A Model of Growth Based on Knowledge Diffusion Within Firms

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Abstract

A model of economic growth is presented in which firm productivity is encoded in manager's knowledge. Knowledge is subject to random shocks and is transmitted to some workers, who then become managers. Some managers start their own firms. In the competition for labor, the most productive firms are favored, so average productivity grows over time. The model predicts that the firm-size distribution rapidly converges to a Pareto distribution with exponent close to one, as seen by researchers looking at firm-size data. The model also predicts the existence of "gazelles": firms that grow rapidly from birth, gradually decelerating as they become large. Finally, the model predicts a scale effect that is strong in small economies but very weak in large economies.

Keywords: Growth, Diffusion of knowledge, Scale effect

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1 Introduction

Economic growth is the outcome of incremental improvements in productivity. Large numbers of people strive against the odds to improve their economic standing by thinking of possible ways to increase output. By the sheer weight of numbers, some surely succeed. Assume there is a mechanism of knowledge transmission between people that favors the most productive ideas. Given such a mechanism even random ideas will be filtered so that output ratchets upward, one positive shock at a time.

This paper presents a model of growth in which productivity is encoded in manager's knowledge within firms. Managers randomly come up with new ideas, and laborers learn those ideas, as in the classic master-apprentice relationship. Once knowledge is transmitted from managers to laborers, laborers become managers, who then come up with new ideas. A selection mechanism is proposed by which the rate of learning is increasing in the productivity associated with the manager's ideas. The combination of random idea generation and selective learning leads to economic growth.

The details of the model are as follows. There are two factors of production, labor and management. The basic unit of production, henceforth called a 'unit' consists of a single manager and several laborers. Firms are collections of units (see Figure 1). Each unit produces homogeneous output that is immediately consumed by the manager and laborers. The productivity of a unit is encoded in the knowledge of the manager, who directs the activities of the laborers. There are diminishing returns to labor and the labor market is competitive, hence laborers earn a market wage. Managers earn the output of their units net of wages paid to labor. Managers choose the number of laborers that maximizes this net income. As a result, the number of laborers in a unit is an increasing function of the productivity of that unit. The model abstracts from the existence of firm owners separate from managers, and hence ignores income-smoothing arrangements that probably exist in the real world, such as managers earning a salary and owners making a profit.[1]

Let us now consider for a moment an economy where firms contain just one production unit. Then we can equate firms with units and just talk about the dynamics of units. The log-productivity of a unit can be pictured as a rung on a quality ladder (Aghion & Howitt, 2001). During each interval of time,

[1] So far, this model is reminiscent of Lucas (1978).

a manager gets a new independent idea, which causes the log-productivity of the unit to either ascend to the next higher rung with some probability p, or to descend to the next lower rung with probability $1 - p$.[2] At each point in time a manager must rationally decide whether to continue managing, or to stop managing and join the labor market. The optimal rule is to exit when income is sufficiently below the market wage that the value of waiting for a future productivity boost is less than zero.

The mechanism of learning is as follows. In the course of working, laborers passively gain access to the knowledge of their managers. As a result, some laborers become managers, who hire new laborers and create new units. The rate of production of managers is an increasing but concave function of the number of laborers in the unit. The concavity arises because laborers (students) are sharing the attention of a single manager (teacher). Since the number of laborers in a unit is increasing in the unit's productivity, the rate of production of new managers is also increasing in the unit's productivity. Hence there is selection for the most productive ideas.

In the case where production is Cobb Douglas and where the concavity of the learning function takes the form of a log function, the above model is analytically tractable. The economy gravitates to a stable path where the rate of production of new units equals the rate of exit of unproductive units, and the distribution of log-productivity across units takes the form of a traveling wave, representing a growing economy. Surprisingly, a calibration to U.S. data suggests that a wage growth-rate of 2% per annum is consistent with *zero-mean* productivity shocks. Given the growing market wage, a manager needs to be exceptionally lucky to maintain a given level of profit for very long.[3]

In the above discussion, units are treated as though they are small independent businesses and all managers are entrepreneurs, coming up with new ideas on their own. But within *firms*, managers do not act independently. Let us define a firm to be a collection of units that experience perfectly correlated productivity shocks. Let us assume that most new managers stay in their "parent" firms and hire new laborers, which leads to firm growth.

[2] According to Harberger (1998), declines in productivity are common. For example, they can occur because of changing factor prices.

[3] The Red Queen said to Alice "Now, *here*, you see, it takes all the running you can do, to keep in the same place." (Carroll, 1871).

In addition, let's assume that a small (fixed) percentage of new managers become independent entrepreneurs and launch spinoff firms with initial productivity equal to that of their parents. The reason that we need spinoffs is that without them the economy would eventually be dominated by a single highly-productive monopoly. An implication of this model is that fast-growing firms produce more spinoffs than slow-growing firms, consistent with the data (Klepper & Sleeper, 2005, Franco & Filson, 2006).

The proposed model of firm growth is reminiscent of Edith Penrose (1959). She wrote that firm growth is constrained by the need to transmit internal knowledge by training new managers. Such encoded knowledge can only be absorbed in house, and cannot be acquired by simply hiring managers from the outside. A modern version of this story is that firms must accumulate firm-specific "organization capital" in order to grow (Prescott & Visscher, 1980).

Turning to spinoffs, the assumption that a fixed percentage of new managers leave their firms to start new firms is very simplistic. It is not the intension of the present study to graft an endogenous model of spinoffs, such as can be found for example in Franco & Filson, (2006) or Chatterjee & Rossi-Hansberg (2012), onto the model of growth. Rather, the intention is to recognize the need to include spinoffs in order to prevent concentration, and to include them in as simple a manner as possible. One way to think about spinoffs is that they occur, by definition, when the behavior of a manager is uncorrelated with the behavior of the other managers in the firm. Klepper & Thompson (2010) find that disagreement between managers is an important trigger of spinoffs. In essence, we assume that most people are comfortable in taking directions from others, but a small percentage are stubbornly independent.

In the new version of the model containing firms, the entry of new firms balances the exit of unproductive firms and the distribution of log-productivity across firms is the same as that across units in the previous model. The growth rate is unchanged. Other predictions are as follows. First, newborn firms are small relative to incumbent firms. Second, growth rates generally decline with age. In particular, some firms are "gazelles": they are very productive at birth and grow rapidly, then slow down as they become large. And third, the firm-size distribution rapidly approaches a Pareto distribution with exponent just above one. These predictions are broadly consistent with the data.[4]

The final prediction of the model is that there is a strong scale effect in small economies but a very weak scale effect in large economies. The model predicts that for a typical-sized modern economy, an increase in population by a factor of ten leads to an increase in the growth rate of only 0.2%. The observed scale effect seems to be somewhat higher than this, but there is so much noise that an absence of scale effects is also consistent with the data (Jones, 1995; Dinopoulos & Thompson, 1999). Hence the prediction of a small scale effect in modern economies is consistent with observation given the uncertainty of the estimates.

The concave scale effect deserves some explanation. Let us assume that the mean of stochastic productivity shocks is zero. If there is only one firm, then stochastic productivity shocks do not lead to long-term growth. Monopoly implies stagnation. If there are several firms, there is competition for labor and selection for the most productive firm. Since the distribution of log-productivity shocks is normal, large positive draws are unlikely. But the more firms there are, the more likely that *one* of them will experience a large positive shock. Hence there is a strong scale effect. This argument is a variation on the idea that the larger the population, the more likely that a genius will be born.[5] However, when the population of firms is large, the finite rate of diffusion of knowledge puts a brake on the scale effect. Laggard managers, who have not yet been trained in the leading firms, spend time inventing things that have already been invented. Hence the scale effect becomes weak once an economy becomes large.

Related Literature The main theme of the present model is that growth is the outcome of selection operating on random ideas. Nelson & Winter (1982) developed an early and influential contribution to growth theory along these lines. The direct ancestor of the present model is described in Staley (2011) and Luttmer (2012a).[6] In that model there is individual random

[4] Axtell (2001) observed a Pareto distribution with exponent close to one for U.S. firms. See Luttmer (2011) for a discussion of the growth path of large firms.

[5] William Petty, writing in London in the seventeenth century, may have been the first economist to point to the strong scale effect when he said: "...it is more likely that one ingenious curious man may rather be found amongst 4 millions than among 400 persons." (as quoted by Dimoploulos & Thompson, 1999).

[6] That model is in turn based on models of idea flows developed by Alvarez *et al.* (2008) and Lucas (2009).

innovation and imitation across individuals, leading to a traveling wave of productivity. Two similar models, also yielding traveling wave solutions, are König *et al.* (2015) and Luttmer (2015). In the model of König *et al.* firms either innovate or imitate other firms, depending on which option yields the higher expected profit. Luttmer (2015) describes a model with random productivity shocks and knowledge transfer *inside firms*. None of these models predict a Pareto firm size distribution with exponent close to one.[7]

Luttmer (2011) describes a model of firm growth that gives rise to a Zipf distribution. The relevant mechanism is blueprint replication that operates at two speeds, an initial high speed and a subsequent low speed. The combination of an initial high speed and random transition to low speed ensures that the size distribution rapidly approaches the Zipf distribution. The present model works in a similar fashion except that there is a continuum of speeds, each corresponding to a different productivity, and blueprint replication is replaced with internal knowledge transfer between managers and workers.

None of the above models, other than Staley (2011), address the scale effect. The scale effect has a role to play in models of long run growth that incorporate preindustrial Malthusian dynamics, the industrial revolution, and modern growth. Jones (2001) assumes an accelerating rate of idea creation as the population grows in early times, based on data in Kremer (1993). Galor (2011) utilizes a scale effect to explain the takeoff from preindustrial stagnation to modern growth. Romer (1990), Grossman & Helpman (1991), Aghion & Howitt (1992) describe endogenous growth models that have strong scale effects. Jones (1995) and Dimopoulos & Thompson (1999) show that there is little evidence for scale effects in modern economies. Jones' critique of the scale-effect prediction of early endogenous growth models prompted the development of second-generation models that avoided the strong scale effect, either by introducing an expanding number of sectors or by introducing a so-called "weak scale effect" that linked productivity growth to population growth (see Jones, 1999 for a summary). The present model leaves open the possibility of a strong scale effect in small economies such as existed prior to industrialization, but avoids the strong scale effect in large economies without the need for an expanding number of sectors, and without having to link

[7] Several other recent models incorporate diffusion of knowledge but not stochastic generation of new ideas. In those models the growth rate depends on the shape of the existing productivity distribution. Prominent examples are Alvarez *et al.* (2012), Lucas (2009), Lucas & Moll (2014), and Perla & Tonetti (2014).

productivity growth to population growth.

The plan of the paper is as follows. Section 2 presents the model. The model with independent units is first described. Then, a nonlinear partial differential equation is derived for the distribution of productivity and an analytic solution is found. Then, firms are introduced and it is shown that the distribution of firm size is Pareto with exponent close to one. In Section 3 the model is calibrated to U.S. data and several predictions are tested against the empirical data. Finally, Section 4 concludes.

2 The Model

2.1 Independent Units

Production: A unit of production (henceforth a 'unit') consists of one manager and several laborers, as in Lucas (1978). The output y_i of unit i is

$$y_i = A_i \, l_i^{\alpha},$$

where A_i is the productivity of the unit, which is encoded in the ideas of the manager, l_i is the labor employed by the unit, and α is the span of control parameter. The integer i takes on values from 1 to M, where M is the number of managers.

Given a market wage w, each manager maximizes income net of wages π_i:

$$\pi_i = \max_{l_i} \left(y_i - w l_i \right). \tag{1}$$

The supply of labor is L, hence $\sum_i^M l_i = L$. The equilibrium solution for the labor market is

$$l_i = B_i \left(\frac{\alpha}{w} \right)^{\frac{1}{1-\alpha}}, \quad w = \alpha \left(\frac{\sum_i^M B_i}{L} \right)^{1-\alpha}, \tag{2}$$

where we have introduced $B_i = A_i^{\frac{1}{1-\alpha}}$. We will be refering to B_i as the productivity. Finally, we have

$$L + M = N,$$

where N is the total number of people in the economy, which is assumed constant.

Productivity Shocks: The productivity of each unit is subject to independent random shocks as follows

$$\mathrm{d}x_i = \mu\,\mathrm{d}t + \sigma\,\mathrm{d}z_i, \tag{3}$$

where $x_i = \ln B_i$ and $\mathrm{d}z_i \sim N(0, \mathrm{d}t)$. The above equation is used in analysis, but in simulation a quality ladder is implemented using binomial shocks Δx_i over a small time step Δt:

$$\Delta x_i = \begin{cases} \sigma\sqrt{\Delta t}, & \text{with probability } p, \\ -\sigma\sqrt{\Delta t}, & \text{with probability } 1-p, \end{cases}$$

where

$$p = \frac{1}{2}\left(1 + \frac{\mu}{\sigma}\sqrt{\Delta t}\right).$$

This implementation is consistent with (3) for very small time steps and prevents unrealistically large shocks from occurring in the simulation.

Exit: There is a fixed cost w per unit time associated with running a unit. If the manager is the owner of the unit, w is the opportunity cost of being a manager. If the owner is separate from the manger then w is the wage paid to the manager. The owner must decide when to exit from the business. Assuming risk neutral behavior, the value function is (dropping the index i)[8]

$$V(\pi, t) = \sup_{\tau} E_t\left[\int_t^{t+\tau} e^{-ra}[\pi(a) - w(a)]\,\mathrm{d}a\right],$$

where τ is the optimal stopping time and r is the interest rate. We will assume that w grows at the rate g_w (to be determined later). From Equation (1) we have the following useful expression for the profit (dropping time dependence)

$$\pi - w = w\left(e^h - 1\right),$$

where

$$h = x - \frac{1}{1-\alpha}\ln w + \ln\left(\alpha^{\frac{\alpha}{1-\alpha}}(1-\alpha)\right),$$

[8]This section follows Luttmer (2012b), Section 3.3. The technique is described in detail in Dixit & Pindyck (1994).

and from (3) we have

$$\mathrm{d}h = \left(\mu - \frac{1}{1-\alpha} g_w\right) \mathrm{d}t + \sigma \, \mathrm{d}z.$$

Defining $V(\pi, t) = w(t) F(h)$, the Bellman equation is

$$(r - g_w)\, F = \left(\mu - \frac{1}{1-\alpha} g_w\right) \frac{\mathrm{d}F}{\mathrm{d}h} + \frac{\sigma^2}{2} \frac{\mathrm{d}^2 F}{\mathrm{d}h^2} + e^h - 1,$$

which has an exact solution. The stopping point h_e is such that $F(h_e) = 0$ and $\mathrm{d}F/\mathrm{d}h|_{h_e} = 0$, which joins smoothly with the region below h_e where the value function must be zero. The solution implies the following lower bound on profit, which we can call $\pi_e(t)$:

$$\pi_e(t) = w(t) \frac{\xi}{1+\xi} \left(1 + \frac{\frac{g_w}{1-\alpha} - \mu - \frac{\sigma^2}{2}}{r - g_w}\right), \tag{4}$$

where

$$\xi = \frac{-\frac{g_w}{1-\alpha} + \mu}{\sigma^2} + \sqrt{\left(\frac{\frac{g_w}{1-\alpha} - \mu}{\sigma^2}\right)^2 + \frac{2(r - g_w)}{\sigma^2}}.$$

When the profit of a unit falls below the quantity on the RHS of (4), the unit exits and the manager enters the labor market.

Learning: So far, the elements of the model described above can be found in other similar models. The unique element of the present model is the learning mechanism. Given l_i laborers in unit i, the rate of production of new managers is assumed to be $\eta \ln l_i$, where η is a constant. New managers start new units with the same productivity as their former units. If there are n_i units with productivity A_i, the rate of production of units having that productivity is

$$\frac{\mathrm{d}n_i}{\mathrm{d}t} = \nu \, n_i \ln l_i, \quad l_i > 1. \tag{5}$$

The equation captures a crowding effect. Even though each laborer is equally adept at learning, the overall rate of production of new managers is less than linear in the number of laborers. As we will see, the choice of the log function enables an analytic solution to be found for the growth rate and for the distribution of productivity. The condition $l_i > 1$ also turns out to be be satisfied because exit occurs when there is more than one employee.

2.2 Solution - Independent Units

A simulation of the above system of equations shows that the economy gravitates to a balanced growth path in which the distribution of units across log-productivity takes the form of a traveling wave. The shape of the traveling wave is fixed and moves to the right at a constant speed, representing a growing economy. Define $\rho(x, t)$ to be the probability density of units with respect to x at time t. That is, $\rho(x, t)\,\mathrm{d}x$ is the proportion of units having log-productivity between x and $x + \mathrm{d}x$ at time t. From (5) and (2), we can express the learning rule in the following form:

$$\frac{\partial \rho}{\partial t} = \nu(x - x_0)\rho,$$

where

$$x_0 = \frac{1}{1-\alpha} \ln\left(\frac{w}{\alpha}\right).$$

The Kolmogorov forward equation for (3) is[9]

$$\frac{\partial \rho}{\partial t} = -\mu \frac{\partial \rho}{\partial x} + \frac{\sigma^2}{2} \frac{\partial^2 \rho}{\partial x^2}$$

Combining the Kolmogorov forward equation with the learning equation we have the following time-evolution equation for the density function:

$$\frac{\partial \rho}{\partial t} = -\mu \frac{\partial \rho}{\partial x} + \frac{\sigma^2}{2} \frac{\partial^2 \rho}{\partial x^2} + \nu(x - x_0)\rho. \tag{6}$$

This nonlinear partial differential equation exhibits traveling wave solutions, similar to those seen in the study of reaction-diffusion systems in physics.[10]

We can use a trick to find the speed of the traveling wave (the growth rate of the economy) in terms of the parameters of the model.[11] Define new variables

[9] For a discussion of the Kolmogorov forward equation (also called the Fokker-Planck equation), see Cox & Miller (1995).

[10] Reaction diffusion equations are reviewed in Grindrod (1996). A good reference on kinematic wave equations such as Equation (6) is Whitham (1974).

[11] This trick is from Tsimring *et al.* (1996), who use a similar differential equation in their model of RNA virus evolution.

as follows

$$\tilde{x} = \left(\frac{2\nu}{\sigma^2}\right)^{\frac{1}{3}} (x - \mu t), \tag{7}$$

$$\tilde{t} = \left(\frac{\sigma^2}{2}\nu^2\right)^{\frac{1}{3}} t. \tag{8}$$

Equation (6) in terms of the new variables is

$$\frac{\partial \rho}{\partial \tilde{t}} = \frac{\partial^2 \rho}{\partial \tilde{x}^2} + (\tilde{x} - \tilde{x}_0)\rho. \tag{9}$$

In this version of the differential equation there are no parameters. Let's say the speed of the wave in terms of these new variables is s. We can translate s back into the original variables as follows

$$g_x = \mu + \sigma^{\frac{4}{3}} \left(\frac{\nu}{4}\right)^{\frac{1}{3}} s. \tag{10}$$

And from (2) the growth rate of the market wage is $g_w = (1 - \alpha)g_x$.

We still need to get s, and it would be nice to know the shape of the traveling wave. We need to find a solution to Equation (9) of the form $\rho(\tilde{x}, \tilde{t}) = u(\tilde{x} - s\tilde{t})$. Substituting this form into (9) and shifting by $\tilde{x}_0$ we have the following ordinary differential equation

$$0 = s\frac{\mathrm{d}u}{\mathrm{d}z} + \frac{\mathrm{d}^2 u}{\mathrm{d}z^2} + zu, \tag{11}$$

where $z = \tilde{x} - \tilde{x}_0$. The solution is obtained by substituting $u(z) = e^{az} f(z)$. The choice of $a = -s/2$ leads to

$$0 = \frac{\mathrm{d}^2 f}{\mathrm{d}y^2} - yf,$$

where $y = s^2/4 - z$. This is the Airy equation, and the solution is called the Airy function $A_i(y)$.[12] So finally, we have

$$u(z) = Ce^{-s\frac{z}{2}} A_i\left(\frac{s^2}{4} - z\right), \tag{12}$$

[12] See Abramowitz & Stegun, 1964, p. 446. There is another solution B_i that blows up (see Figure 10.7).

where C is a normalizing constant. The solution can be re-expressed in term of the original variables. Figure 2 shows what the shape looks like.

We still don't know the value of s. All that equation (12) tells us is that *any* value of s is admissible and that different values of s give rise to different shapes of the log-productivity density function. The analytic solution, although correct, does not help us to pin down the value of s.

It turns out that the above analytic solution is not the whole story. In the far-right tail of the productivity distribution, the density of units becomes very small and the dynamics is dominated by stochastic effects. Of all the possible speeds, there is only one for which the movements of the body and tail are consistent. There is no exact analytic technique for dealing with this kind of dynamics, so we must resort to simulation to obtain s.[13] In simulation studies, s is an increasing but steeply concave function of the number of units (see Figure 3). This subject will be revisited in the next section.

The above solution is expressed in terms of a density function, but this doesn't tell us how many units there are, i.e. how many managers there are. Simulation studies show that the ratio of managers to laborers converges towards a constant (less than one) and fluctuates around that constant as the economy grows. The number of managers is determined by the balance between the creation of new units via learning and the exit of unproductive units. It is not possible to derive an analytic expression for the ratio of managers to laborers, but simulation can be used to obtain this ratio.[14]

[13]Fisher, a biophysicist, discusses traveling waves of fitness seen in microbial populations and offers the amusing analogy of the random snuffing of an exploring dog's nose: "The balance between the irregular snuffling and the inertial motion of the body determine the overall speed; yet, as the owner of a large, headstrong dog knows, predicting its speed is very hard!" (Fisher, 2011). A different Fisher (1937) addressed this issue in his study of gene propagation. Tsimring *et al.* (1996) and Brunet & Derrida (1997) study similar system to ours and derive approximations for the speed.

[14]One can also numerically integrate Equation (11) between the lower bound on z and infinity to derive a flow balance condition. The lower bound on z is determined by the exit condition (4). The flow balance condition implies a relationship between the market wage and average productivity, which can then be used to numerically derive the ratio of managers to laborers using equation (2).

2.3 Firms

The modern economy contains many small firms that are managed by single individuals, and those firms can be thought of as units. But there are also firms that are run by groups of managers, and some of those firms are very large. Let us now consider an extension of the above model in which units gather together within firms and coordinate their productivity changes. In particular, let us assume that most new managers stay within their parent's firms, and only a small percentage ε break away to start new firms, taking knowledge (encoded productivity) with them.

The production function of a multi-unit firm is

$$Y_i = A_i n_i \left(\frac{L_i}{n_i}\right)^\alpha = A_i n_i^{1-\alpha} L_i^\alpha,$$

where n_i is the number of managers and L_i is the total number of laborers in the firm.

In the previous sections, units were characterized by their productivity. Now firms are characterized by their productivity *and* by the number of units sharing that productivity. Let $f(x, \theta, t)$ be the density of firms at time t with respect to log-productivity x and the log of the number of units $\theta = \ln n$. From (5), and the assumption that a proportion $1 - \varepsilon$ of new managers stay in their parent firms, we have[15]

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = (1-\varepsilon)\nu(x - x_0). \tag{13}$$

Using equations (3) and (13), the two-dimensional version of the Kolmogorov forward equation is

$$\frac{\partial f}{\partial t} = -\mu \frac{\partial f}{\partial x} + \frac{\sigma^2}{2}\frac{\partial^2 f}{\partial x^2} - (1-\varepsilon)\nu(x - x_0)\frac{\partial f}{\partial \theta}. \tag{14}$$

A solution to (14) is

$$f(x, \theta, t) = \exp\left(-\frac{1}{1-\varepsilon}\theta\right)\rho(x, t), \tag{15}$$

[15] Equation (13) implies that the growth rate of firm employment should be proportional to the log of firm productivity. Bottazzi *et al.* (2002) provide evidence of this pattern in Italian manufacturing (see their Figure 10).

where $\rho(x,t)$ satisfies Equation (6). This solution implies that the distribution of productivity across firms is given by (12), the unit-count density function is exponential with exponent $1/(1-\varepsilon)$, and that the number of units in a firm is independent of firm productivity. When the firm-size density is translated into a density with respect to unit count n, one obtains a Pareto density function with exponent $1+1/(1-\varepsilon)$. The cumulative distribution function has exponent $1/(1-\varepsilon)$. This solution is verified in simulation studies.

The growth rate is unchanged from Equation (10) but now the scale factor s is a function of the number of firms N_F:

$$g_w = (1-\alpha)\left\{\mu + \sigma^{\frac{4}{3}}\left(\frac{\nu}{4}\right)^{\frac{1}{3}}\right\} s(N_F). \tag{16}$$

The shape of $s(N_F)$ is shown in Figure 3. There is no analytic formula for s so it must be determined by simulation.

Why does n follow a Pareto distribution with exponent $1/(1-\varepsilon)$? A simplified model of firm-growth can be used to answer this question. Consider a population of firms all having the same growth rate of units $g_n = (1-\varepsilon)g$, where $g = \nu(x-x_0)$. The firms exist in a kind of growth corridor. It is useful to imagine that there are discrete corridors. Let's say there are M managers in some corridor, and the managers are scattered amongst different firms. Firms enter with $n=1$, and the rate of entry is $M\varepsilon g$. The number of units in a firm grows as

$$n = e^{(1-\varepsilon)gt} \tag{17}$$

Now consider a group of firms within the corridor having unit count between n and $n+\mathrm{d}n$. Differentiating (17) we have $\mathrm{d}t = \mathrm{d}n/\left[n(1-\varepsilon)g\right]$. So all the firms with unit count between n and $n+\mathrm{d}n$ must have originated from a pool of new firms of count $M\varepsilon g\,\mathrm{d}n/\left[n(1-\varepsilon)g\right]$. Let us also assume that firms get bumped out of the growth corridor at some hazard rate h due to stochastic productivity shocks and the increasing market wage. Call the (non-normalized) density of firms within the corridor $f(n)$. Then the number of firms having a unit count between n and $n+\mathrm{d}n$ must be

$$f(n)\mathrm{d}n = M\varepsilon g\frac{\mathrm{d}n}{n(1-\varepsilon)g}e^{-ht}.$$

Substituting for t from (17) we have

$$f(n) = M\frac{\varepsilon}{1-\varepsilon}n^{-\left(1+\frac{h}{(1-\varepsilon)g}\right)}.$$

The size density function will become stationary when the hazard rate h takes a value such that the flow of units into the corridor, both from entry and from firm growth, is offset by the loss of units due to entire firms leaving the corridor. This occurs when $h = g$. Hence the steady-state density function of n is Pareto with exponent $1 + 1/(1 - \varepsilon)$. The cumulative distribution function is Pareto with exponent $1/(1 - \varepsilon)$.

Define the size of a firm as the total employment $E = n + nl$. Given that the distribution of n is independent of the distribution of l (via x) it is easy show that E also follows a Pareto distribution with exponent $1/(1 - \varepsilon)$.[16] Figure 4 shows what the distribution looks like. This result confirms what is seen in the data for U.S. firms (Axtell, 2001). A value of ε around 0.06 seems to fit the data well.

Since the growth rate of the number of units is a function of productivity, the independence of size and productivity implies that the growth rate of firms is *cross-sectionally* independent of size, which is a form of Gibrat's law (Gibrat, 1931). The evidence for Gibrat's law is mixed, but it is generally accepted as a first-order approximation, especially for large firms. Santarelli *et al.* (2006) provide a comprehensive survey. Note that this result does not imply that firm growth rates are independent of size for a given firm through time. In fact, given that x_0 rises over time due to the growth of the market wage, we expect from Equation (13) that the growth rate of a given firm will show some momentum (positive autocorrelation) but slow down over time.

The autocorrelation prediction seems to be true for established firms but not for young firms, however the prediction that growth rates decline with age is well verified by the data (Coad, 2007). A striking example of the latter is the existence of "gazelles". Firms that grow rapidly right from birth, and gradually slow down as they age. They are predicted to be rare since the density of firms in the right tail of the productivity density function is small. Classic examples of technology gazelles are Intel and AMD, which were spawned by Fairchild Semiconductor, itself a gazelle (Klepper & Thompson, 2010).

[16] In the same vein, the quantity $An^{1-\alpha}$ is Pareto distributed. So if "management" is a hidden variable, productivity may be Pareto distributed as observed by König *et al.* (2015) and Del Gatto (2006).

Parameter	Value	Notes
α	0.813	Atkeson & Kehoe (2005), with α transformed according to Equation (18).
L	64 million	Toossi (2002): American labor force in 1950.
ε	0.06	Pareto exponent as seen by Axtell (2001).
r	0.1	Arbitrary. The interest rate only affects the point of exit.
ν	0.047	Calibrated so that the firm entry rate is 10% per annum as seen in U.S. census data. See Luttmer (2011) and Chatterjee & Rossi-Hansberg (2012).
μ	0	See text.
σ	0.18	Davis *et al.* (2007), figure 2.5. Values range between 0.1 and 0.2. A value of 0.18 is consistent with a GDP per capita growth rate of 2% per annum.

Table 1: *Parameters used in the simulation studies.*

3 Calibration and Tests

The model has been calibrated to data on the U.S. economy in the twentieth century. The goal was to test Equation (16) for the growth rate. Table 1 lists the resulting parameters. A few notes on the calibration are in order. To obtain α, the model of Atkeson & Kehoe (2005) was utilized. In their model, output takes the form

$$Y = An^{1-\alpha}\left(K^{\beta}L^{1-\beta}\right)^{\alpha}$$

where $\alpha = 0.85$ is the span of control parameter, and $\beta\alpha = 0.199$ is the share of physical capital. Assuming a perfectly elastic supply of capital, or alternatively a constant savings rate, one can eliminate physical capital to obtain a production function of the form used in the present model. The resulting share of labour is

$$\alpha' = \frac{(1-\beta)\alpha}{(1-\beta\alpha)} = 0.813. \tag{18}$$

The labor force L determines the scale factor via N_F, as in Figure 3. The labor force in 1950 was chosen. Given the Pareto form for the firm-size distribution, the relationship between L and N_F is linear. A log-linear approximation for the scale effect (determined by regression, $R^2 = 0.92$), applicable for most countries is

$$s(N_F) \approx 0.19 \ln N_F + 2.4.$$

Combining this with the relationship between L and N_F, also obtained numerically, we have

$$s(L) \approx 0.19 \ln L + 1.4. \tag{19}$$

Th parameter ν was calibrated to the entry rate of firms, which is about 10% per year (Luttmer, 2011, Chatterjee & Rossi-Hansberg, 2012). The entry rate is proportional to $\epsilon\nu$, and ϵ is already determined by the observed exponent of the firm-size distribution. There is no analytic expression for the entry rate, so simulation was used. The parameter ν was adjusted until the entry rate was seen to be 10%.

The drift rate of productivity shocks, μ, was set to zero. The was partly to see if we could calibrate the model to U.S. data assuming a purely "Darwinian" economy where the only driver of growth is selection. According to Foster *et al.* (2005), the retail trade segment of the U.S. economy has been very Darwinian in recent decades, the most obvious example being the spread of Wal-Mart units at the expense of local shops. Baily, Hulten & Campbell (1992) find that about half of productivity growth in U.S. manufacturing in the 1980s can be attributed to factor reallocation, or selection.

The last parameter to calibrate in Equation (16) was σ. To calibrate σ a link between productivity growth and size growth was used. From Equation (2), the volatility of labor changes (per unit) must be the same as the volatility of B, which is σ. So we can directly use time-series statistics on firm growth rate volatilities to estimate σ. In Davis *et al.* (2007), Figure 2.5, values of σ between 0.1 and 0.2 are shown for the period 1981 to 1996, with the higher values occurring in the earlier part of the period. Figure 5 shows the dependence of g_w on σ given the other parameters calibrated as above. This figure shows the interesting tradeoff between growth and stability. If $\sigma = 0.18$ (growth volatilities in the early 1980s), the growth rate of the market wage is about 2% per year, as seen during the twentieth century.

Given the above parameters, we can now check that the condition $l_i > 1$ in Equation (5) is satisfied. The smallest firm in the simulation has 4.4 laborers and one manager.[17]

[17]The simulation allows non-integer numbers of laborers, which implies part-time workers. The simulation also allows for a non-integer number of managers in a firm. The

	Slope: g_w **vs** $\ln(L)$*	**Intercept***	**Notes**
Model Prediction	0.0008	0.006	Based on parameters in Table 1. See Figure 3.
Cross-Country Regression	0.0018 $(-0.0007, 0.004)$	-0.009 $(-0.048, 0.029)$	$R^2 = 0.015$. 135 countries. The growth rate of GDP per capita, $1990-2000$, is a proxy for g_w (Maddison, 2008). Labor statistics for 1990 are from The World Bank.
US Historical Regression, 1820-2000	0.0029 $(-10^{-5}, 0.0058)$	-0.033 $(-0.083, 0.018)$	$R^2 = 0.22$. 18 decades. Population and real GDP data from Maddison (2008). Historical labor participation rates are from Carter (2003).

* Quantities in brackets are the boundaries of the 95% confidence intervals.

Table 2: *Measuring the scale effect.*

The firm-size distribution rapidly converges to the Pareto distribution with exponent $1/(1-\epsilon)$. The numbers shown in Figure 4 were obtained after only one hundred years of simulation, starting from firms having equal size. In order to get such rapid convergence, the gazelles must grow rapidly. In the simulation, the largest 0.5% of firms had an average growth rate since birth of 15% per year. This is close to the value of 18% reported by Luttmer (2011).

Finally, we can test the scale effect. From Equations (16) and (19) we have

$$g_w = 0.0008 \ln L + 0.006$$

Two tests are shown in Table 2. The first is a cross-country regression and the second is a times-series regression for the U.S., covering the years 1820 to 2000. In both regressions a positive relationship between the growth rate and size is detected, but the statistical uncertainty is so large that a null hypothesis of no relationship cannot be rejected at the 95% level. The predicted slope of the scale effect from our model fits comfortably within the

interpretation is that as people learn they spend part of their time managing and part of their time "doing".

95% confidence intervals of the empirical slopes. So we can conclude that the predicted scale effect is too small to be seen.

4 Summary and Conclusions

This paper has presented a simple model of growth based on random innovation and selection. The core assumptions are as follows. First, the productivity of a unit of production is a function of the ideas held by the manager of that unit. Second, the span of control of a manager is an increasing function of the productivity of the unit. Third, managers come up with new new ideas that affect productivity and they stop being managers if productivity falls too low. And fourth, laborers passively learn from their managers and become managers. The resulting economy grows at a steady rate, there are constant inflows of new firms and outflows of unproductive firms, and the distribution of productivity across units forms a traveling wave.

A modified version of the model incorporates firms. A firm is a collection of units that experience perfectly correlated productivity shocks. Most new managers stay in their firms and hire new laborers, leading to firm growth, but a small percentage start new firms. In this version of the model the size distribution of firms becomes Pareto with exponent close to one, as seen empirically. Some firms grow especially fast, but slow down as they age. These correspond to "gazelles" and are needed to ensure fast convergence to the Pareto distribution.

There are two new predictions of the model that require verification. The first is that the growth rate of an economy is increasing in the volatility of firm employment growth rates. Second, there is a concave scale effect that is strong in small closed economies, but very week in large modern economies. The last prediction of a weak scale effect is consistent with observation, but the uncertainty is so large that the effect is difficult to confirm.

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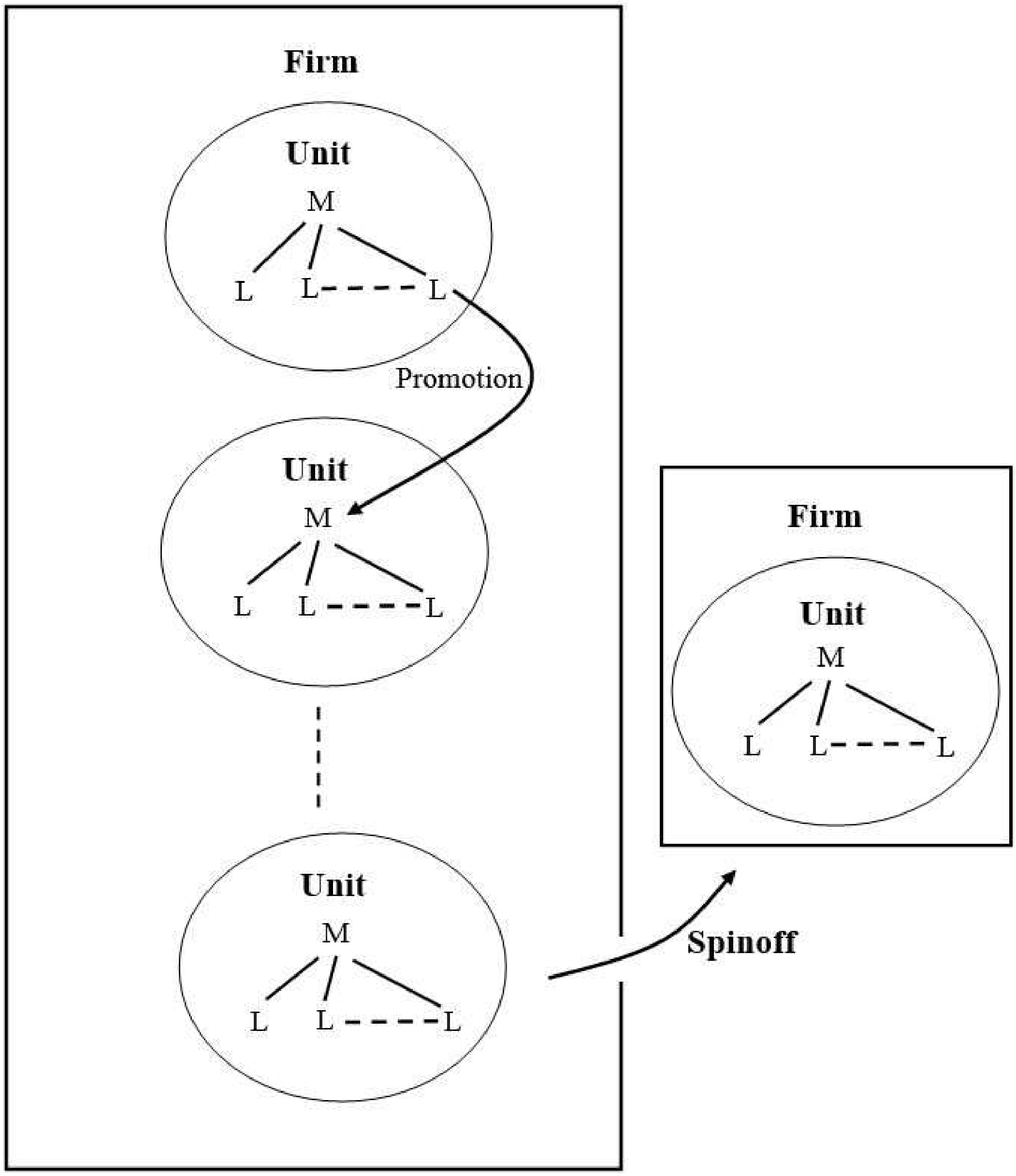

Figure 1: ***Firm Growth and Spinoffs.*** *The letter 'M' stands for manager, and the letter 'L' stands for laborer. Laborers are promoted to managers, who start new production units and hire new laborers. Some managers leave their firm to start new spinoff firms, initially containing one production unit.*

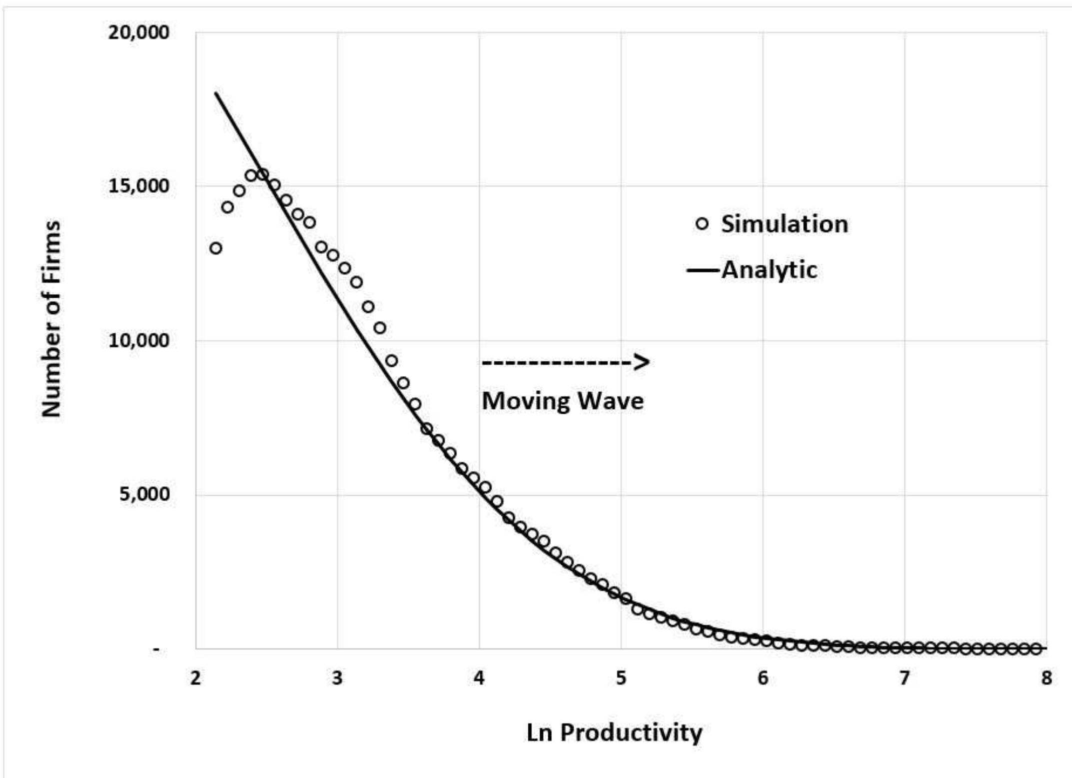

Figure 2: ***Productivity Moving Wave.*** *The shape of the density function of log-productivity maintains its shape and moves to the right at a constant speed. Simulation result are shown after* 100 *years using a time step of* 1/40 *years. The analytic solution is given by Equation* (12)*. Parameters are listed in Table 1.*

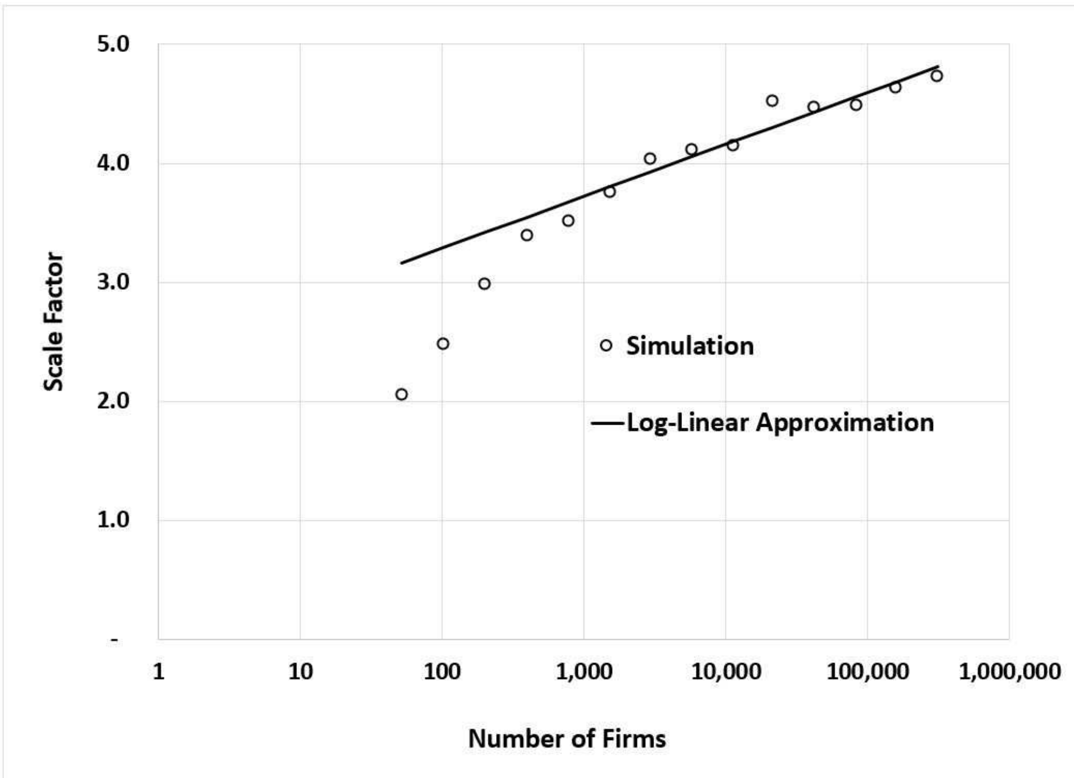

Figure 3: ***Logarithmic Scale Effect.*** *The log-linear approximation, Equation (19), captures the dependence of the growth rate on* $\ln(N)$ *for typical countries. Simulation results are based on averaging over* 100 *seeds of the random number generator.*

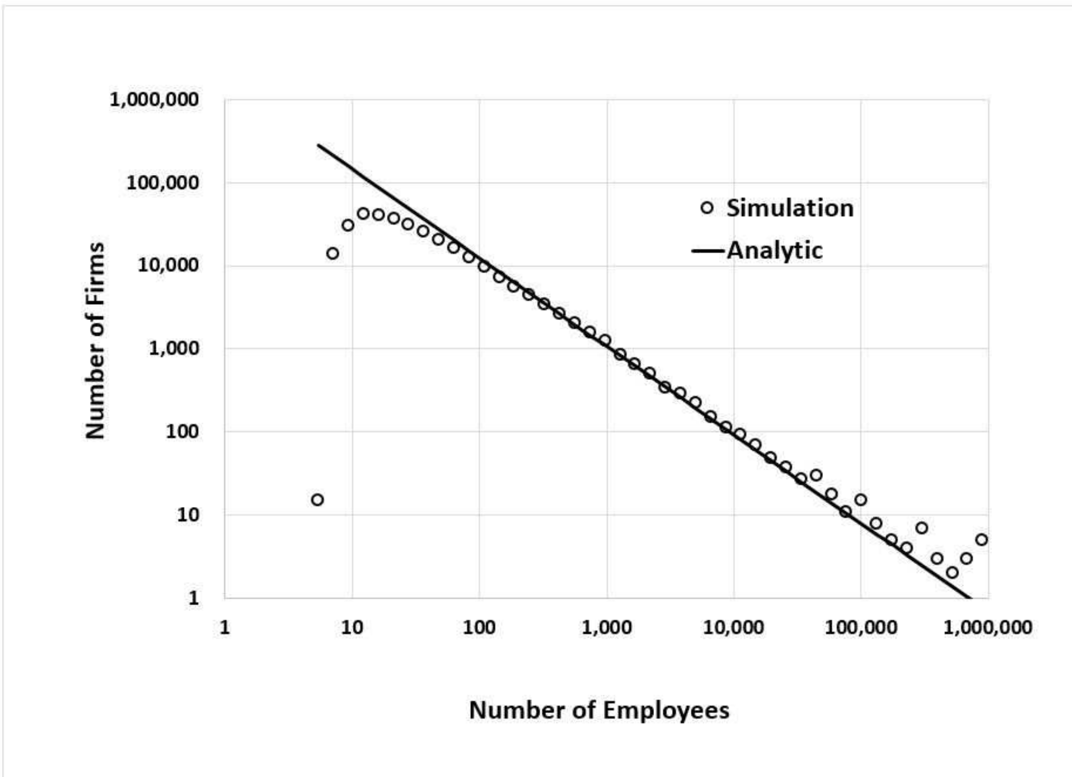

Figure 4: ***Firm Size Density.*** *The simulation starts with 100,000 firms having equal productivity, a labor force of* 64 *million, and* $\varepsilon = 0.06$*. After* 100 *years, the log-size density function (circles) is close to exponential with exponent* $-1/(1 - 0.06)$ *(solid line). Simulation time step is* $1/40$ *years. Parameters are listed in Table 1.*

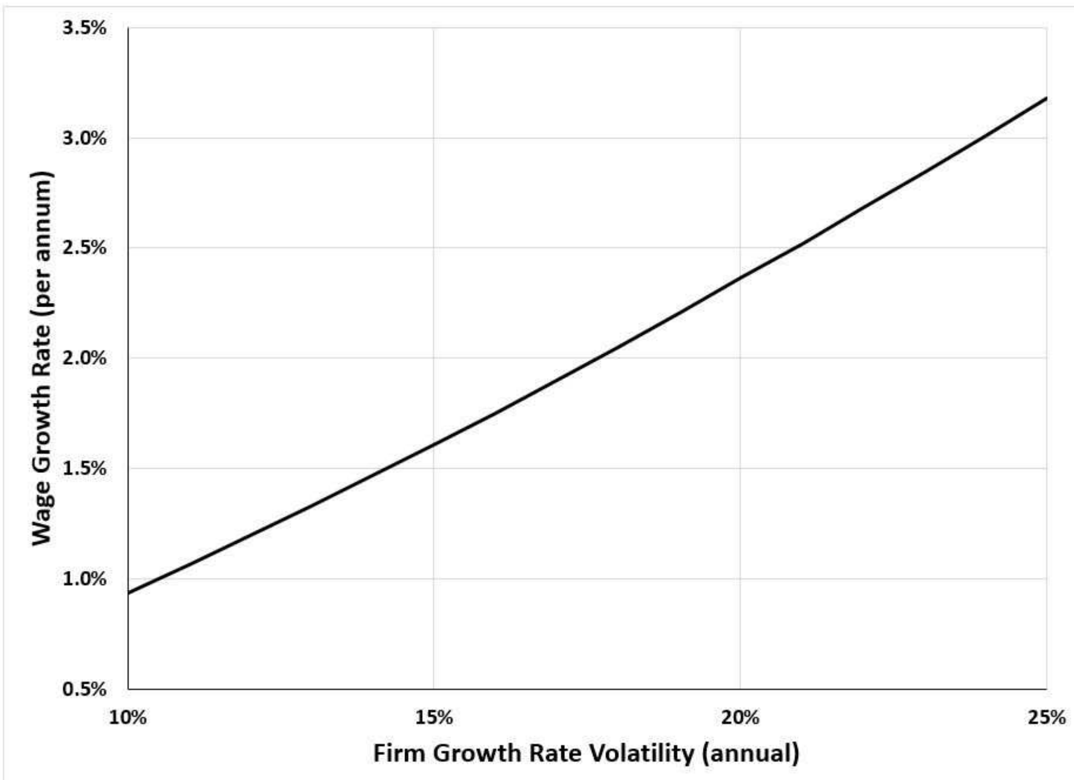

Figure 5: ***Growth rate dependence on*** σ***.*** *There is a tradeoff between growth and employment volatility. The larger the amount of "noise" in productivity shocks, the higher the firm growth volatility, but the higher the growth rate of the economy.*