Distributional Neutral Welfare Ranking-Extending Pareto Principle

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Abstract

We extend standard Pareto criterion for welfare ranking in terms of inequality. We suggest strongly Pareto superior or SPS allocations which are inequality neutral but guarantee higher welfare for everyone. The purpose is to entertain the idea that rising inequality is a major welfare concern and hence one must go beyond standard Pareto superior (PS) allocations that necessarily lead to greater inequality. In the main result of this paper we show that whenever there is aggregate gain in net utility then there exists counterfactual allocation which is eventually a SPS allocation.

1 Introduction

Pareto ranking or Pareto efficiency is a topic economists are exposed to very early in their career. In particular the basic welfare comparison between two social situations starts with the ranking in terms of a principle Pareto have talked about in the nineteenth century. If we compare two social situations A and B, we say B is Pareto superior to A iff everyone is as well of in B as in A and at least one is strictly better off in situation B compared to A. This comparison is done in terms of utility or welfare levels individuals enjoy in A and B. Theory of social welfare has been a widely discussed topic with seminal contributions from De Scitovszky (1941), Samuelson (1958), Arrow (1963) and others to make recent treatments such as Cowels () Stiglitz (1987) Sen (1970).

Pareto's principle provides a nice way to compare situations when some gain and some lose by considering whether transfer from gainers to losers can lead to a new distribution in B such that B turns out to be Pareto superior to A, the initial welfare distribution. It is obvious that if sum of utilities increases in B relative to A, then welfare be the actual distribution in B, a transfer mechanism will always exist such that transfer-induced redistribution will make B Pareto superior to A. The great example is how gains from international trade can be redistributed in favor of those who lose from trade such that everyone gains due to trade. Overall gains from trade lead to a highly level of welfare, under ideal conditions and therefore one can show that under free trade eventually nobody may lose as gainers *'bribe'* the losers. But whatever it is Pareto ranking definitely does not address the inequality issue. There will be situations where B will be Pareto superior to A, but inequality in B can be much greater than A. The purpose of this short paper is to extend the basic principle of Pareto's welfare ranking subjecting it to a stricter condition that keeps the degree of inequality intact between A and B after transfer from gainers to losers, but at the same time guaranteeing that everyone gains in the end.

Thus we coin a Strong-Pareto criterion which not only insists that everyone must be better of in B compared to A, but also requires that degree of inequality must remain the same between A and B. Only then B will be Strongly Pareto Superior (SPS) to A. Concern for such a Strong principle stems from the fact that people do care about inequality and inequality has become a worldwide popular point of debate in public domain (see Stewart, 2004; Stiglitz, 2012; Piketty, 2014; Atkinson and Stiglitz, 2015; Pickett and Wilkinson, 2015, for further readings).

Pareto superior move as such may not contain agitation to change policies further because of rising inequality. We are also motivated by the query as do whether the basic condition that guarantees Pareto superiority of B to A, would also guarantee that B is SPS to A. Apparently it need not be since there can be transfer that make B PS A, but that aggravate inequality.

We show that if total utility in B is grater than total utility of A, we can always construct a counterfactual distribution C which is SPS to A. The counterfactual allocation is obtained by taxing a subset of individual and redistributing the collected tax to the rest of the individuals. In order to keep the inequality level same we redistribute the aggregate gains proportionate to that of individuals utility at the initial stage.

Section 2 describes the environment and the result. Last section concludes.

2 Model

Consider an n ($n > 1$) agent society being observed for two time points. The initial time point is denoted by 0, whereas the final time point is denoted by 1. The utility profile for the set of individuals at time $t(t \in \{0, 1\})$ is defined in the following fashion:

$$U_t = \{u_{t1}, u_{t2}, ..., u_{tn}\}, \ \forall t \in \{0, 1\} \tag{1}$$

We assume that the individual utilities are cardinal. Furthermore, we also assume that the individual utilities are also strictly positive, i.e. $u_{ti} > 0$, $\forall t \in \{0, 1\}$ and $\forall i \in \{1, 2, .., n\}$.[1] Let $\mathbb{D}^n$ be the set of all such n coordinated utility profiles.

Pareto superiority (PS) is defined as the situation where no one looses from the initial to the final period but at least one individual gains. However, PS allocation may aggravate inequality. We thus introduce "*Strong Pareto Superiority*" (SPS). By SPS we mean a situation where the utility of all the individuals increases and the inequality also remains same, compared to that of the initial distribution. Throughout this paper by inequality we restrict our attention to the family of relative inequality indices of the form $I : \mathbb{D}^n \longrightarrow \mathbb{R}$ which are homogeneous of degree 0, i.e.,

$$I(u_1, u_2, .., u_n) = I(\delta u_1, \delta u_2, .., \delta u_n) \tag{2}$$

where $\delta > 0$.

We now formally define SPS allocations in the following fashion:

Definition 1. ***SPS allocation*** *For all $U_0, U_1 \in \mathbb{D}^n$, any counterfactual distribution $\hat{U} = \{\hat{u}_1, \hat{u}_2, .., \hat{u}_n\} \in \mathbb{D}^n$ which is obtained from U_1 is said to be a SPS allocation to U_0 which is denoted by $\hat{U} \succ_{SPS} U_0$, if and only if $\hat{u}_i > u_{0i}$ and $\frac{u_{0i}}{u_{0j}} = \frac{\hat{u}_i}{\hat{u}_j}$, $\forall i, j \in \{1, 2, .., n\}$.*

Note that if we scale up utilities of all the individual's of the initial distribution by any positive scalar greater than 1, we necessarily get a SPS allocation. Nevertheless, such an allocation is not feasible if the aggregate utility of the counterfactual distribution exceeds that of the final distribution. Formally we define a feasible SPS allocation in the following fashion:

Definition 2. ***Feasible SPS allocation:*** *For all $U_0, U_1 \in \mathbb{D}^n$, and $\exists \hat{U} = \{\hat{u}_1, \hat{u}_2, .., \hat{u}_n\} \in \mathbb{D}^n$ which is obtained from U_1 such that $\hat{U} \succ_{SPS} U_0$, then $\hat{U}$ is said to be a feasible SPS allocation if and only if $\sum_{i=1}^{n} \hat{u}_i \leq \sum_{i=1}^{n} u_{1i}$.*

A feasible SPS allocation may not be the most efficient. This is particularly when there is some resource left as a residual which can be further redistributed amongst the agents to make every one better off. We define the most efficient SPS allocation, among the set of feasible SPS allocations in the following fashion:

[1] We make this assumption for mathematical simplicity. We can always allow an utility function which takes negative values. However, in such cases we have to restrict the class of utilities that are invariant to any change in the origin of the utility function's.

Definition 3. ***Most efficient SPS allocation:*** *For all $U_0, U_1 \in \mathbb{D}^n$, and $\exists \hat{U} = \{\hat{u}_1, \hat{u}_2, .., \hat{u}_n\} \in \mathbb{D}^n$ which is obtained from U_1 such that $\hat{U} \succ_{SPS} U_0$, then $\hat{U}$ is said to be the most efficient SPS allocation if and only if $\hat{U}$ is a feasible SPS allocation and $\hat{U}$ is Pareto superior to any other feasible SPS allocation.*

Our next result shows a necessary and sufficient condition for any feasible SPS allocation to become the most efficient SPS allocation. Formally,

Lemma 1. *For all $U_0, U_1 \in \mathbb{D}^n$, and $\exists \hat{U} = \{\hat{u}_1, \hat{u}_2, .., \hat{u}_n\} \in \mathbb{D}^n$ which is obtained from U_1 such that $\hat{U} \succ_{SPS} U_0$, then $\hat{U}$ is the most efficient SPS allocation if and only if $\sum\limits_{i=1}^{n} \hat{u}_i = \sum\limits_{i=1}^{n} u_{1i}$.*

Proof: We prove both the if and the only part considering method of contradiction.

Only if

Let $\sum\limits_{i=1}^{n} \hat{u}_i \neq \sum\limits_{i=1}^{n} u_{1i}$. Now if $\sum\limits_{i=1}^{n} \hat{u}_i > \sum\limits_{i=1}^{n} u_{1i}$ the SPS allocation is infeasible, hence $\hat{U}$ is not the most efficient. On the other hand if $\sum\limits_{i=1}^{n} \hat{u}_i < \sum\limits_{i=1}^{n} u_{1i}$, we can always construct another feasible SPS allocation $\hat{Z} = \{\hat{z}_1, \hat{z}_2, .., \hat{z}_n\} \in \mathbb{D}^n$ where $\hat{z}_i = \hat{u}_i + \left(\frac{\sum_{i=1}^{n} u_{1i} - \sum_{i=1}^{n} \hat{u}_i}{\sum_{i=1}^{n} u_{0i}}\right) u_{0i}$. Clearly $\hat{Z}$ is Pareto superior to $\hat{U}$.

If

We begin with the assumption that there exists a feasible $\hat{V} = \{\hat{v}_1, \hat{v}_2, .., \hat{v}_n\} \in \mathbb{D}^n$ such that $\hat{v}_i = \hat{u}_i + \kappa.u_{0i}$ where $\kappa > 0$. Clearly both $\hat{V}$ and $\hat{U}$ are SPS allocations and $\hat{V}$ is Pareto superior to $\hat{U}$. Since $\hat{V}$ is feasible, this implies $\sum\limits_{i=1}^{n} \hat{v}_i \leq \sum\limits_{i=1}^{n} u_{1i} \implies \sum\limits_{i=1}^{n} \left(\hat{u}_i + \kappa.u_{0i}\right) \leq \sum\limits_{i=1}^{n} u_{1i}$. Now given $\kappa > 0 \implies \sum\limits_{i=1}^{n} \hat{u}_i < \sum\limits_{i=1}^{n} u_{1i}$ which is a contradiction. **Q.E.D.**

We are now ready to introduce the main result of this paper. By SPS allocations we mean a counterfactual distribution which has same inequality as the initial distribution and also is Pareto superior to the initial distribution. Obviously such a distribution will never exist if there is aggregate loss in the society. This is because the net loss must make at least one individual worse off and eventually there does not exist any feasible Pareto Superior allocation. However, if there is net gain a SPS allocation can be obtained by taxing a subgroup of individual and transferring the collected tax to the rest of the population. Our next result characterizes the tax-transfer vector that can be associated with a most efficient SPS allocation, in the following fashion:

Proposition 1. *For all $U_0, U_1 \in \mathbb{D}^n$, $\exists \hat{U} = \{\hat{u}_1, \hat{u}_2, .., \hat{u}_n\} \in \mathbb{D}^n$ which is obtained from U_1 such that $\hat{U} \succ_{SPS} U_0$ and $\hat{U}$ is the most efficient SPS allocation, if and only if $\sum_{i=1}^{n} u_{1i} > \sum_{i=1}^{n} u_{0i}$ and the tax-transfer vector is $T = \{T_1, T_2, .., T_n\} = \{u_{11}-\hat{u}_1, u_{12}-\hat{u}_2, .., u_{1n}-\hat{u}_n\}$, where $T_i = u_{1i}-u_{0i}\left(\frac{\sum_{i=1}^{n} u_{1i}}{\sum_{i=1}^{n} u_{0i}}\right)$, $\forall i \in \{1, 2, .., n\}$.*

Proof: Only if
Given $\hat{U} \succ_{SPS} U_0 \implies \exists \theta$ such that $\hat{u}_i = \theta.u_{0i}$ $\forall i \in \{1, 2, .., n\}$ and $\theta > 1$. Now $\hat{U}$ is most efficient SPS $\implies \sum_{i=1}^{n} \hat{u}_i = \sum_{i=1}^{n} u_{1i} \implies$.

$$\theta = \frac{\sum_{i=1}^{n} u_{1i}}{\sum_{i=1}^{n} u_{0i}} \tag{3}$$

Now $\theta > 1 \implies \sum_{i=1}^{n} u_{1i} > \sum_{i=1}^{n} u_{0i}$. Putting $\hat{u}_i = \theta u_{0i} = u_{0i}\frac{\sum_{i=1}^{n} u_{1i}}{\sum_{i=1}^{n} u_{0i}}$ in the elements of T, (i.e., $T_i = u_{1i} - \hat{u}_i$), we can write $T_i = u_{1i} - u_{0i}\left(\frac{\sum_{i=1}^{n} u_{1i}}{\sum_{i=1}^{n} u_{0i}}\right)$.

If

Given T_i we can write

$$\hat{u}_i = u_{0i}.\left(\frac{\sum_{i=1}^{n} u_{1i}}{\sum_{i=1}^{n} u_{0i}}\right) \tag{4}$$

Furthermore, it is also given that $\sum_{i=1}^{n} u_{1i} > \sum_{i=1}^{n} u_{0i} \implies \hat{u}_i > u_{0i}$. Since, 4 is satisfied implies the distributions $\hat{U}$ and U_0 have same inequality, following any relative inequality measure satisfying property 2. Furthermore, from equation 4 we have $\sum_{i=1}^{n} \hat{u}_i = \sum_{i=1}^{n} u_{1i}$ Combining these three arguments it is straightforward to write that $\hat{U} \succ_{SPS} U_0$ and $\hat{U}$ is also most efficient. **Q.E.D.**

The tax transfer vector for the construction of the SPS allocation is infinite. However, it is unique for the most efficient SPS allocation. This is illustrated formally in our next result.

Proposition 2. *Given Proposition 1, the most efficient SPS allocation $\hat{U}$ and the associated Tax transfer vector T is unique.*

Proof: We begin with the assumption that there exists any arbitrary $\tilde{T} = \{\tilde{T}_1, \tilde{T}_2, .., \tilde{T}_n\} \in \mathbb{R}^n$ and a counterfactual distribution $\hat{V} = \{\hat{v}_1, \hat{v}_2, ..., \hat{v}_n\} = \{u_{11} - \tilde{T}_1, u_{12} - \tilde{T}_2, .., u_{1n} - \tilde{T}_n\} \in \mathbb{D}^n$, such that $\tilde{T} \neq T$ and $\hat{V}$ is also a most efficient SPS allocation. Now any two vectors of the same order are related in the following fashion: $\tilde{T} = T + \epsilon$ where $\epsilon = \{\epsilon_1, \epsilon_2, .., \epsilon_n\} \in \mathbb{R}^n$. Since $T \neq \tilde{T} \implies \exists i \in \{1, 2, .., n\}$ such that $\epsilon_i \neq 0$. Clearly, inequality in $\hat{V}$ is same as U_0 following 2 if and only if $\exists \alpha (\alpha \in \mathbb{R})$ such that $\epsilon_i = \alpha u_{0i}$, $\forall i \in \{1, 2, .., n\}$. Hence we can write $\hat{v}_i = u_{0i}.\bigg(\theta - \alpha\bigg)$, where $\theta = \frac{\sum_{i=1}^{n} u_{1i}}{\sum_{i=1}^{n} u_{0i}}$. Clearly if $\alpha > 0$ the allocation is a feasible SPS but is not the most efficient. On the other hand if $\alpha < 0$ then $\sum_{i=1}^{n} \hat{v}_i > \sum_{i=1}^{n} u_{1i} \implies$ the allocation is not feasible and eventually is not the most efficient. Hence, $\alpha = 0 \implies T \equiv \tilde{T}$ and $\hat{U} \equiv \hat{V}$. **Q.E.D.**

3 Conclusion

We have extended the basic Pareto principle to focus on inequality-neutral or distribution neutral Pareto superior allocation which we call strongly Pareto superior or SPS allocation which guarantees higher individual welfare keeping the degree of inequality same as before. We have shown that whenever there is aggregate gain in the society we can compute a counterfactual distribution obtained by taxing a subgroup of population and redistributing the collected tax to the rest of the population such that the counterfactual allocation is a SPS allocation. In the counterfactual distribution the aggregate gains of utility has been redistributed among the individuals in the proportionate to their utilities of the initial distribution. This keeps the inequality level same and also ensures that the SPS is feasible and is the most efficient one.

A future research problem in this direction is to compare inequality between the counterfactual and the final distributions.

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