_t	*T*	Numerical 3 Grids	Expected
0,08	1	0,03	1,5	1	0,00000	0,00000
0,08	1	0,03	1,25	1	0,00000	0,00002
0,08	1	0,03	1	1	0,00231	0,01884
0,08	1	0,03	0,75	1	0,22694	0,22046
0,08	1	0,03	0,5	1	0,49663	0,47045
0,08	1	0,03	1,5	2	0,00000	0,00000
0,08	1	0,03	1,25	2	0,00000	0,00024
0,08	1	0,03	1	2	0,00677	0,02069
0,08	1	0,03	0,75	2	0,22751	0,19254
0,08	1	0,03	0,5	2	0,50917	0,44176
0,08	1	0,03	1,5	5	0,00000	0,00005
0,08	1	0,03	1,25	5	0,00200	0,00124
0,08	1	0,03	1	5	0,02975	0,01859
0,08	1	0,03	0,75	5	0,25661	0,12889
0,08	1	0,03	0,5	5	0,56729	0,36075

Instead, for American options are given by:

$$C_{j,i} = Max\ [Su^i d^{j-i} - K\ , a\ C_{j+1,i-1} + b\ C_{j+1,i} + c\ C_{j+1,i+1}]$$

$$P_{j,i} = Max\ [K - Su^i d^{j-i}\ , a\ P_{j+1,i+1} + b\ P_{j+1,i} + c\ P_{j+1,i-1}]$$

For American Call options we obtain the same result of European Call options, in fact, it is interesting to compare the results for American Put options, as such, we have the following prospect:

σ_S	*K*	*r*	S_t	*T*	Numerical 5 Grids	Expected
0,08	1	0,03	1,5	1	0,00000	0,00000
0,08	1	0,03	1,25	1	0,00000	0,00007
0,08	1	0,03	1	1	0,03514	0,03191
0,08	1	0,03	0,75	1	0,27636	0,25000
0,08	1	0,03	0,5	1	0,51757	0,50000
0,08	1	0,03	1,5	2	0,00000	0,00001
0,08	1	0,03	1,25	2	0,00000	0,00115
0,08	1	0,03	1	2	0,04934	0,04511
0,08	1	0,03	0,75	2	0,28700	0,25017
0,08	1	0,03	0,5	2	0,52467	0,50000
0,08	1	0,03	1,5	5	0,00000	0,00088

0,08	1	0,03	1,25	5	0,00021	0,01014
0,08	1	0,03	1	5	0,07688	0,07127
0,08	1	0,03	0,75	5	0,30766	0,25352
0,08	1	0,03	0,5	5	0,54398	0,50000

Appendix

To run the simulation we used the following VBA code:

```
Function NumeriCallAmerican(Spot, k, T, r, sigma, n)

Dim dt As Double, u As Double, d As Double, p As Double

   dt = T / n

   u = Exp(sigma * (dt ^ 0.5))

   d = 1 / u

Dim S() As Double

ReDim S(n + 1, n + 1) As Double

 For i = 1 To n + 1

   For j = i To n + 1

      S(i, j) = Spot * u ^ (j - i) * d ^ (i - 1)

   Next j

Next i

Dim Opt() As Double

ReDim Opt(n + 1, n + 1) As Double

For i = 1 To n + 1

    Opt(i, n + 1) = Application.Max(S(i, n + 1) - k, 0)

Next i

Dim a() As Double, b() As Double, c() As Double

ReDim a(n + 1) As Double, b(n + 1) As Double, c(n + 1) As Double

For i = 1 To n + 1

   a(i) = (0.5 * sigma ^ 2 * i ^ 2 * dt) - (dt * r * i)

   b(i) = (1 - (sigma ^ 2 * i ^ 2 * dt) + r * dt)
```

```
   c(i) = (dt * r * i + 0.5 * sigma ^ 2 * i ^ 2 * dt)
Next i
For j = n To 1 Step -1
   For i = 2 To j
   Opt(i, j) =  Application.Max(S(i, j) - k, (a(i) * Opt(i + 1, j + 1) + b(i) * Opt(i, j + 1) + c(i) * Opt(i - 1, j +
1)))
   NumeriCallAmerican = Opt(i, j)
  Next i
 Next j
End Function
Function NumeriCallEuropean(Spot, k, T, r, sigma, n)
Dim dt As Double, u As Double, d As Double, p As Double
   dt = T / n
   u = Exp(sigma * (dt ^ 0.5))
   d = 1 / u
Dim S() As Double
ReDim S(n + 1, n + 1) As Double
 For i = 1 To n + 1
   For j = i To n + 1
      S(i, j) = Spot * u ^ (j - i) * d ^ (i - 1)
   Next j
Next i
Dim Opt() As Double
ReDim Opt(n + 1, n + 1) As Double
For i = 1 To n + 1
    Opt(i, n + 1) = Application.Max(S(i, n + 1) - k, 0)
Next i
Dim a() As Double, b() As Double, c() As Double
```

```
ReDim a(n + 1) As Double, b(n + 1) As Double, c(n + 1) As Double
For i = 1 To n + 1
    a(i) = (0.5 * sigma ^ 2 * i ^ 2 * dt) - (dt * r * i)
    b(i) = (1 - (sigma ^ 2 * i ^ 2 * dt) + r * dt)
    c(i) = (dt * r * i + 0.5 * sigma ^ 2 * i ^ 2 * dt)
Next i
For j = n To 1 Step -1
    For i = 2 To j
    Opt(i, j) =(a(i) * Opt(i + 1, j + 1) + b(i) * Opt(i, j + 1) + c(i) * Opt(i - 1, j + 1))
    NumeriCallEuropean = Opt(i, j)
  Next i
 Next j
End Function
Function NumeriPutAmerican(Spot, k, T, r, sigma, n)
Dim dt As Double, u As Double, d As Double, p As Double
    dt = T / n
    u = Exp(sigma * (dt ^ 0.5))
    d = 1 / u
Dim S() As Double
ReDim S(n + 1, n + 1) As Double
 For i = 1 To n + 1
    For j = i To n + 1
        S(i, j) = Spot * u ^ (j - i) * d ^ (i - 1)
    Next j
Next i
Dim Opt() As Double
ReDim Opt(n + 1, n + 1) As Double
```

```
For i = 1 To n + 1

    Opt(i, n + 1) = Application.Max(k - S(i, n + 1) , 0)

Next i

Dim a() As Double, b() As Double, c() As Double

ReDim a(n + 1) As Double, b(n + 1) As Double, c(n + 1) As Double

For i = 1 To n + 1

   a(i) = (0.5 * sigma ^ 2 * i ^ 2 * dt) - (dt * r * i)

   b(i) = (1 - (sigma ^ 2 * i ^ 2 * dt) + r * dt)

   c(i) = (dt * r * i + 0.5 * sigma ^ 2 * i ^ 2 * dt)

Next i

For j = n To 1 Step -1

   For i = 2 To j

   Opt(i, j) = Application.Max(k - S(i, j) , (a(i) * Opt(i - 1, j + 1) + b(i) * Opt(i, j + 1) + c(i) * Opt(i + 1, j +
1)))

   NumeriPutAmerican = Opt(i, j)

  Next i

 Next j

End Function

Function NumeriPutEuropean(Spot, k, T, r, sigma, n)

Dim dt As Double, u As Double, d As Double, p As Double

   dt = T / n

   u = Exp(sigma * (dt ^ 0.5))

   d = 1 / u

Dim S() As Double

ReDim S(n + 1, n + 1) As Double

 For i = 1 To n + 1

   For j = i To n + 1

      S(i, j) = Spot * u ^ (j - i) * d ^ (i - 1)
```

```
   Next j
Next i
Dim Opt() As Double
ReDim Opt(n + 1, n + 1) As Double
For i = 1 To n + 1
    Opt(i, n + 1) = Application.Max(k - S(i, n + 1) , 0)
Next i
Dim a() As Double, b() As Double, c() As Double
ReDim a(n + 1) As Double, b(n + 1) As Double, c(n + 1) As Double
For i = 1 To n + 1
   a(i) = (0.5 * sigma ^ 2 * i ^ 2 * dt) - (dt * r * i)
   b(i) = (1 - (sigma ^ 2 * i ^ 2 * dt) + r * dt)
   c(i) = (dt * r * i + 0.5 * sigma ^ 2 * i ^ 2 * dt)
Next i
For j = n To 1 Step -1
   For i = 2 To j
   Opt(i, j) = (a(i) * Opt(i - 1, j + 1) + b(i) * Opt(i, j + 1) + c(i) * Opt(i + 1, j + 1))
   NumeriPutEuropean = Opt(i, j)
  Next i
 Next j
End Function
Function BinomialCallAmerican(Spot, k, T, r, sigma, n)
Dim dt As Double, u As Double, d As Double, p As Double
   dt = T / n
   u = Exp(sigma * (dt ^ 0.5))
   d = 1 / u
   p = (Exp(r * dt) - d) / (u - d)
```

```
Dim S() As Double
ReDim S(n + 1, n + 1) As Double
      For i = 1 To n + 1
   For j = i To n + 1
      S(i, j) = Spot * u ^ (j - i) * d ^ (i - 1)
   Next j
Next i
Dim Opt() As Double
ReDim Opt(n + 1, n + 1) As Double
For i = 1 To n + 1
    Opt(i, n + 1) = Application.Max(S(i, n + 1) - k, 0)
Next i
For j = n To 1 Step -1
   For i = 1 To j
   Opt(i, j) = Application.Max(S(i, j) - k, Exp(-r * dt) * (p * Opt(i, j + 1) + (1 - p) * Opt(i + 1, j + 1)))
   BinomialCallAmerican = Opt(i, j)
  Next i
Next j
End Function
Function BinomialCallEuropean(Spot, k, T, r, sigma, n)
Dim dt As Double, u As Double, d As Double, p As Double
   dt = T / n
   u = Exp(sigma * (dt ^ 0.5))
   d = 1 / u
   p = (Exp(r * dt) - d) / (u - d)
Dim S() As Double
ReDim S(n + 1, n + 1) As Double
```

```
    For i = 1 To n + 1
  For j = i To n + 1
     S(i, j) = Spot * u ^ (j - i) * d ^ (i - 1)
  Next j
Next i
Dim Opt() As Double
ReDim Opt(n + 1, n + 1) As Double
For i = 1 To n + 1
   Opt(i, n + 1) = Application.Max(S(i, n + 1) - k, 0)
Next i
For j = n To 1 Step -1
  For i = 1 To j
  Opt(i, j) = Exp(-r * dt) * (p * Opt(i, j + 1) + (1 - p) * Opt(i + 1, j + 1))
  BinomialCallEuropean = Opt(i, j)
 Next i
Next j
End Function
Function BinomialPutAmerican(Spot, k, T, r, sigma, n)
Dim dt As Double, u As Double, d As Double, p As Double
  dt = T / n
  u = Exp(sigma * (dt ^ 0.5))
  d = 1 / u
  p = (Exp(r * dt) - d) / (u - d)
Dim S() As Double
ReDim S(n + 1, n + 1) As Double
    For i = 1 To n + 1
  For j = i To n + 1
```

```
      S(i, j) = Spot * u ^ (j - i) * d ^ (i - 1)
   Next j
Next i
Dim Opt() As Double
ReDim Opt(n + 1, n + 1) As Double
For i = 1 To n + 1
    Opt(i, n + 1) = Application.Max(k - S(i, n + 1) , 0)
Next i
For j = n To 1 Step -1
   For i = 1 To j
   Opt(i, j) = Application.Max(k - S(i, j) , Exp(-r * dt) * (p * Opt(i, j + 1) + (1 - p) * Opt(i + 1, j + 1)))
   BinomialPutAmerican = Opt(i, j)
  Next i
Next j
End Function
Function BinomialPutEuropean(Spot, k, T, r, sigma, n)
Dim dt As Double, u As Double, d As Double, p As Double
   dt = T / n
   u = Exp(sigma * (dt ^ 0.5))
   d = 1 / u
   p = (Exp(r * dt) - d) / (u - d)
Dim S() As Double
ReDim S(n + 1, n + 1) As Double
      For i = 1 To n + 1
   For j = i To n + 1
      S(i, j) = Spot * u ^ (j - i) * d ^ (i - 1)
   Next j
```

```
Next i
Dim Opt() As Double
ReDim Opt(n + 1, n + 1) As Double
For i = 1 To n + 1
    Opt(i, n + 1) = Application.Max(k - S(i, n + 1) , 0)
Next i
For j = n To 1 Step -1
   For i = 1 To j
   Opt(i, j) = Exp(-r * dt) * (p * Opt(i, j + 1) + (1 - p) * Opt(i + 1, j + 1))
   BinomialPutEuropean = Opt(i, j)
  Next i
Next j
End Function
```

Implied Volatility

Now it is interesting to introduce the concept of implied volatility, in practice the implied volatility is the value of volatility that gives you the market prices of the options. The problem is geometric with respect the normal distribution or the cumulative of the normal distribution, as such we may get the implied volatility by using the following formulation:

$$\frac{\sigma T}{2} = \sqrt{2\pi}\,\frac{C(S,K,T)}{S+K}$$

Now if we compute the price of the options with this implied value we get a value that needs Taylor series at first degree to reach the market value, furthermore, the volatility increases as the option goes in the money and decreases as the option goes out money, this is in line with market prices because the values of the options deep out money are nil, otherwise we may assume parity relation Put Call such that prevails the option in the money. The interesting fact is that the value extracted for options at the money usually are the half of the three month volatility. Indeed, if we use the continuous volatility with the numerical model we get the market price of the options along this formulation for the implied volatility by showing again that the price of an option indeed is a geometric problem. The numerical model seems to capture the skew in the implied volatility extracted without varying the volatility. This it is very interesting because the market prices of the options converge to the value computed on the base of the implied value extracted so we may have two target prices: the market & the equilibrium. Indeed, we may obtain the implied volatility value with the following relation:

$$2\,\sigma T = \sqrt{2\pi\frac{C(S,K,T)}{S+K}}$$

We are assuming that the continuous time drops so the implied volatility is twice with respect the instantaneous volatility, this value may be used for numerical model as well and as substitute of the equality presented before. As such we may obtain the market value by doing Taylor series at first order. We have to observe that it is possible to do not assume the hypothesis seen before, such that the market value is given by the implied volatility as jump process such that:

$$\sigma T = \sqrt{2\pi\frac{C(S,K,T)}{S+K}}$$

On its we may estimate the series by doing the following:

$$\sigma T = \sqrt{2\pi\frac{C(S,K,T)}{S+K}} + \sqrt{2\pi[\frac{C(S,K,T)}{S+K}]^2}$$

The main idea is that the problem is geometric and that there is relation between the historical volatility and the implied value, further generalizations are for future research.

Caplet and Floorlet

The price of Caplet and Floorlet may be derived directly from the arbitrage condition between Cap, Floor and Swap that is given by the following relation:

$$Nominal\ x\ \left[\left(1+r(t)\right)+Floor-Cap\right]xT = Nominal\ x\ [1+Swap\ Rate]xT$$

From the relation we may note that we have a fix rate $\left(1+r(t)\right)$ because if the interest rate decreases the Floor goes in the money so to have a fix rate, instead if interest rate increases the Cap goes in the money to subtract the greater earning to obtain a fix rate. Indeed the fix rate is not $\left(1+r(t)\right)$ because we income the price of the Cap less the price of the Floor, As such we have a fix rate that to avoid arbitrage opportunities must give the following prices for Caplet and Floorlet:

$$Caplet = P(T)\left[\,F(T)N[h1] - \,KN[h2]\right]$$

$$Floorlet = P(T)\left[\,KN[-h2] - \,F(T)N[-h1]\right]$$

Where:

$$h1 = \frac{\ln\left(\frac{F(T)}{K}\right)+\frac{1}{2}\sigma_F^2(T-t)}{\sigma_F\sqrt{T-t}}$$

$$h2 = \frac{\ln\left(\frac{F(T)}{K}\right)-\frac{1}{2}\sigma_F^2(T-t)}{\sigma_F\sqrt{T-t}}$$

$$\sigma_F = \ \sigma_r/F(T)$$

F(T) denotes the Forward of the Swap rate given by:

$$-\frac{ln\left(\frac{exp-(Swap(T)xT)}{exp-(Swap(t)xt)}\right)}{T-t}$$

This may be considered the fair value as well but if we go in the OTC market we will not get the market prices because there isn't arbitrage and the price of Cap and Floor are equals, this suggests that they are priced with a geometric martingale such that we have the following pricing formula:

$$Caplet = P(T)\left[\,r(t)N[h1] - \,KN[h2]\right]$$

$$Floorlet = P(T)\left[\,KN[-h2] - \,r(t)N[-h1]\right]$$

Where:

$$h1 = \frac{\ln\left(\frac{r(t)}{K}\right)+\frac{1}{2}\sigma_r^2 T}{\sigma_r\sqrt{T}}$$

$$h2 = \frac{\ln\left(\frac{r(t)}{K}\right) - \frac{1}{2}\sigma_r^2 T}{\sigma_r \sqrt{T}}$$

Now if we use the volatility of the short rate σ_r we will not get the market prices of the options because we have to consider the stochastic volatility, so by doing the Taylor series with respect the volatility by solving the problem of percentage with approximate we may obtain the effective market price of the options. Note that we may get the implied volatility value from the following equality:

$$\sigma = \sqrt{2\pi}\,\frac{Caplet}{2}\,10$$

Structural Model for Credit Risk

The equity value of a levered firm may be seen as contingent claim on the value of assets, in fact, if at the time of maturity of the debts the value of assets is less than the value of debts, the equity value is nil (out of money), and the residual value of assets is shared between alls debt holders as such the liabilities has a short position on a Put option, such that there is parity relation Put Call. As result the American options are priced with the familiar European options, in particular the balance sheet of the company may be decomposed by the following:

$$E(t) = C(A, L, T)$$

$$\mathrm{L(t) = \ L(T)P(T) - P(A, L, T)}$$

The Equity value E(t) is a Call option C(A,L,T) on the firm's under laying value A(t), that represents the asset portfolio, with strike prices equal to the final value of the debts L(T). Instead, the initial value of the debts L(t) is given by the debts value discounted by using the risk free discount factor P(T) with a short position on the Put option P(A,L,T) on firm's under laying value A(t), with strike prices equal to the final value of the debts. It reflects the option of stake holder to walk away if things go wrong by leaving the liabilities holders with the residual value of the company. We may note that the PD probability is given by:

$$PD = N[d1]$$

Where:

$$d1 = \frac{\ln\left(\frac{KP(T)}{A(t)}\right) - \frac{1}{2}\sigma_N^2 T}{\sigma_N \sqrt{T}}$$

The problem of this approach is that it consider just a single maturity, but the problem it is easy to solve, by assuming the following:

$$B(t) = L1P(t) - \ P(A, L1, t) \qquad C(A, L1, t) = A - B(t)$$

$$B1(t) = L1P(t) - \ P(A, L1, t) - \frac{L2}{L1 + L2}\, P(A, L1, t) - \frac{L2}{L1 + L2}\ P(A - B, L2, T)$$

$$B2(T) = L2\, P(T) - \ P(A - B, L2, T) + \frac{L2}{L1 + L2}\, P(A, L1, t) + \frac{L2}{L1 + L2}\ P(A - B, L2, T)$$

$$E(T) = C(A - B, L2, T)$$

Our time horizon is T > t > 0 , We denote with L1, the final value of the debts, with time of maturity t , and with L2, the final value of the debts, with time of maturity T, and the respective initial value of the debts with B1(t) and with B2(T). We can note that the company can declare insolvency at every instance and for every level of the value of assets by exercising the American Put options that for the parity relation in the company balance sheet are equals to the European options. We may note that we are assuming that there is not covenant, so the pricing formula may be used for junk bonds where the company is free to default when it is more convenient for itself. Alternatively we may price the liabilities with a short position on barrier in Put option such that if the value of assets touches the barrier the company will default, this permits to avoid to the bond holders to face a very great loss, in practice we are assuming covenant between share holders and bond holders. We may note that it is possible to assume that when the value of assets crosses the default barrier the company will default, as such we have the following pricing formula:

$$E(t) = A(t) - L(t)$$

$$\mathrm{L(t)} = \mathrm{L(T)P(T)} - Put\ in(A, L > H, T)$$

Where:

$$H = L(T)\, e^{-\gamma T}$$

$$Put\ in(A, K > H, T) = N[\ldots]\, P(T)\, (L(T) - H)$$

$$N[\ldots] = \left(N[d1] - \frac{HP(T)}{A(t)}\, N[h1]\right)$$

$$d1 = \frac{\ln\left(\frac{HP(T)}{A(t)}\right) + \frac{1}{2}\sigma_N^2 T}{\sigma_N \sqrt{T}}$$

$$h1 = \frac{\ln\left(\frac{A(t)}{HP(T)}\right) - \frac{1}{2}\sigma_N^2 T}{\sigma_N \sqrt{T}}$$

From this we may approximate the value of debts with the following:

$$L(t) = L(T)P(T)\, [1 - N[\ldots]\, (1 - e^{-\gamma T})]$$

As such we have that $\mathrm{N}[\ldots]\, (1 - \mathrm{e}^{-\gamma \mathrm{T}})$ denotes the probability of defaults, so by weighting the value of the debts with the survival probability and a risk free discount factor we may obtain its present value, where γ may be considered the credit risk spread given by the instantaneous probability of default $\mathrm{h(t)}$ and the recovery rate R, such that we have the following:

$$\gamma = (1 - R)h(t)$$

we are assuming that the risk free rate and the credit risk spread are independent such that we have the following credit risk yields:

$$Credit\ Risk\ Yield = R(T) + (1 - R)h(t)$$

ALM Strategies

The ALM approach starts from a formulation of liabilities duration derived in the world of numeraire typically of contingent claim terminology that approximate a multi periods model based on compound option approach. We may see the decomposition of structural model with participation to the profits and its duration measure:

$$C(A, L1, t) = A - B1(t)$$

$$B1(t) = L1P(t) - P(A, L1, t) + \alpha_1\beta\, C(A, T, L1/\alpha_1)$$

$$B2(T) = L2\, P(T) - P(A - B, L2, T) + \alpha_2\beta\, C(A - B, T, L2/\alpha_2)$$

$$E(T) = C(A - B, L2, T) - \alpha_2\beta\, C(A - B, T, L2/\alpha_2)$$

Where:

$$C(A, L1, t) = AN[d1] - L1P(T)N[d2]$$

$$P(A, L1, t) = -AN[-d1] + L1P(T)N[-d2]$$

$$C(A - B, L2, T) = \big(A - B1(t)\big)N[h1] - L2P(T)N[h2]$$

$$P(A - B, L2, T) = -\big(A - B1(t)\big)N[-h1] + L2P(T)N[-h2]$$

$$C\left(A, T, \frac{L1}{\alpha_1}\right) = AN[d3] - \frac{L1}{\alpha_1}P(T)N[d4]$$

$$C\left(A - B, T, \frac{L2}{\alpha_2}\right) = (A - B1(t))N[h3] - \frac{L2}{\alpha_2}P(T)N[4]$$

The duration measure is given by the following:

$$\begin{aligned}
D = \frac{L1P(t)}{B1 + B2}\, t + \frac{L2P(T)}{B1 + B2}\, T \\
&+ \frac{A\alpha T}{B1 + B2}\, \big[N(-d1) + \alpha_1\beta N(d3) \\
&+ \big(N(d1) - \alpha_1\beta N(d3)\big)\big(N(-h1) - \alpha_2\beta N(h3)\big)\big] \\
&- \frac{L1P(t)}{B1 + B2}\, t\, \big[N(-d2) + \beta N(d4) + \big(N(d2) - \beta N(d4)\big)\big(N(-h1) - \alpha_2\beta N(h3)\big)\big] \\
&- \frac{L2P(T)}{B1 + B2}\, T[N(-h2) + \beta N(h4)]
\end{aligned}$$

Furthermore, it is presented in combination with a formulation derived in absence of default risk, compare a formulation derived with default risk with one in absence of default risk may seem a paradox but it is possible to show that the two formulations converge for any guaranteed rate of return. The main idea is that the liabilities is protected by the equity value and that the default risk is faced with credit risk premium, instead with the model in absence of default risk it is possible to derive the guaranteed rate of return to avoid the default, the two formulations in academic sense converge for the participation case where obviously the company do not face the default risk because the risk is translated on the clients. The balance sheet of a company may be decomposed in options by using a contingent claim approach, in particular we have:

$$E(t) = C(A, L, T) - \alpha\beta\, C(A, \frac{L}{\alpha}, T)$$

$$\mathrm{L(t) = L(T)P(T) - P(A, L, T)} + \alpha\beta\, C(A, \frac{L}{\alpha}, T)$$

The equity value E(t) is a Call option C(A,L,T) on the firm's under laying value A(t), that represents the asset portfolio, with strike prices equal to the final value of the debts L(T). Instead, the initial value of the debts L(t) is given by the debts value discounted by using the risk free discount factor P(T) with a short position on the Put option P(A,L,T) on firm's under laying value A(t), with strike prices equal to the final value of the debts. Furthermore, there is the participation Call C(A,L/α,T) weighted with the participation coefficient β and the weight of the liabilities on the total value of asset α. The prices of the options may be expressed by making use of the cumulative normal distribution N[...]. As result we have:

$$C(A, L, T) = A(t)N[d1] - L(T)\, P(T)N[d2]$$

$$P(A, L, T) = L(T)P(T)N[-d2] - A(t)N[-d1]$$

$$C\left(A, \frac{L}{\alpha}, T\right) = A(t)N[d3] - \left(\frac{L}{\alpha}\right)P(T)N[d4]$$

Briys, De Varenne (1997) derived a duration measure of liabilities by using a formulation in the world of numerarire, specifically we have:

$$D = T - (T - \alpha T)\frac{A(t)}{L(t)}(N[-d1] + \alpha\beta N[d3])$$

For the elasticity of asset we have chose αT because if we use the beta a high correlation coefficient may mean a high investment in bonds or stocks, thus the solution is to think on the base of time horizon . In fact, if we assume that stocks are independent from interest rates a swapped assets means a high duration and a heavy weight means a high duration. On the other side if we assume that stocks are dependents from interest rates i.e. negative correlation, a high weight on stocks means a high duration, thus the solution is to take the time horizon weighted with leverage factor. The duration measure presented approximate all maturities in fact is computed in the world of interest rate and is independent from different maturities, we think in terms of time horizon that can be the asset duration or the liabilities duration. In Giandomenico (2007) we have a valuation model in absence of default risk so opposite to that presented before. As such, we have:

$$L(t) = L(t)P(T) + \alpha\left[P\left(A, T, \frac{L}{\alpha}\right) + \beta C(A, T, A) - P(A, T, A)\right]$$

$$L(t) = \alpha A(t) + \alpha\, [\, P\left(A, T, \frac{L}{\alpha}\right) - (1 - \beta)C(A, T, A)]$$

The prices of the options are given by:

$$P(A,T,L/\alpha) = P(T)\left(\frac{L}{\alpha}\right)N[-d2] - A(t)N[-d1]$$

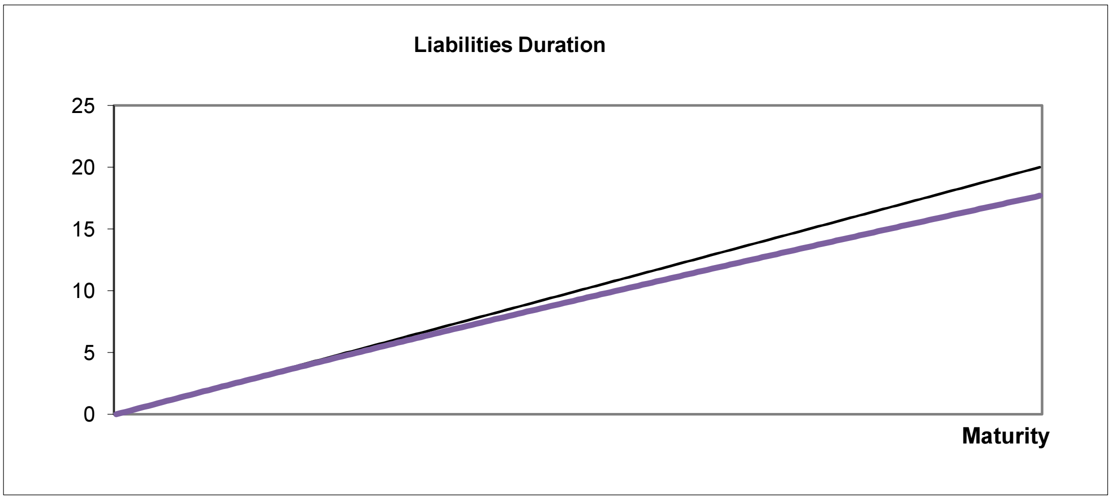
$$C(A,T,A) = A(t)N[h1] - AP(T)N[h2]$$

$$P(A,T,A) = AP(T)N[-h2] - A(t)N[-h1]$$

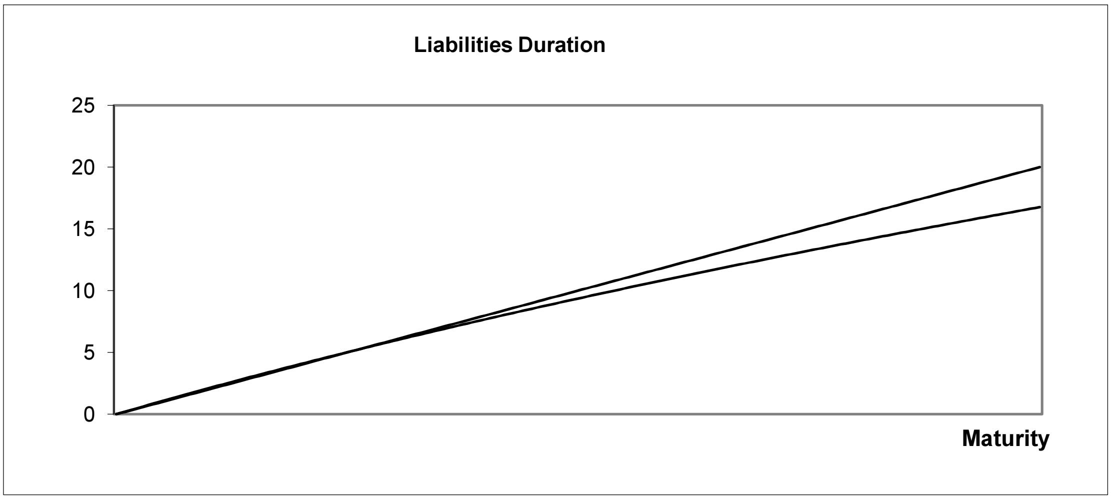
As result, the duration measure derived is given by the following:

$$D = \frac{L(T)P(T)}{L(t)}\, T + (\alpha A(t)\alpha T/L(t))(\,1-(1-\beta)N[h1] - (\beta N[h1]+N[-h1])\, N[-d1])$$
$$-\frac{\alpha AP(T)}{L(t)}\, T\, (\exp r^{*}T\; N[d2] + N[-d1] - (\beta N[h2]+N[-h2])N[-d1]$$
$$- N(h2)(1-\beta))$$

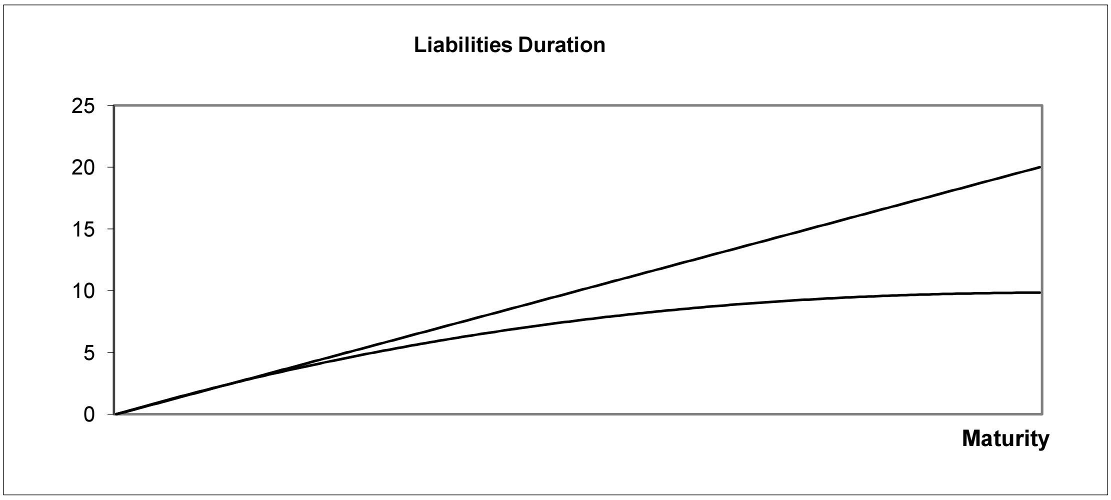
The two formulations are based on the participation of clients to the profits so as result are both in absence of default risk. Furthermore, the model of Briys, De Varenne (1997) without participating yields a credit risk spread due to default risk and is approximately independent from the time horizon of asset duration in the case without participating; instead the formulation in absence of default risk yields a rate of return less than the treasury yields. Although, they are in opposite point of view they yield the same duration measure if we assume risk premium with respect the treasury yields, this is due to the fact that in the model in absence of default risk there is the default option as well and the protective Put option may be seen as protection given by the equity value. We have the following figure for the model of Briys, De Varenne (1997) without participating to the profits:



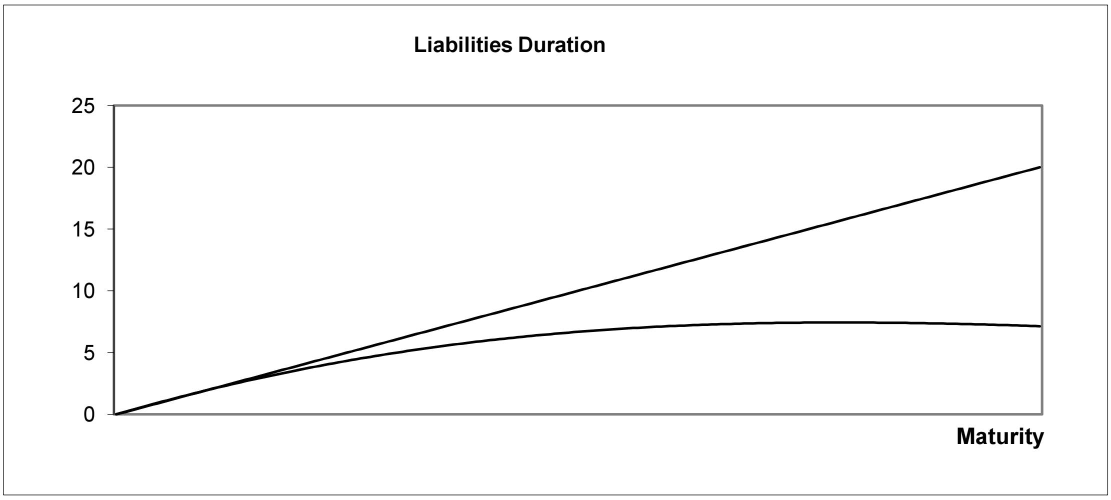
Instead, for the model in absence of default risk by assuming risk premium that is equal to assume that clients buy options, we have the following figure:



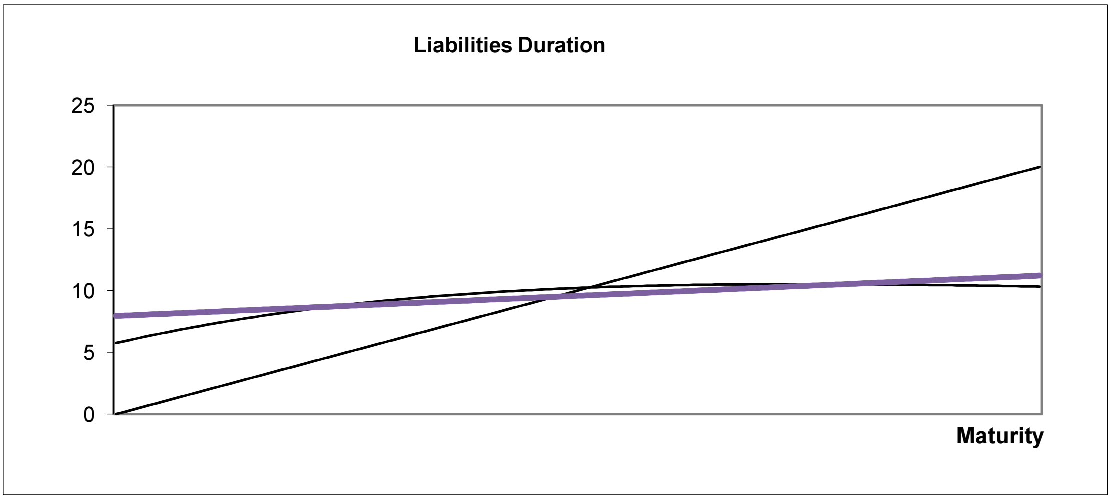
The two formulations converge, instead the rate of return of equilibrium in absence of default risk yields the following figure, i.e. treasury yields 6% and rate of return 4%:



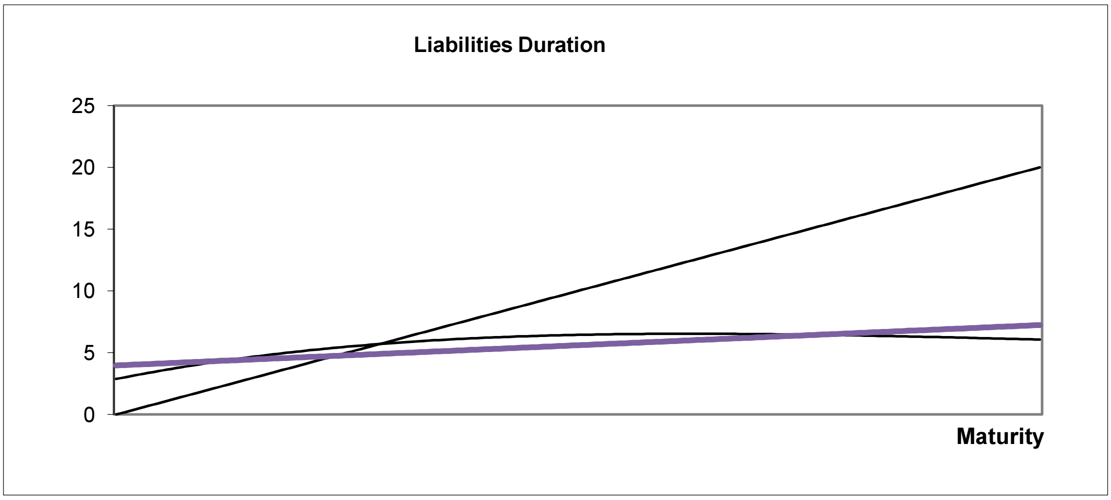
But we can stress the duration measure by assuming a rate of return of 2%, as result we have the following figure:



The formulation if you do not consider the participation permits to measure the effective liabilities duration on the base of the rate of return given to the clients, this characteristic is very important as we will see in the case of participation to the profits because as it is rational to expect the effective liabilities duration will depend from the effective rate of return. Another, characteristic is that the duration measure does not depend from the time horizon of asset duration as in the case of participating. Now, It is important to observe the duration measure in the case of participation, if we think in term of time horizon we get the same duration figure of that of risk premium, i.e. dependents of time horizon of asset duration; instead if we assume a time horizon of 10 years for the asset duration we get the following figure:



Instead, if we assume a time horizon of 5 years for the asset duration we have the following figure:



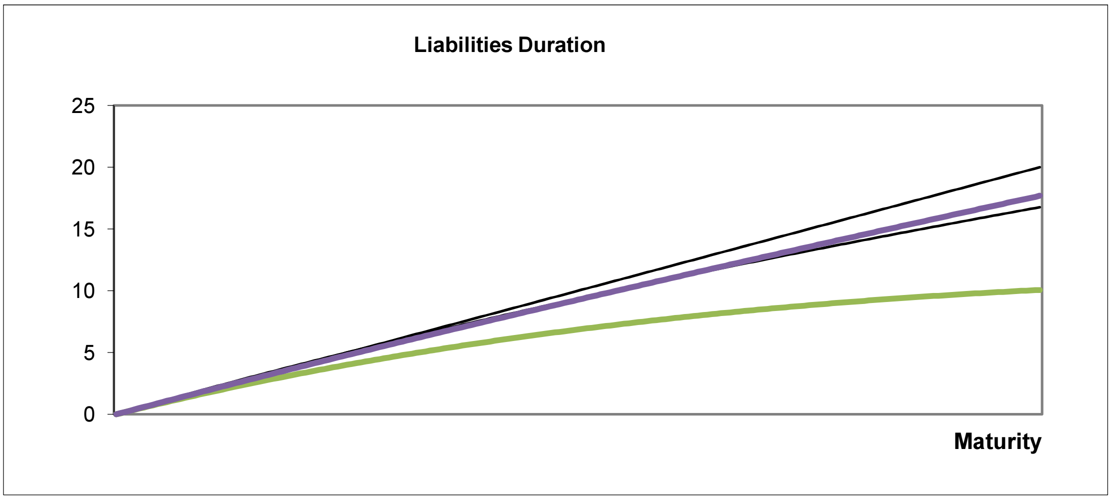
As we may see the two formulations converge exactly in the case of participation to the profits, the effective liabilities duration depends from the asset duration from the mathematical point of view. Now it is interesting to introduce the surrender option in the following way:

$$B(t) = L(t) + \varphi Ps(L, T, B)$$

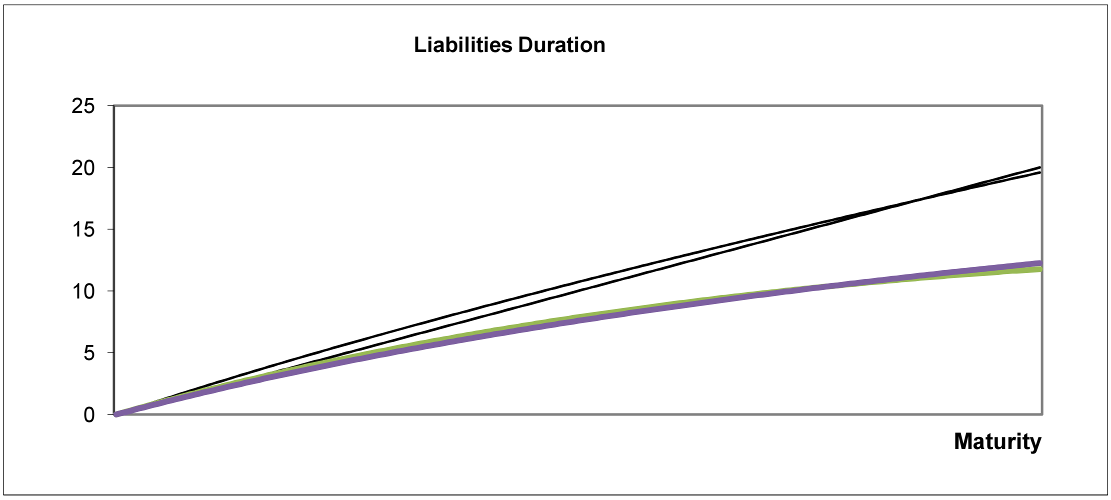
The liabilities has a long position on a surrender option weighted with the surrender probability, as result we have the following duration measure:

$$Ds = D(1 - \varphi\, N[\dots])$$

As suggested before we may assume that clients will not value options as against them but as buying as for example to get protection, liquidity, risk premium, etc. As result we have the following duration measure figure without participating to the profits by assuming the asset duration equal to Ds:

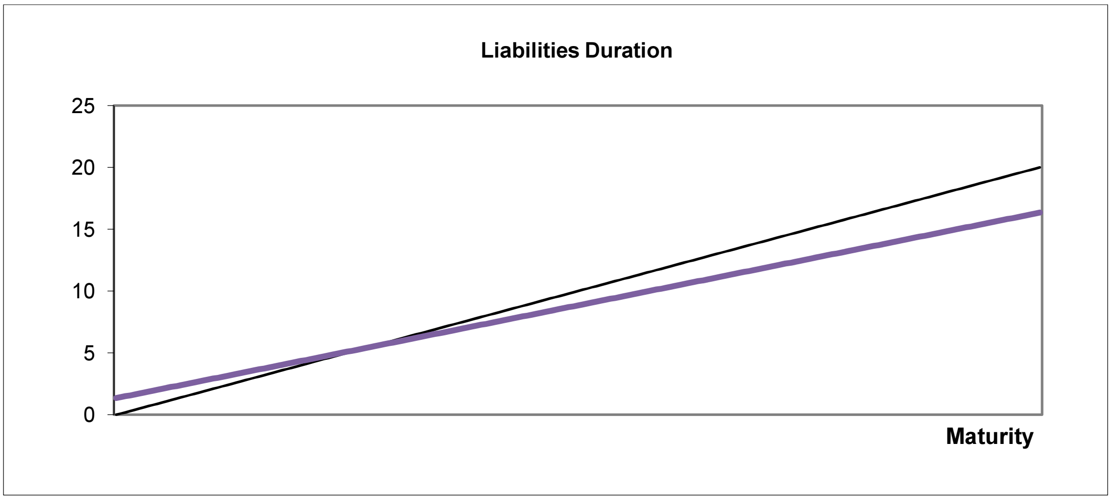


In this case is possible to contemplate the case of banks where clients buy options for the liquidity, in fact the surrender line may be seen as surrender risk faced by the banks as stressed scenario. Now it is interesting to note the duration measure in the case of participating to the profits:

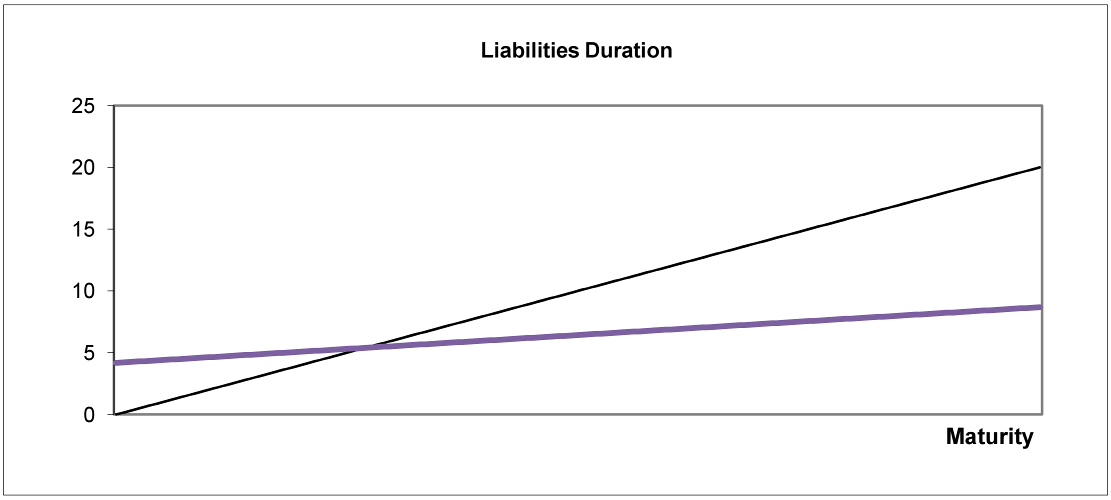


As we may see, in the formulation developed by Briys, De Varenne (1997), for the case of participation, there is the surrender option embedded in the valuation derived. So the best strategies is not to give the risk premium but the participation because with the risk premium clients and companies will face default risk, instead with the participation to the profits clients may get the risk premium as well without facing the default risk, so the effective liabilities duration can be derived on the base of effective rate of return. We may estimate now the duration measure by using the Beta for the asset portfolio with the following value of parameters, $\rho = 0.9$ $a = 0.95$ $\sigma_S =$

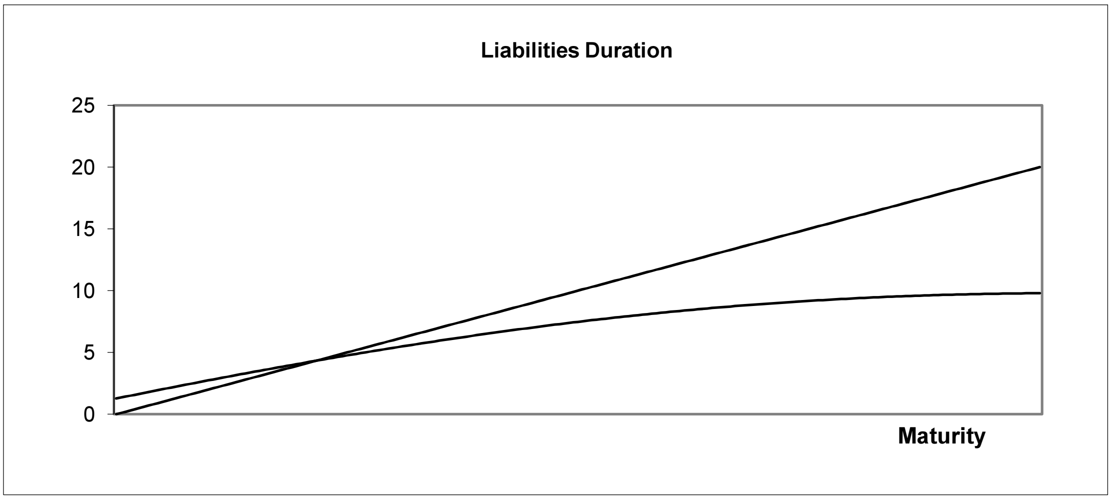
$0.2\ \sigma_r = 0.03$, as such we have the following figure for the model of Briys, De Varenne (1997) without participating:



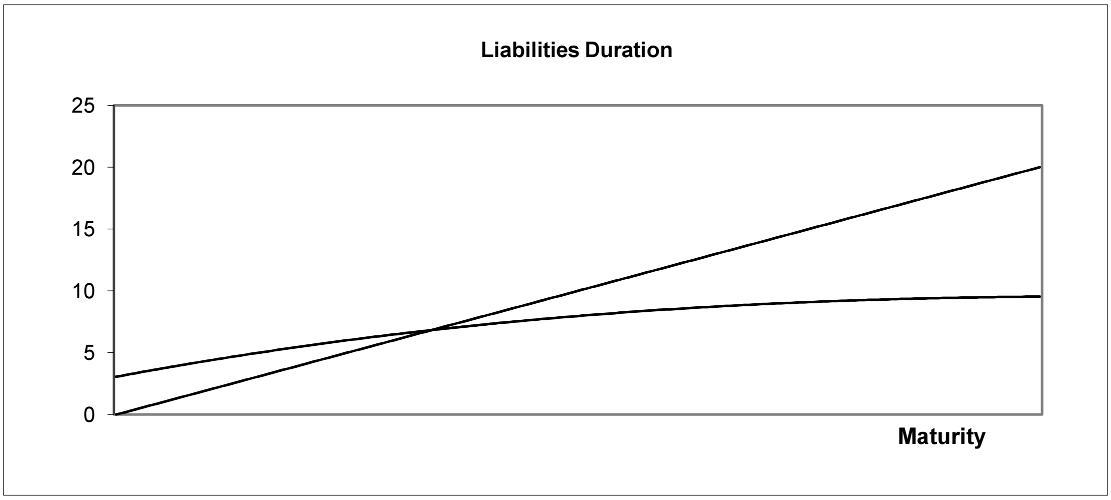
This suggests that with the risk premium the duration measure is approximate equal to maturities but we have to consider the surrender possibility, in fact the increasing on interest rate or the increasing of default probability may bring clients to exercise the surrender option, in fact as we will see and have seen in the case of participating there is the surrender option embedded in the duration measure derived. Furthermore, in the case of absence of default risk the protective Put option valued against the clients measures the surrender possibility as well, so we may see the duration measure in the case there is the surrender option embedded in the duration measure derived. We have the following prospect by assuming participation to the profits:



Instead, for the model in absence of default risk, we have the following prospect in the case without participating that may be considered the surrender line in the case we have credit risk premium:



And the following prospect in the case of participating in absence of default risk:



We definitively conclude by noting that we may stress the duration by using the duration GAP, in fact if we assume that the investment time horizon is equal to the duration the loss on the prices will be rebalanced with greater earning in the reinvestment such that the position is immunized, this may be formalized in the duration GAP by considering the liquidity of the company in combination with the liquidity risk given by the following:

$$Duration\ GAP = Asset\ Duration - \frac{Liquidiy\ \%}{Liquidity\ Risk\ \%}$$

As we may see the duration GAP measures the GAP between the asset duration usually measured in years and out cash flows measured by the liquidity risk in % yearly that will covered by the % of liquidity, the GAP starts when the liquidity ends and the company has to start to liquidate their assets for out cash flows so to face markets risks that from the interest rate may be computed on the GAP to get back the money invested in interest rate sensitive products.

Asset Liability Management

We may note that the equity value may be given by the following

$$E(t) = C(A,L,T) - C(A,H,T) - Call\ in(A > H, L < H, T)$$

We may note that this is the same value of equity value of a bank, in fact it is a Capped Call option, the participation to the profit derived from their investment in the companies that it is limited because the profit after fulfilled the face value of the loan is taken by the borrowers. We may note that the pay off of in option is deterministic and depends from the probability that the underlying will touch the barrier, indeed is a binary option, as such we have the following:

$$Call\ in(A > H, L < H, T) = N[\ldots]\, P(T)\, (H - L)$$

$$N[\ldots] = \left(N[d1] - \frac{HP(T)}{A(t)}\, N[h1]\right)$$

Where:

$$d1 = \frac{\ln\left(\frac{HP(T)}{A(t)}\right) + \frac{1}{2}\sigma_N^2 T}{\sigma_N \sqrt{T}}$$

$$h1 = \frac{\ln\left(\frac{A(t)}{HP(T)}\right) - \frac{1}{2}\sigma_N^2 T}{\sigma_N \sqrt{T}}$$

As result we have the following result for the liabilities:

$$L(t) = A(t) - C(A,L,T) + C(A,H,T) + N[\ldots]\, P(T)\, (H - L)$$

From this we have:

$$L(t) = L(T)P(T) - P(A,L,T) + C(A,H,T) + N[\ldots]\, P(T)\, (H - L)$$

As such we have the following prices for the options:

$$C(A,L,T) = A(t)N[d1] - L(T)\, P(T)N[d2]$$

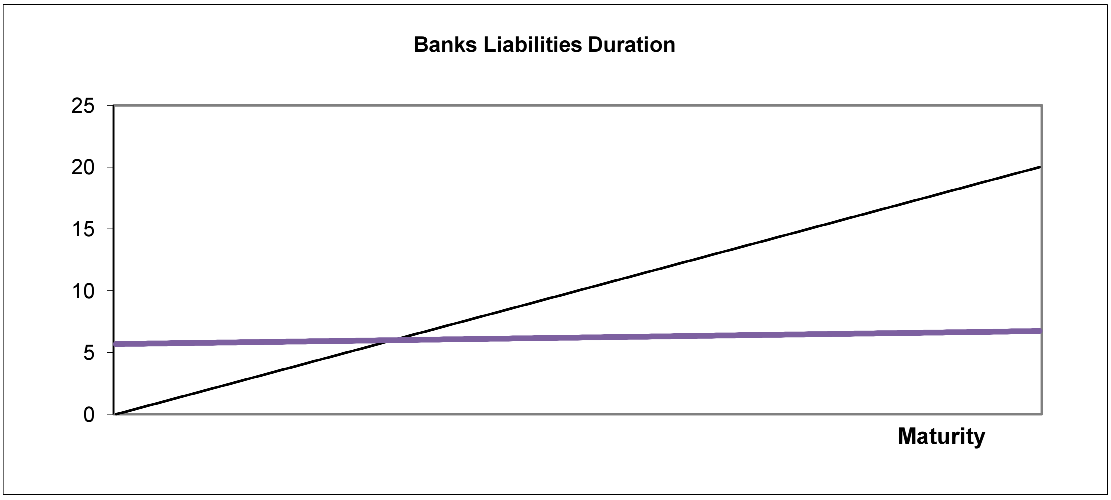
$$P(A,L,T) = L(T)P(T)N[-d2] - A(t)N[-d1]$$

$$C(A,H,T) = A(t)N[d3] - HP(T)N[d4]$$

As result we have the following duration measure of assets and liabilities:

$$D = T - \left(T + \frac{\rho\sigma_A}{\sigma_r}\right)\frac{A(t)}{L(t)}(N[-d1] + N[d3])$$

For the elasticity of assets we have choose the Beta of assets with the following value of parameters, $\rho = -0.9 \quad \frac{L(t)}{A(t)} = 0.95 \; \sigma_A = 0.2 \; \sigma_r = 0.03$ as result we have the following figure:



We are measuring the duration measure by assuming just the protection of equity value, but if we will consider the in option we have a model in absence of default risk as result we have the following duration measure:

$$D = T - \left(T + \frac{\rho\sigma_A}{\sigma_r}\right)\frac{A(t)}{L(t)}(N[-d1] + N[d3]) + T\left(N[d1] - \frac{HP(T)}{A(t)}N[h1]\right)\frac{H-L}{L(t)}$$

It is interesting now to compare the duration measure with one with protection of liabilities, as such we have the following:

$$L(t) = L(t)P(T) + \alpha\left[P\left(A, T, \frac{L}{\alpha}\right) - P(A, T, A)\right]$$

$$L(t) = \alpha A(t) + \alpha\left[P\left(A, T, \frac{L}{\alpha}\right) - C(A, T, A)\right]$$

We may note that there is the default option and the protective put option, from the transformation we may note the liabilities participate to the value of assets and have a long position on a protective Put option and a short position on a Call option that gives the possibility to the shareholders to take the profit from the investment, the liabilities are in equilibrium when:

$$P\left(A, T, \frac{L}{\alpha}\right) = C(A, T, A)]$$

From this we may obtain the equilibrium rate of return. The prices of the options are given by:

$$P(A, T, L/\alpha) = P(T)\left(\frac{L}{\alpha}\right)N[-d2] - A(t)N[-d1]$$

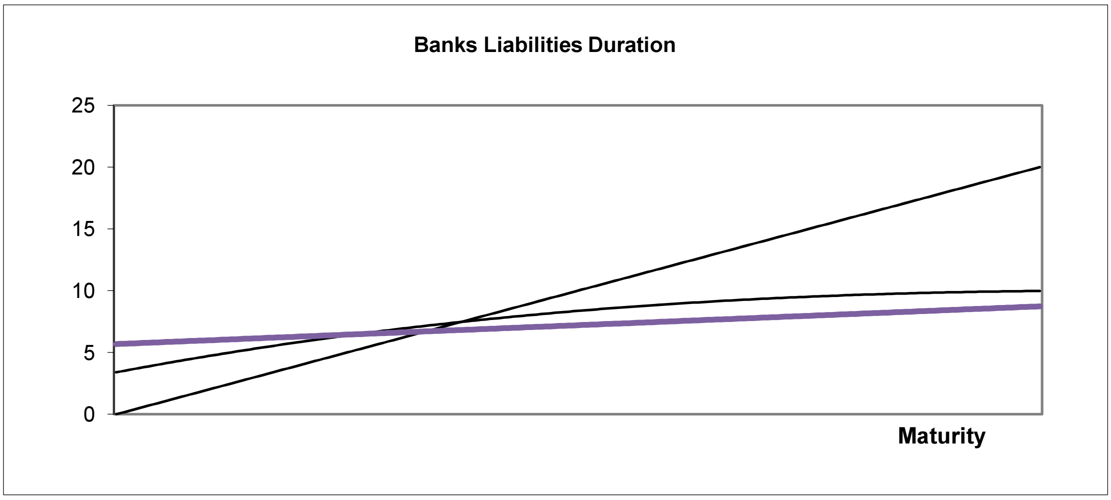
$$C(A, T, A) = A(t)N[h1] - AP(T)N[h2]$$

$$P(A, T, A) = AP(T)N[-h2] - A(t)N[-h1]$$

As result, the duration measure derived is given by the following:

$$D = \frac{L(T)P(T)}{L(t)} T + \left(\frac{\alpha A(t) \frac{-\rho\sigma_A}{\sigma_r}}{L(t)} \right) (1 - N[h1] - N[-h1]N[-d1])$$
$$- \frac{\alpha A P(T)}{L(t)} T \, (\exp r^* T \, N[d2] + N[-d1] - N[-h2]N[-d1] - N[h2])$$

It is interesting now to compare the two duration measures derived, with the following value of parameters: $\rho = -0.9 \quad \alpha = \frac{L(t)}{A(t)} = 0.95 \; \sigma_A = 0.2 \; \sigma_r = 0.03 \; \frac{H-L}{L(t)} = 0.4$ we have the following figure:



We may note that the two formulations converge, for maturities greater than five years, the duration measure converges to ten years maturity, so the model permits to have a measure of liquidity GAP for maturities greater than five, but if we remove the protection the duration measure collapses to five years maturity. We may extend the analysis to insurance companies, in fact the equity value is given by the following:

$$E(t) = C(A, L, T) - a\,\beta\, C\left(A, \frac{L}{a}, T\right) - Call\ in\left(A, L < \frac{L}{a}, T\right)$$

If we will consider the in option we have a model in absence of default risk as result we have the following duration measure:

$$D = T - \left(T + \frac{\rho\sigma_A}{\sigma_r}\right)\frac{A(t)}{L(t)}(N[-d1] + a\beta N[d3]) + T\left(N[d1] - \frac{\frac{L}{a}P(T)}{A(t)} N[h1]\right)\frac{\frac{L}{a} - L}{L(t)}$$

It is interesting now to compare the duration measure with one with protection of liabilities, as such we have the following:

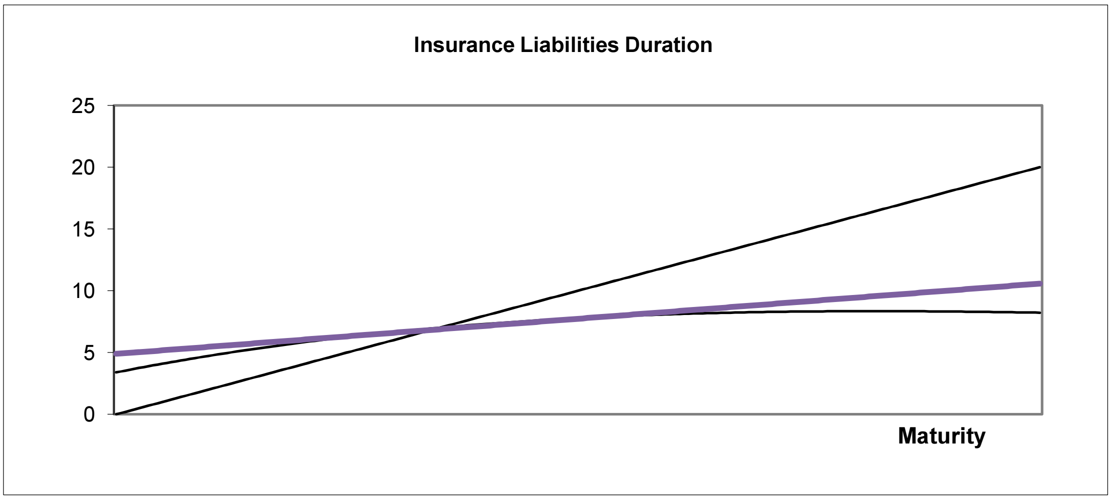
$$L(t) = L(t)P(T) + \alpha \left[P\left(A, T, \frac{L}{\alpha}\right) + \beta C(A, T, A) - P(A, T, A) \right]$$

$$L(t) = \alpha A(t) + \alpha\, [\, P\left(A, T, \frac{L}{\alpha}\right) - (1-\beta) C(A, T, A)]$$

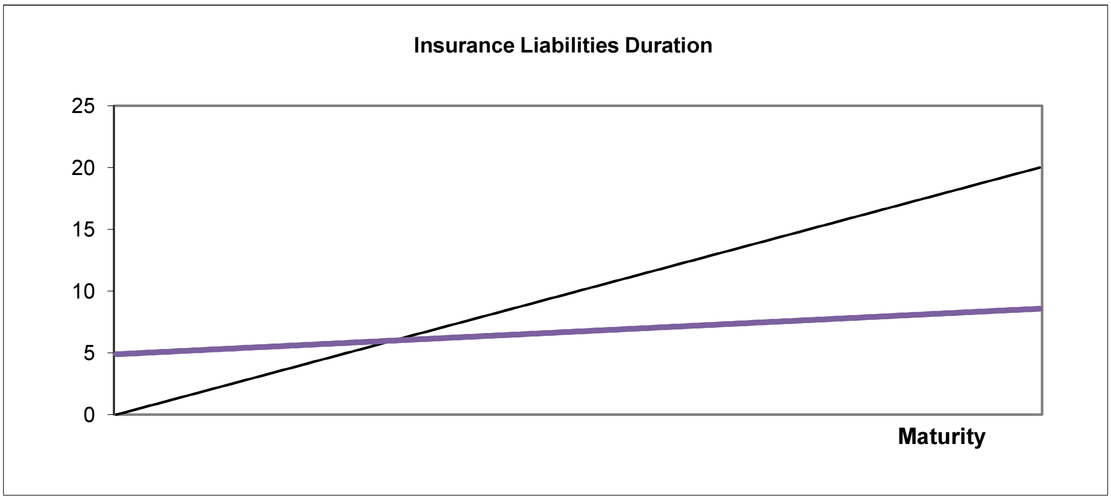
As result, the duration measure derived is given by the following:

$$D = \frac{L(T)P(T)}{L(t)}\, T + \left(\frac{\alpha A(t) \frac{-\rho \sigma_A}{\sigma_r}}{L(t)} \right) (\, 1 - (1-\beta) N[h1] - \; (\beta N[h1] + N[-h1])\, N[-d1])$$
$$- \frac{\alpha A P(T)}{L(t)}\, T\, (\exp r^{*} T\, N[d2] + \; N[-d1] - \; (\beta N[h2] + N[-h2]) N[-d1]$$
$$- N(h2)(1-\beta))$$

It is interesting now to compare the two duration measures derived, with the following value of parameters: $\rho = -0.9 \quad \alpha = \frac{L(t)}{A(t)} = 0.95\; \sigma_A = 0.2 \quad \sigma_r = 0.03 \quad \beta = 0.85 \quad \frac{\frac{L}{\alpha} - L}{L(t)} = 0.4$ we have the following figure:



We may note that in the case of insurance companies the two formulations converge as well, for maturity greater than five years the duration measure converges to ten years maturity, but if we remove the protection we have the following figure less severe than that of the banks:



Advanced Interest Rate Models

The interest rate models developed in the literature are based on arbitrage setting such that the yield curve is based on the expectation of the market on the future short rates. To assume that the expectation is the future does not consider the uncertainty associated with the expectation that may be reassumed in liquidity and risk premium on the average of the short rate. As result we may assume the following:

$$\frac{dP(T)}{P(T)} = r(t) + \frac{1}{2}\,\sigma^2 dt - \,\sigma dWr$$

The expectation of $-\sigma dWr$ is equal to $-\frac{1}{2}\sigma^2 dt$ as result we have the short rate *r*(t) as average, but if we simulate $\frac{dP(T)}{P(T)}$ cannot assume negative value because is a geometric Brown process, as result we have that $-\sigma dWr$ simulated is equal to $+\frac{1}{2}\sigma^2 dt$ that represents the liquidity risk. As result in absence of arbitrage opportunities we have that the yield curve is given by the following;

$$r(T) = \, r(t) + \sigma_r^2 T$$

Because if apply regular Ito's lemma, we have by simulating as well:

$$\frac{dP(T)}{P(T)} = r(t) + \frac{1}{2}\,\sigma^2 dt - \frac{1}{2}\,\sigma^2 dt - \frac{-2}{2}\,\sigma^2 dt$$

As result we have:

$$r(T) = \, r(t) + \sigma_r^2 T$$

Where *r*(T) is the value of the future short rate and represents the yield curve with the liquidity risk, in fact if we take the average of *r*(T) for each maturities we have that the risk free rate is equal to:

$$r(T) = \, r(t) + \frac{1}{2}\sigma_r^2 T$$

For a yield curve decreasing we have:

$$r(T) = Ass(\, r(t) - \sigma_r^2 T)$$

We are assuming that the interest rate cannot assume negative value, we may remove this hypothesis in the case we have deflation. Now by taking the average of r(T) we may obtain the risk free rate, on this we may add liquidity risk to obtain the yield curve.

Again if we take the yield curve as average we have:

$$\frac{dP(T)}{P(T)} = r(T) + \frac{1}{2}\,\sigma^2 dt - \,\sigma dWr$$

As result we have the following:

$$r(T) = \, r(t) + \sigma_r^2 T + \sigma_r^2 T$$

Where $\sigma_r^2 T$ denotes the risk premium, so in the yield curve it is possible to have liquidity risk and risk premium. If we take in consideration the inflation we have the following;

$$\sigma_I^2 T = \sigma_r^2 T + \sigma_r^2 T$$

Where $\sigma_I^2 T$ is the expected inflation for each maturities, where σ_I^2 is the variance of the inflation, that is the drift condition such that we have:

$$r(T) = \, r(t) + \sigma_I^2 T$$

As result we have that:

$$Average\;(\sigma_I^2 T) = \sigma_r^2 T$$

The compounded inflation is equal to $\sigma_r^2 T$, so there is an equilibrium relation between the volatility of inflation and the volatility of the short rate. Now it is interesting to analyse the inflation, as result we have the following for the inflation swap as expectation;

$$Inflation\,Swap = \sigma_I^2 T + \frac{1}{2}\sigma_I^2 T$$

and the following for the inflation simulated:

$$Inflation\,Simulated = \sigma_I^2 T + \sigma_I^2 T$$

We may have an inflation, decreasing, or stable, i.e. there are different kind of equilibrium with the short rate, and it is possible to have just liquidity risk or risk premium as well. Now it is interesting to note that the variation of short rate may be simulated by the following process that is in the midline between a Martingale and a geometric Brown process that does not permit to the short rate to assume negative value, as such we have the following:

$$dr(t) = \frac{k}{2}\big(R - r(t)\big)dt + \,\sqrt{}r(t)\sigma_r dWr$$

Where R denotes the average of the short rate and k is given by $\frac{\sigma_I^2}{\mathrm{Ass}(R - r(t))}$, from this it is possible to derive that the risk premium is given approximate by $\frac{\sigma_r^2 T}{3}$, as such the liquidity risk is given by $\frac{\sigma_r^2 T}{6}$. Furthermore, the short rate may be simulated by using the following modification of the diffusion process $\sqrt{r(t)}\sigma_r dWr$, as such we have the following;

$$dr(T) = \frac{k}{2}\left(R - r(t)\right) T + 100\, r(t)\, \frac{\sigma_r^2 T}{2}$$

We may note that R(T) = *r(t) + dr(T)* converges to the same result of the $r(T) = r(t) + \sigma_r^2 T$. By taking the average we obtain approximate the same risk free rate for rational value of parameters. Again in the case we have a decreasing curve such that r(t) > R we may obtain the yield curve by the following:

$$dr(T) = \frac{k}{2}\left(R - r(t)\right) T - 100\, r(t)\, \frac{\sigma_r^2 T}{2}$$

We may note that the short rate cannot assume negative value. We may see an example of equilibrium between interest rate, inflation and simulation of the short rate:

Vol Inflation	5,94%
Vol Rate	4,20%
Average	2,00%
Rate	1,25%
Parameter	0,47

Maturity (Years)	Rate MKT	Short Rate Down	Risk Free Down	Swap UP & Short Rate UP	Risk Free UP	STD Inflation	Inflation	Short Rate CIR
0	1,25%	1,25%	1,25%	1,25%	1,25%	0,00%	3,53%	1,25%
1	1,60%	1,07%	1,16%	1,43%	1,34%	0,35%	3,18%	1,54%
2	1,96%	0,90%	1,07%	1,60%	1,43%	0,71%	2,82%	1,74%
3	2,31%	0,72%	0,99%	1,78%	1,51%	1,06%	2,47%	1,89%
4	2,66%	0,54%	0,90%	1,96%	1,60%	1,41%	2,12%	2,02%
5	3,01%	0,37%	0,81%	2,13%	1,69%	1,76%	1,76%	2,12%
6	3,37%	0,19%	0,72%	2,31%	1,78%	2,12%	1,41%	2,20%
7	3,72%	0,02%	0,63%	2,48%	1,87%	2,47%	1,06%	2,28%
8	4,07%	0,16%	0,58%	2,66%	1,96%	2,82%	0,71%	2,34%
9	4,43%	0,34%	0,56%	2,84%	2,04%	3,18%	0,35%	2,39%
10	4,78%	0,51%	0,55%	3,01%	2,13%	3,53%	0,00%	2,44%
	Average	0,55%	Average	2,13%	Comp. Inflation	1,76%	1,76%	2,02%

Now we may see an example of simulation of the short rate decreasing:

Vol Inflation	5,94%
Vol Rate	4,20%
Average	0,25%
Rate	1,25%
Parameter	0,35

Maturity (Years)	Rate MKT	Short Rate Down	Risk Free Down	Swap UP & Short Rate UP	Risk Free UP	STD Inflation	Inflation	Short Rate CIR
0	1,25%	1,25%	1,25%	1,25%	1,25%	0,00%	3,53%	1,25%
1	1,60%	1,07%	1,16%	1,43%	1,34%	0,35%	3,18%	0,96%
2	1,96%	0,90%	1,07%	1,60%	1,43%	0,71%	2,82%	0,83%
3	2,31%	0,72%	0,99%	1,78%	1,51%	1,06%	2,47%	0,72%
4	2,66%	0,54%	0,90%	1,96%	1,60%	1,41%	2,12%	0,66%
5	3,01%	0,37%	0,81%	2,13%	1,69%	1,76%	1,76%	0,60%
6	3,37%	0,19%	0,72%	2,31%	1,78%	2,12%	1,41%	0,56%
7	3,72%	0,02%	0,63%	2,48%	1,87%	2,47%	1,06%	0,51%
8	4,07%	0,16%	0,58%	2,66%	1,96%	2,82%	0,71%	0,52%
9	4,43%	0,34%	0,56%	2,84%	2,04%	3,18%	0,35%	0,41%
10	4,78%	0,51%	0,55%	3,01%	2,13%	3,53%	0,00%	0,60%
	Average	0,55%	Average	2,13%	Comp. Inflation	1,76%	1,76%	0,69%

As we may see from the examples the two approaches converge to the same result. We may note now that there are two factors in the determination of the yields curve, the short rate and the inflation in combination with the uncertainty, as such we may assume the following distributions:

$$dr(t) = -a\, r(t) + \sigma_r dWr$$

$$d\, \sigma_r^2 T = -a\, \sigma_r^2 T + \sigma_r dW\sigma$$

As such we have the following simplified pricing formula for a default free zero coupon bond:

$$P(t,T) = Exp\left[-\frac{1-e^{-a(T-t)}}{a} r(t) - \frac{1-e^{-a(T-t)}}{a} \sigma_r^2 T + \mathrm{G(t,T)}\right]$$

Where:

$$\mathrm{G(t,T)} = \frac{\sigma_r^2}{a^2}\left(T-t+\frac{1-e^{-a(T-t)}}{a}-\frac{e^{-2a(T-t)}}{2a}-\frac{3}{2a}\right) + \frac{\sigma_r^2}{2a^2}\left(T-t+\frac{1-e^{-a(T-t)}}{a}-\frac{e^{-2a(T-t)}}{2a}-\frac{3}{2a}\right)$$

Intensity Models

Now we will present an intensity model approach that is based on the instantaneous probability of default. The price of a credit with face value 1 may be expressed by the following formulation:

$$1 - h(t)(1 - R)dt$$

Where *h*(t) denotes the probability of default and R the recovery rate The formulation may be rewritten by the following:

$$e^{-h(t)(1-R)T}$$

This is the survival probability, so by weighting the face value of the credit with the survival probability we may obtain a credit value adjustment (CVA). On the other side we have the following for the probability of default:

$$1 - e^{-h(t)(1-R)T}$$

We have along the interest rate models the following:

$$Credit\ Risk\ Yield = r(T) + h(t)(1 - R) + \frac{1}{2}\sigma_h^2 T + \sigma_h^2 T \quad \sigma_h = \sqrt{PD * (1 - PD)}\,(1 - R)$$

On the credit risk yield we may have liquidity and risk premium as in the interest rate models. We have assumed no drift for the probability of default that is equal to assume that it is stable. From this we may have the CVA by considering the liquidity and risk premium that may be considered the Credit Value at Risk, so the main point is the variance such that we may consider the case of bilateral credit risk and the wrong way risk by taking the variance between the two references and computing the correlation coefficient and the PD spread between the two references. Indeed, we may have a greater credit risk yield due to the systemic risk, but we may obtain the information with the copula approach. The systemic risk may be estimated on the base of deco relation risk, the main idea is that an entity is very correlated with other entities the systemic risk is low because the system will cover each entities with each others, it is the case for example when assets and liabilities between different entities is mixed but for instance if an entity is deco related from the others that have the liabilities of the deco related entity in their assets, i.e. negative correlation, we have the systemic risk, i.e. the deco related entity may bring a systemic risk in the mixed entities or the group, as such we have the following measure of systemic risk:

$$N\left[\frac{\left(N^{-1}\left(1 - e^{-h(t)(1-R)T}\right) - \rho\right)}{\sqrt{1 - \rho^2}}\right]$$

If the correlation with the group increases the systemic risk decreases, instead, if the correlation with the group decreases the systemic risk increases such that we may have negative correlation that is really systemic risk as the deco relation increases the systemic risk increases and after begins to decrease as the entities become totally different from the group, i.e. without relationing.

PD Rating

The problem of intensity models is the probability of default but we may estimate it from the rating class by using the migration matrix as result we have the following:

Sovereign Foreign-Currency Average One-Year Transition Rates (1975-2010)*

		--Rating one year later (%)--									
Ratings as of Jan. 1	#	AAA	AA	A	BBB	BB	B	CCC/CC	SD	NR	PD
AAA	450	97.78	2.22	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0,38%
AA	267	3.37	93.63	2.25	0.00	0.37	0.37	0.00	0.00	0.00	0,47%
A	278	0.00	3.60	92.81	3.60	0.00	0.00	0.00	0.00	0.00	0,94%
BBB	237	0.00	0.00	6.75	89.03	3.38	0.84	0.00	0.00	0.00	1,17%
BB	293	0.00	0.00	0.00	6.14	88.05	4.10	1.02	0.68	0.00	2,75%
B	264	0.00	0.00	0.00	0.00	7.20	86.36	3.41	1.89	1.14	3,41%
CCC/CC	22	0.00	0.00	0.00	0.00	0.00	31.82	31.82	36.36	0.00	>3,41%

The probability of default for each rating class may be computed by using the diagonal as such we have for example for the BBB rating the following:

AVERAGE(3,38;3,38*0,041;3,38*0,041*0,0341)/100

As result we have the following interval:

Ratings	PD
AAA	0% - 0,38%
AA	0,38% -0,47%
A	0,47% -0,94%
BBB	0,94% -1,17%
BB	1,17% -2,75%
B	2,75% -3,41%
CCC/CC	>3,41%

Because we have the following cumulative probability of default:

Sovereign Foreign-Currency Cumulative Average Default Rate Without Rating Modifiers (1975-2010)*

(%)	--Time horizon (years)--														
Rating	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**
AAA	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
AA	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
A	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
BBB	0.00	0.46	1.45	2.53	3.68	4.93	5.61	5.61	5.61	5.61	5.61	5.61	5.61	5.61	5.61
BB	0.68	2.16	3.38	4.28	5.79	7.46	9.35	11.47	12.28	12.28	12.28	13.55	15.21	17.50	17.50
B	1.89	4.85	6.79	9.58	12.18	14.51	17.30	21.83	24.57	28.08	33.22	33.22	33.22	33.22	42.76
CCC/CC	36.36	46.97	58.75	64.65	70.54	77.90	88.95	88.95	N/A	N/A	N/A	N/A	N/A	N/A	N/A
Investment grade	0.00	0.09	0.27	0.46	0.67	0.90	1.02	1.02	1.02	1.02	1.02	1.02	1.02	1.02	1.02
Speculative grade	2.59	5.06	6.98	8.89	11.06	13.23	15.75	18.67	20.14	21.29	22.71	23.62	24.83	26.47	28.63
All rated	0.83	1.66	2.36	3.05	3.81	4.54	5.26	5.95	6.27	6.51	6.77	6.92	7.09	7.27	7.48

As such we have the following historical recovery rate:

Discounted Recovery Rates By Instrument Type (1987-2011)*

	Mean(%)	Median(%)	Dollar-weighted rate (%)	Standard deviation	Coefficient of variation(%)	Count
Nominal recovery (Instrument type)						
Revolving credit	88.5	100.0	74.2	34.4	38.9	672
Term loans	79.1	94.8	73.1	38.3	48.4	689
All loans/facilities	83.7	100.0	73.6	36.7	43.8	1,361
Senior secured bonds	68.9	69.5	65.0	39.2	56.9	317
Senior unsecured bonds	51.8	48.0	47.5	38.9	75.1	1,133
Senior subordinated bonds	34.5	20.0	33.8	37.7	109.1	526
All other subordinated bonds	28.5	12.5	29.2	37.6	132.0	452
All bonds	45.9	34.8	45.6	40.6	88.3	2,428
Total defaulted instruments	59.6	56.9	54.6	43.2	72.5	3,789

Equilibrium Relation

The evolution of the stock prices may be denoted by the following stochastic continuous process:

$$\frac{dS(t)}{S(t)} = (r(t) + \delta - q)dt + \sigma_S \, dWs$$

δ may be the risk premium or the liquidity premium, depends from the equilibrium in the treasury market, instead q denotes the dividend yield, so by putting $\delta = q$ it is possible to get the prices of the stocks for the different kind of risk premiums that are in the interest rate models, by putting:

$$Price = \frac{Dividend}{\delta} \frac{1}{100}$$

Instead, the risk premium is given by:

$$\frac{1}{2}\sigma_S^2 T$$

Usually is given by the average of the rate of return computed with respect the initial value of the prices. Because if we simulate:

$$S(T) = S(t)e^{(r(t)-\frac{1}{2}\sigma^2)dt+ \sigma dWs}$$

We obtain:

$$S(T) = S(t)e^{(r(t)-\frac{1}{2}\sigma^2)T+ \sigma^2 T}$$

Because:

$$\frac{dS(t)}{S(t)} = (r(t) - \frac{1}{2} \sigma^2)dt + \frac{2}{2}\sigma dWs$$

As such by applying Ito's lemma and simulating we obtain the same result of expected value. Now it is possible to note that if GDP increases the dividend yield increases, but there is an equilibrium relation, as such we have:

$$\partial GDP = \partial q = \partial r(T) = \partial \sigma_r^2 T$$

$$\delta = q \implies \partial S$$

As GDP increases, dividend increases, but stocks prices will increase as well, so the dividend yield after increase will decrease, this movement is embedded in the term structure of interest rates through the equilibrium with the inflation, because the increase of the demand for product will increase the consumer prices and the inflation through the increase of GDP. In the term structure of interest rates there is embedded, through the dynamic of GDP and the equilibrium with the term structure of the inflation, the policy on the movement of the short rate as expectation of the market. The equilibrium relation has implications for the portfolio construction; indeed, a high volatility may be a sign of junk securities if the rate of return is low or negative, as such we may simulate the market and by using the Beta we may obtain the rate of return of equilibrium, this permits to have a positive rate of return for all securities, but the junk securities will have a low rate of return and a high volatility such that in the portfolio construction their weight will be very low, indeed, with this approach it is possible to make stock picking as well. If we decide to adopt a stock picking we have to be careful with respect the insider trading, in fact we may have increasing and decreasing skew of the rate of return with respect the initial prices, this suggests that the market has more info in the valuation of a securities with respect just a quantitative approach. We have to note that the market is diversified so its volatility is less than the sum of each single stock in the composite, as result if we simulate the market we undervalue the effective rate of return of each single stocks and the markets in general; this opens the door a Bayesian approach in fact we may have different kinds of equilibrium with the risk premium in the treasury markets, in fact as we have said the risk premium may be substitute with the liquidity premium as well, from this we have different kind of risk premium dependents from the models we will use, but we may say that in the markets there are different kinds of equilibrium from the minimum to the maximum, in fact we may have that the minimum is given by:

$$Price = \frac{Dividend}{\sigma_r^2 T} \frac{1}{100}$$

And the maximum is given by;

$$Price = \frac{Dividend}{\frac{\sigma_r^2 T}{6}} \frac{1}{100}$$

With these relations it is possible to correct the simulated rate of return of the market due the diversification effect. As result we may compute the equilibrium rate of return of a stock on the base of its present price, usually in the dominio of an equilibrium, with respect the upper equilibrium or the maximum equilibrium, as such we may obtain for the portfolio construction the expected rate of return by combining the simulated return with the equilibrium return weighted with their uncertainty in a probability context by doing the following:

$$R_{Exp} = R_{Sim} + (N[-R_{Sim}; \sigma_S] - N[-R_{Equ}; \sigma_S])(R_{Equ} - R_{Sim})$$

The expected return is given by the rate of return simulated more the difference between the rate of return of equilibrium and the rate of return simulated weighted with the probability that the rate of equilibrium is over the rate of return simulated. From this it is easy to construct the efficient frontier by using the expected rate of return, we may note that if the rate of return of equilibrium is equal to the simulated the formulation drops to the rate of return simulated, again if the rate of equilibrium is zero i.e. the price is in the maximum the formula drops to be used with the rate of return simulated, the idea is based on the fact that the rate of return of equilibrium is greater than the rate of return simulated due the diversification effect that opens the door to a Bayesian approach but we may have overvalued stocks as well so in this case again the formulation drops to be used with the rate of return simulated, because the insider trading may suggest that the dividend will increase, in this case usually we have right skew as well.

Time Series

The analysis of time series is based on the relations between different variables, we present the problem and how to solve it by using the regression methods:

$$y_i \ = \ b_0 \ + \ b_{\,1} x_{\,i} + \ e_i$$

y_i denotes the output of the regression and it is the explained variable, b_0 denotes the regression constant and $b_{\,1}$ the regression coefficient of the explaining variable $x_{\,i}$. Instead, e_i denotes the independent standard error with normal distribution, zero average and volatility parameters with zero covariance from one step to another such that $Cov\,(\,e_{i-1}\ e_i\,) = 0$, this avoid to have autocorrelation such that the error explains the output. Indeed, in finance autocorrelation is very useful in many subjects with multi regression approach at the same time series of stocks markets usually show autocorrelation that may be simplified in a trend. Now we will determine the value of the coefficients of the regression approach, as result we may write the error in terms of coefficient:

$$e_i \ = \ y_i - b_0 - \ b_{\,1} x_{\,i}$$

The solution of the problem is to minimize the square root of the standard error of the regression:

$$\text{Min} \sum_{\text{i}=1}^{\text{N}} (y_i - b_0 - \ b_{\,1} x_{\,i})^2$$

As such we take the derivatives with respect the parameters coefficient of the regression b_0 and $b_{\,1}$ respectively, as result we have the following:

$$-2 \sum_{\text{i}=1}^{\text{N}} (y_i - b_0 - \ b_{\,1} x_{\,i}) \ = 0$$

$$-2 \sum_{\text{i}=1}^{\text{N}} x_{\,i}\ (y_i - b_0 - \ b_{\,1} x_{\,i}) = 0$$

By solving we obtain:

$$b_0 = \sum_{\text{i}=1}^{\text{N}} \frac{y_i}{\text{N}} - \ b_{\,1} \sum_{i=1}^{N} \frac{x_{\,i}}{N}$$

$$\sum_{i=1}^{N} b_{\,1} x_i^2 = \sum_{\text{i}=1}^{\text{N}} x_{\,i} y_i - \sum_{i=1}^{N} x_i b_0$$

By substitute b_0 we obtain the following:

$$b_1 = \frac{\sum_{i=1}^{N} x_i y_i - \sum_{i=1}^{N} \frac{x_i y_i}{N}}{\sum_{i=1}^{N} x_i^2 - (\frac{\sum_{i=1}^{N} x_i}{N})^2}$$

Now it is interesting to see the multi regression series.

$$\mathbf{Y} = \mathbf{XB} + \mathbf{e}$$

where **Y** is an N×1 column matrix of cases' scores, **X** is an N×(k+1) matrix of cases' scores (where the first column is a placeholder column of ones for the constant and the remaining columns correspond to each part), **B** is a (k+1)×1 column matrix containing the regression constant and coefficients, and **e** is an N×1 column matrix of cases' errors of prediction. The quantity that we are trying to minimize can be expressed as follows:

$$\mathbf{e'e}$$

$$\mathbf{e} = \mathbf{Y} - \mathbf{XB}$$

$$(\mathbf{Y} - \mathbf{XB})' (\mathbf{Y} - \mathbf{XB})$$

$$(\mathbf{Y}' - \mathbf{B}' \mathbf{X}')(\mathbf{Y} - \mathbf{XB})$$

$$\mathbf{Y'Y} - \mathbf{Y'XB} - \mathbf{B'X'Y} + \mathbf{B'X'XB}$$

$$\mathbf{Y'Y} - 2\mathbf{Y'XB} + \mathbf{B'X'XB}$$

$$0 = -2\mathbf{X'Y} + 2\mathbf{X'XB}$$

$$\mathbf{X'XB} = \mathbf{X'Y}$$

$$\mathbf{B} = [\mathbf{X'X}]^{-1}\mathbf{X'Y}$$

Portfolio

Portfolio optimization was pioneered by Markowitz (1952) with its mean variance approach and further developed by Sharpe (1994), we will present the approach and its implications. The return on the *i* asset is given by the following vector:

$$R_t = [\, R_{1,t} R_{2,t} \, \ldots . \, R_{N,t}]'$$

The portfolio weight is given by the following vector:

$$w_t = [\, w_{1,t} w_{2,t} \, \ldots . \, w_{N,t}]'$$

The portfolio return is given by:

$$R_p = \sum_{i=1}^{N} w_{i,t} R_{i,t} = w_t' R_t$$

Instead, the portfolio variance is given by:

$$\sigma_p^2 = \sum_{i=1}^{N} \sum_{j=1}^{N} w_{i,t} w_{j,t} \, Cov\,(R_{i,t} R_{j,t}) = w_t' \sum w_t$$

Where $\sum$ denotes the covariance matrix *N x N*. The problem is termed by minimizing the variance and maximizing the return, the solving of this portfolio optimization gives the efficient frontier, i.e. for a given variance portfolio return there is a maximum portfolio return. If we assume that the market is efficient we have that the efficient portfolio is the market portfolio as such we have the CAPM:

$$R_{i,t} = R(T) + \beta_{i,MKT} \left(R_{MKT} - R(T) \right)$$

Where:

$$\beta_{i,MKT} = \frac{Cov\,(R_{i,t} R_{MKT})}{\sigma_{MKT}^2}$$

The return of the *i* asset is given by its sensitivity with the excess return of the market with respect the risk free rate. The Sharpe ratio instead is given by the following:

$$\frac{\left(R_{i,t} - R(t) \right)}{\sigma_{i,t}}$$

The excess return is weighted by the standard deviation, i.e. the Sharpe ratio is the excess return per unit of risk. The market portfolio is the highest Sharpe ratio by combining the risk free rate with the efficient frontier, in fact, the market portfolio is the tangent of the risk free rate with the efficient frontier. If the market is efficient we can't beat it but if we assume that the market portfolio in not efficient we can vary its weight on the base of a view on the future return, this opens the door to a Bayesian approach. By denoting:

$$\mu_{i,t} = \frac{R_{MKT,t} - R(T)}{\sigma^2_{MKT}}$$

We have:

$$E_{i,t} = \mu_{i,t}\, Cov\,(R_{i,t}R_{MKT})w_{i,equ}$$

$$w_{i,equ} = \frac{1}{\mu_{i,t}} Cov\,(R_{i,t}R_{MKT})^{-1}\, E_{i,t}$$

At this point, it is easy to note how we may vary the market weight by changing the expected return but to keep the market equilibrium we must have excess expected return equal to lack expected return. The market portfolio will have a different rate of return on the base of view expressed, Black, Litterman (1992) formalized the theory by using an example of three assets:

$$\mu_{1,t} = \pi_1 + b_1 Z + e_1$$

$$\mu_{2,t} = \pi_2 + b_2 Z + e_2$$

$$\mu_{3,t} = \pi_3 + b_3 Z + e_3$$

π denotes the equilibrium risk premium, b denotes the impact of the Z common factor and e the independent shock. The covariance matrix $\sum$ is determined by the impact of common factor and independent shock. As such we have:

$$E[\mu_{i,t}] = \pi_i + b_i E[Z] + E[e_i]$$

$E[\mu_{i,t}]$ is distributed with covariance matrix proportional to $\sum$ as $\tau\sum$ because the uncertainty in the average is much smaller than the uncertainty in the return itself. They assume the linear restriction $(\mu_{1,t} - \mu_{2,t}) = Q$ where $E[\mu_{i,t}]$ is given by normal $[\pi_{i,}\tau\sum]$, as result they calculate the conditional distribution for the expected return that is the solution of the minimizing problem of:

$$(E[\mu_{i,t}] - \pi_i)\, \tau\sum{}^{-1}(E[\mu_{i,t}] - \pi_i)'$$

Subject to the restriction of expected return:

$$P\ x\ E[\mu_{i,t}]' = Q$$

Where P is the vector $[1, -1, 0]$. The conditional distribution has the following average:

$$\pi_i'\ \tau\sum\ P'[\,P\ \tau\sum\ P']^{-1}\,(Q - P\ \pi_i{}')$$

They use this vector as expected excess return with 100% of confidence on the view. They introduce uncertainty with the following:

$$P\ x\ E[\mu_{i,t}] = Q + e$$

Here P denotes K x N matrix and Q denotes K view vector matrix, e denotes a normal random vector variable with zero mean and a diagonal covariance matrix φ. The first example is solved by using the lagrangian, but it is much more interesting give the proof in the case of uncertainty view, as such we have the following by using the generalized least square methods:

$$Y = \begin{matrix} \pi_i \\ Q \end{matrix} \qquad X = \begin{matrix} I \\ P^T \end{matrix} \qquad W = \begin{matrix} \tau\sum & 0 \\ 0 & \varphi \end{matrix}$$

$$E[\mu_{i,t}] = [X^T W^{-1} X]^{-1} X^T W^{-1} Y$$

The resulting conditional distribution by considering the equilibrium market has the following average:

$$((\tau\sum)^{-1} + \mathrm{P}'\varphi^{-1}\mathrm{P})^{-1}(\,(\tau\sum)^{-1}E_{i,t} + \ \mathrm{P}'\varphi^{-1}\mathrm{Q})$$

In the portfolio optimization we use this vector as expected excess return.

Value at Risk

In finance there is the problem to value the risk associated to a portfolio of securities given an interval of confidence that gives the measure of risk in a probability context. The problem is more complicated that as it may seem because it depends from the distribution of the prices, in fact we may have a distribution with right or left side skew. The problem was solved by using the percentile that does not assume any distribution. If we have a series of observations and we order them from the lesser to the greater we may chose the observation that for example put the 99% observations over it, simply by using the following relation:

$$2 : 100 = Perc : Number\ of\ Observations$$

The problem of this approach is that if the number of observations increase the 99% may undervalue the risk, so the solution is to take the 99,9%. If we take the daily return of stocks we may observe that the distribution is skewed as such we may obtain the 99,9 percentile of a normal distribution by using a mixture of normal distribution where the parameter of skew may be calibrated to the market value of daily return such that we have:

$$N[\dots] = N[0,1] + \ a\,N[0,1]$$

We may simulate the market price with geometric Brown process, from this we may calibrate the parameter skew *a* such that the percentile percentage of the simulated distribution is equal to the historical percentile of the daily return. After been calibrated the simulated distribution we may obtain the tails risks by taking the average between the tail given by -3 + *a* -3 and the 99,9 percentile of the mixture normal distribution given by -2,57 + *a* -2,57 because the 99,9 percentile of a normal distribution is given by -2,57. This approach may be used as alternative to the historical percentile by using the tail risk with respect the expected short fall given by the 99,9 percentile. Another way is to use the static VaR the solution is to take the rate of return with respect the average price so to normalize the distribution. Along this approach we may have the following relation:

$$Percentile\ \% = \sigma^{\frac{1}{x}}$$

For volatility greater than 20% the relation may be approximate by using the square root. The relation holds sure for distribution that show a normal distribution although they have fats tails. In fact we may have that for normal distribution the following relation:

$$Percentile\ \% = 2\sigma$$

These relations may be used as targets and for value the tails risks, in fact we have that the 99,9 percentile of a standard normal distribution is equal to -2,57, if we take the tail -3 and we make the average we obtain -2,78 that may be considered the tail risk. We have to observe that a geometric Brown process converges to zero as such -2,78 is not much different from -2,57, indeed, with the relation saw before it is easy to get this value for the tail by simulating, in fact, if we take the square root of the exponential of a standard normal distribution N[0;1,5] we obtain 2,78, so by simulating with the geometric Brown process in term of percentage we may obtain the tail risk by taking the square root of the simulated distribution given by the following:

$$e^{(\mu(t)-\frac{1}{2}\sigma^2)dt+\ \sigma N[0;1,5]\sqrt{dt}}$$

This approach has more characterizes, in fact if we make back test it is possible to observe that the percentile is equal to the target or more or less 10%, so the target is an average but if we take the 99% of the simulation for the tail risk N[0;1,5] we obtain that the result is +10% with respect to the target so it is the back test tails risks. We may note that if we use this approach for the daily return the tails risks are equal to the historical 99,9 percentile in normal condition, i.e. growing economies, by showing that the distributions are skewed because the tails risks are equal to the historical expected shortfall; instead, if we take in considerations periods of crisis the tails risks of the normalized distribution is much less than the historical expected shortfall by showing that the distributions are really skewed so to get the simulated VaR by using the skew normal distribution. Indeed, we may make dynamic the static VaR by computing the rate of return with respect the moving average, along this approach is possible to note that in normal condition usually the risk is given by the historical 99,9 percentile of the daily return for one day risk; instead, in period of crisis the risk is given by the historical 99,9 percentile of the daily return for ten days risks.

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