Equilibrium Concepts in the Large Household Model

Zhu, Tao

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Tao Zhu[*]

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Abstract

This paper formulates equilibrium concepts in the large (non atomic) household model under the team interpretation, characterizes a class of equilibrium allocations, explores whether an equilibrium allocation in the large-household model has a foundation in the finite-household model, and establishes the existence of equilibrium allocations generated by generalized Nash bargaining.

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1 Introduction

Search models play a dominant role in labor economics and a prominent role in monetary economics. In such models, meeting-specific shocks are obvious sources of heterogeneity in wealth. Because such heterogeneity precludes closed-form solutions, efforts have been made to create models in which equilibria have degenerate distributions of wealth. One such model is the so-called *large-household* model initiated by Merz [11] in labor economics and by Shi [15] in monetary economics. In this model, each household consists of a nonatomic measure of agents and each agent from a household meets someone from outside the household—a firm in [11] or an agent from another household in [15]. If all households start with the same wealth, then it is feasible for all households to experience the same distribution of meeting

[*]Department of Economics, Cornell University, Email: tz34@cornell.edu.

outcomes, and, by a law of large numbers argument, to end up with the same wealth. But feasibility should not be confused with equilibrium.

There are two conceivable interpretations of the household construct, each of which leads to a distinct notion of equilibrium. A household can be viewed as a *team* in the sense of Marschak and Radner [10]: each agent in the household is a decision maker, but all share the same objective function. Alternatively, a household can be viewed as a programmer and the agents as automata: the programmer, the unique decision maker, chooses what the agents do in pairwise meetings. Here, I adopt the team interpretation. It permits us to adopt an off-the-shelf equilibrium concept that provides a unified treatment—one that applies independent of the size of the household. In particular, the standard single-agent household is a special case. In contrast, as is well known, the programmer-automata interpretation turns a sequential game into a simultaneous-move game,[1] and, consequently, as demonstrated in appendix II, gives rise to a plethora of equilibria.[2]

Although not made explicit, the team interpretation seems to be the interpretation [11, 15]. However, the authors do not correctly treat the implied interdependence of trading outcomes among members of the same household. Rauch [13], in a lengthy comment on [15], pointed out the problem.[3] But his suggested correction is incomplete. In particular, neither he nor the authors whose work he criticizes present a complete and consistent definition of equilibrium. The missing ingredient is a complete and consistent description of the agent's off-equilibrium payoffs, and, as a consequence, they do not prove that an equilibrium exists.

Here I work with the team notion in the context of a money model. In order to distinguish the role of *large* to ensure the *feasibility of degeneracy* from the role of *large* to produce the *linearity of payoffs* to a household member (linearity is implied by the infinitessimal contribution of that member to the household's money holdings), I set up the model in a way that the feasibility

[1] As a simple example, think about two players play an ultimatum game through automata. Also, see Abreu and Rubinstein [2, pg. 1256] for a discussion in the context of the prisoner's dilemma game.

[2] Following Shi [16], recent literature (e.g., Head and Shi [16], Shi and Wang [17]) seems to adopt the programmer-automata interpretation, but fails to recognize the implied multiplicity of equilibria.

[3] The problem in [11], as far as I know, has not been explicitly pointed out, but it appears to be recognized by researchers. Seemingly to avoid this problem in an application, for instance, den Haan, Ramery and Watson assume that the utility function of each household member depends on the household's aggregate consumption (see [3, footnote 7]).

of degeneracy does not depend on the size of the household. In particular, search is directed—a buyer always meets a seller, and buyers and sellers in each household are indexed by the same set. Then I define equilibrium (section 3), characterize a class of equilibrium allocations (section 4), explore whether an equilibrium allocation in the large-household setting has a foundation in the finite-household setting (section 5), and prove the existence of equilibrium allocations generated by generalized Nash bargaining (section 6).

One remarkable existence result is the equilibrium in which agents from the same large household do not jointly deviate (Proposition 4). The essential role of *large* in this equilibrium is to produce the above indicated linearity, the exact role that quasi-linearity plays in the single-agent household model of Lagos and Wright [9]. As it turns out, the in-equilibrium trade of this equilibrium is the one purported by some of the literature, so after suitable reformulation results in this part of literature may be justified. My proof technique, however, does not cover all existing applications. Indeed, my equilibrium definition does not permit a general approach to existence. As a matter of fact, the feasibility of degeneracy alone does not ensure existence of certain type of degenerate equilibrium (e.g., Proposition 6 (ii)). Therefore, I do not intend to justify the entire literature.

2 The physical environment

Time is discrete. There is a non atomic measure of infinitely lived households. All households are ex ante identical. Each household is identified with a probability space $(I, \mathcal{I}, \mu)$: there are a set of buyers and a set of sellers in the household, both indexed by I. The set I is either finite with n elements or uncountable infinite. If the former, then μ is uniform over I and the household is referred to as *the finite household*; if the latter, then μ is non atomic and the household is referred to as *the large household*.[4]

There is one produced and perishable good per date. At each date, agents from different households are matched in pairs; matching is random but in a way that a buyer always meets a seller.[5] In each meeting, the buyer can

[4] The household with n buyers and n sellers can be viewed as the n-fold replica of the household with 1 buyer and 1 seller; the large household can be viewed as the limit of the finite household as $n \to \infty$. This is analogous to treating the large economy as the limit of the sequence of replica economies (see Hildenbrand [5, Ch 2]).

[5] Search is not directed in [13, 15] so the household must be large to make degeneracy feasible.

consume but cannot produce the good; the seller can produce but cannot consume the good; and the good produced must be consumed in the end of the meeting.[6] Agents from the same household cannot communicate in pairwise meetings.

Agents from the same household share the same objective with the period return in form of

$$\int_{i\in I} u(q_b^i)\mu(di) - \int_{i\in I} c(q_s^i)\mu(di),$$

where q_b^i is consumption of buyer i from the household, and q_s^i is production of seller i from the household. Each agent as an independent decision maker (or player) maximizes expected discounted utility with discount factor $\beta \in (0, 1)$. As is standard, u is bounded, $u' > 0$, $u'' < 0$, $u(0) = 0$, and $u'(0) = \infty$; it is without loss of generality to set $c(q) = q$.

There is another durable and intrinsically useless object called money. The stock of money is constant. Each household starts at date 0 with one unit of money. There is an upper bound $M > 1$ on the household's money holdings. When M is finite, it is *non binding* in the sense given below (finiteness of M is only used in Propositions 3-4, and its roles are discussed there).

In each pairwise meeting, agents may exchanges the good for money; the trading outcome is determined by a way described below. Agents from a household are anonymous to agents from other households, so each agent's trading history is unknown to agents of other households (but known to agents from the same household).

Two more assumptions about the physical environment are as follows. First, within each household, money is evenly redistributed among its buyers at the start of each date (note then sellers hold zero). This assumption is explicitly made in [13, 15]; here it permits me to simplify the individual agent's state space (see more in footnote 9). Second, in each meeting, one agent's money holding and his household's start-of-date money holding are common knowledge to the relevant pair. This assumption is consistent with the treatment in [13, 15]; here it permits me to avoid dealing with asymmetric information. Without these two assumptions, I cannot establish the existence results in section 6.[7]

[6] Inspired by Kiyotaki and Wright [6], [13, 15] consider multiple types of goods and households. In this setting, one can further assume that buyers pool goods together after search. It is easy to adapt formulations and results below for these variants.

[7] Admittedly, it is restrictive that money must be redistributed in the special way, and it is peculiar that one agent can observe another agent's household money holding. One might think, however, that they are due to some unmodelled features of the environment.

3 Allocation, trading mechanism, and equilibrium

My first goal is to examine conditions under which an allocation can be an equilibrium allocation under some trading mechanism. An allocation prescribes a trade for each pairwise meeting conditional on certain factors; a trading mechanism specifies for each pairwise meeting sets of actions for agents and a mapping from actions to trading outcomes; the matching process and the trading mechanism imply a game so one can define equilibrium; and an allocation is an equilibrium allocation if its prescribed trades coincide with trades implied by some equilibrium strategy profile.

I focus on *stationary allocation* in that it prescribes the trade of a meeting only conditional on states of the pair in the meeting; here and below, an agent is said in state m if his household's beginning-of-date money holding is m. A generic pairwise trade is denoted by (q, l): q is the transfer of the good (from a seller to a buyer), ln^{-1} is the transfer of money (from a buyer to a seller) when the household is finite, and l is the transfer of money when the household is large. Therefore, an allocation A is a pair real-valued functions $(q(.), l(.))$ on $[0, M]^2$, where $(q(m_b, m_s), l(m_b, m_s))$ is the prescribed trade between a buyer in state m_b and a seller in state m_s. (It is important to remember the following convention: q and l are scalers; $q(.)$ and $l(.)$ are functions over $[0, M]^2$; and $q(m_b, m_s)$ and $l(m_b, m_s)$ are evaluations of $q(.)$ and $l(.)$ at (m_b, m_s).)

While allocations in concern need not be those generated by surplus-splitting rules standard in the literature (Nash bargaining, the ultimatum game, price taking, etc.), they satisfy two properties that allocations generated by those rules satisfy. That is, if an allocation A is an equilibrium allocation under some trading mechanism, then its prescribed pairwise trade, in equilibrium or off equilibrium, satisfies the *sequential individual rationality* (SIR)—the trade weakly dominates autarky for each agent, and *pairwise efficiency* (PE)—the trade is in the pairwise Pareto frontier.

Given $A = (q(.), l(.))$, I specify the following trading mechanism, denoted T^A, which can be viewed as a generalized version of the direct mechanism in Kocherlakota [7] and Kocherlakota and Wallace [8].[8] As a buyer in state m_b meets a seller in state m_s, T^A indicates two stages of actions. In stage 1, the buyer and seller simultaneously announce a number from $\{0, 1\}$. If both

[8]These authors show that any incentive-feasible (or equilibrium) allocation under any arbitrary trading mechanism is an incentive-feasible allocation under the direct mechanism associated with the allocation. The direct mechanism only assures SIR.

announce 1, then they move to stage 2; otherwise, the trade is $(0,0)$ and the meeting is over. In stage 2, the buyer first proposes some (q,l) and then the seller announces a number from $\{0,1\}$. If 1 is announced, then (q,l) is carried out and the meeting is over; if 0 is announced, then $(q(m_b,m_s), l(m_b,m_s))$—the trade prescribed by A—is carried out and the meeting is over. As it shall be clear later, if A is an equilibrium allocation under T^A then it satisfies SIR (ensured by stage 1) and PE (ensured by stage 2); moreover, if A is an equilibrium allocation under other trading mechanism and satisfies SIR and PE, then it is an equilibrium allocation under T^A.

In the game implied by T^A, I consider strategies by which each individual agent does not condition his actions on his private information and calendar time. Denote by f_b a generic strategy of a buyer, and by f_s a generic strategy of a seller. To simplify notation, I express f_b and f_s over the relevant domains, so that $f_b = (f_{b1}, f_{b2})$ and $f_s = (f_{s1}, f_{s2})$ are represented by

$$f_{b1} \;:\; [0,M]^2 \to \{0,1\},\; f_{b2} : [0,M]^2 \to \mathbb{R}_+^2, \tag{1}$$

$$f_{s1} \;:\; [0,M]^2 \to \{0,1\},\; f_{s2} : [0,M]^2 \times \mathbb{R}_+^2 \to \{0,1\}. \tag{2}$$

In words, as a buyer in state m_b meets a seller in state m_s, $f_{b1}(m_b,m_s)$ and $f_{s1}(m_b,m_s)$ are numbers announced in stage 1 by the buyer and seller, respectively; $f_{b2}(m_b,m_s)$ is the offer the buyer makes in stage 2; and $f_{s2}(m_b,m_s,q,l)$ is the number announced by the seller in stage 2 when seeing the offer (q,l).[9] I restrict attention to *symmetric* equilibrium in that all buyers (from all households) choose the same f_b and all sellers choose the same f_s.

Definition 1 *Given the trading mechanism T^A, an equilibrium is a strategy profile represented by $f = (f_b, f_s)$ (see (1)-(2)) such that it is optimal for one agent to follow actions indicated by f currently and in the future, provided that all other agents, including those from the same household, follow actions indicated by f.*

By the initial distribution of money, symmetry implies that any Definition-1 equilibrium is *degenerate* in that all in-equilibrium households hold one unit of money at the start of each date. From now on, I refer to a household with

[9] In the absence of the special money-redistribution assumption, the individual agent's state is a probability distribution regarding how money is distributed among the household members, and so domains of A and f pertain to a set of such distributions (instead of the interval $[0,M]$).

one unit of money as a *regular* household, an agent from a regular household as a regular agent, and a meeting between two regular agents as a regular meeting.

I also consider a stronger notion of equilibrium, referred to as strong equilibrium, which rules out possibility of beneficial joint deviation by members from the same household.[10] The rationale for this notion is that if members from the same household can agree on a beneficial joint deviation, there is no problem of implementation. Because the household members do no communicate in search, it is sensible to restrict joint deviations to those in meeting regular agents.

Definition 2 *A Definition-1 equilibrium f is strong if agents from the same household cannot improve from any joint deviation when meeting regular agents.*

Given $A = (q(.), l(.))$, by a trivial application of Blackwell's sufficient conditions, there is a unique bounded function on $[0, M]$ satisfying

$$v(m) = u(q(m, 1)) - q(1, m) + \beta v(g(m)) \text{ with } g(m) = m - l(m, 1) + l(1, m). \tag{3}$$

If A is an equilibrium allocation then v is the value function on the household's money holdings; in that equilibrium, if the household starts by m, then with probability one, each of its buyers consumes $q(m, 1)$, each of its sellers produces $q(1, m)$, and the household ends up with $g(m)$. In what follows, I restrict attention to $A = (q(.), l(.))$ satisfying (**C1**) $q(1, 1) > 0$; and (**C2**) the function v given by (3) is *non decreasing*, *continuous* and *concave.*

If A is an equilibrium allocation, then by (C1) that equilibrium is *monetary.* Non decreasing of v is equivalent to free disposal of money. In the large-household setting, I identify the payoff to an individual agent from his own action in any pairwise meeting as the marginal contribution to the agent's objective function,[11] and given concavity and continuity of v, I can express the marginal contribution in terms of sided derivatives of v (c.f. [14,

[10] Note, as is standard, the notion of equilibrium only rules out possibility of beneficial unilateral deviation. Simply put, the individual agent's optimality in equilibrium need not ensure the household's optimality as a team in strong equilibrium. One can see the analogy from a person-by-person satisfactory team decision function and the best team decision function in [10].

[11] This resembles the way that Aumann and Shapley [2] define Shapley value of a non atomic agent; note the measure of the agent is an infinitesimal so is the payoff in concern.

p 213, Theorem 23.1]). Those properties of A, as is not unusual in dynamic models, cannot be ensured in equilibrium by primitives. I study in section 4 conditions for A with such properties to be an equilibrium allocation; I show in section 6 the existence of equilibrium allocations with such properties.

4 Characterization of equilibrium allocation

Now fix $A = (q(.), l(.))$ satisfying (C1)-(C2) and suppose that it is an equilibrium allocation under the trading mechanism T^A. In order to describe no deviation from trades prescribed by A, I first describe how an agent in arbitrary state m evaluates a trade in a meeting, taking as given that all other agents from the same household obtain trades prescribed by A currently and that v in (3) is the value function defined on the household's money holdings.

When the household is finite and the agent in state m is a buyer, with probability one, each of n sellers from his household trades $(q(1,m), l(1,m))$, and each of other $n-1$ buyers from his household trades $(q(m,1), l(m,1))$. Therefore, if the agent trades $(0,0)$, then the evaluation of his objective function is $u(q(m,1))(1-n^{-1}) - q(1,m) + \beta v(g(m) + l(m,1)n^{-1})$ (the first term is the contribution to the objective function from other buyers' consumption, the second term is from all seller's production, and the last term is from the household's end-of-meeting money holding). It follows that the additional contribution from trading (q,l) to his objective function is

$$B_0(q,l,m) = u(q)n^{-1} + \beta[v(g(m) + l(m,1)n^{-1} - ln^{-1}) - v(g(m) + l(m,1)n^{-1})], \tag{4}$$

and this contribution in term of per unit of his measure in the set I (which is n^{-1}) is

$$B(q,l,m) = u(q) + n\beta[v(g(m) + l(m,1)n^{-1} - ln^{-1}) - v(g(m) + l(m,1)n^{-1})]. \tag{5}$$

The analogy applies to the agent in state m is a seller. The household's end-of-meeting money holding is $g(m) - l(1,m)n^{-1}$ if the seller trades $(0,0)$, so the additional contribution from (q,l) to his objective function is

$$S_0(q,l,m) = -qn^{-1} + \beta[v(g(m) - l(1,m)n^{-1} + ln^{-1}) - v(g(m) - l(1,m)n^{-1})], \tag{6}$$

and this contribution in term of per unit of his measure in I is

$$S(q,l,m) = -q + n\beta[v(g(m) - l(1,m)n^{-1} + ln^{-1}) - v(g(m) - l(1,m)n^{-1})]. \tag{7}$$

When the household is large, as indicated above, I define the payoff from trading (q, l) as the marginal contribution to the agent's objective function. As $n \to \infty$, the limit of $B(q, l, m)$ in (5) ($S(q, l, m)$ in (7), resp.) defines this contribution when the agent in state m is a buyer (seller, resp.). The limit of $B(q, l, m)$ in (5), still denoted $B(q, l, m)$, is

$$\begin{aligned}&B(q, l, m) = u(q) - \beta w_b(l, m) \text{ with} \\ &w_b(l, m) = \min\{l, l(m, 1)\} v'_+(g(m)) + \max\{0, l - l(m, 1)\} v'_-(g(m)),\end{aligned} \tag{8}$$

and the limit of $S(q, l, m)$ in (7), still denoted $S(q, l, m)$, is

$$\begin{aligned}&S(q, l, m) = -q + \beta w_s(l, m) \text{ with} \\ &w_s(l, m) = \min\{l, l(1, m)\} v'_-(g(m)) + \max\{l - l(1, m), 0\} v'_+(g(m)),\end{aligned} \tag{9}$$

where $v'_-(z)$ is the left derivative and $v'_+(z)$ is the right derivative of v at z.[12]

I have assumed that when M is finite it is *non binding*. Precisely, this means that for any feasible transfer of money from a buyer in an arbitrary state m_b to a seller in an arbitrary state m_s, given all other agents follow trades prescribed by A, the buyer's and seller's households end the date with holdings less than M. That is, when the household is finite, for all (m_b, m_s), $g(m_s) - l(1, m_s)n^{-1} + m_b n^{-1} < M$ and $g(m_b) - l(m_b, 1)n^{-1} + m_b n^{-1} < M$;[13] when the household is large, for all m, $g(m) < M$. With this assumption, I can apply the payoffs in (4)-(9) below with no restriction.

For A to be an equilibrium allocation under T^A, as a buyer in state m_b meets a seller in state m_s, no one announces 0 at stage 1, so

$$B(q(m_b, m_s), l(m_b, m_s), m_b) \geq 0, \tag{10}$$

$$S(q(m_b, m_s), l(m_b, m_s), m_s) \geq 0; \tag{11}$$

the buyer cannot find an offer at stage 2 leading to pairwise improvement, so

$$\begin{aligned}&(q(m_b, m_s), l(m_b, m_s)) \in \arg\max_{(q,l)} B(q, l, m_b) \\ &\text{s.t. } 0 \leq l \leq m_b \text{ and } S(q, l, m_s) \geq S(q(m_b, m_s), l(m_b, m_s), m_s).\end{aligned} \tag{12}$$

[12] Note if $v'_-(g(m)) = v'_+(g(m))$ then $w_b(l, m) = w_s(l, m) = lv'(g(m))$. To see (8), write $v(g(m) + l(m, 1)n^{-1} - ln^{-1}) - v(g(m) + l(m, 1)n^{-1})$ as $v(g(m) + l(m, 1)n^{-1} - ln^{-1}) - v(g(m)) + v(g(m)) - v(g(m) + l(m, 1)n^{-1})$. The similar treatment leads to (9).

[13] If the buyer transfers all his holding $m_b n^{-1}$ to the seller, the seller's household ends up with $g(m_s) - l(1, m_s)n^{-1} + m_b n^{-1}$. If the buyer transfers zero, the buyer's household ends up with $g(m_b) - l(m_b, 1)n^{-1} + m_b n^{-1}$.

Now let the strategy profile $f^A = (f_b^A, f_s^A)$ be given by

$$f_{b1}^A(m_b, m_s) = f_{s1}^A(m_b, m_s) = 1, \tag{13}$$
$$f_{b2}^A(m_b, m_s) = (q(m_b, m_s), l(m_b, m_s)), \tag{14}$$

$$f_{s2}^A(m_b, m_s, q, l) = 1 \tag{15}$$
$$\Leftrightarrow \ [0 \leq l \leq m_b \text{ and } S(q, l, m_s) \geq S(q(m_b, m_s), l(m_b, m_s), m_s)].$$

If A satisfies (10)-(12) (i.e., (10)-(12) hold for all (m_b, m_s)), then by the one-stage-deviation principle, f^A is a Definition-1 equilibrium (so A is an equilibrium allocation).

Now suppose that f^A is a Definition-1 equilibrium. Then for a buyer in state m in meeting a regular seller, the set of admissible trades is

$$\overline{\Gamma}_b(m) = \{(0,0)\} \cup \{(q,l) : 0 \leq l \leq m, S(q,l,1) \geq S(q(m,1), l(m,1), 1)\};$$

this buyer either obtains $(0,0)$ at stage 1, or any (q,l) at stage 2 that gives the seller at least the same payoff as $(q(m,1), l(m,1))$. Also, for a seller in state m in meeting a regular buyer, the set of admissible trades is

$$\underline{\Gamma}_s(m) = \{(0,0)\} \cup \{(q(1,m), l(1,m))\};$$

this seller either obtains $(0,0)$ at stage 1, or $(q(1,m), l(1,m))$ at stage 2. It follows that $[\overline{\Gamma}_b(m) \times \underline{\Gamma}_s(m)]^I$ is the set of admissible joint trades for agents from the household with m in meeting regular agents; here and below, I denote by $[K]^I$ the product set $\prod_{i \in I} K_i$ with $K_i = K$ for all i. Then there is no beneficial joint deviation in equilibrium f^A if and only if for all m,

$$v(m) = \max W(\gamma(m)) \text{ s.t. } \gamma(m) \in [\overline{\Gamma}_b(m) \times \underline{\Gamma}_s(m)]^I, \tag{16}$$

where $\gamma(m) = \{(q_b^i, l_b^i), (q_s^i, l_s^i)\}_{i \in I}$ is μ-measurable and where

$$W(\gamma(m)) = \int_{i \in I} u(q_b^i)\mu(di) - \int_{i \in I} q_s^i \mu(di) + \beta v(m - \int_{i \in I} l_b^i \mu(di) + \int_{i \in I} l_s^i \mu(di)). \tag{17}$$

To summarize, we have

Proposition 1 *Suppose that A satisfies (C1)-(C2). When the household is finite, let B and S be as given in (5) and (7); when the household is large, let B and S be as given (8) and (9).*

(i) f^A (see (13)-(15)) is a Definition-1 equilibrium if and only if A satisfies (10)-(12).

(ii) f^A is a Definition-2 strong equilibrium if and only if A satisfies (10)-(12) and (16).

Now consider T', an arbitrary trading mechanism (e.g., the one that differs from T^A by switching roles of buyers and sellers at stage 2), and again restrict attention to strategies by which each individual agent does not condition his actions on his private information and calendar time. We can define (symmetric) equilibrium as above. Suppose that A is an equilibrium allocation under T' *and* satisfies SIR and PE. Then it is straightforward to verify that it satisfies (10)-(12). We can also define strong equilibrium as above. But in the absence of details of T', we can say little more about A from knowing that the equilibrium supporting A is strong (e.g., we do not know if (16) holds), because we only know that for agents from the household with m, the set of admissible joint trades in meeting regular agents includes $[\underline{\Gamma}_b(m) \times \underline{\Gamma}_s(m)]^I$, and is included in $[\overline{\Gamma}_b(m) \times \overline{\Gamma}_s(m)]^I$, where

$$\begin{aligned} \underline{\Gamma}_b(m) &= \{(0,0)\} \cup \{(q(m,1), l(m,1)\} \\ \overline{\Gamma}_s(m) &= \{(0,0)\} \cup \{(q,l) : 0 \le l \le 1, B(q,l,1) \ge B(q(1,m), l(1,m), 1)\}. \end{aligned}$$

However, a sharp characterization can be made when A satisfies hypotheses in the next proposition: In the equilibrium supporting A, there is no beneficial joint deviation even if for agents from the household with any m, $[\overline{\Gamma}_b(m) \times \overline{\Gamma}_s(m)]^I$ is the set of admissible joint trades in meeting regular agents; that is, for all m,

$$v(m) = \max W(\gamma(m)) \text{ s.t. } \gamma(m) \in [\overline{\Gamma}_b(m) \times \overline{\Gamma}_s(m)]^I, \tag{18}$$

where $\gamma(m) = \{(q_b^i, l_b^i), (q_s^i, l_s^i)\}_{i \in I}$ is μ-measurable and $W(\gamma(m))$ is defined by (17).

Proposition 2 *Suppose that A satisfies (C1)-(C2) and v is strictly increasing and differentiable. Let the household be large. If A satisfies (10)-(12), then A satisfies (18).*

All proofs are in Appendix I. To illustrate uses of properties of A and the large household in Proposition 2, I sketch the proof here. By strict monotonicity of v and (12), I show (i) if $(q,l) \in \overline{\Gamma}_b(m) \backslash \underline{\Gamma}_b(m)$ then $B(q,l,m) < B(q(m,1), l(m,1), m)$; and (ii) if $(q,l) \in \overline{\Gamma}_s(m) \backslash \underline{\Gamma}_s(m)$ then $S(q,l,m) < S(q(1,m), l(1,m), m)$ (here I also use concavity of u and v). With (10), (11), (i) and (ii), by concavity of v and that the household is large, I show that there is an optimal $\gamma(m)$ with $g(m)$ as the implied household's end-of-meeting holding. With this result, by applying $v'_-(g(m)) = v'_+(g(m))$ to different combinations of (10), (11), (i) and (ii), I rule out all possible compositions of the $\gamma(m)$ that may lead to $W(\gamma(m)) \neq v(m)$.

5 An approximation result

In this section, I take an ϵ-equilibrium approach to exploring whether an equilibrium allocation $A = (q(.), l(.))$ in the large-household setting has a foundation in the finite-household setting.[14] Given the trading mechanism T^A, I consider two notions of ϵ equilibrium when the household is finite.

Definition 3 *An ϵ equilibrium is a strategy profile represented by f (see (1)-(2)) such that for one agent, the (expected lifetime) payoff from any sequence of his own actions in meeting regular agents does not exceed by ϵ the payoff from the sequence of actions indicated by f, provided that all other agents, including those from the same household, follow actions indicated by f.*[15]

Definition 4 *A strengthened ϵ equilibrium is a strategy profile represented by f such that for agents from the same household, the (expected lifetime) payoff from any sequence of their joint actions in meeting regular agents does not exceed by ϵ the payoff from the sequence of joint actions indicated by f, provided that agents outside the household follow actions indicated by f.*

In the rest of this section, I adopt the following notation: (a) B in (5) and S in (7) are denoted by B_n and S_n, respectively; (b) f^A in (13)-(15) with $S = S_n$ is denoted by f_n^A; (c) for $a \in \{b, s\}$, $\overline{\Gamma}_a$ and $\underline{\Gamma}_a$ with $B = B_n$ and $S = S_n$ are denoted by $\overline{\Gamma}_a^n$ and $\underline{\Gamma}_a^n$, respectively; (d) W in (17) with finite I is denoted by W_n; and (e) B in (8), S in (9), and all terms in (b)-(d) related to the large household are maintained as they are.

The next proposition gives the main result of this section.

Proposition 3 *Suppose that A satisfies (C1)-(C2) and that M is finite.*

(i) If f^A is a Definition-1 equilibrium when the household is large, then for any $\epsilon > 0$, there exists N such that f_n^A is a Definition-3 ϵn^{-1} equilibrium when the household is finite with $n > N$.

(ii) If f^A is a Definition-2 strong equilibrium when the household is large and $v'_+(1) > 0$, then for any $\epsilon > 0$, there exists N such that f_n^A is a Definition-4 strengthened ϵ equilibrium when the household is finite with $n > N$.

[14] As an alternative, one may address this issue by studying whether there exists a sequence of equilibrium allocations in the finite-household setting which converge (in some sense) to A as $n \to \infty$. I do not have any general result in this line.

[15] One may set ϵ as an upper bound on gainings from the individual deviations each of which starts in an arbitrary meeting, that is, the meeting partner need not be a regular agent. Such defined ϵ equilibrium does not affect anything substantial.

In Proposition 3 (i), I am interested in ϵn^{-1} equilibrium instead of ϵ equilibrium because when n increases, the measure of an agent in the household decreases so in any pairwise meeting the payoff from any action of the agent decreases (see (4) and (6)). The proof of Proposition 3 is built on the following lemma, which is closely related to Proposition 1.

Lemma 1 *Suppose that A satisfies (C1)-(C2). Let the household be finite.*

(i) If for all m, all $(q_b, l_b) \in \overline{\Gamma}_b^n(m)$ and all $(q_s, l_s) \in \underline{\Gamma}_s^n(m)$,

$$\begin{aligned} B_n(q_b, l_b, m) - B_n(q(m,1), l(m,1), m) &< \epsilon(1-\beta), \quad (19) \\ S_n(q_s, l_s, m) - S_n(q(m,1), l(m,1), m) &< \epsilon(1-\beta), \quad (20) \end{aligned}$$

then f_n^A is a Definition-3 ϵn^{-1} equilibrium.

(ii) If for all m and all $\gamma(m) \in [\overline{\Gamma}_b^n(m) \times \underline{\Gamma}_s^n(m)]^I$,

$$W_n(\gamma(m)) - v(m) < \epsilon(1-\beta), \quad (21)$$

then f_n^A is a Definition-4 strengthened ϵ equilibrium.

The key to showing part (i) of Proposition 3 is the upper bound in (19), and the key to part (ii) is the upper bound in (21). The basic idea behind the first bound is as follows. For any fixed m, when n is sufficiently large, any trade in $\overline{\Gamma}_b^n(m)$ is not far away from some trades in $\overline{\Gamma}_b(m)$, and, therefore, if the first bound is violated by (q_b, l_b), then from this (q_b, l_b) some (q, l) in $\overline{\Gamma}_b(m)$ with $B(q, l, m) > B(q(m,1), l(m,1), m)$ can be constructed. Finiteness of M ensures uniformity. The similar idea works for the second bound; here to construct a suitable (q, l) in $\overline{\Gamma}_b(m)$, I need $v'_+(1) > 0$ in one scenario.

Now suppose that when the household is finite, the set of admissible trades for a buyer (a seller, resp.) in any state m is $\overline{\Gamma}_b^n(m)$ ($\overline{\Gamma}_s^n(m)$, resp.). First, maintain hypotheses in Proposition 3 (i). Then it can be shown that when n is large, (19)-(20) hold for all m, all $(q_b, l_b) \in \overline{\Gamma}_b^n(m)$ and all $(q_s, l_s) \in \overline{\Gamma}_s^n(m)$, and, consequently, that for any agent, the (expected lifetime) payoff from any sequence of admissible trades does not exceed by ϵn^{-1} the payoff from the sequence of trades prescribed by A. Second, strengthen hypotheses in Proposition 3 (ii) such that A satisfies (18) when the household is large. Then it can be shown that when n is large, (21) holds for all m and all $\gamma(m) \in [\overline{\Gamma}_b^n(m) \times \overline{\Gamma}_s^n(m)]^I$, and, consequently, that for agents from any household, the (expected lifetime) payoff from any sequence of admissible joint trades does not exceed by ϵ the payoff from the sequence of joint trades prescribed by A. From those results and the related discussion in section 4, it shall become clear how to adapt Proposition 3 for an arbitrary trading mechanism.

6 Generalized Nash bargaining

For the existence of equilibrium allocations studied above, following the existing literature, I focus on those generated by generalized Nash bargaining. We say that $A = (q(.), l(.))$ satisfying (C1)-(C2) is *generated by generalized Nash bargaining* if there exists some $\lambda \in (0, 1]$ such that for all (m_b, m_s),

$$(q(m_b, m_s), l(m_b, m_s)) \in \arg\max_{(q,l)} [B(q, l, m_b)]^{\lambda}[S(q, l, m_s)]^{1-\lambda} \text{ s.t. } 0 \leq l \leq m_b; \tag{22}$$

that is, if the buyer in state m_b and the seller in state m_s take the meeting-specific Pareto frontier as the one implied by $B(q, l, m_b)$ and $S(q, l, m_s)$, then $(q(m_b, m_s), l(m_b, m_s))$ is the Nash bargaining solution with λ as the buyer's bargaining weight. If A satisfies (C1)-(C2) and (22), then by Proposition 1, f^A is a Definition-1 equilibrium.[16]

First, I consider the large household. If the transfer of money is l, $w_b(l, m_b) = l\omega_b$ and $w_s(l, m_s) = l\omega_s$ (see (8)-(9)), then the Nash solution implies that

$$y(l, \omega_b, \omega_s) = \arg\max_{q \geq 0} [u(q) - \omega_b l]^{\lambda}[-q + \omega_s l]^{1-\lambda} \tag{23}$$

is the transfer of good, which satisfies

$$\lambda u'(y(l, \omega_b, \omega_s))[-y(l, \omega_b, \omega_s) + \omega_s l] = (1 - \lambda)[u(y(l, \omega_b, \omega_s)) - \omega_b l]. \tag{24}$$

I use this function $y(.)$ to construct two allocations satisfying (C1)-(C2) and (22). For the construction to go through, I need the following assumption,

(**U**) Either $\lambda = 1$, or $\lambda < 1$ and $u''u'' \geq u'u'''$.

For the first allocation, let y_λ satisfy $\beta u'(y_\lambda) \geq 1$ and

$$1 = \frac{\beta[\lambda u'(y_\lambda) + (1 - \lambda)]}{1 - (1 - \lambda)u''(y_\lambda)[u(y_\lambda) - \omega]/[u'(y_\lambda)u'(y_\lambda)]}, \tag{25}$$

where

$$\omega = \frac{\lambda u'(y_\lambda)y_\lambda + (1 - \lambda)u(y_\lambda)}{\lambda u'(y_\lambda) + (1 - \lambda)}. \tag{26}$$

[16] In case $\lambda = 1$, one can verify that this A is also an equilibrium allocation when agents play the ultimatum game in pairwise meetings (buyers make offers). In case $\lambda < 1$, if one applies a suitable version of the Rubinstein-Stähl alternating-offer game in the meeting where the Pareto frontier is determined by $B(q, l, m_b)$ and $S(q, l, m_s)$, then after taking the limit one obtains $(q(m_b, m_s), l(m_b, m_s))$ as the trade outcome.

Existence and uniqueness of y_λ follow from (U).[17] Let A be defined by

$$q(m_b, m_s) = y(m_b, \omega, \omega) \text{ and } l(m_b, m_s) = m_b. \tag{27}$$

The function v implied by the allocation in (27) is

$$v(m) = u(y(m, \omega, \omega)) - y_\lambda + \beta(1-\beta)^{-1}[u(y_\lambda) - y_\lambda]. \tag{28}$$

Proposition 4 *Suppose that $M > 1$ satisfies $u'(y(M, \omega, \omega)) \geq 1$ and that (U) holds. Let the household be large. Then A given by (27) satisfies (C1)-(C2) and (22), and f^A is a Definition-2 strong equilibrium.*

Letting $(\omega_b, \omega_s, l) = (\omega, \omega, 1)$ in (24) and comparing to (26), we see $y(1, \omega, \omega) = y_\lambda$ so y_λ is the regular-meeting output of the Proposition-4 allocation.[18] By (25), $\beta u'(y_\lambda) \geq 1$, and as is shown in the proof, $l \mapsto y(l, \omega, \omega)$ is strictly increasing and differentiable, so the hypothesis about M is not vacuous. Note that there is no beneficial joint deviation even if for agents from the household with any m, the set of admissible joint trades is $[\overline{\Gamma}_b(m) \times \overline{\Gamma}_s(m)]^I$, because v is strictly increasing and differentiable (Proposition 2).

The remarkable feature of the Proposition-4 allocation is $l(m, 1) = m$ and $l(1, m) = 1$ so $g(m) = 1$ all m; that is, the household's end-of-meeting money holding does not depend on its start-of-date money holding. As a consequence, $v(.)$ in (28) is completely determined by its derivative at 1, $v'(1)$. Indeed, as shown in the proof, $\omega = \beta v'(1)$, and note y_λ and ω are determined by (25)-(26) without referring to any other evaluations of functions $v(.)$ or $v'(.)$. The restriction on M ensures that $l(m, 1) = m$ holds for all m.

The Proposition-4 equilibrium resembles the equilibrium in the Lagos-Wright model [9]. In [9], agents trade in a centralized market after random matching, and preferences over centralized-trade goods are quasi-linear. For an internal solution in the centralized market, the agent must enter the centralized market with money holdings that are not too large. In that case, the assumed quasi-linear preferences imply that the value function for the agent's

[17] Substituting ω in (26) into (25), we obtain $[\lambda u' + (1-\lambda)][\beta\lambda u' - \beta\lambda + \beta - 1] = -\lambda(1-\lambda)(u''/u')(u-y)$. Then use $\lambda - \beta\lambda + \beta - 1 \leq 0$ and that $-u''/u'$ is non decreasing (implied by (U)).

[18] In dealing with symmetric Nash bargaining, Rauch [13] provides a regular-meeting output comparable to $y_{0.5}$; see section 6 for how Rauch obtains his result. Also, in a seemingly programmer-automata setup, Shi [16] provides a regular-meeting output comparable to y_1; see Appendix II for how Shi obtains his result.

end-of-match money holdings is affine, and that, in turn, implies that in a pairwise meeting, the buyer's and seller's payoff functions are quasi linear, linear in end-of-match money holdings. Moreover, those functions have the same linear coefficient, provided that the sum of the buyer and seller money holdings is consistent with an internal solution in the centralized market. In the Proposition-4 equilibrium, the buyer's and seller's payoffs are linear in each agent's end-of-match money holdings (so the individual agent's payoff functions are quasi linear). The upper bound on the household money holding ensures that all households have the same end-of-meeting money holding, which in turn ensures that the linear coefficients regarding money for the buyer and seller in a meeting are identical. Finally, (U) has the same uses in [9] as here.

Now I turn to the second allocation. Let $\tilde{y}_\lambda$ satisfy $\beta u'(\tilde{y}_\lambda) \geq 1$ and

$$1 = \frac{\beta \lambda u'(\tilde{y}_\lambda)}{1 - (1-\lambda) u''(\tilde{y}_\lambda) u(\tilde{y}_\lambda) / [u'(\tilde{y}_\lambda) u'(\tilde{y}_\lambda)]}, \tag{29}$$

where

$$\tilde{\omega} = \frac{\lambda u'(\tilde{y}_\lambda)\tilde{y}_\lambda + (1-\lambda) u(\tilde{y}_\lambda)}{\lambda u'(\tilde{y}_\lambda)}. \tag{30}$$

Existence and uniqueness of $\tilde{y}_\lambda$ again follow from (U). Let A be defined by

$$q(m_b, m_s) = y(\min\{m_b, 1\}, 0, \tilde{\omega}) \text{ and } l(m_b, m_s) = m_b. \tag{31}$$

The function v implied by the allocation in (31) is

$$v(m) = u(y(\min\{m, 1\}, 0, \tilde{\omega})) - \tilde{y}_\lambda + \beta(1-\beta)^{-1}[u(\tilde{y}_\lambda) - \tilde{y}_\lambda]. \tag{32}$$

Proposition 5 *Suppose that (U) holds. Let the household be large. Then A given by (31) satisfies (C1)-(C2) and (22), and f^A is a Definition-1 (but not Definition-2 strong) equilibrium.*

Letting $(\omega_b, \omega_s, l) = (0, \tilde{\omega}, 1)$ in (24) and comparing to (30), we see $y(1, 0, \tilde{\omega}) = \tilde{y}_\lambda$ so $\tilde{y}_\lambda$ is the regular-meeting output of the Proposition-5 allocation. This allocation is similar to the one in Proposition 4 except that here v is constant over $[1, M]$ (so $v'_+(1) = 0$). Now again $g(m) = 1$, and $v(.)$ in (32) is completely determined by $v'_-(1)$—as shown in the proof, $\tilde{\omega} = \beta v'_-(1)$, and note $\tilde{y}_\lambda$ and $\tilde{\omega}$ are completely determined by (29)-(30).

The reason that the function v is flat is simple. Let $m \geq 1$. Given all buyers from the same household to spend m, the suitable value of $v'_-(1)$

induces a seller in state m to acquire 1. Given all sellers from the same household to acquire 1, $v'_+(1) = 0$ induces a buyer in state m to spend m, even in case $m > 1$, spending m gets the same amount of good as spending 1 (note then M need not be restricted as in Proposition 4). In equilibrium f^A, if all buyers from the household with $m > 1$ offer $(\tilde{y}_\lambda, 1)$ to regular sellers and all sellers from the household accept $(\tilde{y}_\lambda, 1)$ from regular buyers, then the household obtains a higher payoff than following f^A.[19]

Next I turn to two results for the finite household. Both results can be extended to general λ; I restrict to $\lambda = 1$ for the sake of simplicity.

Proposition 6 *Suppose that $\lambda = 1$. Let the household be finite.*

(i) If $n > 1$ and $M > \frac{n}{n-1}$, then there exists A with $l(1,1) = 1$ satisfying (C1)-(C2) and (22) and with v constant over $[1, M]$, and f^A is a Definition-1 (but not Definition-2 strong) equilibrium.

(ii) If A with $l(1,1) = 1$ satisfies (C1)-(C2) and (22), then $n > 1$.

In Proposition 6 (i), $M > \frac{n}{n-1}$ ensures non bindingness when M is finite. The allocation here resembles the one in Proposition 5. In contrast to Proposition 5, here no single evaluation of $v(.)$ or $v'(.)$ (e.g., $v'_-(1)$) can determine the whole $v(.)$ (because the household is finite), so the proof is not constructive; instead, the proof uses a fixed-point argument.

I have attempted to adapt this argument to establish some A for large n that resembles the Proposition-4 allocation with f^A being a strong equilibrium, but I fail to obtain a positive increment of v over a neighborhood of 1. This difficulty, of course, has nothing to do with the feasibility of degeneracy.

The key behind Proposition 6 (ii) is as follows. When $n = 1$, $l(1,1) = 1$ gives rise to the dependence of the current payoff to agents in state m on the value function v in the form of $v(m) = u(\beta v(m))$ for m in a neighborhood of 0—that is, v is a (strict) concave transformation of itself over that neighborhood. This result suggests that the feasibility of degeneracy mean little to existence of certain type of degenerate equilibrium.[20]

[19] One may regard any non strong equilibrium uninteresting in the current context. For such a view, Proposition 5 provides an example about importance of the notion of strong equilibrium. On the other hand, because joint deviation implicitly requires that the household members communicate before search, existence of a non strong equilibrium may be an interesting implication of the team model when communication among the team members is difficult (difficult communication is emphasized by [10]).

[20] Wallace and Zhu [18, section 2] deliver the same message by a similar result obtained from a standard single-agent household model.

7 Comparison to the literature

As indicated above, aside from details, Shi [15] and Rauch [13] make all important assumptions I make about the physical environment.[21] They both consider symmetric Nash bargaining. Shi [15], who initiated the use of the large household model for money applications, describes the household's problem in terms of sequences of the household's choices. In his formulation, each household takes as given that the regular-meeting trade is the trade that its buyers and sellers will make—independent of the household's start-of-date money holding. However, such trade is not feasible for a household with $m < 1$, which leaves $v(m)$ for $m < 1$ undefined. It also implies that $v(m) = v(1)$ for $m \geq 1$. As Rauch [13] points out in a comment on [15], neither is satisfactory. He proposes an alternative formulation.

Following [15], Rauch describes the household problem in sequence form. He proposes a special Lagrangian

$$\mathcal{L}(\{m_t\}, \{\omega_t\}) = \sum_t \beta^t \{F(m_t, \omega_t) + \omega_t[m_{t+1} - m_t - \Delta(m_t, \omega_t)]\}, \qquad (33)$$

where m_t is the household's money holding at the start of t, $F(m_t, \omega_t)$ is the return to the household from consumption and production at t, $\Delta(m_t, \omega_t)$ is the net money inflow to the household at t, and ω_t is the Lagrangian multiplier associated with the constraint $m_{t+1} = m_t + \Delta(m_t, \omega_t)$.[22] Notably, Rauch treats ω_t as a function of m_t (see [13, Eqs (22)-(23)]). This function cannot be arbitrary. It ought to be determined by some equilibrium conditions. But Rauch provides no such conditions, so his formulation is incomplete.[23]

One may complete Rauch's formulation by distinguishing the Lagrangian multipliers in the above $\mathcal{L}$ from a function describing the individual agent's

[21] In [13, 15], the buyer's ratio may be endogenous and there may be a constant rate of lump sum money transfer. I can extend the above formulations to these variants, but I can only extend the above existence proof to deal with the variant with money transfer.

[22] This Lagrangian is recovered from [13, Eqs (10)-(15) and (17)] in case the buyer's ratio is exogenous. Although $\{m_t\}$ is not included as a choice in [13, Eq (10)], it should be; otherwise, when the buyer's ratio is exogenous, the problem is absent of choices, which is not the case according to the context. By [13, Eq (1)], ω_t is taken by the individual agent as his marginal value of money in the Nash bargaining problem in date t meeting; it affects F and Δ by the way given in [13, Eqs (6)-(7)].

[23] It is not a way to obtain completeness by assuming that ω_t is a free variable as in the usual Lagrangian formulation, because here the period return F (also the money flow Δ) depends on ω_t. For comparison, think about the Lagrangian in the planner's version of the growth model where the period return does not depend on multipliers.

marginal value of money. For instance, let $\hat{\omega}(.)$ be such a function so that if an agent's household starts by m at $t+1$, $\hat{\omega}(m)$ is the agent's marginal value of money at date-t meeting. Then for each m_0, let

$$\hat{\mathcal{L}}(\{m_t\},\{\omega_t\}) = \sum_t \beta^t \{F(m_t, \hat{\omega}(m_{t+1})) + \omega_t[m_{t+1} - m_t - \Delta(m_t, \hat{\omega}(m_{t+1})]\} \tag{34}$$

so the maximum of the expected discounted utility for a household with m_0 is $v(m_0) = \max \hat{\mathcal{L}}(\{M_t\},\{\omega_t\})$. The equilibrium conditions are for all m, (a) $\hat{\omega}(m) = \beta v'(m)$ and (b) $\{m_+ : m_+ = m + \Delta(m, \hat{\omega}(m_+)\}$ is nonempty. Of course, now the existence of an equilibrium means the existence of a function $\hat{\omega}(.)$ satisfying (a)-(b).[24,25]

8 The concluding remarks

The equilibrium concepts developed in the above money model can be applied to the labor search model of Merz [11]. They can also be adapted to deal with the large firm's decision problem in the labor search literature. The large firm has many job positions, and the wage in each position is determined by bargaining with a worker. But in the literature (see Pissarides [12, Ch 3.1]), the firm takes the prevailing wage as given. This seems problematic. Instead, following the approach used above, it could be assumed that the wage in each position is determined by bargaining between a firm's agent and a worker, while taking as given the bargaining outcome between other agents of the firm and workers.

[24] From such a $\hat{\omega}(.)$, one can construct the above defined equilibrium allocation as follows. Fix m and set $m_0 = m$ in (34) and then set an optimal m_1 as $g(m)$. Then let $(q(m_b, m_s), l(m_b, m_s)) \in \arg\max[u(q) - \hat{\omega}(g(m_b))l][-q + \hat{\omega}(g(m_s))l]$ subject to $0 \leq l \leq m_s$ (note Rauch deals with symmetric Nash bargaining).

[25] In his analysis, Rauch ignores the effect from the change in m_t on ω_t in [13, Eqs (22)-(23)] but keeps it in [13, Eq (18)]. Remarkably, this inconsistency turns the Lagrangian in concern from $\mathcal{L}$ in (33) into $\hat{\mathcal{L}}$ in (34). As a result, he actually solves an equilibrium $\hat{\omega}(1)$ comparable to ω in (25)-(26). The reason that $\hat{\omega}(1)$ can be solved without referring to the function $\hat{\omega}(.)$ is the same as that $\omega = \beta v'(1)$ in the Proposition-4 equilibrium can be solved without referring to any other evaluations of functions $v(.)$ or $v'(.)$.

Appendix I: Proofs

The proof of Proposition 2

Let Π be the set of all $\pi = (\pi_b, \pi_b^*, \pi_b^0, \pi_s, \pi_s^*, \pi_s^0) \in \mathbb{R}_+^6$ with $\pi_b + \pi_b^* + \pi_b^0 = 1$ and $\pi_s + \pi_s^* + \pi_s^0 = 1$. Fix m, and by concavity of u and v and Jensen's inequality, for any $\gamma(m)$ in (16), there exist some $\pi \in \Pi$ and $(q_b, l_b, q_s, l_s) \in [\overline{\Gamma}_b(m) \backslash \underline{\Gamma}_b(m)] \times [\overline{\Gamma}_s(m) \backslash \underline{\Gamma}_s(m)] \equiv \Gamma$ such that

$$\begin{aligned} & \pi_b u(q_b) + \pi_b^* u(q_b^*) - \pi_s q_s - \pi_s^* q_s^* + \beta v(m - \pi_b l_b - \pi_b^* l_b^* + \pi_s l_s + \pi_s^* l_s^*) \\ \equiv \;& w(\pi, q_b, l_b, q_s, l_s) \geq W(\gamma(m)), \end{aligned}$$

where $(q_b^*, l_b^*) = (q(m,1), l(m,1))$ and $(q_s^*, l_s^*) = (q(1,m), l(1,m))$. So if $v(m)$ is the maximum of $w(\pi, q_b, l_b, q_s, l_s)$ over $\Pi \times \Gamma$, then (16) holds. Denote by $(\bar{\pi}, q_b, l_b, q_s, l_s)$ a maximizer of $w(.)$ with $\bar{\pi} = (\bar{\pi}_b, \bar{\pi}_b^*, \bar{\pi}_b^0, \bar{\pi}_s, \bar{\pi}_s^*, \bar{\pi}_s^0)$. Now it suffices to show $\bar{\pi}_b^* = \bar{\pi}_s^* = 1$, which clearly implies $w(\bar{\pi}, q_b, l_b, q_s, l_s) = v(m)$. I first establish some intermediate results, mainly properties of $\bar{\pi}$. In what follows, let $\hat{w}(\pi) \equiv w(\pi, q_b, l_b, q_s, l_s)$ and $h(\pi) \equiv m - \pi_b l_b - \pi_b^* l_b^* + \pi_s l_s + \pi_s^* l_s^*$.

Claim 1: (i) $B(q_b, l_b, m) < B(q_b^*, l_b^*, m)$; (ii) If $\bar{\pi}_b > 0$ then $l_b \neq l_b^*$; (iii) $S(q_s, l_s, m) < S(q_s^*, l_s^*, m)$; and (iv) If $\bar{\pi}_s > 0$ then $l_s \neq l_s^*$.

For part (i) of claim 1, first note by (12), $B(q_b, l_b, m) \leq B(q_b^*, l_b^*, m)$. By strict monotonicity of v and $g(m) < M$, $v'(g(m)) > 0$. So if $B(q_b, l_b, m) = B(q_b^*, l_b^*, m)$ then $q_b \neq q_b^*$ and $l_b \neq l_b^*$, but then any interior linear combination of (q_b, l_b) and (q_b^*, l_b^*), denoted (q, l), satisfies $B(q, l, m) > B(q_b^*, l_b^*, m)$ and $S(q, l, 1) \geq S(q_b^*, l_b^*, 1)$ (recall u is strictly concave), contradicting (12). For part (ii), if $l_b = l_b^*$ then part (i) implies $u(q_b) < u(q_b^*)$, but then $(\bar{\pi}, q_b, l_b, q_s, l_s)$ cannot be a maximizer of $w(.)$. Analogously, we can establish parts (iii)-(iv).

Next, without loss of generality, we can assume (**A1**) If $l_b^* = 0$ then $\bar{\pi}_b^0 = 0$; (**A2**) If $l_s^* = 0$ then $\bar{\pi}_s^0 = 0$; (**A3**) If $\bar{\pi}_b^0 > 0$ then $h(\bar{\pi}) \leq g(m)$; and (**A4**) If $\bar{\pi}_s^0 > 0$ then $h(\bar{\pi}) \geq g(m)$.

For (A1), if $\bar{\pi}_b^0 > 0$ then $\hat{w}(\bar{\pi}) = \hat{w}(\pi)$, where $\pi = (\bar{\pi}_b, \bar{\pi}_b^* + \bar{\pi}_b^0, 0, \bar{\pi}_s, \bar{\pi}_s^*, \bar{\pi}_s^0)$, so we can replace $\bar{\pi}$ by this π. Analogously, we can justify (A2); now use $\pi = (\bar{\pi}_b, \bar{\pi}_b^*, \bar{\pi}_b^0, \bar{\pi}_s, \bar{\pi}_s^* + \bar{\pi}_s^0, 0)$. For (A3), if $h(\bar{\pi}) > g(m)$ then by (10) and concavity of v, $d\hat{w}(\pi(x))/dx \geq 0$ if $h(\pi(x)) \geq g(m)$, where $\pi(x) = (\bar{\pi}_b, \bar{\pi}_b^* + x, \bar{\pi}_b^0 - x, \bar{\pi}_s, \bar{\pi}_s^*, \bar{\pi}_s^0)$, so we can replace $\bar{\pi}$ by the $\pi(x)$ with $h(\pi(x)) = g(m)$. Analogously, we can justify (A4); now use (11) and $\pi(x) = (\bar{\pi}_b, \bar{\pi}_b^*, \bar{\pi}_b^0, \bar{\pi}_s, \bar{\pi}_s^* + x, \bar{\pi}_s^0 - x)$.

Claim 2: (i) If $l_b < l_b^*$ and $\bar{\pi}_b > 0$ then $h(\bar{\pi}) \leq g(m)$; (ii) If $l_b > l_b^*$ and $\bar{\pi}_b > 0$ then $h(\bar{\pi}) \geq g(m)$; (iii) If $l_s > l_s^*$ and $\bar{\pi}_s > 0$ then $h(\bar{\pi}) \leq g(m)$; and (iv) If $l_s < l_s^*$ and $\bar{\pi}_s > 0$ then $h(\bar{\pi}) \geq g(m)$.

For part (i) of claim 2, suppose $h(\bar{\pi}) > g(m)$, but then by claim 1 (i) and concavity of v, $d\hat{w}(\pi(x))/dx > 0$ if $h(\pi(x)) > g(m)$, where $\pi(x) = (\bar{\pi}_b - x, \bar{\pi}_b^* + x, \bar{\pi}_b^0, \bar{\pi}_s, \bar{\pi}_s^*, \bar{\pi}_s^0)$. The exact argument rules out $h(\pi) < g(m)$ in part (ii). Analogously, we can establish parts (iii) and (iv); now use claim 1 (iii) and $\pi(x) = (\bar{\pi}_b, \bar{\pi}_b^*, \bar{\pi}_b^0, \bar{\pi}_s - x, \bar{\pi}_s^* + x, \bar{\pi}_s^0)$.

Claim 3: $h(\bar{\pi}) = g(m)$.

For claim 3, first suppose $h(\bar{\pi}) < g(m)$. Then by (A4), $\bar{\pi}_s^0 = 0$; by claim 1 (ii) and claim 2 (ii), if $\bar{\pi}_b > 0$ then $l_b < l_b^*$; by claim 1 (iv) and claim 2 (iv), if $\bar{\pi}_s > 0$ then $l_s > l_s^*$. But then it must be $h(\bar{\pi}) \geq g(m)$, a contradiction. Analogously, we can rule out $h(\bar{\pi}) > g(m)$; now use (A3), claim 1 (ii), claim 2 (i), claim 1 (iv), and claim 2 (iii).

Next, without loss of generality, we can assume (**A5**) $\bar{\pi}_b^0 \bar{\pi}_s^0 = 0$.

For (A5), if $\bar{\pi}_b^0 \bar{\pi}_s^0 > 0$ then first by (A1)-(A2), $l_b^* l_s^* > 0$; then by claim 3 and (10)-(11), $d\hat{w}(\pi(x))/dx \geq 0$, where $\pi(x) = (\bar{\pi}_b, \bar{\pi}_b^* + x l_s^*, \bar{\pi}_b^0 - x l_s^*, \bar{\pi}_s, \bar{\pi}_s^* + x l_b^*, \bar{\pi}_s^0 - x l_b^*)$ (note $h(\pi(x)) = g(m)$), so we can replace $\bar{\pi}$ by the $\pi(x)$ with $x = \min\{\bar{\pi}_b^0 / l_s^*, \bar{\pi}_s^0 / l_b^*\}$.

Claim 4: (i) If $\bar{\pi}_b \bar{\pi}_s > 0$ then $(l_b - l_b^*)(l_s - l_s^*) < 0$; (ii) If $\bar{\pi}_b^0 \bar{\pi}_b > 0$ then $l_b < l_b^*$; and (iii) If $\bar{\pi}_s^0 \bar{\pi}_s > 0$ then $l_s < l_s^*$.

For part (i) of claim 4, suppose $(l_b - l_b^*)(l_s - l_s^*) > 0$, but then by claim 3 and claim 1 (i) and (iii), $d\hat{w}(\pi(x))/dx > 0$, where $\pi(x) = (\bar{\pi}_b - x, \bar{\pi}_b^* + x, \bar{\pi}_b^0, \bar{\pi}_s - x\delta, \bar{\pi}_s^* + x\delta, \bar{\pi}_s^0)$ with $\delta = (l_b - l_b^*)/(l_s - l_s^*)$ (note $h(\pi(x)) = g(m)$). For part (ii), by claim 1 (ii), $l_b \neq l_b^*$. Suppose $l_b > l_b^*$, but then by claim 3 and (10) and claim 1 (i), $d\hat{w}(\pi(x))/dx > 0$, where $\pi(x) = (\bar{\pi}_b - x l_b^*/l_b, \bar{\pi}_b^* + x, \bar{\pi}_b^0 + x l_b^*/l_b - x, \bar{\pi}_s, \bar{\pi}_s^*, \bar{\pi}_s^0)$ (note $h(\pi(x)) = g(m)$). Analogously, we can establish part (iii); now use claim 1 (iv), claim 3, (11), claim 1 (iii), and $\pi(x) = (\bar{\pi}_b, \bar{\pi}_b^*, \bar{\pi}_b^0, \bar{\pi}_s - x l_s^*/l_s, \bar{\pi}_s^* + x, \bar{\pi}_s^0 + x l_s^*/l_s - x)$.

Finally, I show $\bar{\pi}_b^* = \bar{\pi}_s^* = 1$ by ruling out the following three cases.

(a) $\bar{\pi}_b^0 = \bar{\pi}_s^0 = 0$ and $\bar{\pi}_b + \bar{\pi}_s > 0$. If $\bar{\pi}_b \bar{\pi}_s = 0$ then $h(\bar{\pi}) = g(m)$ cannot hold. If $\bar{\pi}_b \bar{\pi}_s > 0$, then by claim 4 (i), $h(\bar{\pi}) = g(m)$ cannot hold either.

(b) $\bar{\pi}_b^0 > 0$. By (A5), $\bar{\pi}_s^0 = 0$. If $\bar{\pi}_b + \bar{\pi}_s = 0$, then $h(\bar{\pi}) = g(m)$ cannot hold. If $\bar{\pi}_b > 0$, then by claim 4 (ii), $l_b < l_b^*$ so $h(\bar{\pi}) = g(m)$ can hold only if $\bar{\pi}_s > 0$ and $l_s < l_s^*$, which contradicts claim 4 (i). So the remaining possibility is $\bar{\pi}_b = 0$ and $\bar{\pi}_s > 0$. Now $h(\bar{\pi}) = g(m)$ only if $l_s < l_s^*$. But then by (10) and claim 1 (iii), $d\hat{w}(\pi(x))/dx > 0$, where $\pi(x) = (\bar{\pi}_b, \bar{\pi}_b^* + x(l_s^* - l_s), \bar{\pi}_b^0 - x(l_s^* - l_s), \bar{\pi}_s - x l_b^*, \bar{\pi}_s^* + x l_b^*, \bar{\pi}_s^0)$ (note $h(\pi(x)) = g(m)$).

(c) $\bar{\pi}_s^0 > 0$. We can rule out case (c) analogously to case (b); here use (A5), claim 4 (iii) and (i), (11), claim 1 (i), and $\pi(x) = (\bar{\pi}_b - x l_s^*, \bar{\pi}_b^* + x l_s^*, \bar{\pi}_b^0, \bar{\pi}_s, \bar{\pi}_s^* + x(l_b^* - l_b), \bar{\pi}_s^0 - x(l_b^* - l_b))$. This completes the proof.

The proof of Lemma 1

For part (i) of the lemma, let $v_n^b(m)$ be the supremum of the (expected lifetime) payoffs to the buyer from sequences of his actions starting from a date when his household holds m, provided that all other agents follow f^A. Note the function $v_n^b : [0, M] \to \mathbb{R}$ is bounded above (recall u is bounded). Without loss of generality, assume $v_n^b(m)$ is attained so that

$$\begin{aligned} & v_n^b(m) - v(m) \\ = \ & [B_n(q_b, l_b, m) + \beta v_n^b(m_+) - \beta v(m_+)] - B_n(q(m,1), l(m,1), m) \\ < \ & \epsilon n^{-1}(1-\beta) + \beta[v_n^b(m_+) - v(m_+)], \end{aligned}$$

where $(q_b, l_b) \in \overline{\Gamma}_b^n(m)$ is the buyer's present-date trade in meeting a regular seller, m_+ is the implied household's end-of-meeting holding (i.e., $m_+ = g(m) + l(m,1)n^{-1} - l_b n^{-1}$), and the inequality follows from the hypothesis. Iterating this and using boundedness of v_n^b, we obtain $v_n^b(m) - v(m) < \epsilon n^{-1}$, as desired. If the agent in concern is a seller, we define v_n^s analogously and by the similar argument, we obtain $v_n^b(m) - v(m) < \epsilon n^{-1}$.

For part (ii) of the lemma, let $v_n^h(m)$ be the (expected lifetime) payoffs to the household from sequences of joint actions of the household members starting from a date when the household holds m, provided that all agents outside the household follow f^A. Note the function $v_n^h : [0, M] \to \mathbb{R}$ is bounded above. Without loss of generality, assume $v_n^h(m)$ is attained so that

$$\begin{aligned} v_n^h(m) - v(m) &= [W_n(\gamma(m)) + \beta v_n^h(m_+) - \beta v(m_+)] - v(m) \\ &< \epsilon(1-\beta) + \beta[v_n^h(m_+) - v(m_+)], \end{aligned}$$

where $\gamma(m) \in [\overline{\Gamma}_b^n(m) \times \underline{\Gamma}_s^n(m)]^I$ is the household's present-date trades in meeting regular agents, m_+ is the implied household's end-of-meeting holding, and the inequality follows from the hypothesis. Iterating this and using boundedness of v_n^h, we obtain $v_n^h(m) - v(m) < \epsilon$, as desired.

The proof of Proposition 3

Let A satisfy (C1)-(C2) and let M be finite. Then we have,

Claim 1: $\forall m$, (a) $B_n(q, l, m) - B_n(q(m,1), l(m,1), m) \leq B(q, l, m) - B(q(m,1), l(m,1), m)$; (b) $S_n(q, l, m) - S_n(q(1,m), l(1,m), m) \leq S(q, l, m) - S(q(1,m), l(1,m), m)$.

Claim 2: $\forall \epsilon > 0$, $\exists N$ s.t. $\forall n > N$, $m \in [0, M]$, $q \geq 0$, and $l \in [0, m]$, $S_n(q, l, 1) - S_n(q(m,1), l(m,1), 1) < S(q, l, 1) - S(q(m,1), l(m,1), 1) + \epsilon$.

For claim 1, by concavity of v and definitions of v'_+ and v'_-, $\forall z \geq 0$, if $\delta > 0$ then $v'_+(z) \geq [v(z+\delta) - v(z)]\delta^{-1}$; also, $\forall z > 0$, if $\delta \in (0, z]$ then $[v(z) - v(z-\delta)]\delta^{-1} \geq v'_-(z)$. Applying these inequalities into (5), (7), (8), and (9), we obtain claim 1; here let $z = g(m)$ and $\delta = ln^{-1}$ and note $n[v(z + ln^{-1}) - v(z)] = [v(z+\delta) - v(z)]\delta^{-1}l$. For claim 2, by concavity of v and definitions of v'_+ and v'_-, $\forall \epsilon > 0$, $\exists \delta_\epsilon < 1$ s.t. $\forall \delta \in (0, \delta_\epsilon)$, $v'_+(1) < [v(1+\delta) - v(1)]\delta^{-1} + \epsilon$ and $v'_-(1) > [v(1) - v(1-\delta)]\delta^{-1} - \epsilon$. Applying these inequalities into (5), (7), (8), and (9), we obtain claim 2; here let $\delta_1 = [l - l(1,1)]n^{-1}$, $\delta_2 = [l(1,m) - l(1,1)]n^{-1}$ and $\Delta_i = v(1+\delta_i) - v(1)$, and note $v(1+\delta_1) - v(1+\delta_2) = \Delta_1 - \Delta_2$, $n\Delta_1 = (\Delta_1\delta_1^{-1})[l - l(1,1)]$ and $n\Delta_2 = (\Delta_2\delta_2^{-1})[l(1,m) - l(1,1)]$.

Fix $\epsilon > 0$. Let $\bar{\epsilon} \equiv \epsilon(1-\beta)$ and let q_ϵ and ζ_ϵ satisfy $u(q_\epsilon) = 0.5\bar{\epsilon}$ and $u(q_\epsilon) - u(q_\epsilon - \zeta_\epsilon) = 0.25\bar{\epsilon}$.

Now I prove part (i) of the proposition. Fix N such that (**N**) $\forall n > N$ claim 2 holds as ϵ is replaced by ζ_ϵ. Fix $n > N$, m, $(q_b, l_b) \in \overline{\Gamma}^n_b(m)$, and $(q_s, l_s) \in \underline{\Gamma}^n_s(m)$. It suffices to show (19) and (20) hold. First, by the hypothesis, $B(q^*_b, l^*_b, m) \geq 0$ and $S(q^*_s, l^*_s, m) \geq 0$, where $(q^*_b, l^*_b) = (q(m,1), l(m,1))$ and $(q^*_s, l^*_s) = (q(1,m), l(1,m))$, so setting $(q,l) = (0,0)$ in claim 1 (a) and (b), we have $B_n(q^*_b, l^*_b, m) \geq 0$ and $S_n(q^*_s, l^*_s, m) \geq 0$. Next, suppose (20) does not hold, i.e., $S_n(q_s, l_s, m) - S_n(q^*_s, l^*_s, m) \geq \bar{\epsilon}$ so $(q_s, l_s) = (0,0)$. But then setting $(q,l) = (0,0)$ in claim 1 (b), we have $S(0,0,m) - S(q^*_s, l^*_s, m) \geq \bar{\epsilon}$, a contradiction to (11) when the household is large. Next, suppose (19) does not hold, i.e., $B_n(q_b, l_b, m) - B_n(q^*_b, l^*_b, m) \geq \bar{\epsilon}$. Again (q_b, l_b) cannot be $(0,0)$ so $(q_b, l_b) \notin \overline{\Gamma}^n_b(m) \backslash \underline{\Gamma}^n_b(m)$ and so $S_n(q_b, l_b, 1) - S_n(q^*_b, l^*_b, 1) \geq 0$, which by (N) implies $S(q_b, l_b, 1) - S(q^*_b, l^*_b, 1) > -\zeta_\epsilon$. Also, by setting $(q,l) = (q_b, l_b)$ in claim 1 (a), we have $B(q_b, l_b, m) - B(q^*_b, l^*_b, m) \geq \bar{\epsilon}$, $B_n(q_b, l_b, m) \geq \bar{\epsilon}$ and $B(q_b, l_b, m) \geq \bar{\epsilon}$. By $B(q_b, l_b, m) \geq \bar{\epsilon}$, $u(q_b) > \bar{\epsilon}$ so $q_b > q_\epsilon$. Setting $(q,l) = (q_b - \zeta_\epsilon, l_b)$, then $S(q,l,1) - S(q^*_b, l^*_b, 1) > 0$, and $B(q_b, l_b, m) - B(q,l,m) = u(q_b) - u(q_b - \zeta_\epsilon) < 0.25\bar{\epsilon}$ so $B(q,l,m) - B(q^*_b, l^*_b, m) > 0$, a contradiction to (12) when the household is large.

Next I prove part (ii) of the proposition. Let z_ϵ and l_ϵ satisfy $v(z_\epsilon) - v(0) = 0.5\bar{\epsilon}/\beta$ and $v(z_\epsilon) - v(z_\epsilon - l_\epsilon) = 0.25\bar{\epsilon}/\beta$ and let $\eta_\epsilon \equiv \beta v'_+(1) l_\epsilon > 0$. Fix N such that (**N'**) $\forall n > N$ claim 2 holds as ϵ is replaced by $\min\{\zeta_\epsilon, \eta_\epsilon\}$. Fix $n > N$, m and $\gamma(m) \in [\overline{\Gamma}^n_b(m) \times \underline{\Gamma}^n_s(m)]^I$. It suffices to show (21) holds. First, by concavity of u and v and Jensen's inequality, $W_n(\gamma(m)) \leq w = \pi_b u(q_b) + \pi^*_b u(q^*_b) - \pi^*_s q^*_s + \beta v(m_+)$, where (q^*_b, l^*_b) and (q^*_s, l^*_s) are given as above, $(q_b, l_b) \in \overline{\Gamma}^n_b(m) \backslash \underline{\Gamma}^n_b(m)$, each π term is non negative, $\pi_b + \pi^*_b + \pi^0_b = 1$, $\pi^*_s + \pi^0_s = 1$, and $m_+ = m - \pi_b l_b - \pi^*_b l^*_b + \pi^*_s l^*_s$. Now suppose (21) does not hold so $w - v(m) \geq \bar{\epsilon}$. Because (16) holds when the household is large, $\pi_b = 0$ implies $v(m) \geq w$. So $\pi_b > 0$, and by concavity of u and v and Jensen's inequality, we can assume $\pi_b^* = 0$. By $(q_b, l_b) \in \overline{\Gamma}_b^n(m) \backslash \underline{\Gamma}_b^n(m)$, $S_n(q_b, l_b, 1) - S_n(q_b^*, l_b^*, 1) \geq 0$, which by (N') implies $S(q_b, l_b, 1) - S(q_b^*, l_b^*, 1) > -\min\{\zeta_\epsilon, \eta_\epsilon\}$. Let $\delta_1 = \pi_b u(q_b) - u(q_b^*)$, $\delta_2 = (1-\pi_s^*)q_s^* + \beta[v(m'_+) - v(g(m))]$ and $\delta_3 = \beta[v(m_+) - v(m'_+)]$, where $m'_+ = m - l_b^* + \pi_s^* l_s^*$, so $w - v(m) = \sum_i \delta_i$. Note, as in the above, $S_n(q_s^*, l_s^*, m) \geq 0$; this and concavity of v imply $\delta_2 \leq 0$. Hence either $\pi_b u(q_b) - u(q_b^*) \geq 0.5\bar{\epsilon}$ or $v(m_+) - v(g(m)) \geq 0.5\bar{\epsilon}/\beta$. If the former then set $(q, l) = (q_b - \zeta_\epsilon, l_b)$; if the latter then set $(q, l) = (q_b, l_b + l_\epsilon)$. Either way, $S(q, l, 1) - S(q_b^*, l_b^*, 1) > 0$ and $\pi_b u(q) - q_s^* + \beta v(m - \pi_b l + l_s^*) > u(q_b^*) - q_s^* + \beta v(g(m))$, a contradiction to (16) when the household is large.

The proof of Proposition 4

First, I claim that $u(y(m, \omega, \omega))$ is strictly increasing, concave and differentiable in m, and $\beta v'(1) = \omega$. To see this, set $\omega_b = \omega_s = \omega$ in (24). It is clear that $l \mapsto y(l, \omega, \omega)$ is strictly increasing and so is $l \mapsto u(y(l, \omega, \omega))$. By the implicit function theorem, $l \mapsto y(l, \omega, \omega)$ is continuously differentiable and so is $l \mapsto u(y(l, \omega, \omega))$. Differentiating (24) w.r.t. l at $(\omega_b, \omega_s) = (\omega, \omega)$ and substituting $-y(l, \omega, \omega) + \omega l = (1-\lambda)[u(y(l, \omega, \omega)) - \omega l]/[\lambda u'(y(l, \omega, \omega))]$ (which is from rewriting (24) with $(\omega_b, \omega_s) = (\omega, \omega)$), we have

$$y'(l, \omega, \omega) = \frac{\lambda u'(y)\omega + (1-\lambda)\omega}{u'(y) - (1-\lambda)u''(y)[u(y) - \omega l]/u'(y)}, \tag{35}$$

where $y'(l, \omega, \omega)$ denotes the derivative of $y(.)$ w.r.t. its first argument and $y = y(l, \omega, \omega)$. By (35) and (U), some algebra confirms $u''y' + u'y'' \leq 0$ so $l \mapsto u(y(l, \omega, \omega))$ is concave. By (28), $v'(1) = u'(y(1, \omega, \omega))y'(1, \omega, \omega)$. Using this and (35) with $l = 1$ and (25), we have $\beta v'(1) = \omega$ (recall $y_\lambda = y(1, \omega, \omega)$). Next, it follows from $y_\lambda > 0$ and the claim that A in (27) satisfies (C1)-(C2). Substituting $g(m_b) = g(m_s) = 1$ and $\beta v'(1) = \omega$ (all implied by this A) into (22) and referring to (8)-(9), we see that A satisfies (22) if and only if (i) $\lambda u'(q(m_b, m_s))[-q(m_b, m_s) + \omega m_b] = (1-\lambda)[u(q(m_b, m_s)) - \omega m_b]$, and (ii) $u'(q(m_b, m_s)) \geq 1$. But (i) follows from (24) and (ii) from $u'(y(M, \omega, \omega)) \geq 1$. Finally, by Propositions 1-2 and the claim, f^A is a strong equilibrium.

The proof of Proposition 5

First, I claim that $u(y(m, 0, \tilde{\omega}))$ is continuous, concave and strictly increasing in m, $\beta v'_-(1) = \tilde{\omega}$ and $v'_+(1) = 0$. Continuity, monotonicity and $v'_+(1) = 0$ are obvious. Differentiating (24) w.r.t. l at $(\omega_b, \omega_s) = (0, \tilde{\omega})$ and substituting $-y(l,0,\tilde{\omega})+\tilde{\omega}l=(1-\lambda)u(y(l,0,\tilde{\omega}))/[\lambda u'(y(l,0,\tilde{\omega}))]$ (which is from rewriting (24) with $(\omega_b,\omega_s)=(0,\tilde{\omega})$), we have

$$y'(l,0,\tilde{\omega})=\frac{\lambda u'(y)\tilde{\omega}}{u'(y)-(1-\lambda)u''(y)u(y)/u'(y)}, \tag{36}$$

where $y'(l,0,\tilde{\omega})$ denotes the derivative of $y(.)$ w.r.t. its first argument and $y=y(l,0,\tilde{\omega})$. By (36) and (U), some algebra confirms $u''y'+u'y''\leq 0$ so $l\mapsto u(y(l,0,\tilde{\omega}))$ is concave. By (32), $v'_{-}(1)=u'(y(1,0,\tilde{\omega}))y'(1,0,\tilde{\omega})$. Using this, (36) with $l=1$ and (29), we have $\beta v'_{-}(1)=\tilde{\omega}$ (recall $\tilde{y}_\lambda=y(1,0,\tilde{\omega})$). Next, it follows from $\tilde{y}_\lambda>0$ and the claim that A in (31) satisfies (C1)-(C2). Substituting $g(m_b)=g(m_s)=1$ and $\beta v'_{-}(1)=\tilde{\omega}$ and $\beta v'_{+}(1)=0$ (all implied by this A) into (22) and referring to (8)-(9), we see that A satisfies (22) if and only if $\lambda u'(q(m_b,m_s))[-q(m_b,m_s)+\tilde{\omega}m_b]=(1-\lambda)u(q(m_b,m_s))$. But this follows from (24). Finally, see the main text for a beneficial joint deviation in equilibrium f^A.

The proof of Proposition 6

When the household is finite, A satisfies (22) with $\lambda=1$ (refer to (5) and (7)) if and only if

$$q(m_b,m_s)=n\beta[v(g(m_s)-l(1,m_s)n^{-1}+l(m_b,m_s)n^{-1})-v(g(m_s)-l(1,m_s)n^{-1})]; \tag{37}$$

$$\begin{aligned}[l(m_b,m_s) &= m_b]\Rightarrow[u'(q(m_b,m_s))v'_{-}(g(m_s)-l(1,m_s)n^{-1}+l(m_b,m_s)n^{-1})\\ &\geq v'_{+}(g(m_b)+l(m_b,1)n^{-1}-l(m_b,m_s)n^{-1})];\end{aligned} \tag{38}$$

$$\begin{aligned}[l(m_b,m_s) &< m_b]\Rightarrow[u'(q(m_b,m_s))v'_{+}(g(m_s)-l(1,m_s)n^{-1}+l(m_b,m_s)n^{-1})\\ &\leq v'_{-}(g(m_b)+l(m_b,1)n^{-1}-l(m_b,m_s)n^{-1})].\end{aligned} \tag{39}$$

Also notice that $g(1)=1$, which is used frequently below.

Now I prove part (i) of the proposition. Let Δ satisfy $u(\beta\Delta)<\Delta$, and let $\mathbf{V}$ be the set of functions $\nu:[0,M]\to\mathbb{R}$ that is continuous, non decreasing, concave, and with $\nu(1)-\nu(0)\leq\Delta$. Let $K=[1-n^{-1},1]$ and let $\mathbf{W}$ be the set of functions $w:K\to\mathbb{R}$ with $w=\nu$ on K for some $\nu\in\mathbf{V}$. For $w\in\mathbf{W}$ and $m\in[0,1]$, let

$$y(m,w)=n\beta[w(1-n^{-1}+mn^{-1})-w(1-n^{-1})]. \tag{40}$$

Then let $Gw : K \to \mathbb{R}$ be defined by

$$Gw(m) = u(y(m, w)) - y(1, w) + \beta w(1). \tag{41}$$

Suppose that w is a fixed point of G with $y(1, w) > 0$. Let the allocation A be defined by

$$q(m_b, m_s) = y(\min\{m_b, 1\}, w),\ l(m_b, m_s) = m_b. \tag{42}$$

This A determines the value function

$$v(m) = u(y(\min\{m, 1\}, w)) - y(1, w) + \beta v(1), \tag{43}$$

and note by $w = Gw$, $v = w$ on K. Clearly A satisfies (C1)-(C2). Substituting $g(m_b) = g(m_s) = 1$ and $v'_+(1) = 0$ (all implied by A) into (37)-(39), we immediately see that A satisfies (37)-(39). It is also easy to verify that f^A is not a Definition-2 strong equilibrium.

It remains to find a fixed point of G with $y(1, w) > 0$. To this end, let $\mathbf{W}$ be equipped with the sup norm topology, and I claim (i) $\mathbf{W}$ is compact and convex; and (ii) $G\mathbf{W} \subset \mathbf{W}$. For (i), it is obvious that $\mathbf{W}$ is convex and closed. Fix w and let $\nu \in \mathbf{V}$ satisfy $\nu = w$ on K. By $\nu(1) - \nu(0) \leq \Delta$, $w'_+(1 - n^{-1})$ must be bounded above by Δ. This and concavity of w imply that $\mathbf{W}$ is equicontinuous and hence compact (Arzelà-Ascoli theorem). For (ii), fix w, and it is obvious that Gw is non decreasing, continuous and concave. For this w, let $v \in \mathbf{V}$ be defined by (43) and it is obvious that $v = Gw$ on K; then by $y(1, w) \leq n\beta[w(1) - w(1 - n^{-1})] \leq \beta\Delta$ and $y(0, w) \geq 0$, we have $v(1) - v(0) \leq u(\beta\Delta) < \Delta$.

Because $\mathbf{W}$ is compact, $y(.,.)$ is uniformly continuous on $[0, 1] \times \mathbf{W}$ and so $w_j \to w$ implies $y(., w_j) \to y(., w)$ uniformly, that is, $G : \mathbf{W} \to \mathbf{W}$ is continuous. To rule out the trivial fixed point of G (the zero function), I introduce a sequence of auxiliary mappings. In specific, choose sufficiently large integer i so that $\forall w \in \mathbf{W}$, $w^i : K \to \mathbb{R}$ with $w^i(m) = w(m) + m/i$ is in $\mathbf{W}$. Then let $G^i w : K \to \mathbb{R}$ be defined by $G^i w(m) = Gw^i(m)$. Because the mapping $w \mapsto w^i$ is continuous, G^i is continuous, and so by Sauder's fixed point theorem, G^i has a fixed point w_i, that is, $w_i = G^i w_i = Gw^i_i$. Because $\mathbf{W}$ is compact, the sequence $\{w_i\}$ has a convergent subsequence. To simplify notation, denote this subsequence by $\{w_i\}$ and let w be its limit point. Because w is also the limit of point of $\{w^i_i\}$ and G is continuous and $w_i = Gw^i_i$, it follows that w is a fixed point of G. Because w^i_i is concave and strictly increasing, it follows that $w_i = Gw_i^i$ is concave and non decreasing, and hence that $w'_{i-}(1)$ is defined and nonnegative. It follows from $Gw_i^i = w_i$ and (40)-(41) that $w'_{i-}(1) = \beta u'(y(1, w_i^i))[w'_{i-}(1) + 1/i]$, which implies $y(1, w_i^i) \geq y_1$ (y_1 is defined by (25) and note $\beta u'(y_1) = 1$). By continuity of $y(1, .)$, $y(1, w) \geq y_1$. (It can actually be shown that $y(1, w) = y_1$.)

Next I prove part (ii) of the proposition. Suppose that A satisfies hypotheses but $n = 1$. Setting $(m_b, m_s) = (1, 1)$ in (37) and using $l(1, 1) = 1$ and $q(1, 1) > 0$, we have $v(1) > v(0)$. So by continuity and concavity of v, v is strictly increasing in a neighborhood of 0. By continuity, we can find $\bar{m} \in (0, 1)$ close to 0 so $v(\bar{m})$ close to $v(0)$ and so $\forall m \in [0, \bar{m}]$, $q(m, 1)$ close to 0 (again use (37) and note $l(m, 1) \leq m$) and $u'(q(m, 1)) > 1$.

Let $z \equiv \min\{m : v(m) = v(1)\} = 1$. First consider $z \geq 1$. I claim that $\forall m \in [0, \bar{m}]$, $m = l(m, 1)$ and $l(1, m) = 1$ so that $g(m) = 1$. By this claim and (37) and $l(1, 1) = 1$, $q(m, 1) = \beta v(m) - \beta v(0)$ and $q(1, m) = \beta v(1) - \beta v(0)$ so $v(m) = u(q(m, 1)) + \beta v(0)$. Then by $q(0, 1) = 0$ and $u(0) = 0$, we have $v(0) = 0$ so $v(m) = u(q(m, 1))$ or $v(m) = u(\beta v(m))$. But this cannot hold for all $m \in [0, \bar{m}]$, because u is strictly concave and v is concave.

To see the claim, fix $m \in [0, \bar{m}]$. First suppose $l(1, m) < 1$. Setting $(m_b, m_s) = (1, m)$ in (39) and using $l(1, 1) = 1$, we have

$$u'(q(1, m))v'_+(g(m)) \leq v'_-(2 - l(1, m)). \tag{44}$$

Setting $(m_b, m_s) = (1, 1)$ in (38) gives $u'(q(1, 1))v'_-(1) \geq v'_+(1)$. Comparing this with (44) and using $l(1, m) < 1$ and $q(1, m) < q(1, 1)$ (implied by (37) and $l(1, m) < l(1, 1)$), we have $g(m) \geq 1$ so $l(m, 1) < m$. Then setting $(m_b, m_s) = (m, 1)$ in (39) gives $u'(q(m, 1))v'_+(l(m, 1)) \leq v'_-(g(m))$, which by $u'(q(m, 1)) > 1$ implies $l(m, 1) \geq g(m)$, contradicting $g(m) \geq 1$ because $l(m, 1) < m \leq \bar{m} < 1$. So $l(1, m) = 1$. Now suppose $l(m, 1) < m$. Then again $u'(q(m, 1))v'_+(l(m, 1)) \leq v'_-(g(m))$ so $l(m, 1) \geq g(m)$. Given $l(1, m) = 1$, $l(m, 1) < m$ implies $g(m) > 1$, contradicting $l(m, 1) \geq g(m)$ because $m < 1$.

In case $z < 1$, $v'_+(z) = 0$. Using this and by the similar argument to establish the above claim, we can verify that $\forall m \in [0, \bar{m}]$, $l(m, 1) = m$ and $l(1, m) \in [z, 1]$, which again imply $v(m) = u(\beta v(m))$.

Appendix II: The programmer-automata interpretation

Here I study the same physical environment as in section 2. But now I assume that there is a unique decision maker in each household, labelled as the programmer, and that each agent (buyer or seller) is an automaton.

As indicated above, Shi [16] and most recent literature seemingly adopt this interpretation of the household construct. Following this literature, I restrict attention to the ultimatum game (buyers make offers) in pairwise meetings.

First, I provide a formal definition of equilibrium. For a generic household, the programmer's set of actions in a date is $K = \prod_{i \in I} K_b^i \times \prod_{i \in I} K_s^i$, where $K_b^i = K_b$ is the set of offering programs, and $K_s^i = K_s$ is the set of responding programs; that is, the programmer chooses for each of its buyers an offering program from K_b, and for each of its sellers a responding program from K_s. An offering program is a pair of real-valued functions $(q(.), l(.))$ on $[0, M]^2$; a responding program is a function $\sigma(.)$ from $[0, M]^2 \times \mathbb{R}_+^2$ to $\{0, 1\}$.

To be specific, consider a meeting between a buyer from a household whose start-of-date money holding is m_b, and a seller from a household whose start-of-date money holding is m_s. Let $(q(.), l(.))$ be the offering program carried by the buyer and let $\sigma(.)$ be the responding program carried by the seller. Then the buyer's offer is $(q(m_b, m_s), l(m_b, m_s))$. If the $0 \leq l(m_b, m_s) \leq m_b$ and $\sigma(m_b, m_s, (q(m_b, m_s), l(m_b, m_s)) = 1$, then this offer is carried out; otherwise, there is no trade.

Let F denote a generic strategy of a generic programmer. I restrict attention to F that does not depend on the programmer's private information and the calendar time, so I can express F as a mapping from $[0, M]$ to K. I restrict attention to symmetric equilibrium. Formally, an equilibrium is a strategy profile represented by F so that it is optimal for a programmer to choose programmes indicated by F provided that all other programmers to choose programmes indicated by F currently and in the future. Note any equilibrium is degenerate (in the same sense as in the main text).

Next, I show that there exists a continuum of equilibria when the household is large. To this end, let y_1 be the one given by (25)-(26) and let $M > 1$ satisfy $u'(My_1) \geq 1$. Fix $y \in (0, y_1]$ and let $F(m) = \{(q_m^i(.), l_m^i(.)), \sigma_m^i(.)\}_{i \in I}$ be such that $(q_m^i(.), l_m^i(.)) = (q_m(.), l_m(.))$ and $\sigma_m^i(.) = \sigma_m(.)$ $\forall i$, where

$$\begin{aligned}
&\forall (m_b, m_s),\ (q_m(m_b, m_s), l_m(m_b, m_s)) = (m_b y, m_b)\,, \\
&\sigma_m(q, l, m_b, m_s) = 1 \text{ if } q \leq ly, \text{ and } \sigma_m(q, l, m_b, m_s) = 0 \text{ otherwise.}
\end{aligned}$$

If all programmers follow F, then the value function defined on the household's money holdings is the unique continuous, strictly increasing, and concave function satisfying $v(m) = u(my) - y + \beta v(1)$. Using $u'(My) \geq 1$, one can verify that this v also satisfies

$$v(m) = \max_{0 \leq l \leq m, 0 \leq \rho \leq 1} u(ly) - \rho y + \beta v(m + \rho - l). \tag{45}$$

Then by (45) and concavity of u and v and Jensen's inequality, one programmer does not gain from deviations provided that all other programmers follow F.

Therefore, for each $y \in (0, y_1]$, there exists an equilibrium when the household is large. In the equilibrium, $(y, 1)$ is the regular-meeting trade (in the same sense as in the main text), and the value function v on the household's money holdings is concave. In contrast, under the team interpretation, if A with $l(1, 1) = 1$ and with a concave v is an equilibrium allocation generated by the ultimatum game when the household is large, then it can be shown that the regular-meeting output equals y_1.

Finally, I provide a brief comparison with Shi [16]. Shi does not give explicit description about the household's set of actions and strategies. He formulates a value function v comparable to the one in (45). He argues that the regular-meeting output y must equal $\beta v'(1)$, which implies $y = y_1$. Shi's argument for $y = \beta v'(1)$ is based on a reasoning applied to perfect equilibrium in a sequential-move game, i.e., a regular seller accepts any offer (q, l) no worse than no trade. But this reasoning has no bite in the automata game (which is a simultaneous-move game).

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