A Theory of Continuum Economies with Idiosyncratic Shocks and Random Matchings

Karavaev, Andrei

Pennsylvania State University

25 February 2008

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Andrei Karavaev[2]

This version: February 2008

"Entities should not be multiplied beyond necessity."

Occam's razor

Many economic models use a continuum of negligible agents to avoid considering one person's effect on aggregate characteristics of the economy. Along with a continuum of agents, these models often incorporate a sequence of independent shocks and random matchings. Despite frequent use of such models, there are still unsolved questions about their mathematical justification. In this paper we construct a discrete time framework, in which major desirable properties of idiosyncratic shocks and random matchings hold. In this framework the agent space constitutes a probability space, and the probability distribution for each agent is replaced by the population distribution. Unlike previous authors, we question the assumption of known *identity* — the location on the agent space. We assume that the agents only know their *previous history* — what had happened to them before, — but not their identity.

The construction justifies the use of numerous dynamic models of idiosyncratic shocks and random matchings.

Key Words and Phrases: random matching, idiosyncratic shocks, the Law of Large Numbers, aggregate uncertainty, mixing.

JEL Classification Numbers: C78, D83, E00.

[1]I am indebted to my advisor Kalyan Chatterjee who guided this work. This paper would not be possible without his encouragement. I also thank Edward Green, James Jordan, and Sophie Bade for their helpful comments.

[2]Department of Economics, 608 Kern Graduate Building, The Pennsylvania State University, University Park, PA 16802. E-mail: akaravae@psu.edu.

1. Introduction

The Problem

A large number of economic models consider an uncountable number of negligible agents who experience idiosyncratic shocks and randomly meet each other. Some examples of such models are given by Aliprantis et al. [2], Alós-Ferrer [4], and Boylan [7]. The assumptions in these models are made in the spirit of the Law of Large Numbers. For example, the mixing property assumption is often used, which states that the fraction of the agents from one set who are matched with the agents from another set equals the measure of the second set.[3] Until recently there was no formal mathematical evidence that such models exist. Moreover, some economists pointed out serious contradictions among the standard assumptions used.

The existence of many economic models with both idiosyncratic shocks and random matchings[4] makes it impossible to discuss all the discrepancies that arise. Therefore, we give here only two most famous examples of contradictions in the standard assumptions.

One of the most famous contradictions was described by Judd [19].[5] It has been assumed that the average of a continuum of independent and identically distributed random variables (idiosyncratic shocks) is nonrandom (no aggregate uncertainty property). Judd shows that the nonrandomness of the average neither contradicts nor follows from the independence and identical distribution of the shocks. Moreover, Judd proves that if the population is represented by the unit interval with the Borel σ-algebra, then most of the shock realizations on the agent space are not measurable and therefore the average shock cannot be calculated. Feldman and Gills [12] notice that measurable functions on the unit interval are in some sense "almost continuous." However, "almost continuity" of shock

[3]Shi [21] describes the following matching process with the mixing assumption: "... the distribution of different types of matches for each household is almost surely nonrandom, although each member in the household is uncertain about the kind of agent he will meet."

[4]The word "meeting" will be used with respect to one agent. The one-time process of all the agents of the population being paired up with each other will be called "matching."

[5]This and the random matching example will be considered in details in Section 3.3. "Standard Model Inconsistencies."

realizations contradicts the independence of shocks for different agents implied by the idiosyncratic nature of the shocks.[6]

A second contradiction is about random matching assumptions. It was described by McLennan and Sonnenschein [20] in footnote 4. Using Proposition 1 from Feldman and Gilles [12], the authors show that a measure preserving matching can not be mixing. The source of the contradiction is also measurability: the agents who are "close" to each other on the agent space should be paired up with the agents who are also "close" to each other,[7] which contradicts the randomness of the meetings.

There exist several solutions to the idiosyncratic shocks problem (see the literature review for the details.) There are also several papers dealing with the random matching problem. At the same time, to our knowledge, there is no paper dealing with both idiosyncratic shocks and random matching. Furthermore, no existing paper completely justifies the mixing property of a random matching simultaneously for all the measurable subsets of the agents, which is the cornerstone of many economic models. In this paper we build a mathematically valid discrete time model of idiosyncratic shocks and random matchings, which resolves the conflicting issues.

The Causes of the Problem

All the contradictions among the standard assumptions stem from an uncountable number of the agents. The very idea of considering a non-atomic agent space comes from the convenient properties of a finite but large population of agents. For example, because of the Law of Large Numbers, in a large but finite population the average shock is close to the average shock for each agent. Thus, by using a continuum of agents, one wants to achieve two important goals. The first one is to have negligible agents who have no influence on the aggregate characteristics of the economy. The second goal is to have an analogue of the Law of Large Numbers with respect to shocks and meetings.[8]

[6] Measurable functions on the unit interval have a good approximation by the continuous functions (Luzin's theorem); Al-Najjar [1] in Theorem 8 shows that "a typical realization of an i.i.d. process can not be approximated by a continuous function."

[7] As Alós-Ferrer [4] puts it, "the very concept of matching destroys the most basic independence aspirations."

[8] As Gilboa and Matsui [15] wrote, "...there are uncountably many agents of various types, each of which has no effect whatsoever on the aggregate behavior, thus eliminating strategic considerations which

A replacement of a large but finite agent space with an uncountable space of negligible agents has several hidden problems arising from significant differences in the structures of these spaces. For a finite space, the measure comes naturally (the counting measure) and in the unique way. One-to-one matching automatically guarantees the measure preserving property. The natural discrete σ-algebra consists of all the subsets, making all the functions measurable. For an uncountable space, the choice of the measure is ambiguous. Additional assumptions should be made to guarantee the measure preserving property of the matching. Measurability seriously restricts the set of functions and matchings.[9]

The Main Idea

The existing models assume that the agents know their identity — the exact location on the agent space. For example, the requirement that the shocks are independent for different agents is only necessary if the agents know who they are. Because of the known identity, the standard setup requires the space of the states of the world, which allows us to model the uncertainty the agents face about their future shocks and meetings. Therefore, two different spaces are needed: the space of the agents and the space of the states of the world. The shocks and matchings are defined on their product.

However, in many economic models the knowledge of identity is excessive and has no influence on the results. What the models really require is that the agents have their own attributes, like type, initial endowment, etc.[10] Another requirement is that the agents perceive their future as random.[11] With these assumptions, the agents use strategies

extend beyond a specific encounter." Alós-Ferrer [5] notices that "...oftentimes, agents have to be modeled as being negligible." Alós-Ferrer [4] also mentions convenience in using a continuum of agents because of analytical simplicity and anonymity properties.

[9]Al-Najjar [1] discusses the measurability problem in Section 4. As he notices, this problem emerges because the agents who are ex ante similar have to have independent shocks ex post.

[10]Kandori [19] describes the following rules: "1. A label is attached to each agent. 2. Before executing trade each agent observes his and his partner's label. 3. A player and his partner's actions and labels today determine their labels tomorrow."

[11]Gale [14] requires the following form of randomness: "the probability of an active agent meeting an agent whose history <belongs to some set> is independent of the first agent's history."

which depend only on the history and some initial attributes.[12] The idea of such history-dependent strategies implies that the agents do not know their identities; otherwise they would include it in their strategies.

Based on the idea of unknown identity, we suggest to simplify the setup and consider only one space — the agent space, which also serves as the probability space. With this approach the agents still face some randomness. It does not come from the realization of the state of the world, but from the unknown identity. Each agent believe that he was placed randomly (and uniformly) on the agent space. The shocks and meetings are predetermined.[13] However, the agents do not know their identities and therefore perceive thir shocks and meetings as random. The events help the agents to refine the knowledge of their identities, but do not resolve it completely. Based on the previous shocks and meetings, the agents update their beliefs about the future.[14]

The Results

The main result of the paper (Theorem 2) states that there exists a mathematically correct dynamic discrete time model of negligible agents with idiosyncratic shocks and random matchings. The shocks and matchings are measurable. The matchings are measure preserving. For any agent his shocks and meetings are independent of the past events. The equivalent of the Law of Large Numbers holds with respect to the σ-algebras generated by the histories.

Many discrete time economic models with idiosyncratic shocks and random matchings can be reformulated in the new setup. The model is very flexible. It can incorporate

[12]Green and Zhou [17] write: "an agent's strategy is a function of only his own trading history and initial money holdings..."

[13]This makes the shocks and meetings of different agents to be dependent. This idea is not new; it was explored in Feldman and Gilles [12] and Alós-Ferrer [5] and [4]. Boylan [7] and Alós-Ferrer [4] showed that the matching scheme should be dependent on the assignment of types (shocks). However, in these papers the agents know their identities and therefore this dependence can influence the choice of the strategies.

[14]This approach is similar to the Kolmogorov definition of the probability space. Namely, in Kolmogorov's definition random variables are some measurable functions on the probability space Ω. The realization $\omega \in \Omega$ is not known to the observer(s); initially it is chosen at random. The observers(s) can update the belief about ω based on the events.

some additional characteristics of the agents, such as type, product consumed, etc. It also permits shocks and matchings that depend on the agents' histories or other characteristics.

Plan of the Paper

The rest of the paper is organized as follows. Section 2 contains the literature review. Section 3 discusses the standard assumptions used in economic models and shows the problems arising from the use of these assumptions. In section 4, we give the formal definition and prove the existence of idiosyncratic shocks and random matchings. Section 5 concludes. All the proofs are given in the Appendix.

2. Literature Review

All the related literature naturally falls into two different categories: idiosyncratic shocks and random matchings.

2.1. Idiosyncratic Shocks

Judd [19] and Feldman and Gilles [12] were among the first economists to notice that for the unit interval of agents with the Borel σ-algebra and Lebesgue measure either the shocks are not measurable across the population, or the equivalent of the Law of Large Numbers does not hold.[15] Several remedies to the problem of idiosyncratic shocks were offered by Judd, Feldman and Gilles, and other economists.

Judd [19] solves the problem of idiosyncratic shocks on the whole agent space. The idea is to build an extension of the agent space so that the Law of Large Numbers is satisfied. The author notices that the extension is not unique, and for other extensions the Law of Large Numbers might fail.[16] Feldman and Gilles [12] suggest relaxing the assumption of shocks independence or considering finite or countably infinite approximations of the

[15]Judd proves that the measure of the realizations for which the sample distribution function on the unit interval of agents does not exist has inner measure zero and outer measure one. He also shows that there is an extension of the whole space for which the Law of Large Numbers holds for the whole agent space. Feldman and Gilles show that there is no extension for which "a law of large numbers is valid simultaneously for all members of σ-algebra." As Sun [22] mentions, Doob [9] noticed long ago that "the sample functions of such a process are usually too irregular to be useful."

[16]Several other authors, including Green [16] and Alós-Ferrer [5], mention non-uniqueness of the extension. From our point of view, non-uniqueness does not really constitute a problem. The goal is to find a

agent space. A countably infinite population provides an appropriate idealization of a large economy. The authors show that there exists a density charge on the agent space such that the Law of Large Numbers is satisfied. Al-Najjar [1] also considers a sequence of finite but increasingly large economies. The continuum-like laws, including the Law of Large Numbers for any subinterval of the $[0, 1]$ set of agents, hold in important aspects, although the measure is not countably additive and the integral might not coincide with the Lebesgue integral.

Developing the idea of dependent shocks, Alós-Ferrer [5] suggests to consider a population extension. To illustrate the population extension, consider a randomly rotated circumference of agents which is naturally mapped onto the original circumference. The shock (chosen from a finite set) of an agent is the one that was originally at the point of the circumference before the rotation. The shocks in this model are not independent, but the Law of Large Numbers holds on the whole space.

Instead of the Lebesque integral, Uhlig [23] uses the Pettis integral to calculate the average shock, which captures the idea of a countable normalized sum of shocks. On the unit interval of agents the Pettis integral is equivalent to the L_2-Riemann integral. If the shocks are pairwise uncorrelated and have the same mean and uniformly bounded variances, the Pettis integral exists. Uhlig shows that the Law of Large Numbers holds: the Pettis integral on any measurable subset of the agents is almost everywhere constant.

Green [16] changes the agent space by endowing the unit interval with a σ-algebra richer then the Borel σ-algebra. He constructs a family of iid variables on this nonatomic space so that for any subset of the population with a positive measure the Law of Large Numbers holds almost surely. Sun [22] uses hyperfinite Loeb spaces to demonstrate the same type of no aggregate uncertainty.[17] The results on the hyperfinite space can be routinely translated into large finite populations models.

viable mathematical setup, and the question we want to answer is if we can justify using an uncountable number of agents."

[17]Judd [19] suggested using hyperfinite discrete models from nonstandard analysis to solve the problem of idiosyncratic shocks.

2.2. Random Matching

The problems of random matching and idiosyncratic shocks have the same roots. Nevertheless, the random matching problem is more challenging because of the additional assumptions on the structure of the matching. Several authors suggested their remedies to the problem, which essentially use the same ideas as the solutions of the idiosyncratic shocks problem.

Gilboa and Matsui [15] were the first to approach the random matching problem. They consider a continuous time model with only a few individuals out of a countable population meeting at each period of time. Each individual meets someone only once. The probability measure in the model is finitely additive. The authors satisfy the properties of no aggregate uncertainty for any measurable set of the agents and randomness of the meetings. Boylan [7] also considers countably many agents. The Law of Large Numbers is formulated with respect to a finite set of agent types. The randomness implies that the type proportions evolve in accordance with the mixing property. The author develops a matchings scheme in which the mixing property holds, however the matching scheme can not be independent of the assignment of types. In [8] Boylan discusses the conditions under which a finite deterministic matching approximates a random matching process with continuous time as the population grows to infinity.

Alós-Ferrer [4] considers a random matching of a continuum of agents. He uses the tool of population extension developed in Alós-Ferrer [5] and proves the existence of a random matching satisfying the Law of Large Numbers properties with respect to a finite set of types. In [6], the author extends the results to several populations. Duffie and Sun [11] employ the framework of hyperfinite Loeb spaces (see Sun [22] above). The matching agents are assigned a finite number of types. The authors find an agent space and a random matching satisfying the following properties: measure preserving, uniform distribution, and mixing (for any two given sets of the agents the mixing property holds with probability one). For any two different agents their partners' types are independent. The matching scheme is independent in types for any assignment of types. In [10] the authors study the type distribution evolvement induced by the matchings and random mutations.

Although Aliprantis et al. [2] and [3] do not suggest any solution to the random matching problem, they use a set-theoretical approach to build a foundation for the random matching models. The basic object they consider is a *cluster* — several agents meeting at some period of time. The matching rule consists of non-intersecting k-element clusters. The stochastic matching rule is defined as a probability measure over all k-clustering matching rules. Although the authors mainly concentrate on finite-agent models, they define such general concepts as direct and indirect partners, along with anonymous and strongly anonymous sequences of k-clustering matching rules.

2.3. Why Further Solution?

Although the papers mentioned partially solve the problems of idiosyncratic shocks and random mmatchings, no solution guarantees no aggregate uncertainty/mixing property simultaneously for all the measurable subsets of the agent space (many of them also do not have other important characteristics, like countably additive measure, anonymous meetings, Lebesgue integral, independent shocks/meetings, etc.) No paper deals with both idiosyncratic shocks of a general distribution and random matchings.

Duffie and Sun [11] suggest a random matching in which the properties do not hold almost surely for all the subsets; they hold for a given set/sets of agents almost surely. For example, the mixing property is formulated in the following way: for any two sets of agents the mixing property (or an equivalent of it with respect to a finite set of types) holds with probability one. It means that with non-zero probability one may be able to find two sets for which the mixing property does not hold. McLennan and Sonnenschein in [20] emphasize that the properties should hold almost surely simultaneously on all the measurable subsets of the agents.[18]

3. Standard Assumptions

In this section we list the assumptions usually made about idiosyncratic shocks and random matchings. Particular assumptions may vary, therefore we give here the most

[18]For a discussion of different formulations of the properties, see subsection 3.2. "Different Formulations of the Properties." In this subsection we provide different formulations of the properties and discuss which formulation was used by which author.

general setup. For simplicity and without loss of generality, time is not taken into account in this section. After defining the assumptions, we discuss their possible alternative formulations. Several problems that arise under the standard assumptions conclude this section.

3.1. The Model

Let A be the agent space with σ-algebra $\mathcal{A}$ and probability measure μ. Let $(\Omega, \mathcal{F}, \mathbf{P})$ be the state space. The assumption of an agent negligibility is usually made to ensure that no particular agent can influence the aggregate characteristics of the economy, i.e. measure μ is non-atomic. The unit interval $[0, 1]$ of the real line $\mathbb{R}$ is often taken as the agent space A.

Every agent $a \in A$ first experiences shock ξ and then meets with other agent in accordance with some rule $\mathbf{M}$. The rest of the assumptions describe those two objects and fall into three different categories: assumptions about the shocks (shocks are idiosyncratic), assumptions about the matching (meetings are random), and assumptions about the joint properties of the shocks and matching.

Idiosyncratic Shocks[19]

An idiosyncratic shock[20] is a function $\xi : A \times \Omega \to \mathbb{R}$ such that the following properties hold:

A1. Measurability: for any $a \in A$ function $\xi_a(\cdot)$ is $\mathcal{F}$-measurable; for any $\omega \in \Omega$ function $\xi_\omega(\cdot)$ is $\mathcal{A}$-measurable;

A2. Identical distribution: for any $a \in A$ shock $\xi_a(\cdot)$ has cdf $F(x)$;

A3. Independence: for any different $a_1, a_2, \ldots, a_l \in A$ corresponding shocks $\xi_{a_1}(\cdot)$, $\xi_{a_2}(\cdot), \ldots, \xi_{a_l}(\cdot)$ are independent;

[19]Some parts of this definition are given in many papers, including Alós-Ferrer [5], Feldman and Gilles [12], Green [16], Judd [19], and Sun [22].

[20]The singular form "shock" is used because this is just one function. Plural "shocks" will be used for many functions at different periods of time, or to refer to the shocks of different agents at one period of time. The exact meaning will be clear from the context.

A4. No aggregate uncertainty:[21] the sample distribution of the shock equals $F(x)$ for any positive-measured $B \in \mathcal{A}$, i.e.

$$\forall \omega \in \Omega,\ \forall x \in \mathbb{R} \quad \mu(\{a \in B : \xi_\omega(a) \le x\}) = F(x)\mu(B).$$

We used the formulation of the no aggregate uncertainty property with respect to the distribution (Glivenko-Kantelly type). Judd [19], Green [16], Alós-Ferrer [5] use the same formulation. Some authors prefer to formulate it with respect to the averages, see Feldman and Gilles [12]:[22]

$$\int\limits_B \xi_\omega(a)\, d\mu(a) = \mu(B) \int\limits_{\mathbb{R}} x\, dF(x).$$

Random Matching[23]

A random matching, which determines whom everyone meets for each $\omega \in \Omega$, is a mapping $\mathbf{M} : A \times \Omega \to A$ such that the following properties hold:

B1. Everyone is met by someone:

$$\forall \omega \in \Omega \qquad \mathbf{M}_\omega(A) = A;$$

B2. No agent meets himself:

$$\forall \omega \in \Omega,\ \forall a \in A \qquad \mathbf{M}_\omega(a) \ne a;$$

B3. The partner's partner is the agent himself:

$$\forall \omega \in \Omega,\ \forall a \in A \qquad \mathbf{M}_\omega(\mathbf{M}_\omega(a)) = a;$$

B4. Measurability: for any $a \in A$ operator $\mathbf{M}_a(\cdot)$ is $\mathcal{F}$-measurable; for any $\omega \in \Omega$ operator $\mathbf{M}_\omega(\cdot)$ is $\mathcal{A}$-measurable;

[21]Sun [22] says that this property with respect to the average is often referred to as "aggregation removes uncertainty" and Feldman and Gilles say that "risks disappear in the aggregate." Alós-Ferrer [5] writes "individual uncertainty vanishes upon aggregation." Green [16] calls it the "idealized Glivenko-Cantelly property."

[22]Alós-Ferrer [5] calls this formulation "less demanding."

[23]Gale [14] and McLennan and Sonnenschein [20] define random matching in a similar way; Alós-Ferrer [4] and Boylan [7] give an analogous list of properties, including some properties with respect to types. Duffie and Sun [11] give the closest definition.

B5. Measure preserving:

$$\forall \omega \in \Omega,\ \forall B \in \mathcal{A} \qquad \mu(\mathbf{M}_\omega(B)) = \mu(B);$$

B6. Uniform distribution: every agent has equal probability of meeting everyone else:[24]

$$\forall a \in A,\ \forall B \in \mathcal{A} \qquad \mathbf{P}(\{\omega : \mathbf{M}_a(\omega) \in B\}) = \mu(B);$$

B7. Independence: for any different $a_1, a_2, \ldots, a_l \in A$ random variables $\mathbf{M}_{a_1}(\cdot)$, $\mathbf{M}_{a_2}(\cdot), \ldots,\ \mathbf{M}_{a_l}(\cdot)$ are independent;[25]

B8. Mixing: the fraction of the agents from one set who meet the agents from another set equals the measure of the second set:

$$\forall \omega \in \Omega, \forall B_1, B_2 \in \mathcal{A} \qquad \mu(\mathbf{M}_\omega(B_1) \cap B_2) = \mu(B_1)\mu(B_2),$$

i.e. sets $\mathbf{M}_\omega(B_1)$ and B_2 are independent.[26]

Joint Independence

Joint independence between shocks and meetings may be formulated in numerous ways.[27] Many papers have the following common element in the formulation: based on their past histories the agents can not infer any informative conclusion about current event. Different formulations of the property come from different kinds of information the

[24]Boylan [7] shows that in an economy with countably many agents it is impossible to satisfy this property directly; he therefore formulates it (Property II) with respect to types. Alós-Ferrer [4] requires that "given a fixed individual, any other agent were equiprobable as its partner." Alós-Ferrer [4] calls this property with respect to a finite set of types "Type's proportional law," and the general property "General proportional law." Duffie and Sun [11] give the closest formulation.

[25]Duffie and Sun [11] formulate this property with respect to a finite set of types.

[26]Boylan [7] formulates this property (Properties III or IV) with respect to a finite set of types. Alós-Ferrer [4] requires that "the proportion of matches between agents of two given types is equal to (twice) the product of the proportions of agents of those types." Alós-Ferrer [4] calls this property with respect to a finite set of types "Type's mixing", and the general property "Strong mixing." McLennan and Sonnenschein [20] and Duffie and Sun [11] give the closest formulation.

[27]Green and Zhou [17] give the following kind of joint independence: "the probability distribution of the trading partners bid and offer should be identical to the sample distribution."

agents remember or share during the meetings. For example, possible information available to the meeting parties can be the payment during the transaction, the identification of the partner, or the payments in all the partner's previous transactions, and so on.[28]

Assuming that the agents first learn their shock and then meet and learn their partner's shock, joint independence means that the partner's shock is independent of the agent's own shock. Suppose that the agents have types as in Boylan [7], Duffie and Sun [11], and Alós-Ferrer [4]. We see the types as shocks. Boylan [7] considers the following form of independence: (1) for any agent his probability of being matched with an agent of some type equals the fraction of the agents of this type; (2) the fraction of the agent of one type who meet the agents of another type equals the product of the fractions of the agents for both types.[29] Boylan shows that a random matching scheme can not be independent of the assignment of types.[30] Duffie and Sun [11] require that (1) for almost every agent the probability of meeting a partner with some type equals the fraction of the agents of this type, and (2) for almost every agent his partner's shock is pairwise independent of other agent partner's shocks for almost all other agents. Alós-Ferrer [4] considers the following forms of independence: (types' proportional law) for all agents the probability of meeting a partner with some type equals the fraction of the agents of this type; (types' mixing) the fraction of the agent of one type who meet the agents of another type equals the product of the fractions of the agents for both types.

3.2. Different Formulations of the Properties

We formulated the properties for the idiosyncratic shocks and random meetings for all $\omega \in \Omega$, although we also could use "for almost all ω" formulations. Significant differences in the formulations may arise for the properties dealing with the subset of the agents. For

[28]Kandori [19] describes the following information rules: "1. A label is attached to each agent. 2. Before executing trade each agent observes his and his partner's label. 3. A player and his partner's actions and labels today determine their labels tomorrow."

[29]Boylan [7] describes it as a "subpopulation facing the distribution of types equal to the population distribution."

[30]Boylan [7] shows that in his setup of countably many agents the independence of the random matching scheme from the assignment of types contradicts the condition that for any agent his probability of being matched with an agent of some type equals the fraction of the agents of this type.

example, consider no aggregate uncertainty property. The following different formulations, in addition to A4, are possible:

A4'. For any measurable subset of the agent space with probability one there is no aggregate uncertainty, i.e.

$$\forall B \in \mathcal{A} \qquad \mathbf{P}\left(\{\omega : \ \forall x \in \mathbb{R} \quad \mu(\{a \in B : \xi_\omega(a) \leq x\}) = \mu(B)F(x)\}\right) = 1.$$

A4". With probability one there is no aggregate uncertainty for any measurable subset of the agent space, i.e.

$$\mathbf{P}\left(\{\omega : \ \forall B \in \mathcal{A}, \forall x \in \mathbb{R} \quad \mu(\{a \in B : \xi_\omega(a) \leq x\}) = \mu(B)F(x)\}\right) = 1.$$

The difference between A4' and A4" is in the location of the clause "for almost all ω." In definition A4' we first fix the subset of the agents, and then say that for this subset for almost all ω there is no aggregate uncertainty on this subset. For A4", for almost all ω, for any measurable subset of the agents there is no aggregate uncertainty. Obviously, A4" follows from A4, and A4' follows from A4", but not a vice versa:

$$\text{A4} \Rightarrow \text{A4"} \Rightarrow \text{A4'}.$$

The following example shows the difference between A4' and A4".[31] Consider a set of independent random variables $\{\tau_i\}_{i \in \mathbb{N}}$, uniformly distributed on the unit interval $[0, 1]$. For a given $B \subset [0, 1]$ define $\nu_\omega(B) = \lim_{n \to \infty} \frac{\#\{i < n : \tau_i \in B\}}{n}$, if it exists. From the Law of Large Numbers, for any Borel-measurable B with Lebesgue measure $\mu(B)$ we have $\nu_\omega(B) = \mu(B)$ with probability 1. However, for any particular ω sequence τ_i consists of a countable number of elements (with Lebesgue measure 0), therefore there are plenty sets B with non-zero measure such that $\nu_\omega(B) = 0$ (and many measurable sets B for which $\nu_\omega(B)$ is not defined.)

We can use similar different formulations of other properties, like measure preserving (B5), uniform distribution (B6), etc. Some properties allow additional interpretations. For example, independence (A3 or B7) allows the following formulation: for almost every

[31] This example has the same idea as the one given by Al-Najjar [1] in footnote 17.

agent his shock (partner) is independent of the shock (partner) of almost every other agent.[32]

Different authors use different formulations of no aggregate uncertainty property in their papers. Feldman and Gilles [12] use several of them: in equation 1 they use A4'-type property; in Proposition 1 they switch to A4"-type; in Proposition 2 (the existence of idiosyncratic shocks) they use no aggregate uncertainty on the whole space; in Proposition 3 they again adopt A4'-type. Judd [19] and Alós-Ferrer [5] require no aggregate uncertainty the whole space only. Al-Najjar [1] provides a model in which the Law of Large Numbers holds only on all subintervals.[33] Green [16] satisfies the no aggregate uncertainty A4' with respect to any measurable subset of the population. Uhlig [23] and Sun [22] also use property A4'.

No paper resolving the random matching problem deals with A4"-type of mixing property. Gilboa and Matsui use A4'-type property (they fix the sets, and then define the probabilities.) Duffie and Sun [11] also deal with A4'-type properties. At the same time, some authors need for their models A4"-type and not A4'. In particular, McLennan and Sonnenschein [20] (footnote 4) require (using our notation) that "with probability one, $\mu(\mathbf{M}(B_1, \omega) \cap B_2) = \mu(B_1)\mu(B_2)$ for all Borel sets B_1, B_2."

The question is, which formulation suits better a discrete time economic model with idiosyncratic shocks and random matchings? Consider an arbitrary agent. This agent should not be able to infer any informative conclusion about his future from the past. The agent has history — what had happened to him before. Therefore, he can associate himself with many subsets of the agents (those who had some shock at the previous period of time, those whose partner had specific shock, etc.) Formulation A4' guarantees that for a particular subset there is no aggregate uncertainty with probability one. But it does not guarantee that no agent belongs to a set for which aggregate uncertainty exists. For formulation A4', with probability one there is no aggregate uncertainty for any of the sets to which the agent belongs. And exactly this formulation will be used in our main result.

[32] Duffie and Sun [11] use this formulation.

[33] Al-Najjar provides an example in Footnote 17, showing that although the Law of Large Numbers holds for all the subintervals, for any ω there exists a subset of agents on which the Law of Large Numbers fails. The reason for this is the example considered above.

3.3. Standard Model Inconsistencies

Although the standard assumptions given above look intuitively natural, some of them contradict each other. Here we consider several possible problems which one can face while using them.

There Always Exists a Subset with Aggregate Uncertainty[34]

It is often assumed that the no aggregate uncertainty property holds simultaneously on all the measurable subsets of the agent space (property A4”). The following proposition shows that this is impossible for any non-degenerate $F(\cdot)$.

Proposition 1.[35] *Suppose that $\xi_\omega(a)$ is $\mathcal{A}$-measurable and is not constant $\mu(\cdot)$-almost surely. Then there exists a positive-measured $B \in \mathcal{A}$ such that*

$$\int\limits_B \xi_\omega(a)\, d\mu_B(a) \neq \int\limits_A \xi_\omega(a)\, d\mu(a),^{36} \tag{1}$$

where $\mu_B(\cdot)$ stands for the measure on B induced by $\mu(\cdot)$: for any $X \subset B$, $X \in \mathcal{A}$

$$\mu_B(X) = \mu(X)/\mu(B).$$

The result is a generalization of Proposition 1 from Feldman and Gilles [12]. (We do not specify the distribution function $F(\cdot)$ and consider a general agent space A.) Although Feldman and Gilles claim that the problem is in “almost” continuity of measurable functions, the proposition holds for an arbitrary space. This result demonstrates that the standard model does not allow no aggregate uncertainty simultaneously on all the measurable subsets of the agent space. The other authors bypass this difficulty by considering other types of no aggregate uncertainty: A4’-type no aggregate uncertainty, no aggregate uncertainty on the whole space, or no aggregate uncertainty on a countable set of agent subsets.

[34] Alós-Ferrer [5] refers to this problem as “Absence of homogeneity.”

[35] All the proofs are given in the Appendix on page 33.

[36] Feldman and Gilles [12] in Proposition 1 and Uhlig [23] in Theorem 1 provide similar results for the unit interval of agents: they show that if the average shock on any agent subset is constant, then the shocks are the same for all the agents. Uhlig’s prove relies on the Radon-Nikodim theorem; Feldman and Gilles use the underlying logic of the same theorem.

Measure Preserving Matching is not Mixing [37]

It is assumed (assumption B8) that an agent is matched with the agents from some set B_2 with probability $\mu(B_2)$, or exactly $\mu(B_2)$ fraction of the agents from set B_1 are matched with the agents from set B_2. Formally, for any $\omega \in \Omega$ and for any $B_1, B_2 \in \mathcal{A}$ holds

$$\mu(\mathbf{M}_\omega(B_1) \cap B_2) = \mu(B_1)\mu(B_2). \tag{2}$$

Take any B_1 such that $\mu(B_1) \in (0,1)$ and $B_2 = \mathbf{M}_\omega(B_1)$. Then

$$\mu(\mathbf{M}_\omega(B_1) \cap B_2) = \mu(B_1) > \mu(B_1)^2 = \mu(B_1)\mu(B_2),$$

which says that the mixing property does not hold simultaneously on all the measurable subsets of the agent space.[38]

This result shows that we can not provide a random matching satisfying the measure preserving and mixing properties. To prove it, we found two such sets of agents on which the mixing property does not hold. Therefore, for any measure preserving random matching for any realization there always exist two subsets on which the mixing property does not hold. However, we did not rule out the existence of a measure preserving random matching such that for any two sets of agents the mixing property holds almost surely for these two sets.[39]

No Aggregate Uncertainty is not the Only Option

Consider no aggregate uncertainty in the form of equality of the sample distribution on the whole space and the theoretical distribution. The probability space $(\Omega, \mathcal{F}, \mathbf{P})$ does not impose any restrictions on measure μ. Independence of the shocks does not restrict measure μ as well. Therefore, if no aggregate uncertainty follows from other properties, it should not matter which measure on $\mathcal{A}$ we consider. For a given $\omega \in \Omega$ consider $\mathcal{A}_\omega$ — the minimal σ-algebra in which $\xi_\omega(\cdot)$ is measurable as a function of $a \in A$.

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[37] This example is close to the one described in Alós-Ferrer [4], and was originally taken from McLennan and Sonnenschein [20].

[38] As Alós-Ferrer [4] proves (Corollary 3.2.), even weaker mixing contradicts measure preserving.

[39] The random matching which was found by Duffie and Sun [11].

Proposition 2. *Suppose that there is no aggregate uncertainty for measure μ. Then there exists aggregate uncertainty for measure $\overline{\mu}$ for the same ω if and only if there exists $B \in \mathcal{A}_\omega$ such that $\mu(B) \neq \overline{\mu}(B)$.*

The proposition states that if we change measure μ so that at least one of the sets from $\mathcal{A}_\omega$ changes its measure, then there exists aggregate uncertainty for the given ω. If measure μ is changed in such a way that with non-zero probability there exists aggregate uncertainty for some $B \in \mathcal{A}$, $\mu(B) > 0$, then there exists aggregate uncertainty in terms of definition A4'. Therefore, each set of idiosyncratic shocks requires its own measure μ (if it exists). Also, arbitrary measure μ might require construction of its own shocks.

The fact that idiosyncratic shocks with no aggregate uncertainty is just one of many options was mentioned in several papers, including Judd [19] and Green [16].

Randomness and σ-Algebra

Suppose that there exist two agents such that for any $B \in \mathcal{A}$ if one of them belongs to B, then another one also belongs to B. Obviously, the shocks for these two agents have to be the same, which follows from the measurability of the shocks across the agents. This leads us to the conclusion that independent nondegenerate shocks do not exist for all σ-algebras, if measurability of these shocks over the agent space is required for any $\omega \in \Omega$. Therefore, the choice of the agent space can not arbitrary.

3.4. Intuitiveness of the Random Matching Properties

Green and Zhou [17] describe the following model. "Agents are nonatomic. Each agent has a type in $(0, 1]$. The mapping from the agents to their types is a uniformly distributed random variable, independent of all other random variables in the model. Similarly, there is a continuum of differentiated goods, each indexed by a number $j \in (0, 1]$.<...> Each agent of type i receives an endowment of one unit of i good in each period. An agent can consume his own endowment and half of the other brands in the economy; agent i consumes goods $j \in [i, i + \frac{1}{2}]$ (mod 1)<...> He prefers other goods in his consumption range to his endowment good <...> Agents randomly meet pairwise each period. By the assumed pattern of endowments and consumption sets, there is no double coincidence of wants in any pairwise meeting." The last statement means that agents whose indexes differ by .5 (for example, .3 and .8) never meet. The authors explain it by the following argument: “Strictly speaking, there is a double coincidence of wants only when types i and j are matched, with $i \equiv j + 1/2 \pmod 1$. Such a match occurs with probability zero. Hence, we ignore this possibility.”

Indeed, for a particular agent i the probability to be matched with agent $j = i + 1/2$ (mod 1) equals zero. However, this does not guarantee that event “there is no agent i who is matched with agent $j = i + 1/2$ (mod 1)” has probability zero (if measurable), because this event is a union of uncountable number of probability zero events (for each i). Simulations show that for a large finite population model, probability of the event “there exists agent i who is matched with agent $i + n/2$ (mod n)” is about .4 (100000 simulations with 100 agents.) Therefore, in finite models the probability of the event in the exact Green and Zhou’s setup does not converge to zero. There are many ways to bypass this difficulty. Therefore, all the results of the the Green and Zhou’s paper hold. Nevertheless, it is obvious that we should be very careful with “intuitive” random matching properties in nonatomic models.

In the conclusion of the section, we want to emphasize that the inconsistencies considered here and nonintuitive properties of idiosyncratic shocks and random matchings demonstrate that the standard model needs to be reconsidered. Different authors tried to solve the problem by considering different deviations from the standard setup, but no solution satisfies what we believe numerous economic models require.

4. Examples

The standard setup needs to be reconsidered in order to satisfy the main requirement — the formulation of the properties in terms “for almost all realizations for all the subsets of the agents space.” Before proceeding to our solution of the problem, we provide here two simple examples showing the main idea.

One-Period Model with Discrete Shocks

Agent space A consists of eight agents a_1,a_2,a_3,a_4,a_5,a_6,a_7, and a_8. At the beginning every agent learns his shock. After that, the agents meet and learn their partner’s shock. Suppose that the shocks and partners are determined by Table 1.

Agent a	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8
Shock $\xi(a)$	1	-1	1	-1	1	1	-1	-1
Partner $\mathbf{M}(a)$	a_5	a_6	a_7	a_8	a_1	a_2	a_3	a_4
Partner's shock $\xi(\mathbf{M}(a))$	1	1	-1	-1	1	-1	1	-1

TABLE 1. One-period model with discrete shocks.

The a priori probability of being one of the agents equals 1/8, and based on the shock and the partner's shock the agent modifies his belief about who he actually is. Suppose that the shock equals 1. Only agents a_1, a_3, a_5, and a_6 have shock 1, therefore his new belief of being each of these agents equals 1/4, and 0 for all the other agents. Half of these four agents have a partner with shock 1 (a_1 and a_5), and half of them have a partner with shock -1 (a_3 and a_6). Therefore, for the agent with shock 1 the probability of having a partner with shock 1 equals 1/2. Suppose that the agent learns that his partner's shock is 1, too. Only two agents have both their own and their partner's shock 1 — agents a_1 and a_5. Therefore, the agent concludes that with equal probability 1/2 he is one of these two agents.

In general, for any $x_1, x_2 \in \{-1, 1\}$ we have

$$\mathbf{P}\left(\xi(\mathbf{M}(a)) = x_1 \middle| \xi(a) = x_2\right) = 1/2,$$

i.e. the probability of an agent with shock x_1 having a partner with shock x_2 equals 1/2 (a priory probability of having shock x_2), and depends neither on x_1 nor on x_2. For example,

$$\begin{aligned}\mathbf{P}\left(\xi(\mathbf{M}(a)) = 1 \middle| \xi(a) = 1\right) &= \frac{\mathbf{P}\left(\xi(\mathbf{M}(a)) = 1, \xi(a) = 1\right)}{\mathbf{P}\left(\xi(a) = 1\right)} \\ &= \frac{\mathbf{P}\left(\{a_1, a_5\}\right)}{\mathbf{P}\left(\{a_1, a_3, a_5, a_6\}\right)} = 1/2.\end{aligned}$$

σ-algebra $\sigma(\xi)$ generated by shock ξ consists of four elements:

$$\sigma(\xi) = \{\emptyset, \{a_1, a_3, a_5, a_6\}, \{a_2, a_4, a_7, a_8\}, A\}.$$

Matching operator $\mathbf{M}$ maps σ-algebra $\sigma(\xi)$ into σ-algebra

$$\mathbf{M}(\sigma(\xi)) = \{\emptyset, \{a_5, a_7, a_1, a_2\}, \{a_6, a_8, a_3, a_4\}, A\}.$$

It is easy to show that any two sets from $\sigma(\xi)$ and $\mathbf{M}(\sigma(\xi))$ are independent, which means that the agent's shock is independent of the partner's shock, and the value of the agent's shock does not change his belief about the shock of his partner.

One-Period Model with Continuous Shocks

For any distribution function $F(\cdot)$ there always exists a probability space $\tilde{A}$ with two independent random variables ξ_1 and ξ_2 distributed in accordance with cdf $F(\cdot)$. Take A_1 and A_2 — two copies of the probability space $\tilde{A}$ with random variables ξ_1 and ξ_2. Let A be a disjoint union of A_1 and A_2 with ξ defined as ξ_1 on A_1 and ξ_2 on A_2 (see Figure 1.) Let $\mathbf{M} : A \to A$ be the natural mapping from A_1 onto A_2 and vice versa, and let σ-algebra $\mathcal{A} = \sigma(\xi, \xi \circ \mathbf{M})$.

For any $B \in \mathcal{A}$ define measure $\mathbf{P}$ as

$$\mathbf{P}(B) = \frac{\mathbf{P}_{A_1}(B \cap A_1) + \mathbf{P}_{A_2}(B \cap A_2)}{2}.$$

Shock ξ has a cdf $F(\cdot)$.

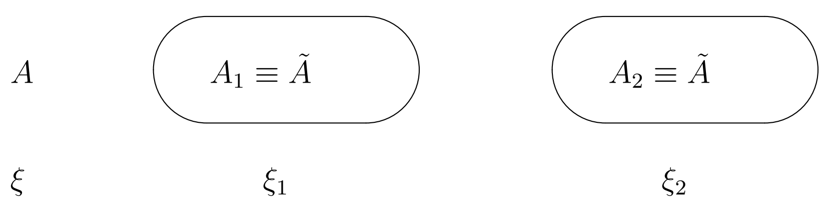

Figure 1. One-period model with continuous shocks.

Take any two $\mathcal{A}$-measurable sets C_1 and C_2. Then

$$\{a \in A_1 : \xi(\mathbf{M}(a)) \in C_1, \xi(a) \in C_2\} = \{a \in A_1 : \xi_2(\mathbf{M}(a)) \in C_1, \xi_1(a) \in C_2\},$$

$$\{a \in A_2 : \xi(\mathbf{M}(a)) \in C_1, \xi(a) \in C_2\} = \{a \in A_2 : \xi_1(\mathbf{M}(a)) \in C_1, \xi_2(a) \in C_2\},$$

and therefore ξ and $\xi \circ \mathbf{M}$ are independent because

$$\begin{aligned}
\mathbf{P}(\xi(\mathbf{M}(a)) \in C_1, \xi(a) \in C_2) &= \sum_{i=1,2} \mathbf{P}(a \in A_i : \xi(\mathbf{M}(a)) \in C_1, \xi(a) \in C_2) \\
&= \frac{\mathbf{P}_{\tilde{A}}(\xi_1 \in C_1, \xi_2 \in C_2) + \mathbf{P}_{\tilde{A}}(\xi_2 \in C_1, \xi_1 \in C_2)}{2} \\
&= \mathbf{P}(\xi_1 \in C_1)\mathbf{P}(\xi_2 \in C_2).
\end{aligned}$$

The examples considered in this section demonstrate the main idea of the construction of idiosyncratic shocks and random matchings: the agents do not know who they are, and based on the previous event (their own shocks) the agents are not able to infer any additional information about their future (partners' shocks).

5. Idiosyncratic Shocks and Random Matchings

This section defines idiosyncratic shocks and random matchings using independence of the history. We prove that there exists a space with a sequence of idiosyncratic shocks and random matchings (the random matchings are anonymous and commutative). We show how the old assumptions relate to the new ones.

5.1. Basic Definitions

Consider an agent space $(A, \mathcal{A}, \mu)$. The same space serves as the probability space: $\Omega = A$, $\mathcal{F} = \mathcal{A}$, and $\mathbf{P} = \mu$. The uncertainty an agent faces comes from his unknown identity $a \in A$.

Definition. A *shock* is a random variable on $(A, \mathcal{A}, \mu)$.

The definition of a random matching consists of three parts: matching operator, measure preservation, and independence.

Definition. A *matching operator* is an operator $\mathbf{M} : A \to A$ with the following properties:

C1. $\mathbf{M}$ is a bijection:

$$\forall\ a \in A\ \exists!\ a' \in A : \mathbf{M}(a') = a;$$

C2. No agent meets himself:

$$\forall\ a \in A\ \ \mathbf{M}(a) \neq a;$$

C3. The partner's partner is the agent himself:

$$\mathbf{M}^{-1} = \mathbf{M}.$$

Definition. A matching operator $\mathbf{M}$ is *measurable* if

C4. It maps measurable sets into measurable sets:

$$\forall B \in \mathcal{A}\ \ \mathbf{M}^{-1}(B) \in \mathcal{A}.$$

Definition. A measurable matching operator $\mathbf{M}$ is *measure-preserving* if

C5. It does not change the measure of any measurable subset:

$$\forall B \in \mathcal{A} \;\; \mu(\mathbf{M}^{-1}(B)) = \mu(B).$$

Before proceeding with defining the concepts of idiosyncratic shocks and random matchings, we need the concept of history in order to capture the idea of independence of the current events from the past.

5.2. History

Time is discrete, $t \in \mathbb{T} \equiv \mathbb{Z}$. A sequence of shocks $\{\xi_t\}_{t\in\mathbb{T}}$ and measure-preserving matching operators $\{\mathbf{M}_t\}_{t\in\mathbb{T}}$ is given. At the beginning of each period, the agents experience shocks, and at the end of each period they meet.

Wither the agents can infer any additional information about the future depends on how much they remember from the past. The information available to an agent at each period of time is called *history*. We assume that the agent's own shocks and the information sharing during the meetings is the only possible source for the history. We denote the agent a's history measured right before the meeting by $H_t(a)$, and right before the shock by $H'_t(a)$.

Let $\mathcal{A}_t$ be the minimal σ-algebra in which history $H_t(a)$ is a measurable function: $\mathcal{A}_t = \sigma(H_t(\cdot))$, and $\mathcal{A}'_t$ be the minimal σ-algebra in which history $H'_t(a)$ is a measurable function: $\mathcal{A}'_t = \sigma(H'_t(\cdot))$. We say that sigma-algebras $\mathcal{A}_t$ and A'_t are generated by the history.[40] The shocks and matchings are $\mathcal{A}$-measurable, therefore $\mathcal{A}_t, \mathcal{A}'_t \subset \mathcal{A}$.

If the agents remember everything and during the meetings they share their full histories, then history $H_{Mt}(a)$ is called *maximal history* and includes:

1. Current shock $\xi_t(a)$;
2. History at the previous period of time $H_{Mt-1}(a)$;
3. Previous period partner's history $H_{Mt-1}(\mathbf{M}_{t-1}(a))$.

[40]This definition helps us to avoid the problem of dynamic coalition formation mentioned by Alós-Ferrer [5] "it would be expected that any large coalition of traders could be able to form a risk-pooling coalition." The same problem is stated in footnote 4 "it is important to keep track of the sets of agents that have experienced a specific realization." The problem is avoided by defining a sequence of σ-algebras.

Therefore,

$$H_{Mt}(a) = (\xi_t(a), H_{Mt-1}(a), H_{Mt-1}(\mathbf{M}_{t-1}(a)). \tag{3}$$

The maximal history before the shock equals

$$H'_{Mt}(a) = (H_{Mt-1}(a), H_{Mt-1}(\mathbf{M}_{t-1}(a)). \tag{4}$$

Since the shocks and meetings is the only source for the history, from equation 3 follows

$$\begin{aligned}\mathcal{A}_{Mt} \equiv \sigma(H_{Mt}(\cdot)) &= \sigma(\xi_t, \mathcal{A}_{Mt-1}, \mathbf{M}_{t-1}(\mathcal{A}_{Mt-1})) \\ &= \sigma\left(\xi_t, \{\xi_{t_0} \circ \mathbf{M}_{t_1} \circ \mathbf{M}_{t_2} \circ \ldots \circ \mathbf{M}_{t_l}\}_{t_0 \le t_1 < t_2 < \ldots < t_l < t}\right).\end{aligned}$$

Example 1. Suppose that Agent 3 meets with Agent 4 at period 1 and with Agent 2 at period 3 (see Figure 2.) Suppose also that Agent 2 meets with Agent 1 at Period 2, and Agent 4 meets with Agent 5 at period 2.

Then for the maximal history at the end of period 3 (after the matching) Agent 3 knows Agent 4's history up to period 1 (when they met), Agent 2's history up to period 3, and Agent 1's history up to period 2 (when Agent 2 met Agent 1). At the same time, Agent 3 at the end of period 3 does not know anything about Agent 5 because their common partner — Agent 4 — met with Agent 3 before he met Agent 5.

Denoting the agents as a_1, a_2, a_3, a_4, and a_5 correspondingly, we can write:

$$\mathbf{M}_1(a_3) = a_4;\ \mathbf{M}_2(a_1) = a_2;\ \mathbf{M}_2(a_4) = a_5;\ \mathbf{M}_3(a_2) = a_3,$$

and maximal histories

$$H_{M3}(a_2) = (\xi_3(a_2), \xi_2(a_2), \xi_1(a_2), \xi_2(a_1), \xi_1(a_1));$$

$$H_{M3}(a_3) = (\xi_3(a_3), \xi_2(a_3), \xi_1(a_3), \xi_1(a_4)).$$

5.3. Independence

Mixing property B8 was not defined correctly because no measure-preserving matching operator can be mixing on the whole non-trivial σ-algebra $\mathcal{A}$.[41] At the same time, if we consider a family of σ-algebras $\{\mathcal{A}_t\}$ generated by the history, then the sequence of matching operators $\{\mathbf{M}_t\}$ can be mixing in the sense of $\mathbf{M}_t$ being mixing on $\mathcal{A}_t$ for each t.

[41] See "A Measure Preserving Matching can not be Mixing" inconsistency, page 16.

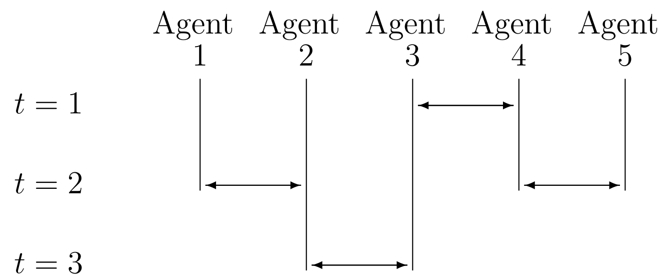

FIGURE 2. Scheme of matchings $\mathbf{M}_t$ for example 1.

To define mixing correctly, we need the concept of independence of a matching operator from a σ-algebra.

Definition. A matching operator $\mathbf{M}$ is *independent of a σ-algebra* $\mathcal{A}' \subset \mathcal{A}$ if σ-algebras $\mathcal{A}'$ and $\mathbf{M}\mathcal{A}'$ are independent.

The definition is a reformulation of the mixing property and says that for any $B, C \in \mathcal{A}'$ events B and $\mathbf{M}C$ are independent, or

$$\mu(B \cap \mathbf{M}C) = \mu(B)\mu(\mathbf{M}C) = \mu(B)\mu(C).$$

The definition is similar to the definition of independence of a random variable from a σ-algebra.

Definition. A measure-preserving matching operator $\mathbf{M}_t$ is called a *random matchings* if

C6. Matching operator $\mathbf{M}_t$ is independent of $\mathcal{A}_t$.

Definition. A shock ξ_t is called *idiosyncratic* if

C7. Shock ξ_t is independent of σ-algebra $\mathcal{A}'_t$.

Both definitions require current event (partner's history or own shock) to be independent of the past. For the matching, the past is $H_t(\cdot)$ and the event is $H_t(\mathbf{M}(\cdot))$ — the partner's history. For the shock, the past is $H'_t(\cdot)$ and the event is $\xi_t(\cdot)$ — the shock.

5.4. Old and New Assumptions

Now we want to demonstrate (see Table 2) how the old assumptions A1-A4 and B1-B7 relate to new assumptions C1-C7.

Measurability of the shocks (A1) follows from the definition of a shock. Identical distribution of the shocks (A2) comes from the fact that an agent does not know who he

Old Assumption	Explanation of the Assumption	Replaced by
Assumptions about the Shocks		
A1	Measurability	Definition
A2	Identical distribution	Unknown identity
A3	Independence	Obsolete, C7
A4, A4'	No aggregate uncertainty	Definition, C7
A4"	No aggregate uncertainty	Obsolete
Assumptions about the Matchings		
B1-B5	Everyone meets someone, no agent meets himself, partner's partner is the agent himself, measurability, measure preserving	C1-C5
B6	Uniform distribution	Obsolete
B7	Independence	Obsolete
B8	Mixing	C6
Assumption of Joint Independence		
Independence of the shocks and matchings		C6, C7

Table 2. Equivalence of the Old and New Assumptions

is, and that he sees the future as random. Independence of the shocks becomes obsolete. No aggregate uncertainty comes from the definition of the shocks and the independence of the current and future shocks from the past. Everyone meets someone, no agent meets himself, the partner's partner is the agent himself, measurability of the matchings, and measure preserving (B1-B5) come from C1-C5 respectively. Uniform distribution of the partners (C6) and independence of the partners (C7) become obsolete because of the assumption of unknown identity. Mixing (B8) follows from the measure preserving and independence of σ-algebras $\mathcal{A}_t$ and $\mathbf{M}_t(\mathcal{A}_t)$.

5.5. Anonymity in Matchings

The "anonymity" concept, when any two meeting agents do not have any chance of meeting later, directly or through their future partners, plays an important role in many economic models. [42] The anonymity implies that the current action of an agent can not influence his future. We want to establish how idiosyncratic shocks and random matchings are related to the concept of anonymity.

Following Aliprantis et al. [2], we define the concepts of indirect partners and strongly anonymous matchings.[43]

Definition. We define the *set of indirect partners* of agent a at time t (before the matching) as follows:

$$\tilde{H}_t(a) = \{a\} \cup \{\mathbf{M}_{t_1} \circ \mathbf{M}_{t_2} \circ \ldots \circ \mathbf{M}_{t_l}(a)\}_{t_1<t_2<\ldots<t_l<t}.$$

The agent knows some of the shocks of his indirect partners. For the matching scheme from example 1, one may find that

$$\tilde{H}_3(a_5) = \{a_3, a_4, a_5\}; \quad \tilde{H}_3(a_3) = \{a_3, a_4\}.$$

Definition. A sequence of matchings $\mathbf{M}_t$ is called *strongly anonymous*, if for any time t no meeting agents have common indirect partners:

$$\forall t \in \mathbb{T}, \forall a \in A \qquad \tilde{H}_t(a) \cap \tilde{H}_t(\mathbf{M}_t(a)) = \emptyset.$$

The concept of strongly anonymous matchings is very strict; we can allow for a possibility of the future influence if it has zero probability. To do this, we introduce the concept of μ-strongly anonymous matchings.

Definition. A sequence of matchings $\{\mathbf{M}_t\}_{t\in\mathbb{T}}$ is called *μ-strongly anonymous*, if for any $t \in \mathbb{T}$ with probability one the meeting agents do not have common indirect partners:[44]

$$\forall t \in \mathbb{T} \;\; \mu\left(\{a : \tilde{H}_t(a) \cap \tilde{H}_t(\mathbf{M}_t(a)) = \emptyset\}\right) = 1.$$

[42] For example, Kocherlakota [19] requires that "there is no possibility of any direct or indirect contact between the two agents before the current match." Green and Zhou [17] define anonymity as "no pair meets more than once and also that each agent knows the variety of good which his partner is endowed but nothing else."

[43] By indirect partners, we understand both common and indirect partners from Aliprantis et al. [2].

[44] Obviously, all strongly anonymous matchings are μ-strongly anonymous.

The next theorem demonstrates that the random matchings for the maximal history should be μ-strongly anonymous.[45] If the history differs from the maximal history, the requirement of μ-strong anonymity can be weakened.

Theorem 1. *Let $F(\cdot)$ be a continuous distribution function. Then the matchings are random for the maximal history only if they are μ-strongly anonymous.*

5.6. Existence

The following theorem constitutes the main result of the paper. It establishes the existence of a sequence of idiosyncratic shocks and random matchings for the maximal history as we defined them before.

Theorem 2. *For any $F(\cdot)$ there exists a probability space $(A, \mathcal{A}, \mu)$ with a continuum of elements. On this probability space there exists a sequence of idiosyncratic shocks ξ_t and random matchings $\mathbf{M}_t$ for the maximal history. The matchings are strongly anonymous and commutative: for any t_1, t_2 holds*

$$\mathbf{M}_{t_1} \circ \mathbf{M}_{t_2} = \mathbf{M}_{t_2} \circ \mathbf{M}_{t_1}.$$

The idea of the proof is the following. We construct a probability space with a sufficient number of random independent variables. Then, we create countably many copies of this space and allocate the random variables to these spaces. The agent space is the union of all the copies, and the shock at each period of time is a particular random variable on the corresponding copy of the original probability space. At each period of time, the agents from one copy of the probability space meet with the corresponding agents from some other copy. The copies whose agents meet are chosen in such a way that the matchings are strongly anonymous; to achieve this, we use Aliprantis et al. [3] mechanism of recursive block-partition on the copies of the probability space. The σ-algebra is generated by the shocks and matchings; the σ-algebras at each period of time are generated by the histories. It turns out that for any measurable set on the agent space its original measure on any of the copies does not depend on the copy. Therefore, we define the measure of a

[45] In Theorem 2 we show that spaces with idiosyncratic shocks and strongly anonymous random matchings exist. Hence, we do not need to consider any other types of anonymity because, as was shown in Aliprantis et al. [3], all of them follow from strong anonymity.

measurable set as the measure of its part on any of the copies of the original probability space. Then we prove that the shocks constructed are idiosyncratic and the matchings are random.

If the history is not maximal, then the results of theorem 2 still holds because of the following proposition.

Proposition 3. *Suppose that the shocks $\{\xi_t\}_t$ are idiosyncratic and matchings $\{\mathbf{M}_t\}_t$ are random for the maximal history. Then they constitute a system of idiosyncratic shocks and random matchings for any function of the maximal history.*

6. Discussion

In this paper, we have formulated the model that enables us to give the complete justification of the assumptions related to a continuum of agents with no aggregate uncertainty in shocks and mixing in matchings. To our knowledge, no paper combines these in one framework. Duffie and Sun [11] [11] come closest but we differ from them in the following respects. We manage to combine both arbitrary idiosyncratic shocks and random matchings in our model. The matchings satisfy two important properties: strong anonymity and commutativity. The most important result of the paper is in the formulation of the properties for almost all realizations, for all history-measurable subsets of the agents, which was not done in any of the previous papers.

The model suggested in this paper is very flexible and allows us to explore a wide range of extensions. Some extensions worth exploring which can be often met in the literature are:

(1) Dependent shocks. The agents can experience shocks which depend on the histories. Many papers incorporated Markov shocks. The approach suggested in this paper allows to explore more general dependence;
(2) Dependent partners. The agents might not meet randomly: the agents with high shocks might have a higher chance to meet similar agents with high shocks;
(3) Partial meetings. Not all the agents can meet at a given period of time. The model allows at each period of time to choose a fraction of the agents with some measure who do not meet;

(4) Non-anonymous matchings. The agents might have a non-zero chance of meeting the current partner in the future.

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Appendix

Proof of proposition 1.

As $\xi_\omega(\cdot)$ is measurable, then for any $x \in \mathbb{R}$ set $\{a : \xi_\omega(a) \leq x\}$ is measurable. Take $m = \int\limits_A \xi_\omega(a)\, d\mu(a)$ — the sample average of the shock, and

$$B = \{a \in A : \xi_\omega(a) < m\}$$

— all the agents who have the shock not greater than the average. The shock is not constant, therefore B has a positive measure, and

$$\int\limits_B \xi_\omega(a)\, d\mu_B(a) < \int\limits_B m\, d\mu_B(a) = m,$$

which proves inequality 1.

□

Proof of proposition 2.

Notice that the Borel σ-algebra on $\mathbb{R}$ is the minimal σ-algebra containing all the intervals $(-\infty, x]$. σ-algebra $\mathcal{A}_\omega$ is generated by the shocks, therefore there exists $B \in \mathcal{A}_\omega$ such that $\mu(B) \neq \overline{\mu}(B)$ if and only if there exists $x \in \mathbb{R}$ such that

$$\mu(\{a : \xi_\omega(a) \in (-\infty, x]\}) \neq \overline{\mu}(\{a : \xi_\omega(a) \in (-\infty, x]\}). \tag{5}$$

There is no aggregate uncertainty for measure μ, therefore

$$\mu(\{a : \xi_\omega(a) \leq x\}) = F(x) \qquad \forall x \in \mathbb{R}.$$

Then, using inequality 5,

$$\exists\ B \in \mathcal{A}_\omega : \mu(B) \neq \overline{\mu}(B) \Leftrightarrow$$

$$\exists\ x : \mu(\{a : \xi_\omega(a) \leq x\}) \neq \overline{\mu}(\{a : \xi_\omega(a) \leq x\}) \Leftrightarrow$$

$$\exists\ x : \overline{\mu}(\{a : \xi_\omega(a) \leq x\}) \neq F(x).$$

The last conditions means the existence of aggregate uncertainty for measure $\bar{\mu}$.

□

Proof of theorem 1.

Suppose that there exists a positive-measured subset of agents $B \in \mathcal{A}$ such that the matching at time period t for the agents from this set is not strongly anonymous. Consequently, for any $a \in B$ there exist $t_1 < t_2 < \ldots < t_l < t$ and $t'_1 < t'_2 < \ldots < t'_m < t$ such that

$$\mathbf{M}_{t_1} \circ \mathbf{M}_{t_2} \circ \ldots \circ \mathbf{M}_{t_l}(a) = \mathbf{M}_{t'_1} \circ \mathbf{M}_{t'_2} \circ \ldots \circ \mathbf{M}_{t'_m}(\mathbf{M}_t(a)).$$

Denote

$$B^{t'_1 t'_2 \ldots t'_m}_{t_1 t_2 \ldots t_l} = \{a \in B : \mathbf{M}_{t_1} \circ \mathbf{M}_{t_2} \circ \ldots \circ \mathbf{M}_{t_l}(a) = \mathbf{M}_{t'_1} \circ \mathbf{M}_{t'_2} \circ \ldots \circ \mathbf{M}_{t'_m}(\mathbf{M}_t(a))\}.$$

Note that

$$B = \bigcup_{\substack{t_1,t_2,\dots,t_l,\\ t'_1,t'_2,\dots,t'_m}} B^{t'_1 t'_2 \dots t'_m}_{t_1 t_2 \dots t_l}.$$

Set B can be represented as a countable union of sets $B^{t'_1 t'_2 \dots t'_m}_{t_1 t_2 \dots t_l}$. If B has a positive measure, then for some indexes $t_1 < t_2 < \ldots < t_l < t$ and $t'_1 < t'_2 < \ldots < t'_m < t$ set $B^{t'_1 t'_2 \dots t'_m}_{t_1 t_2 \dots t_l}$ also has a positive measure.

Denote $\eta_{t_1 t_2 \dots t_l} = \xi_{t_1}(\mathbf{M}_{t_1} \circ \mathbf{M}_{t_2} \circ \ldots \circ \mathbf{M}_{t_l}(a))$ and $\eta_{t'_1 t'_2 \dots t'_m} = \xi_{t'_1}(\mathbf{M}_{t'_1} \circ \mathbf{M}_{t'_2} \circ \ldots \circ \mathbf{M}_{t'_m}(a))$. To complete the proof, we need to use probability theory lemma 1 (the lemma is given after this proof). The matchings are independent, therefore $\mathbf{M}_t$ is independent of $\mathcal{A}_t$ and random variables $\eta_{t_1 t_2 \dots t_l}$ and $\eta_{t'_1 t'_2 \dots t'_m} \circ \mathbf{M}_t$ are independent. From Lemma 1 follows that $\mu\left(\{a : \eta_{t_1 t_2 \dots t_l}(a) = \eta_{t'_1 t'_2 \dots t'_m} \circ \mathbf{M}_t(a)\}\right) = 0$. However, these random variables coincide on $B^{t'_1 t'_2 \dots t'_m}_{t_1 t_2 \dots t_l}$, and therefore the measure of this set is not equal to zero. Contradiction. Thus, set B has measure zero for any time period t.

□

Lemma 1. *Suppose that random variables ζ_1 and ζ_2 are independent and that one of them has a continuous distribution. Then*

$$\mu(\{a : \zeta_1(a) = \zeta_2(a)\}) = 0.$$

Proof of lemma 1.

Suppose that ζ_2 has a continuous distribution. Then for any $\epsilon > 0$ there exists l and a set of real numbers $\{h_i\}_{i=0}^{l}$, such that $h_0 \equiv -\infty < h_1 < h_2 < \ldots < h_{l-1} < h_l \equiv +\infty$ and

$$\mu(\{a : \zeta_2(a) \in [h_i, h_{i+1})\}) < \epsilon \qquad \forall i = 0, \dots, l-1.$$

Note that

$$\{a : \zeta_1 - \zeta_2 = 0\} \subset \bigcup_{i=0}^{l-1} \{a : \zeta_1 \in [h_i, h_{i+1}), \zeta_2 \in [h_i, h_{i+1})\}.$$

Therefore,

$$\begin{aligned} \mu(\zeta_1 - \zeta_2 = 0) &\leq \sum_{i=0}^{l-1} \mu(\zeta_1 \in [h_i, h_{i+1}), \zeta_2 \in [h_i, h_{i+1})) \\ &= \sum_{i=0}^{l-1} \mu(\zeta_1 \in [h_i, h_{i+1}))\mu(\zeta_2 \in [h_i, h_{i+1})) \\ &< \epsilon \sum_{i=0}^{l-1} \mu(\zeta_1 \in [h_i, h_{i+1})) = \epsilon. \end{aligned}$$

As ϵ is arbitrary, then

$$\mu(\zeta_1 - \zeta_2 = 0) = 0.$$

$\square$

Proof of theorem 2.

Agent Space

By Kolmogorov theorem, for any distribution function $F(\cdot)$ there exists a probability space $(\Theta, \mathcal{Q}, \nu)$ with a countable number of independent random variables $\{\xi_t^i\}_{i\in\mathbb{N},t\in\mathbb{T}}$, each distributed in accordance with $F(\cdot)$. Consider spaces A_i, $i \in \mathbb{N}$, of which every space is an exact copy of $(\Theta, \mathcal{Q}, \nu)$. Assume that functions S_i naturally map set Θ onto A_i (see Figure 3).

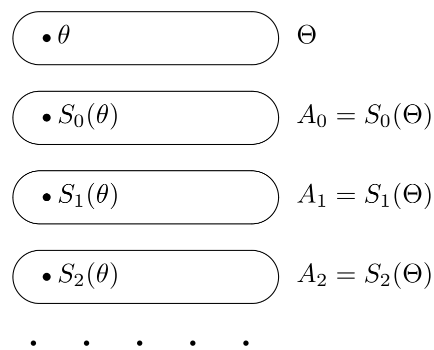

FIGURE 3. Different Instances of Probability Space $(\Theta, \mathcal{Q}, \nu)$.

Define agent space $A = A_0 \sqcup A_1 \sqcup A_2 \sqcup \ldots$ and function $s(a)$ — the space number to which $a \in A$ belongs: $a \in A_{s(a)}$. By $S^{-1}(a)$ we understand the corresponding $S_i^{-1}(a)$, where $i = s(a)$.

Random variables ξ_t^i on probability space $(\Theta, \mathcal{Q}, \nu)$ have only two indexes, $i \in \mathbb{N}$ and $t \in \mathbb{T}$. Therefore, by Kolmogorov theorem we can take $[0,1]^{\mathbb{T}*\mathbb{N}}$ as space Θ (see Wentzell [24]), and $\bigsqcup\limits_{i=0}^{\infty} [0,1]^{\mathbb{T}*\mathbb{N}}$ as the agent space A.

By using cardinal arithmetic (see Halmos [18]), one can show that $\left|\bigsqcup\limits_{i=0}^{\infty} [0,1]^{\mathbb{T}*\mathbb{N}}\right| = |[0,1]|$, i.e. the agent space A can have a continuum of elements.[46]

Shocks

Define shocks $\xi_t : A \to \mathbb{R}$ in the following way: $\xi_t|A_i = \xi_t^i$. Random variables $\xi_t(a)$ consist of components $\xi_t^i(S^{-1}(a))$, depending on the space A_i to which a belongs. We also can write $\xi_t(a) = \xi_t^{s(a)}(S^{-1}(a))$. Figure 4 illustrates the construction.

Matchings

To define the matchings, consider any bijective index $k(\cdot) : \mathbb{T} \to \mathbb{N}$ (for example, $k(0) = 0$, $k(1) = 1$, $k(-1) = 2, \ldots$). All the results can be equivalently formulated in term of time t or index k. From the context it will be clear which concept, time or index, is used.

[46]We showed that there exists an agent space A with a continuum of elements. However, there also might exist spaces with different cardinality.

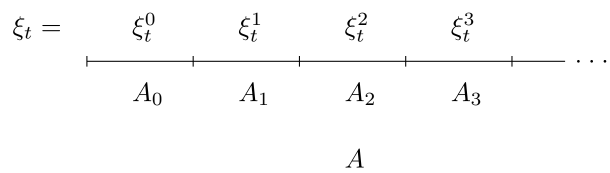

FIGURE 4. Construction of Random Variables ξ_t.

Take any A_i. For any k there exists the unique representation $i = m2^{k+1} + \delta 2^k + n$, where $m \in \mathbb{N}$, $\delta \in \{0, 1\}$, and $n \in \{0, 1, \ldots, 2^k - 1\}$. We define the matching rule $\mathbf{M}_k$ at the time period with index k so that agent $a \in S_i$ meets with agent

$$\mathbf{M}_k(a) = S_{m2^{k+1}+(1-\delta)2^k+n}(S_i^{-1}(a)). \tag{6}$$

The scheme of the matchings is represented at Figure 5.[47] At $k = 0$, agents from A_0 meet with the corresponding agents from A_1, agents from A_2 meet with A_3, agents from A_4 meet with A_5, and so on. At $k = 1$, agents from $\{A_0, A_1\}$ meet with the corresponding agents from $\{A_2, A_3\}$, agents from $\{A_4, A_5\}$ meet with $\{A_6, A_7\}$, and so on. At $k = 2$, agents from $\{A_0, A_1, A_2, A_3\}$ meet with the corresponding agents from $\{A_4, A_5, A_6, A_7\}$, and so on. In other words, if we denote the binary expansion of i by $X(i) = \overline{\ldots x_2 x_1 x_0}$, then at a period with index k agents from $A_{\overline{\ldots x_k \ldots x_1 x_0}}$ meet with the corresponding agents from $A_{\overline{\ldots \overline{x}_k \ldots x_1 x_0}}$, where $\overline{x}_k = 1 - x_k$.

If we consider

$$a' = \mathbf{M}_{k_1} \circ \mathbf{M}_{k_2} \circ \ldots \circ \mathbf{M}_{k_l}(a),$$

then the binary expansion of number $s(a')$ of the copy of the space to which a' belongs differs from the binary expansion of $s(a)$ in digits $k_1, k_2, \ldots, k_l$. In other words,

$$a' = S_{G_{k_1 k_2 \ldots k_l}(s(a))}(S^{-1}(a)), \tag{7}$$

where $G_{k_1 k_2 \ldots k_l}(\cdot)$ denotes the function of changing $k_1, k_2, \ldots, k_l$-th digits of the binary expansion of the argument.

The matchings, as we defined them, satisfy the commutativity property. For example, the agent's t_1-partner is meeting at any other time the agent partner's t_1-partner. The situation is depicted at Figure 6. At time $t = 1$, agent 0 meets agent 2, and agent 1 meets agent 3. At time $t = 2$, agent 0 meets agent 1, and agent 2 meets agent 3.

Whom agent $a \in A$ meets at time with index k_i is determined by changing the k_i-th digit in the binary expansion of number $s(a)$. The result of multiple matchings at times $k_1, k_2, \ldots, k_l$ is changing

[47]This scheme was introduced by Aliprantis et al. in [3]. The authors call it *recursive block-partition.*

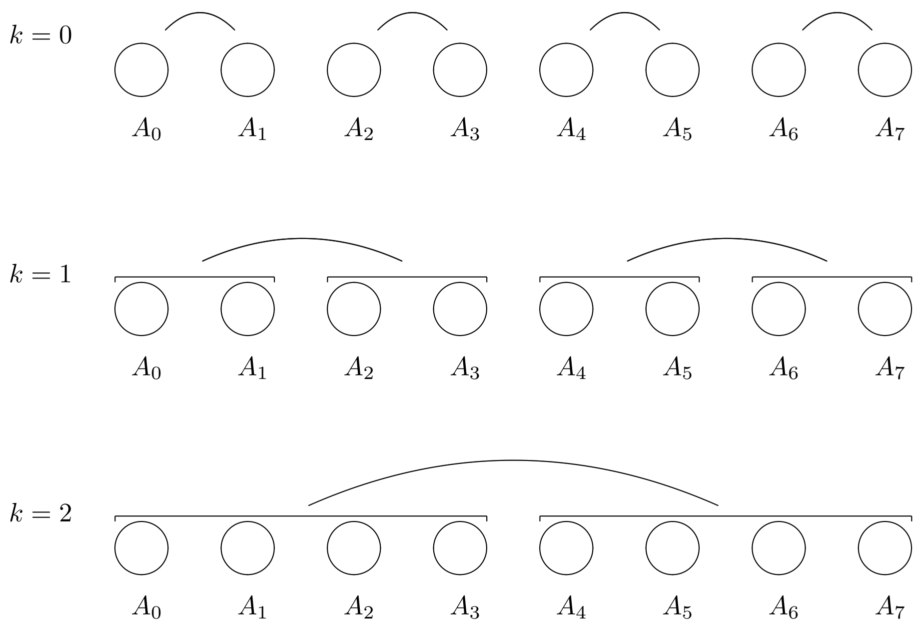

FIGURE 5. The Structure of the Matchings.

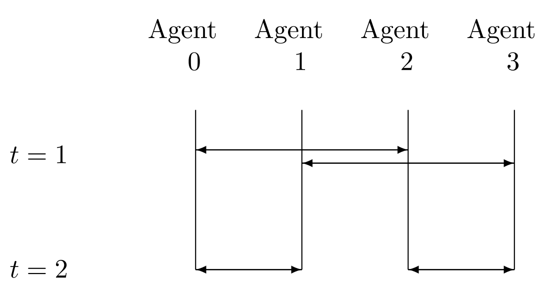

FIGURE 6. Commutativity of matchings.

digits $k_1, k_2, \ldots, k_l$ in the binary expansion of $s(a)$. For any permutation of the times the result is the same because it does not matter in which order to change the digits. Therefore, for any set of time indexes $k_1, k_2, \ldots, k_l$ and for any permutation τ of numbers $1, 2, \ldots, l$, the following equation (commutativity of matchings) holds:

$$\mathbf{M}_{k_1} \circ \mathbf{M}_{k_2} \circ \ldots \circ \mathbf{M}_{k_l} \equiv \mathbf{M}_{k_{\tau(1)}} \circ \mathbf{M}_{k_{\tau(2)}} \circ \ldots \circ \mathbf{M}_{k_{\tau(l)}}.$$

The matchings are strongly anonymous because of the commutativity: if agent a meets agent $a' = M_{k_0}(a)$ at time with index k_0 and they have a common previous indirect partner, then

$$\mathbf{M}_{k_1} \circ \mathbf{M}_{k_2} \circ \ldots \circ \mathbf{M}_{k_l}(a) = \mathbf{M}_{k'_1} \circ \mathbf{M}_{k'_2} \circ \ldots \circ \mathbf{M}_{k'_m}(a');$$

$$\mathbf{M}_{k'_1} \circ \mathbf{M}_{k'_2} \circ \ldots \circ \mathbf{M}_{k'_m} \circ \mathbf{M}_{k_1} \circ \mathbf{M}_{k_2} \circ \ldots \circ \mathbf{M}_{k_l}(a) = a' \equiv M_{k_0}(a) \tag{8}$$

for some sets $\{k_1, k_2, \ldots, k_l\}$ and $\{k'_1, k'_2, \ldots, k'_m\}$. Matching at time k_i corresponds to changing k_i-digit in the binary expansion of the space number to which a belongs. Therefore equation 8 holds only on such sets $\{k_1, k_2, \ldots, k_l\}$ and $\{k'_1, k'_2, \ldots, k'_m\}$ that differ in one element k_0, which is impossible because the previous partners could not meet at time k_0.

σ-Algebra

Following the definition, σ-algebras $\mathcal{A}_t$ and $\mathcal{A}'_t$ are generated by the maximal history:

$$\mathcal{A}_t = \sigma\left(\xi_t, \{\xi_{t_0} \circ \mathbf{M}_{t_1} \circ \mathbf{M}_{t_2} \circ \ldots \circ \mathbf{M}_{t_l}\}_{t_0 \le t_1 < t_2 < \ldots < t_l < t}\right);$$

$$\mathcal{A}'_t = \sigma\left(\mathcal{A}_{t-1}, \mathbf{M}_{t-a}(\mathcal{A}'_{t-1})\right).$$

Note that $\mathcal{A}_t \subseteq \mathcal{A}'_{t+1} \subseteq \mathcal{A}_{t+1}$. Define σ-algebra $\mathcal{A}$ as the minimal σ-algebras in which all the shocks and matchings are measurable:

$$\mathcal{A} = \sigma\left(\{\xi_{t_0} \circ \mathbf{M}_{t_1} \circ \ldots \circ \mathbf{M}_{t_l}\}_{t_0, t_1, \ldots, t_l}\right).$$

Obviously,

$$\mathcal{A}_t, \mathcal{A}'_t \subseteq \mathcal{A}.$$

Probability

In order to define probability measure on $(A, \mathcal{A})$, we will use the probability measure ν on $(\Theta, \mathcal{Q})$. Namely, for any $B \in \mathcal{A}$ define

$$\mu(B) = \lim_{j \to \infty} \frac{1}{j+1} \sum_{i=0}^{j} \nu(S^{-1}(B \cap A_i)). \tag{9}$$

Measure $\mu(B)$ is defined correctly. Indeed, consider any finite set of indexes $W = \{k_0, k_1, \ldots, k_l\}$ and any measurable sets $B_{k_0, k_1, \ldots, k_l}$. From equation 7, for any $a \in A_i$

$$\xi_{t_0} \circ \mathbf{M}_{t_1} \circ \ldots \circ \mathbf{M}_{t_l}(a) = \xi_{t_0}^{G_{k_1 k_2 \ldots k_l}(i)}(S^{-1}(a)).$$

Hence, values

$$\begin{aligned} \nu^i &= \nu\left(\bigcap_{(k_0, k_1, \ldots, k_l) \in W} \{\theta \in \Theta : \xi_{k_0} \circ \mathbf{M}_{k_1} \circ \ldots \circ \mathbf{M}_{k_l}(S_i(\theta)) \in B_{k_0, k_1, \ldots, k_l}\}\right) \\ &= \nu\left(\bigcap_{(k_0, k_1, \ldots, k_l) \in W} \left\{\theta \in \Theta : \xi_{t_0}^{G_{k_1 k_2 \ldots k_l}(i)}(\theta) \in B_{k_0, k_1, \ldots, k_l}\right\}\right) \end{aligned}$$

do not depend on i because of the independence of the random variables $\left\{\xi_{t_0}^{G_{k_1 k_2 \ldots k_l}(i)}(\cdot)\right\}$. Therefore, for any i and j and for any $B \in \mathcal{A}$ holds

$$\nu(S^{-1}(B \cap A_j)) = \nu(S^{-1}(B \cap A_i)), \tag{10}$$

which means that measure $\mu(\cdot)$ was defined correctly, and for any i

$$\mu(B) = \nu(S^{-1}(B \cap A_i)).$$

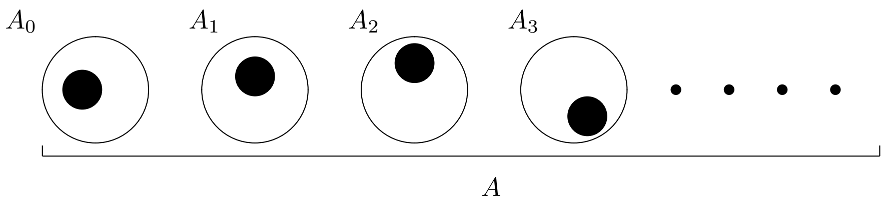

FIGURE 7. A Typical Measurable Set.

Formula 10 explains why we need the agent set A to consist of a countable number of sets A_i. In order to define a countable sequence of measure-preserving matchings, we need to find a countable number of equally probable sets A_i. Because of the equal probability of these sets and countability, the measure of each of these sets should equal zero (and the words "equally relatively probable" are not applicable to these sets). The measure of a union of a countable number of sets with measure zero itself has measure zero, which is impossible because it constitutes the whole space. We overcome this paradox by constructing a σ-algebra $\mathcal{A}$ which does not contain any of A_i by itself (see Figure 7): if $B \in \mathcal{A}$ contains some part of A_0, then it contains the equivalent parts of all the other A_i's, in accordance with the measure ν on $S_i^{-1}(A_i)$.

Random variables $\xi_t(\cdot)$ have distribution function $F(X)$ because

$$\mu(\xi_t \leq x) = \nu(S^{-1}(A_i \cap \{a : \xi_t \leq x\})) = \nu(\xi_t^i \leq x) = F(x).$$

By definition, a matching operator $\mathbf{M}_k$ naturally maps A_i onto A_j, where j's binary expansion differs from the i's binary expansion in the k-th digit. Therefore, for any $B \in \mathcal{A}$ measure $\nu(S^{-1}(\mathbf{M}_k(B) \cap A_i))$ also does not depend on i, which by formula 9 gives us the measure preserving of the matchings.

Independence

Now we can show that for the maximal history the shocks $\{\xi_t\}$ are idiosyncratic and the matchings $\{\mathbf{M}_t\}$ are random, i.e. assumptions C6 and C7 hold.

First we want to prove independence of ξ_t from $\mathcal{A}'_t$ (assumption A6). Consider an arbitrary A_i, for example A_0. Note that the history of an agent before the shock at time t is a function of the shocks at the previous periods of time. Therefore

$$\mathcal{A}'_t \cap A_0 \subset \sigma(\{\xi_{t'}^i\}_{t'<t, i\in\mathbb{N}});$$

$$\sigma(\xi_t) \cap A_0 = \sigma(\xi_t^0).$$

Since random variables $\{\xi_t^i\}$ are independent, we have

$$\sigma(\{\xi_t^i\}_{t<t,i\in\mathbb{N}})\perp\sigma(\xi_t^0);$$

$$\mathcal{A}_t'\cap A_0\perp\sigma(\xi_t)\cap A_0,$$

and therefore ξ_t is independent of $\mathcal{A}_t'$ on A_0, which in accordance with formula 9 means that ξ_t is independent of $\mathcal{A}_t'$, and therefore shocks $\{\xi_t\}$ are idiosyncratic.

To prove independence of $\mathbf{M}_t$ from $\mathcal{A}_t$ (assumption A7), note that the history of an agent from A_0 includes his current shock and the shocks of his previous direct or indirect partners. These previous direct or indirect partners belong to such A_i that $x_{k(t)}(i)=0$. The history of an A_0 agent's partner at time t includes this partner's current shock and the shocks of this partner's indirect partners. These previous indirect partners belong to such A_i that $x_{k(t)}(i)=1$. Therefore

$$\mathcal{A}_t\cap A_0\subset\sigma(\{\xi_{t'}^i\}_{t'\leq t,x_{k(t)}(i)=0});$$

$$(\mathcal{A}_t\circ\mathbf{M}_t)\cap A_0\subset\sigma(\{\xi_{t'}^i\}_{t'\leq t,x_{k(t)}(i)=1}).$$

Since random variables $\{\xi_t^i\}$ are independent, we have

$$(\mathcal{A}_t\cap A_0)\perp(\mathcal{A}_t\circ\mathbf{M}_t\cap A_0);$$

$$\mathcal{A}_t\perp(\mathcal{A}_t\circ\mathbf{M}_t);$$

$$\mathbf{M}_t\perp\mathcal{A}_t,$$

i.e. the matching operators are random.

Proof of proposition 3.

Any functions of two independent variables are independent, too. Therefore, the shocks are idiosyncratic and the matchings are random under any history which is a function of the maximal history.

□