# Uncertainty, learning and growth

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**Abstract**

The paper extends Blackburn and Galindev (2003)' s stochastic growth model in which productivity growth entails both external and internal learning behaviour with a Constant Relative Risk Aversion utility function and productivity shocks. Consequently, the relationship between long-term growth and short-term volatility depends not only on the relative importance of each learning mechanism but also on a parameter measuring individuals' attitude towards risk.

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## 1 Introduction

The nature of the relationship between cyclical volatility (uncertainty) and secular growth has generated a growing interest both empirically and theoretically.[1] The topic is particularly important for the long-run implications of economic policies designed to mitigate short-run cyclical volatility.[2] Empirical evidence based on individual time series and cross section data shows an ambiguity in the sign of the relationship (see Kneller and Young (2001) for a review of the evidence). This has inspired theorists to explore potential mechanisms that could generate the ambiguity in the underlying relationship.

Contributions by De Hek (1999), Jones et al. (2005), Smith (1996) and Varvarigos (2007) consider the Constant Relative Risk Aversion utility function and uncertainty to production technology and show that the effect of uncertainty on growth depends fundamentally on the individuals' attitude towards risk. According to these analyses,

[1] Although volatility and uncertainty are two different terminologies: the former implies *ex post* while the latter implies *ex ante* fluctuations in a variable, it is common practice to use them interchangeably.

[2] Martin and Rogers (1997), Blackburn (1999), Blackburn and Pelloni (2005) and Galindev (2006), for example, all develop theoretical models and discuss the relationship between long-run growth and short-run stabilisation policy.

the more (less) risk averse an agent, more likely that increased uncertainty will lead to a higher (lower) growth by increasing (decreasing) the amount on which productivity growth depends. In addition, De Hek (2005) shows that not only the individuals' attitude towards uncertainty but also returns to scale in knowledge creation matter for the relationship between growth and uncertainty.[3] Blackburn and Pelloni (2004) show that the nature of the shocks (whether they are real or nominal) can explain the ambiguity on the relationship between growth and volatility in a stochastic monetary growth model. The contribution particularly interesting to us is Blackburn and Galindev (2003). They consider a model with a logarithmic utility function and preference shocks in which productivity growth entails both external (serendipitous) and internal (deliberate) learning behaviour. They show that the relationship between growth and volatility depends on the relative importance of each learning mechanism. More precisely, when external learning is more important than internal learning, the underlying relationship is negative, and vice versa. One disadvantage of the models with a logarithmic utility function, especially the additive structure of utility of consumption and leisure, is that technology shocks do not have any effects on the optimal allocation of time endowment. This seems to be the reason why preference shocks are commonly used in order to facilitate an environment to study the relationship between growth and volatility in analytically tractable models. A recent contribution by Blackburn and Varvarigos (2006), however, overcomes this problem by considering an utility function that the marginal rate of substitution between consumption and labour is independent of the level of consumption. In their analysis based on a model with both technology and preference shocks and both internal and external learning, they find that the relationship between growth and volatility depends on not only the relative importance of each learning but also the relative dominance of the source of fluctuations.

The present paper extends the model of Blackburn and Galindev (2003) by specifying a CRRA utility function, and productivity shocks rather than preference shocks.[4] The main conclusions from this analysis are two fold. First, this introduces the added dimension of individuals' attitude towards risk to the framework. More precisely, the results of Blackburn and Galindev (2003) resemble those derived from a special case of the general model in which preferences are more curved than the logarithmic utility function, providing to satisfy an additional condition. However, their results are reversed for the case where the utility function is less curved than the logarithmic utility – i.e., agents with a sufficiently low degree of risk aversion. The implication is that the relative dominance of one learning mechanism over another is not sufficient to unambiguously predict the sign of the relationship between growth and volatility.

[3] For other contributions on the relationship between growth and volatility, see the first model in De Hek (1999), Martin and Rogers (1997), (2000) and Aghion and Saint-Paul (1998a), (1998b).

[4] The present model is different from that in Varvarigos (2007) in terms of the determinants for productivity growth. I consider both external and internal learning mechanisms while Varvarigos assumes internal learning.

Second, I show the case where the optimal allocation of time spent learning could decrease in response to an increase in uncertainty even if agents have a sufficiently high degree of risk aversion. This result has not, to my knowledge, been established in the literature.

The remainder of this paper is organised as follows. Section 2 sets up the model. Section 3 solves for the stochastic dynamic general equilibrium and discusses the main results. Section 4 concludes.

# 2 Model

Time is discrete and indexed by $t = 0, 1, ..., \infty$. The economy consists of a fixed population (normalised to one) of identical, infinitely-lived agents who are both producers and consumers of a single commodity. The representative agent maximises the following expected lifetime utility:

$$U = E_0 \sum_{t=0}^{\infty} \beta^t \left\{ \frac{\left(C_t L_t^{\lambda}\right)^{1-\tau}}{1-\tau} \right\}, \quad \beta \in (0,1); \quad \lambda > 0; \quad \tau > 0. \tag{1}$$

where $C_t$ denotes consumption, $L_t$ denotes leisure and the term $\tau$ measures the household's attitude towards risk (and $1/\tau$ is the elasticity of intertemporal substitution). When $\tau = 1$, the preferences converge to the logarithmic function which is considered in Blackburn and Galindev (2003). Since we abstract from saving and government spending, consumption is equal to output. The agent produces output, $C_t$, by combining his labour, $N_t$, and knowledge or productivity, $Z_t$, in accordance with the production function:

$$C_t = \Omega Z_t N_t^{\alpha}, \quad \Omega > 0, \quad \alpha \in (0,1]. \tag{2}$$

Knowledge is endogenous and evolves according to:

$$Z_{t+1} = A_{t+1} \Psi Z_t \overline{N}_t^{\theta} S_t^{\phi}, \quad \Psi > 1, \quad \phi, \theta \in (0,1]. \tag{3}$$

Productivity growth in Eq. (3) is the result of both internal (deliberate) and external (non-deliberate) learning behaviour. The former is represented by $S_t$, the amount of time that the agent devotes intentionally to improving his own productivity (e.g., through formal education, training and research). The latter is approximated by $\overline{N}_t$, the aggregate level of employment which captures the extent of knowledge spillovers among agents and which each agent takes rationally as given. The relative importance of each learning mechanism for productivity growth is measured by the parameters $\phi$ and $\theta$. The critical value of these parameters in the face of uncertainty such that both types of learning behaviour are equally important for productivity growth will be discussed below. The term, $A_{t+1}$, is a productivity shock which is an identically and independently distributed random variable with unit mean and a constant variance, $\sigma_a^2$.

The agent is assumed to have one unit of time that can be allocated between leisure, producing output and improving productive efficiency:

$$1 = L_t + S_t + N_t. \tag{4}$$

The agent allocates this endowment to maximise the expected lifetime utility in Eq. (1) subject to Eqs. (2), (3) and (4). The first order condition with respect to $N_t$ yields the following relationships:

$$N_t = \frac{\alpha}{\alpha + \lambda}(1 - S_t), \tag{5}$$

$$L_t = \frac{\lambda}{\alpha + \lambda}(1 - S_t). \tag{6}$$

The first order condition with respect to time spent learning is:

$$\lambda C_t^{1-\tau} L_t^{\lambda(1-\tau)-1} = \beta\phi E_t \left( \frac{C_{t+1}^{1-\tau} L_{t+1}^{\lambda(1-\tau)}}{S_t} \right), \tag{7}$$

where $E_t$ is the expectations operator. The left hand side expression in Eq. (7) represents the marginal cost of spending one unit of time on learning in terms of forgone leisure while the right hand side expression shows its marginal benefit in terms of an increase in expected marginal utility of consumption. Substituting Eqs. (2) and (3) into Eq. (7), one can obtain the following expression:

$$\frac{S_t}{L_t} = \frac{\beta\phi\Psi^{1-\tau}}{\lambda} E_t \left( \frac{A_{t+1}^{1-\tau} S_t^{\phi(1-\tau)} \overline{N}_t^{\theta(1-\tau)} N_{t+1}^{\alpha(1-\tau)} L_{t+1}^{\lambda(1-\tau)}}{N_t^{\alpha(1-\tau)} L_t^{\lambda(1-\tau)}} \right). \tag{8}$$

According to the expression in Eq. (8), the optimal policies for time spent learning, $S_t$, and working, $N_t$, hence leisure, $L_t$, are not affected by the realisation of the shock in the current period, $A_t$. The reason is that $A_t$ is deliberately assumed to be an iid shock to the growth rate of productivity in Eq. (3), for simplicity as it enables us to get the results that follow analytically. The intuition is that a positive shock acts as a permanent shock, implying a level effect on all future output through its direct effect on productivity growth hence a level effect on the optimal time allocation in the current period is neutral. For example, a positive shock, $A_t$, increases $N_t$ by increasing its marginal productivity which leads to a decrease in $S_t$. On the other hand, the same shock also increases $S_t$ by increasing through an increase in $Z_t$. The opposite effects cancel each other out hence and $S_t$ and $N_t$ are left unaffected. This implies that the optimal policies for leisure, time spent working and learning are constant over time – i.e., $L_t = L$, $N_t = N$ and $S_t = S$ for any $t$ for a given variance of the shock, $\sigma_a^2$. Substituting these policies together with the equilibrium condition, $\overline{N}_t = N_t$, and Eq. (6) into Eq. (8), we obtain:

$$\frac{S^{1-\phi(1-\tau)}}{(1-S)^{1+\theta(1-\tau)}} = \frac{\beta\phi\Psi^{1-\tau}\alpha^{\theta(1-\tau)}}{(\alpha+\lambda)^{1+\theta(1-\tau)}} E(A_{t+1}^{1-\tau}). \tag{9}$$

The expression in Eq. (9) shows the relationship between uncertainty, captured by $E(A_{t+1}^{1-\tau})$ due to $\sigma_a^2 > 0$, and the optimal policies for time spent learning, $S$, hence time spent working, $N$, in Eq. (5), and leisure, $L$, in Eq. (6). The results are summarised in the following Proposition.

**Proposition 1** *if $\tau = 1$, an increase in uncertainty has no effect on $S$, $N$ and $L$. If $0 < \tau < 1$, increased uncertainty has a negative effect on $S$ hence a positive effect on $N$ and $L$. If $1 < \tau \leq \tau^*$ where $\tau^* = (\theta+1)/\theta$, increased uncertainty has a positive effect on $S$ hence a negative effect on $N$ and $L$. If $\tau > \tau^*$, there will be $S^*$ such that for $S < S^*$ an increase in uncertainty has a positive effect on $S$ hence a negative effect on $N$ and $L$ while the opposite happens for $S > S^*$.*

**Proof.** It is obvious that if $\tau = 1$ – i.e., the instantaneous utility function converges to $\log(C_t) + \lambda\log(L_t)$, then uncertainty has no influence on the optimal policy for $S$ as $E(A_{t+1}^{1-\tau}) = 1$. The optimal policies for time spent learning, working and leisure in this case are constant. Consider now $\tau \neq 1$. Let us rewrite the expression in (9) as $\Pi = \Phi E(A_{t+1}^{1-\tau})$ where $\Pi = \frac{S^{1-\phi(1-\tau)}}{(1-S)^{1+\theta(1-\tau)}}$ and $\Phi = \frac{\beta\phi\Psi^{1-\tau}\alpha^{\theta(1-\tau)}}{(\alpha+\lambda)^{1+\theta(1-\tau)}}$. It is straightforward to verify that $\Pi$ is a monotonically increasing function of $S$ for any $\phi \in (0,1]$, $\theta \in (0,1]$ and $0 < \tau \leq \tau^*$, but a non-monotonic function for $\tau > \tau^*$. In the case where $0 < \tau < 1$, a mean preserving spread in the distribution of the shock $A_{t+1}$ leads to a decrease in $E(A_{t+1}^{1-\tau})$. In response to this, $S$ must decrease to restore the equilibrium. If $1 < \tau \leq \tau^*$, a mean preserving spread in the distribution of the shock $A_{t+1}$ increases $E(A_{t+1}^{1-\tau})$. In response to this, $S$ must increase to restore the equilibrium. There is an interesting new case to the literature that arises for $\tau > \tau^*$. Taking the first order derivative of with respect to $S$ and setting the result equal to zero yields $S^* = \frac{1+\phi(\tau-1)}{(\phi+\theta)(\tau-1)}$ which lies between zero and one as long as $\tau > \tau^*$. The second order derivative of $\Pi$ with respect to $S$ at $S^*$ is found to be negative. This implies that $\Pi$ exhibits an inverted-U shape. Moreover, there exist multiple equilibria for $S$. The first equilibrium is in the interval, $S < S^*$ while the second is in $S > S^*$. Since $E(A_{t+1}^{1-\tau})$ is an increasing function of $\sigma_a^2$, the economy will find it optimal to increase (decrease) time spent learning to restore the equilibrium in the first (second) equilibrium. The effect of uncertainty on the optimal policies for leisure and time spent working can be easily proven using Proposition 1 and Eqs. (5) and (6). ■

Intuitively, for $0 < \tau \leq \tau^*$ and $\tau > \tau^*$ with $S < S^*$, Proposition 1 is consistent with the well-known results in the literature – the more (less) anxious an agent is to smooth consumption over time, it is more likely that increased uncertainty leads to an increase (decrease) in precautionary investments in human or physical capital (e.g., De Hek, 1999; Smith, 1996; Jones et al., 2005; Varvarigos, 2007). The result in the case of $\tau > \tau^*$ with $S > S^*$, however, conflicts the above results and has not, to my knowledge, been established in the literature. It is surprising to think that an increase in uncertainty leads to a fall in the precautionary investment in human capital for an agent who has a sufficiently high degree of risk aversion. This scenario arises because of the existence of external learning, captured by $\theta$. It can be seen that $\theta$ as converges to zero, $\tau^*$ converges to infinity. In the limit when $\theta = 0$ – i.e., external learning for productivity growth is unimportant, $\tau^* = \infty$, implying that Proposition 1 is totally consistent with the above results in the literature. Empirically, common choice of values for $\tau$ range between 1 and 4. These values can be covered if $\theta \leq 1/3$. However, the greater the value of $\theta$, it is more likely that the non-monotonic effect of uncertainty on time spent learning will exist. For example, $\theta = 1$, $\tau^* = 2$ which implies that for half of the commonly used values of $\tau$, we could observe two equilibria and the effect of uncertainty on time spent learning is different from one equilibrium to the other.

# 3 Uncertainty and Output Growth

We have now arrived at the point where we are able discuss the effect of uncertainty on the growth rate of knowledge and output. Let us substitute the optimal policies for $S$ and $N$ into Eqs. (2) and (3) and obtain the actual growth rate of output between two consecutive periods:

$$g_{t+1} \equiv \frac{C_{t+1}}{C_t} = \Psi A_{t+1} S^{\phi} N^{\theta}. \tag{10}$$

Since the optimal policies for $S$ and $N$ are independent of the realisation of the shock, $A_{t+1}$, the mean of this growth can be written as follows:

$$Mean(g) = \Psi S^{\phi} N^{\theta}. \tag{11}$$

According to the expression in Eq. (19), uncertainty can have ambiguous effects on $Mean(g)$ through its opposite and ambiguous effects on $S$ and $N$. Suppose that we obtain $\widetilde{S}$ and $\widetilde{N}$ hence $Mean(\widetilde{g})$ for every $\tau$, for a given $\sigma_a^2$, using Eq. (9). By log-linearising the expression in Eq. (11) around these values we can determine the effects of an increase in $\sigma_a^2$ on $Mean(g)$. Defining $\widehat{g} = \log\left[Mean(g)\right] - \log[Mean(\widetilde{g})]$, $\widehat{s} = \log(S) - \log(\widetilde{S})$ and $\widehat{n} = \log(N) - \log(\widetilde{N})$, we then obtain

$$\widehat{g} = \phi\widehat{s} + \theta\widehat{n}. \tag{12}$$

In order to express the right hand side of Eq. (12) in terms of one variable, we can log-linearise the time constraint in Eq. (5) around $\widetilde{S}$ and $\widetilde{N}$. Accordingly,

$$\widehat{n} = -\frac{\alpha\widetilde{S}}{(\alpha+\lambda)\widetilde{N}}\widehat{s}. \tag{13}$$

Substituting Eq. (13) into Eq. (12) yields

$$\widehat{g} = \left(\phi - \theta \frac{\alpha \widetilde{S}}{(\alpha+\lambda)\widetilde{N}}\right)\widehat{s}. \tag{14}$$

The expression in Eq. (14) says that $\widehat{g}$ could be positive, negative or zero depending on the value of $\left(\phi - \theta \frac{\alpha \widetilde{S}}{(\alpha+\lambda)\widetilde{N}}\right)$ and the value of $\widehat{s}$. From Proposition 1, we know that $\widehat{s} < 0$ for $0 < \tau < 1$ and $\tau > \tau^*$ with $S > S^*$, $\widehat{s} = 0$ for $\tau = 1$, and $\widehat{s} > 0$ for $1 < \tau \leq \tau^*$ and $\tau \geq \tau^*$ with $S < S^*$. From Eq. (9), we see that both $\widetilde{S}$ and $\widetilde{N}$ are determined by the model parameters, including $\phi$ and $\theta$ for the given variance of the shock, $\sigma_a^2$. For the values of the parameters characterising an economy, we could always determine the sign of the expression, $\phi - \theta \frac{\alpha \widetilde{S}}{(\alpha+\lambda)\widetilde{N}}$. For the predetermined values of $\widetilde{S}$ and $\widetilde{N}$, the case in which $\phi > \theta \frac{\alpha \widetilde{S}}{(\alpha+\lambda)\widetilde{N}}$ implies that internal learning is relatively more important for productivity growth than external learning whenever $S$ deviates from $\widetilde{S}$. If the opposite is true, external learning is relatively more important. For the rest of the analysis, we employ the following simple conjecture. For $\widetilde{S}$, $\widetilde{N}$ and $\theta$, we can obtain a critical value, $\phi^* = \theta \frac{\alpha \widetilde{S}}{(\alpha+\lambda)\widetilde{N}}$, such that $\widehat{g} = 0$ – i.e., each type of learning is equally important. Under such circumstances, the case in which $\phi > \phi^*$ implies that internal learning is relatively more important for productivity growth than external learning. The case where $\phi < \phi^*$ implies that external learning is more important.[5]

The effects of uncertainty on growth are summarised in the following Proposition.

**Proposition 2** *An increase in uncertainty leads to a higher growth, $\widehat{g} > 0$, if (i). $1 < \tau \leq \tau^*$ and $\tau > \tau^*$ with $S < S^*$ hence $\widehat{s} > 0$ and $\phi > \phi^*$ and (ii). $0 < \tau < 1$ and $\tau > \tau^*$ with $S > S^*$ hence $\widehat{s} < 0$ and $\phi < \phi^*$. Increased uncertainty decreases growth, $\widehat{g} < 0$, if (iii). $1 < \tau \leq \tau^*$ and $\tau > \tau^*$ with $S < S^*$ hence $\widehat{s} > 0$ but $\phi < \phi^*$ and (iv). $0 < \tau < 1$ and $\tau > \tau^*$ with $S > S^*$ hence $\widehat{s} < 0$ but $\phi > \phi^*$. An increase in uncertainty has no effect on growth, $\widehat{g} = 0$, if (v). $\phi = \phi^*$ for any $\tau > 0$ or (vi). $\tau = 1$ hence $\widehat{s} = 0$ for any $\phi > 0$.*

**Proof.** It directly follows from Eq. (14) in conjunction with Proposition 1. ■

The intuition of Proposition 2 is following. In case (i), internal learning is relatively more important than external learning for productivity growth. When agents allocate more time towards deliberate learning due to increased uncertainty, growth is positively affected. In case (ii), agents allocate less time towards learning hence more time to working in response to increased uncertainty. Since external learning is relatively more important for productivity growth than internal learning, uncertainty

[5] Alternatively, for $\widetilde{S}$, $\widetilde{N}$ and $\phi$, we obtain a critical value, $\theta^* = \phi \frac{(\alpha+\lambda)\widetilde{N}}{\alpha \widetilde{S}}$, such that $\widehat{g} = 0$. Then we conjecture that if $\theta > \theta^*$, external learning is more important for productivity growth and vice versa.

has a positive effect on growth. In case (iii), agents increase time spent learning by decreasing time spent working in response an increase in uncertainty which leads to a lower growth as external learning happens to be a dominant determinant of productivity growth. Case (iv) shows the possibility that agents decrease time spent learning by increasing time spent working in the face of high uncertainty which leads to a lower growth when internal learning is more important for productivity growth. Case (v) shows that both learning mechanism are equally important hence changes in time spent working or learning due to changes in uncertainty has no effect on growth. Case (vi) shows that agents do not respond to changes in uncertainty hence growth is unaffected regardless of the relative importance of each learning mechanism for productivity growth.

The result in the cases of $1 < \tau \leq \tau^*$ and $\tau > \tau^*$ with $S < S^*$ resemble those reached by Blackburn and Galindev (2003): an increase in the variance of the shock has a positive effect on growth if productivity growth is predominantly determined by internal learning – e.g., case (i), and vice versa – e.g., case (iii). These results can be reversed in the cases of $0 < \tau < 1$ and $\tau > \tau^*$ with $S > S^*$.

# 4 Conclusion

The paper extends the model of Blackburn and Galindev (2003) by considering the CRRA utility function and productivity shocks. Under these more general circumstances, it shows that the effect of uncertainty on growth depends not only on the relative importance of alternative learning mechanisms for productivity growth suggested by Blackburn and Galindev (2003), but also on individuals' attitude towards risk suggested by De Hek (1999), Smith (1996), Jones et al. (2005) and Varvarigos (2007). Blackburn and Galindev (2003)'s findings are the same as those in the case where the utility function is more curved than logarithmic (expect for the controversial case where $\tau > \tau^*$ with $S > S^*$): an increase in the volatility of the shock increases (decreases) growth when internal (external) learning is more important for productivity growth. Their results are, however, reversed if the utility function is less curved than the logarithmic function. For example, internal learning is relatively more important for technological change but agents have a sufficiently low degree of risk aversion. An increase in uncertainty has a negative effect on time spent learning and growth.

# References

[1] Aghion, P. and Saint-Paul, P. (1998a) On the virtue of the interaction between economic fluctuations and productivity growth, Macroeconomic Dynamics, 2, 322-44.

[2] Aghion, P. and Saint-Paul, P. (1998b) Uncovering some causal relationships between productivity growth and the structure of economic fluctuations: a tentative survey, Labour, 12, 279-303.

[3] Blackburn, K., 1999. Can stabilisation policy reduce long-run growth? Economic Journal, 109, 67-77.

[4] Blackburn, K., Galindev, R., 2003. Growth, volatility and learning, Economics Letters 79, 417-421.

[5] Blackburn, K., Pelloni, A., 2004. On the relationship between growth and volatility in learning-by-doing economies, Economics Letters 83, 123-127.

[6] Blackburn, K., Pelloni, A., 2005. Growth, cycles and stabilisation policy, Oxford Economic Papers, 57, 262-282.

[7] Blackburn, K., Varvarigos, D., 2006. Human capital accumulation in a stochastic environment: some new results on the relationship between growth and volatility, Economic Theory, forthcoming.

[8] De Hek, P.A., 1999. On endogenous growth under uncertainty, International Economic Review 40, 727-744.

[9] De Hek, P.A., 2005. Uncertain technological change under capital mobility, Tinbergen Institute, Discussion Paper TI 2005-033/2.

[10] Galindev, R., 2006. Note on "Growth, cycles, and stabilisation policy", Oxford Economic Papers, forthcoming.

[11] Jones, L.E., Manualli, R.E., Siu, H., Stacchetti, E., 2005. Fluctuations in convex models of endogenous growth I: Growth Effects, Review of Economic Dynamics 8, 780-804.

[12] Kneller, R. and Young, G. (2001) Business cycle volatility, uncertainty and long-run growth, Manchester School, 69, 534-52.

[13] Levhari, D., Srinivasan, T. N., 1969. Optimal savings under uncertainty, Review of Economic Studies 106, 153-163.

[14] Martin, P., Rogers, C. A., 1997. Stabilisation policy, learning-by-doing, and economic growth. Oxford Economic Papers, 49, 152-66.

[15] Martin, P., Rogers, C. A., 2000. Long-term growth and short-term economic instability, European Economic Review, 44, 359-81.

[16] Smith, T.R., 1996. Cyclical uncertainty, precautionary saving and economic growth, Economica 63, 477-494.

[17] Varvarigos, D., 2007. Sustained output growth under uncertainty: a simple model with human capital, Loughborough University, Discussion Paper 2007/20.