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A TOPOLOGICAL APPROACH TO STRUCTURAL CHANGE ANALYSIS AND AN APPLICATION TO LONG-RUN LABOR ALLOCATION DYNAMICS

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Abstract. A great part of economic literature deals with structural changes, i.e. long-run changes in the structure of economic aggregates. While the standard literature relies on the mathematical branches of analysis and algebra for modeling structural change and describing the relevant empirical evidence, we choose a topological approach, which relies on the notions of self-intersection and mutual intersection of trajectories. We discuss all the methodological and mathematical aspects of this approach and show that it is applicable to a wide range of classical topics and papers of growth and development theory. Then, we apply it for studying a specific type of structural change, namely, the long-run labor re-allocation across sectors: we (a) elaborate new empirical evidence stating that mutual intersection and non-self-intersection are stylized facts of long-run labor re-allocation, (b) suggest and discuss theoretical explanations of non-self-intersection, and (c) discuss mathematical methods for explaining mutual intersection by using standard structural change models. Overall, our approach generates new evidence, new critique points of the previous structural change literature, new theoretical arguments, and a wide range of new research topics.

JEL Codes. C61, C65, O41

Keywords. Structural change, dynamics, long run, trajectory, intersection, self-intersection, differential equations, geometry, topology, labor, allocation, savings, functional income distribution.

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1. INTRODUCTION

Structural change, i.e. long-run change in the structure of economic aggregates[1] (such as GDP, aggregate employment, aggregate income, aggregate consumption expenditures), is a central aspect of growth and development theory[2] and has been analyzed in numerous models and empirical studies over the last centuries.[3] While the standard literature relies on the mathematical branches of analysis and algebra for modeling structural change and describing the relevant empirical evidence, we suggest a *topological approach* for studying structural change.

The first part of our paper deals with the conceptual, methodological, and mathematical aspects of the topological approach. As discussed there, structural change (in a country) can be described by a trajectory on the standard simplex, where the trajectories (of different countries) can be characterized by the topological notions of self-intersection and (mutual) intersection. Thus, empirical evidence and (existing) theoretical models can be classified by using these notions; moreover, the models can be compared with the empirical evidence on (self-)intersection. We discuss the two key aspects of this comparison: the type of economic law that the models represent (cf. Section 4.2.4) and the different ways to generate (non-)(self-)intersecting (families of) trajectories in continuous dynamical systems (cf. Section 5.2), where we focus on differential equation systems for discussing the latter aspect. The definition of structural change and the topological approach developed in the first part of our paper are relatively general and cover many core topics (among others, savings rate dynamics, functional income distribution, personal wealth distribution, and cross-sector labor re-allocation) and classical literature contributions of growth and development theory (cf. Section 2.2). In general, the topological approach proves useful for studying lower dimensional structures (e.g. *three*-sector models), i.e. structures representable on two- and one-dimensional simplexes, since in this case, it is relatively simple to identify the points of (self-)intersection in empirical data (cf. Section 3.4).

In the second part of our paper, we demonstrate how to apply the topological approach developed in the first part of our paper. Due to space restrictions, we focus on a specific sort of structural change, namely, the long-run labor re-allocation in the three-sector framework

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[1] Every aggregate index can be divided into its components. Then, the contribution of the components to the aggregate index (i.e., the components' shares in the aggregate index) can be calculated. In our paper, structural change refers to the long-run dynamics of these contributions/shares. For a formal definition of the term "structural change", see Section 2.1.

[2] For various examples of topics and papers that are covered by our structural change definition, see Section 2.2.

[3] For an overview of structural change literature, see Section 2.2 and, e.g., Silva and Teixeira (2008) and Stijepic (2011, Chapter IV).

referring to the agricultural, manufacturing, and services sector. To demonstrate how to use the topological approach to derive stylized facts, we analyze the data on the long-run labor allocation dynamics in the OECD countries and formulate two new stylized facts stating that (a) the labor allocation trajectories intersect mutually in the long run and (b) self-intersection seems to be a short-run phenomenon and, thus, non-self-intersection is characteristic for the long run. To demonstrate how use the topological approach to classify theoretical models and compare them with empirical evidence, we study the Kongsamut et al. (2001) model (which is a major example of the modern labor re-allocation literature) and discuss under which (parameter) conditions it can generate (self-)intersections.

Since we are not aware of any literature that discusses or tries to theoretically explain the stylized facts derived in the second part of our paper,[4] we devote the third part of our paper to this topic. While (mutual) intersections seem to be easily explainable by cross-country parameter variation and parameter perturbations (cf. Section 5.2.1), the long-run non-self-intersection seems to be an interesting theoretical puzzle. Therefore, we focus on it and elaborate different theoretical and intuitive/economic explanations of non-self-intersection of the long-run labor re-allocation trajectories and of the trajectories associated with some other topics (cf. Section 2.2) covered by our approach. In part, we discuss these aspects by relying on topological concepts (in particular, homeomorphisms).

As a byproduct of the main discussion, our paper provides (a) an overview and discussion of the applicability of different mathematical dynamic models (parameter perturbations, smooth autonomous differential equations, non-autonomous differential equation systems, coverings, homeomorphisms, etc.) in structural change modeling and (b) a classification of various central topics of growth and development theory under the headline of structural change. Overall, our approach generates new evidence, new critique points of the previous structural change literature, new theoretical arguments, and numerous topics for further research (which are summarized in Section 9).

The rest of the paper is set up as follows. The *first part* of our paper encompasses Sections 2-5. In Section 2, we define the term “structural change” and provide examples of topics and literature covered by this definition. Section 3 explains the geometrical interpretation of structural change and the topological classification of structural trajectories. Section 4

[4] Stijepic (2015b) suggests a meta-model of non-self-intersecting trajectories and studies the transitional dynamics in this model. In contrast to Stijepic (2015b), our paper is devoted to the topological classification and comparison of empirical evidence and theoretical models by using the concepts of self-intersection *and* mutual intersection. Furthermore, a significant part of our paper is devoted to the intuitive/economic explanation of non-self-intersection and the mathematical explanation of mutual intersection, whereas Stijepic (2015b) does not discuss these aspects.

discusses the methodological aspects of the theoretical explanation of trajectory-related empirical evidence. Section 5 joins these methodological results with some standard results of the mathematical differential equation theory to elaborate approaches for explaining the observed (self-)intersection of structural change trajectories by using standard structural change models (which are representable by differential equation systems). The *second part* of our paper encompasses Sections 6 and 7. In Section 6, we present the evidence on labor re-allocation focusing on OECD countries and the data from The WorldBank and Maddison (1995, 2007) and formulate the stylized facts regarding the topological properties of labor allocation trajectories. Section 7 discusses the Kongsamut et al. (2001) model. The *third part* of our paper (Section 8) is devoted to the development of a theoretical intuitive/economic explanation of non-self-intersection. A summary of our findings and a discussion of the topics for further research are provided in Section 9.

2. A MATHEMATICAL DEFINITION OF STRUCTURAL CHANGE AND EXAMPLES OF TOPICS/LITERATURE COVERED BY IT

We suggest mathematical definitions of the terms "structure" and "structural change" in Section 2.1 and discuss various examples of topics and classical growth and development theory papers covered by these terms in Section 2.2.

2.1 Mathematical Definition of Structure and Structural Change

Let y denote an aggregate index (e.g. aggregate employment). Every aggregate index can be divided into its components (e.g. employment in agriculture, employment in manufacturing, and employment in services), such that that it is equal to the sum of its components. Let y_1, $y_2, \ldots y_n$ be the components of the index y. Thus, $y = y_1 + y_2 + \ldots y_n$ (e.g. aggregate employment = employment in agriculture + employment in manufacturing + employment in services).

The importance of a component y_i (where $i \in \{1,2,\ldots n\}$) with respect to the aggregate index (y) can be measured by the share y_i/y (e.g. the importance of agricultural employment with respect to aggregate employment can be indicated by the agricultural-employment-to-aggregate-employment ratio, i.e. the agricultural employment share). Let x_i denote the share of component y_i in the aggregate index y, i.e. $x_i := y_i/y$ for $i = 1,2,\ldots n$. Note that $x_1 + x_2 + \ldots x_n = 1$, since $y_1 + y_2 + \ldots y_n = y$. Furthermore, we consider here only the economic variables (y_i) that cannot be negative; thus, $x_i \geq 0$ for $i = 1,2,\ldots n$. (For example, employment shares cannot be negative.)

We define the term *"structure (of the index y)"* such that it refers to the tuple ($x_1, x_2, \ldots x_n$). In other words, the "structure (of the index y)" is given by the shares of the index components in the index. (For example, the structure of employment in our example is given by the tuple of three numbers: agricultural employment share, manufacturing employment share, and services employment share.) In general, the term "structural change" (as it is used in the economic literature) refers to the long-run changes in the structure of some aggregate index (cf. Footnote 1). Thus, according to our definition of the term "structure", *"structural change"* means that at least some of the shares $x_1, x_2, \ldots x_n$ are not constant in the long run. For example, x_1 may grow over time, x_2 may decline over time, x_3 may decline over time, x_4 may be constant over time,…x_n may grow over time.

Definitions 1 and 2 summarize this discussion, where we do not implement the facts that structural change refers to the long run and that we focus on low-dimensional structures (cf. Section 1), since in this way the mathematical formulations are simpler (we do not need a mathematical definition of the long run) and more general (i.e. referring to higher dimension). However, whenever the time frame and dimension become relevant (e.g. in Section 6) we take account of them.

Definition 1. *Let y be an aggregate index and $y_1, y_2, \ldots y_n$ be the components of the index, where n is a natural number. Let $y(t)$ and $y_1(t), y_2(t), \ldots y_n(t)$ denote the values of the index y and its components $y_1, y_2, \ldots y_n$ at time t, respectively, where $t \in \boldsymbol{D} \subseteq \boldsymbol{R}$ and $\boldsymbol{R}$ is the set of real numbers. Define $x_i(t) := y_i(t)/y(t)\ \forall\, t \in \boldsymbol{D}\ \forall\, i \in \{1,2,\ldots n\}$. The "**(n-dimensional) structure**" (of the index y) at time $t \in \boldsymbol{D}$ is represented by the vector $X(t) := (x_1(t), x_2(t), \ldots x_n(t)) \in \boldsymbol{R}^n$, where $X(t)$ satisfies the following conditions*

(1) $\forall t \in \mathbf{D}\ \forall i \in \{1,2,\ldots n\}\ 0 \le x_i(t) \le 1$

(2) $\forall t \in \mathbf{D}\ \ x_1(t) + x_2(t) + \ldots + x_n(t) = 1$.

Thus, Definition 1 states that an n-dimensional structure (of the index y) is simply a vector in n-dimensional real space that satisfies the conditions (1) and (2). Structures, as defined in Definition 1, are often used in economics. In particular, Definition 1 covers many standard topics in growth and development theory, as shown in Section 2.2.

Definition 2. Structural change *(over the period [a,b]) refers to the change (or: dynamics) of X(t) (over the period [a,b]); cf. Definition 1. In particular, the structure has changed over the period [a,b], if* $\exists t \in (a,b]\ X(t) \neq X(a)$.

Simply speaking, Definition 2 states that structural change takes place if X(t) is not constant.

2.2 Examples of Topics and Literature Covered by Definition 2

Since the discussion in Section 2.1 seems quite abstract, we provide now some examples of topics and structural change literature covered by Definition 2. In this way, we can give our structural change definition an intuitive/economic meaning and, thus, facilitate the understanding of the rest of the paper and, in particular, of Sections 3-5. We have tried to choose the topics of Examples 1-8 such that the significance of structural change (as defined in Definition 2) as a core topic of growth and development theory is emphasized. Due to this significance, we have chosen Definition 2 over the many alternative structural change definitions[5] as the basis for our topological approach to structural change analysis. Note that we refer to Examples 1-8 throughout the paper and, in particular, in Section 8 (for elaborating a theoretical explanation of non-self-intersection). Furthermore, the topics discussed in Examples 1-8 imply in association with the results of our paper numerous topics for further research, e.g. testing for (self-)intersection of trajectories in each field of literature discussed in Examples 1-8. For these reasons, it makes sense to explain the examples carefully.

Example 1. One of the most obvious application fields of Definition 2 is the literature on *long-run labor re-allocation in multi-sector growth models*, e.g. Kongsamut et al. (2001), Ngai and Pissarides (2007), Foellmi and Zweimüller (2008), and Herrendorf et al. (2014). These models can be represented here by the following assumptions: $l_i(t)$ stands for the employment in sector i at time t, where i = 1,2,…n; $l(t) := l_1(t) + l_2(t) + \ldots l_n(t)$ is the aggregate employment; $x_i(t) := l_i(t)/l(t)$ is the employment share of sector i at time t and, thus, $X(t) \equiv (x_1(t), x_2(t), \ldots x_n(t))$ indicates the cross-sector labor allocation at time t. Obviously, these assumptions imply that the cross-sector labor allocation X(t) satisfies conditions (1) and (2) (among others since employment cannot be negative) and is, therefore, a "structure" according to Definition 1. Finally, Definition 2 states that structural change takes place if the labor allocation X(t) changes over time. That is, structural change refers here to cross-sector

[5] For example, it is possible to define structural change very restrictively as "labor re-allocation across sectors" (cf. Example 1). Our definition is much more general and covers many other topics.

labor re-allocation. Thus, we have shown that the long-run labor re-allocation models are covered by Definition 2.

Example 2. The three-sector framework is a well-known special case of Example 1. Most of the papers (e.g. Kongsamut et al. (2001), Ngai and Pissarides (2007), and Foellmi and Zweimüller (2008)) refer in some way to this framework. We obtain the three-sector framework if we assume in addition to the assumptions made in Example 1 that: n = 3, i.e. there are only three sectors; sector 1 (i = 1) represents the primary/agricultural sector, sector 2 (i = 2) represents the secondary/manufacturing sector, and sector 3 (i = 3) represents the tertiary/services sector. Then, it follows immediately that: X(t) represents the labor allocation across agriculture, manufacturing, and services at time t; X(t) is a structure, i.e. satisfies (1) and (2); changes in X(t), i.e. labor re-allocation across agriculture, manufacturing, and services, represent structural change, according to Definition 2.

Example 3. The *long-run dynamics of the savings rate* are a central topic of the neoclassical growth theory, where the Ramsey-(1928)/Cass-(1965)/Koopmans-(1967) model assumes that at every point in time t, income (y(t)) can only be used for savings (s(t)) and consumption (c(t)), i.e. y(t) = s(t) + c(t). Let $x_1(t) := s(t)/y(t)$ denote the savings rate and $x_2(t) := c(t)/y(t)$ denote the consumption rate at time t, respectively; thus, the vector $X(t) \equiv (x_1(t), x_2(t))$ indicates the savings and consumption rate dynamics. Obviously, (if we assume that there is no negative savings,) the savings-consumption rate vector X(t) satisfies (1) and (2) and, therefore, represents a "structure" per Definition 1, where n = 2 (cf. Definition 1). Then, structural change takes place according to Definition 2, if the savings/consumption rate changes over time. That is, the term "structural change" refers here to the dynamics of the savings and consumption rate.

Example 4. The *long-run dynamics of the functional income distribution* play a central role in (neoclassical) growth theory. In particular, the question whether the labor income share is constant or not is a central aspect of the discussion of the applicability of Kaldor-facts, Cobb-Douglas production functions and balanced growth paths in growth theory (see, e.g., Stijepic 2015a, p.3f.). Neoclassical growth models (e.g. the Solow (1956) and the Ramsey-(1928)/Cass-(1965)/Koopmans-(1967) model) assume among others that capital and labor are the only input factors and the aggregate income is equal to the factor income. Thus, y(t) = r(t) + w(t), where y(t) is the aggregate income, r(t) is the capital income, and w(t) is the labor income at time t, respectively. In this type of model the capital income share ($x_1(t)$) and the labor income share ($x_2(t)$) are defined as follows: $x_1(t):= r(t)/y(t)$ and $x_2(t):= w(t)/y(t)$. Thus, $X(t) \equiv (x_1(t),x_2(t))$ indicates the functional income distribution. It is obvious that the functional income distribution X(t) satisfies conditions (1) and (2) and, thus, is a structure per Definition 1, where n = 2. Structural change refers here to the dynamics of the functional income distribution X(t), according to Definition 2.

Example 5. While the previous example refers to the dynamics of functional income distribution, the *dynamics of personal income distribution* is covered by Definition 2 as well. (This topic is studied among others by Caselli and Ventura (2000) in the neoclassical framework.) Assume that: $y_i(t)$ stands for the income of household i, where i = 1,2…n; $y(t):= y_1(t) + y_2(t) +...y_n(t)$ is the aggregate income; $x_i(t):= y_i(t)/y(t)$ is the share of household i in aggregate income. Thus, $X(t) \equiv (x_1(t),x_2(t),\ldots x_n(t))$ represents the personal income distribution. Again, it is obvious that the personal income distribution X(t) satisfies conditions (1) and (2) and, thus, is a structure according to Definition 1. Structural change refers here to the dynamics of the (discrete) income distribution X(t), according to Definition 2.

Example 6. The aspects of the Caselli and Ventura (2000) model that deal with the *dynamics of personal wealth distribution* can be described here as follows. $w_i(t)$ stands for the wealth of household i, where i = 1,2…n. $w(t):= w_1(t) + w_2(t) +...w_n(t)$ is the aggregate wealth. $x_i(t):= w_i(t)/w(t)$ is the share of aggregate wealth possessed by household i. It is obvious that the personal wealth distribution $X(t) \equiv (x_1(t),x_2(t),\ldots x_n(t))$ satisfies conditions (1) and (2) and, thus, is a structure according to Definition 1. Structural change refers here to the dynamics of the (discrete) wealth distribution X(t).

Example 7. The dynamics of the consumption and capital sector play a central role in the recent multi-sector growth modeling literature, which includes, e.g., Kongsamut et al. (2001), Ngai and Pissarides (2007), Acemoglu and Guerrieri (2008), Herrendorf et al. (2014), and Boppart (2014). These models focus their analysis on specific dynamic equilibrium paths that are consistent with the Kaldor facts (cf., e.g., Kongsamut et al. (2001)). These paths have different names in the literature, e.g., "generalized balanced growth paths" (cf. Kongsamut et al. (2001)), "aggregate balanced growth paths" (cf. Ngai and Pissarides (2007)), and "constant growth paths" (cf. Acemoglu and Guerrieri (2008)). Nevertheless, they have a common characteristic: they exist only if the dynamics of the consumption and capital sector are balanced among others (cf. Stijepic (2011)). Thus, the discussion of the structural change related to the capital-consumption structure is a central aspect of the modern multi-sector growth literature. This structure can be described here as follows. Assume that c(t) is the value of consumption (i.e. the value of the output of the consumption sector), dk(t) is the value of investment (i.e. the value of the output of the capital sector), and y(t):= c(t) + dk(t) is the value of aggregate output at time t, respectively. Define $x_1(t) := c(t)/y(t)$ and $x_2(t) := dk(t)/y(t)$; thus, $X(t) \equiv (x_1(t), x_2(t))$ indicates the consumption-capital structure at time t. It is obvious that the consumption-capital structure X(t) satisfies (1) and (2) and is, thus, a structure according to Definition 1, where n = 2 (cf. Definition 1). Structural change refers here to the change in the capital-consumption structure X(t), according to Definition 2.

Example 8. The dynamics of the consumption structure play a central role in the multi-sector literature discussed in Examples 1 and 6 (cf., e.g., Kongsamut et al. (2001) and Boppart (2014)). These dynamics can be studied as follows. Let $x_i := c_i(t)/c(t)$ denote the consumption share of sector i at time t for i = 1,2,…n, where $c_i(t)$ stands for the consumption expenditures on goods/services produced by sector i at time t and $c(t) := c_1(t) + c_2(t) + \ldots c_n(t)$ stands for the aggregate consumption expenditures at time t. It is then obvious that $X(t) \equiv (x_1(t), x_2(t), \ldots x_n(t))$, which indicates the consumption structure of the economy at time t, satisfies (1) and (2) and, thus, represents a structure according to Definition 1. Furthermore, structural change takes place according to Definition 2 if the consumption shares change over time. That is, structural change refers here to the changes in the consumption structure.

Overall, these examples show that our structural change definition (i.e. Definition 2) covers a wide range of classical topics from growth and development theory.

3. GEOMETRICAL INTERPRETATION OF STRUCTURAL CHANGE AND TOPOLOGICAL CHARACTERIZATION OF (FAMILIES OF) TRAJECTORIES

In this section, we discuss the geometrical and topological concepts that can be used to describe and characterize a large set of structural change models (cf. Section 2.2) and the empirical evidence on structural change (cf. Section 6). We discuss (a) the geometrical representation of structural change (models) by using simplexes and (families of) trajectories (cf. Section 3.1), (b) some topological concepts that can be used to characterize the (families of) trajectories and, thus, structural dynamics (cf. Section 3.2), and (c) the fact that (self-

)intersection is easily identified in low-dimensional structures (cf. Section 3.3). In Sections 6-8, we use the results of Section 3 to classify the empirical evidence and the theoretical literature and to compare theory with evidence.

3.1 Geometrical Interpretation of Structure and Structural Change: Simplexes and Families of Trajectories

In this section, we recapitulate some geometrical concepts for analyzing structural change, as introduced by Stijepic (2015b).

The set of all points X (in n-dimensional real space) that satisfy Definition 1 is given as follows

(3) $\{X \equiv (x_1, x_2, ... x_n) \in \mathbf{R}^n : \; x_1 + x_2 + ... x_n = 1 \wedge \forall i \in \{1,2,...n\} \; 0 \le x_i \le 1\} =: \mathbf{S}_{n-1}$

It is well known that (3) is the definition of a (n-1)-dimensional standard simplex ($\mathbf{S}_{n-1}$). The 0-dimensional simplex is a point, the 1-dimensional simplex is a line, the 2-dimensional simplex is a triangle, the 3-dimensional simplex is a pyramid, etc. Since the greatest part of our empirical evidence deals with the 2-dimensional standard simplex (henceforth, standard 2-simplex), we depict it in Figures 1 and 2, where we omit the coordinate axes in Figure 2.

Figure 1. *The standard 2-simplex in the Cartesian coordinate system (x_1,x_2,x_3).*

- insert Figure 1 here -

Figure 2. *The standard 2-simplex (without coordinate axes).*

- insert Figure 2 here -

Note that in the Cartesian coordinate system (x_1,x_2,x_3), the vertices of the standard 2-simplex are given by the coordinates/points

(4) $(1,0,0) =: V_1$

(5) $(0,1,0) =: V_2$

(6) $(0,0,1) =: V_3$

This discussion and Definition 1 imply the following geometrical interpretation of the term structure: an n-dimensional structure (cf. Definition 1) can be represented by a point on the (n-1)-dimensional standard simplex. This (n-1)-dimensional simplex contains all the points that satisfy the definition of the term "n-dimensional structure" (i.e. Definition 1). Two different points on the simplex represent two different structures. Thus, if, e.g., $X(1) \neq X(2)$

(cf. Definition 1), where $X(1),X(2) \in \mathbf{S}_{n-1}$, then the structure at t = 2 is not the same as the structure at t = 1, i.e. structural change took place over the time interval (1,2).

We turn now to a more detailed discussion of the representation of structural change via functions and trajectories on the standard simplex.

Let us assume the following function:

(7) $\phi : \mathbf{D} \times \mathbf{P} \to \mathbf{S}_{n-1}$

(8) $\phi : (t, P) \mapsto X \equiv (x_1, x_2, \ldots x_n)$

where P is a parameter vector taking values in the set **P**. (7) and (8) state that the function $\phi(t, P)$ maps time (t) and the parameter vector (P) to the (n-1)-dimensional standard simplex. In particular, for a given parameter vector $P \in \mathbf{P}$, the function $\phi(t, P)$ assigns a point on the standard simplex ($\mathbf{S}_{n-1}$), which is located in the coordinate system $(x_1,x_2,\ldots x_n)$, to each time point $t \in \mathbf{D}$.

Assume an economic model that generates a function $\phi(t, P)$ of the type (7)/(8) describing the structure of the economy $\forall\, t \in \mathbf{D}$. Since this function assigns a structure to each point in time of the domain **D** (cf. (3), (7), and Definition 1), we can derive all the information about structural change (cf. Definition 2) in this economic model from this function. In particular, by studying $\phi(t, P)$ we can derive how the structure changes over time for a given setting of the model parameters P. Therefore, we focus on the analysis of this function, henceforth.

To study the properties of the structural function $\phi(t, P)$ geometrically, we use the concept of trajectory (**T**(P)), which we define as follows (cf. Definition 1):

(9) $\forall P \in \mathbf{P} \quad \mathbf{T}(P) := \{\phi(t, P) \in \mathbf{S}_{n-1} : t \in \mathbf{D}\}$

In fact, **T**(P) is simply the set of states (or: structures) that the economy experiences (or: goes through) over the time period **D** for the given parameter setting P. Geometrically speaking, the economy moves along **T**(P) over the time period **D** if the parameter setting is P. Note that definition (9) implies that the structural trajectory **T**(P) is always located on the standard simplex $\mathbf{S}_{n-1}$. Thus, we can say that $\mathbf{S}_{n-1}$ is the *domain of the structural trajectory*.

Figure 3 depicts an example of a trajectory given by (7)-(9) and n = 3, where we assume that $\phi(t, P)$ is continuous in t for the given parameter setting P. Note that the arrow in Figure 3 indicates the direction of the movement along the trajectory. Let $\phi(a, P) \equiv (x_1^a, x_2^a, x_3^a)$ denote the initial point and $\phi(b, P) \equiv (x_1^b, x_2^b, x_3^b)$ be the end-point of the trajectory depicted Figure 3. Obviously, Figure 3 shows that these points differ. Thus, the trajectory in Figure 3 depicts structural change, according to Definition 2. In more detail, by recalling the position of the standard 2-simplex in the Cartesian coordinate system (x_1,x_2,x_3) (cf. Figure 1), we can see that the trajectory in Figure 3 implies that $x_1^a > x_1^b$, $x_2^a < x_2^b$ and $x_3^a < x_3^b$. That is, x_1 declined and x_2(x_3) inclined over the time period [*a,b*].

Figure 3. *An example of a (continuous) trajectory on the standard 2-simplex.*

- insert Figure 3 here -

In general, an economic model and, in particular, a structural change model does not generate only one trajectory but a family of trajectories, where each family member corresponds to a different initial state/condition of the economy. This is also a well-known characteristic of (well-behaving) differential equation systems, where, in general, such a system generates a family of solutions/trajectories and where each solution/trajectory corresponds to a different initial condition of the differential equation system. We define such a family of solutions/functions as follows:

(10) $\phi^I : \mathbf{D} \times \mathbf{P} \to \mathbf{S}_{n-1}$

(11) $\phi^I : (t, P) \mapsto X \equiv (x_1, x_2, \ldots x_n)$

(12) $I \in \mathbf{I}$

where I is an index (representing the initial condition of the system) taking values in the set **I**. (10)-(12) state that there is a family of functions indexed by I, where for each index value $I \in \mathbf{I}$ and each parameter setting $P \in \mathbf{P}$, there is a function $\phi^I(t,P)$, which assigns to each time point t from **D** a structure $\phi^I(t,P)$ from the simplex $\mathbf{S}_{n-1}$, which is located in the coordinate system (x_1,x_2,…x_n). Analogously, we define a family of trajectories by (12) and

(13) $\forall I \in \mathbf{I} \ \ \forall P \in \mathbf{P} \quad \mathbf{T}^I(P) := \{\phi^I(t,P) \in \mathbf{S}_{n-1} : t \in \mathbf{D}\}$

We can see that for a given parameter vector P, the trajectory $\mathbf{T}^I$(P) corresponds to one function (10) from the family **I**.

Figure 4 depicts a family of trajectories for n = 3, where we assume that $\phi^I(t,P)$ is continuous in t for the given parameter vector P and $\mathbf{I} \subset \mathbf{N}$.

Figure 4. *A family of (continuous) trajectories on the standard 2-simplex.*

- insert Figure 4 here –

Overall, in this section, we have elaborated all the mathematical concepts that we need to interpret a structural change model as a family of (parameter dependent) trajectories on the standard simplex.

3.2 Topological Characterization of (Families of) Trajectories: Continuity and (Self-)Intersection

Trajectories can be characterized by using the concepts of continuity, self-intersection, and in the case of a family of trajectories, (mutual) intersection. In Sections 5-8, we use these concepts to characterize the trajectories generated by the theoretical models of the previous structural change literature and the empirically observable trajectories and to compare theory with evidence.

The intuitive/geometrical notion of a *continuous* trajectory is more or less obvious: it is a curve without interruptions (see, e.g., Figure 3). In contrast, Figure 5 depicts an example of a non-continuous trajectory.

Figure 5. *An example of a non-continuous trajectory on the 2-simplex.*

- insert Figure 5 here -

The following definition of a continuous trajectory is obvious.

Definition 3. *The trajectory (9) is continuous on $\mathbf{S}_{n-1}$ (for the parameter setting P), if the corresponding function $\phi(t,P)$ (cf. (7)/(8)) is continuous (in t) on the interval $\mathbf{D}$ (for the parameter setting P). The family of trajectories (13) is continuous on $\mathbf{S}_{n-1}$ (for the parameter setting P), if $\forall\, I \in \mathbf{I}$, $\mathbf{T}^I(P)$ is continuous on $\mathbf{S}_{n-1}$ (for the parameter setting P).*

For a definition of a continuous function see some introductory book on analysis.

The geometrical/intuitive meaning of the self-intersection of a trajectory is more or less obvious: the trajectory in Figure 3 does not intersect itself, whereas the trajectory in Figure 6 intersects itself.

Figure 6. *An example of a self-intersecting (continuous) trajectory on the 2-simplex.*

- insert Figure 6 here –

We use the following formal definition of non-self-intersection (cf. Stijepic (2015b), p.82).

Definition 4. *The (continuous) trajectory (9) is non-self-intersecting (for a given parameter setting P), if*

(14) $\nexists\,(t_1,t_2,t_3)\in \mathbf{D}^3 : t_1 < t_2 < t_3 \wedge \phi(t_1,P)=\phi(t_3,P)\neq \phi(t_2,P)\wedge P\in \mathbf{P}$.

Note that per Definition 4, a self-intersection requires that the economy leaves the point $\phi(t_1, P)$ at least for some instant of time (t_2) before it returns to it (at t_3). Thus, according to our definition, a self-intersection does not occur if the economy reaches some point on $\mathbf{S}_{n-1}$ (in finite time) and stays there forever.

A second possibility to define a non-self-intersecting trajectory is a topological one: a non-self-intersecting trajectory is homeomorphic to the real line (cf. Section 8.1).

Finally, we define a non-intersecting family of trajectories, as follows.

Definition 5. *The (continuous) family of trajectories (12)/(13) is non-intersecting (for the parameter setting P), if*

(15) $\nexists\,(G,H)\in \mathbf{I}^2 : G\neq H \wedge \mathbf{T}^G(P)\cap \mathbf{T}^H(P)\neq \emptyset \wedge P\in \mathbf{P}$.

That is, if we take two different trajectories ($G\neq H$) from the family **I**, they must not have a point of intersection (i.e., they must not occupy a common point on $\mathbf{S}_{n-1}$) for a given parameter setting P. An alternative way to express (15) is: $\forall (s,r)\in \mathbf{D}^2 \nexists\,(G,H)\in \mathbf{I}^2 : G\neq H$ $\wedge\, \phi^G(s,P)=\phi^H(r,P)\wedge P\in \mathbf{P}$. Figure 7 depicts an intersecting family of trajectories (for a given P), whereas Figure 4 depicts a non-intersecting family of trajectories (for a given P).

Figure 7. *An intersecting family of (continuous) trajectories on the 2-simplex.*

- insert Figure 7 here –

3.3 On (Self-)Intersection and Dimension of the Domain of the Trajectory

In this section, we discuss briefly the difference between (self-)intersecting and non(-self)-intersecting trajectories in relation to the dimension of the space (simplex) in which the trajectory is located. This discussion shows that (self-)intersection is particularly useful for characterizing (empirical) trajectories associated with low-dimensional structures (cf. Definition 1).

We focus here on self-intersections. The same arguments apply to (mutual) intersections. Imagine a trajectory (**T**) in the three-dimensional space and assume that the trajectory intersects itself at the coordinate point S (cf. Figure 8).

Figure 8. *A self-intersecting (**T**) and a nearly identical non-self-intersecting trajectory (**T'**).*

- insert Figure 8 here -

It is easy to construct a non-self-intersecting trajectory (**T**') that is nearly identical to the trajectory **T**: we can marginally deform the trajectory **T** at the coordinate point S such that there is no self-intersection at this point; the trajectory (**T'**) resulting from this deformation is nearly identical to the trajectory **T** (cf. Figure 8). Therefore, it is in some sense "difficult" to distinguish between the self-intersecting trajectory **T** and the non-self-intersecting trajectory **T'**. Exactly speaking, whether it is "difficult" or not to distinguish between **T** and **T'** depends on the mathematical method used. In terms of topology, it is not "difficult" to distinguish between **T** and **T'**: they are not homeomorphic. However, in numerical/quantitative analyses and, in particular, in empirics, where the limits to measurement accuracy and measurement errors do not allow for a precise determination/construction of trajectories describing real-world processes, the "difficulties" are significant. In general, it is not possible to determine whether the process measured by the data generates a(n) (self-)intersection. For example, if our data implies that there is a(n) (self-)intersection, we could argue that there would not be a(n) (self-)intersection, if we increased the accuracy of measurement (i.e. the number of digits after the decimal point).

In contrast, (self-)intersection of trajectories in two or one-dimensional space is easier to detect. In general, trajectories (of significant length) partition the two-dimensional space significantly, such that a(n) (self-)intersection is easy to detect (cf. Stijepic (2015b), p.82f). This fact becomes obvious in Section 6 where we identify (self-)intersection of empirical trajectories on two-dimensional simplexes.

4. ON STRUCTURAL CHANGE MODELS AND THEIR TRAJECTORIES AS EXPLANATIONS OF EMPIRICAL OBSERVATIONS

In this section, we discuss how the structural dynamics of a country or a group of countries can be explained by using the meta-model (10)-(13), which covers a wide range of structural change models. This discussion does neither refer to a specific empirically observed characteristic of structural change trajectories nor does it discuss a specific structural change model from the previous literature, but is rather of methodological character: while it is quite straight forward how to explain the dynamics of one country by using a structural change model (cf. Section 4.1), there are different ways (or approaches) to explain the dynamics of a group of countries by using a structural change model (cf. Section 4.2); as we will see in Section 4.2.4, these ways reflect different (methodological) views on the notion of economic law underlying the structural change models. In Section 5, we use these (methodological) results to develop approaches for explaining a specific sort of empirical evidence, namely the (self-)intersection of trajectories.

4.1 Explanation of a Country's Dynamics

Assume that we have data on the dynamics of a structure (e.g. dynamics of labor allocation) over some period of time (e.g. 1820-2003) in a country (e.g. the US). Furthermore, assume that we construct this country's structural trajectory on the simplex by using this data (cf. Section 6). Figure 9 depicts an example of such a trajectory.

Figure 9. *Trajectory of labor allocation across agriculture, manufacturing, and services in the US between 1820 and 2003.*

- insert Figure 9 here –

Notes: *Data source: Maddison (2007). See Section 6.2 for method description.*

Assume now that we would like to have a theoretical explanation of the dynamics depicted by the trajectory (in Figure 9). To do so, we can choose an existing structural change model (e.g. the Kongsamut et al. (2001) model) and analyze first, whether the model can explain (certain characteristics of) the observed trajectory. This can be done as follows. First, solve the model equations and obtain in this way a family of functions of the type (10)-(12). Note that for a given parameter vector P, (10)-(12) imply a family of trajectories corresponding to different initial values of the system/economy; cf. (13). Thus, among the family members (**I**), we must choose the trajectory that goes through the empirically observed initial state[6] of the (US) economy. Second, choose the model parameters (P) such that the model trajectory

[6] The initial state of the country may refer to the earliest data point in the sample of structures observed for the country.

corresponding to the observed initial state of the country is as *similar*[7] as possible to the empirically observed trajectory of the country. The term “similar” may here refer to *qualitative aspects*, e.g. the shape and orientation of the trajectory on the simplex, or *quantitative aspects*, where the latter refer to the question whether the model generates changes in the structure that are of similar (numerical) magnitude as the changes observed in reality for the given initial value of the country considered.

That is, to analyze whether the model can explain (certain characteristics of) the empirically observed structural trajectory of a country, we compare the (most suitable) trajectory generated by the model and the empirically observed trajectory of the country. If the model trajectory is sufficiently similar to the observed trajectory we can say (under many restrictions) that the model is a theoretical explanation of the country’s dynamics.

4.2 Explanation of the Dynamics of a Group of Countries and Relation to Economic Laws

In this section, we discuss how observable cross-country differences regarding the qualitative (and quantitative) properties of the structural trajectories can be modelled by using a structural change model. In particular, the validity of the statements made in this section is not restricted to (only) one of the specific standard structural change models (discussed in Section 2.2), but has rather general character, since we rely again on the mathematical meta-model (10)-(13), which covers a wide range of specific structural change models. Examples of specific structural change models are discussed in Sections 2.2, 7, and 8.1. Furthermore, in Section 4.2.4 we discuss the implied methodological view of models as representing laws.

Now, assume that we depict the empirically observed trajectories of different countries (e.g. OECD countries) on one and the same simplex (see, e.g., Figure 10) and aim to provide a joint explanation for the dynamics of these countries by using a structural change model (e.g. the Kongsamut et al. (2001) model). Since the empirically observed structural dynamics and, thus, the trajectories of the countries differ significantly (as shown, e.g., in Section 6), we cannot explain the dynamics of all countries by only one model trajectory. That is, we need a model that generates multiple trajectories that differ from each other.

The meta-model (10)-(13) implies three approaches of generating multiple/different trajectories in a model. (We use these approaches later in Section 5.2.)

[7] Note that many parameters of structural change models cannot be observed in reality. Thus, given the theoretical/intuitive restrictions on the parameters, most authors set the model parameters such that that the model fits the data best.

4.2.1 Approach 1

As implied by (13), the dynamic system (10)-(12) generates a family (of different) trajectories for a given parameter setting (P), where each trajectory corresponds to a different initial value of the system. Thus, to model cross-country heterogeneity regarding trajectories, we can assume that (a) all the countries have the same parameter values, i.e. the parameter vector (P) does not differ across countries, and (b) the countries differ by initial conditions. In this case, the countries belong to the same family (**I**) of trajectories, where each $\mathrm{I} \in \mathbf{I}$ represents a country and, in particular, a different initial condition. Example 9 may elucidate these explanations.

Example 9 (Approach 1). Assume that we aim to explain the dynamics of US, UK, and Japan by using a model that generates a trajectory family of the type (10)-(13). It is possible to assign (qualitatively and quantitatively) different trajectories of this model to the different countries, if we assume that the dynamics of US, UK, and Japan can be described by (10)/(11)/(13) and choose the function $\phi^{A}(t,P)$ for US, $\phi^{B}(t,P)$ for UK, and $\phi^{C}(t,P)$ for Japan, where $\mathrm{P} \in \mathbf{P}$, $\mathrm{A,B,C} \in \mathbf{I}$ and $\mathrm{A} \neq \mathrm{B} \neq \mathrm{C} \neq \mathrm{A}$. As we can see, the index I differs across countries, whereas P is the same for all countries.

We apply Approach 1 in Section 5.2 to derive a method for generating trajectory intersections in standard structural change models via parameter perturbations.

4.2.2 Approach 2

As implied by (13), cross-country differences in (qualitative and quantitative) trajectory characteristics can arise if we assume that parameter values (P) differ across countries. In this case, cross-country differences in initial conditions are not necessary to create heterogeneous trajectories within a model (although due to empirical evidence, it may be reasonable to assume that cross-country differences in initial conditions exist). In other words, Approach 2 assumes that all countries have the same index I (cf. (12)), but differ in parameters P. Example 10 elucidates Approach 2. The discussion in Section 5.2 implies that Approach 2 is useful for explaining the structural change evidence when relying on standard structural change models.

Example 10 (Approach 2). Assume that we aim to explain the dynamics of US, UK, and Japan by using a model that generates the trajectory family (10)-(13). It is possible to assign (qualitatively and quantitatively) different trajectories of this model to the different countries, if we assume that the dynamics of US, UK, and Japan can be described by (10)/(11)/(13) and choose the function $\phi^I(t,A)$ for US, $\phi^I(t,B)$ for UK, and $\phi^I(t,C)$ for Japan, where $\mathrm{I} \in \mathbf{I}$, $\mathrm{A,B,C} \in \mathbf{P}$ and $\mathrm{A} \neq \mathrm{B} \neq \mathrm{C} \neq \mathrm{A}$. As we can see, the parameter values (A,B,C) differ across countries, whereas the index I is the same for all countries.

4.2.3 Approach 3

Approaches 1 and 2 refer to the explanation of structural change in different countries by using only *one* structural change model, e.g. the Kongsamut et al. (2001) model. A third approach could be developed by going beyond initial condition differences (Approach 1) and parameter differences (Approach 2) and assuming that each country follows its own model. This may make sense when the structural change determinants differ strongly across countries such that, e.g., US structural change is best described/explained by the Kongsamut et al. (2001) model and UK structural change is best described/explained by the Ngai and Pissarides (2007) model. We can express such model differences by using the mathematical formalism introduced in Section 3.1 as follows. By referring to our US-UK example, assume that US structural change is described by the system (10)-(12) and UK structural change is described by the system

(10’) $\varphi^J : \mathbf{D} \times \mathbf{Q} \rightarrow \mathbf{S}_{n-1}$

(11’) $\varphi^J : (t,Q) \mapsto X \equiv (x_1, x_2, \ldots x_n)$

(12’) $Q \in \mathbf{Q}$

That is, the UK and US systems follow different functional forms (ϕ^I vs. φ^J) and depend on different parameter vector spaces (P vs Q).

Three aspects of Approach 3 are noteworthy.

First, very strong differences in economic assumptions can be represented as differences in model parameters (Approach 2). Recall that the changes in only one parameter value (e.g. the elasticity of substitution) in economic models can cause very strong changes in economic assumptions (e.g. Leontief-type vs. Cobb-Douglas-type utility/production function).

Second, in many cases, it is possible to generate meta-models that cover many different models as parameter special cases. That is, in many cases, Approach 2 covers Approach 3. For example, Stijepic (2011) and Herrendorf et al. (2014) suggest (meta-)models that transform into the Kongsamut et al. (2001) model or the Ngai and Pissarides (2007) model under certain parameter constellations. That is, the latter models are special cases of the former models that arise for certain parameter values (P). This example proves that it is possible to cover the cases belonging to Approach 3 by Approach 2 (and 1).

Third, Approach 3 implies/presumes that the structural change models represent "ad hoc laws", which may be a point of critique for methodological reasons, as discussed in Section 4.2.4.

4.2.4 The relation between the three approaches and the types of economic law

The general notion of "a law" as used in natural sciences (and economics) refers to a regularity that is valid/persistent across time and space. If we use this notion in economics, we would refer to a (general) economic law as a regularity that is persistent across time and countries. Thus, this regularity can be used for predicting future dynamics in different countries. More generally speaking, the existence of some sort of economic law is the basis for any prediction of economic dynamics. For a discussion of laws in economics and natural sciences, see, e.g., Jackson and Smith (2005) and Reutlinger et al. (2015).

Our discussion of Approaches 1-3 is closely related to the methodological discussion of the economic models regarding the economic laws they represent.

Approach 1, assuming that one and the same model and one and the same parameter vector can explain structural change in all time periods (considered) and in all countries, corresponds to the general notion of a (natural) law, i.e. a regularity that is valid/persistent across time ("all periods") and space ("all countries").

In contrast, Approach 2 assumes that empirical observations can be explained by one and the same model, only if we allow that parameters vary across countries. Thus, Approach 2 corresponds to the view that economic models represent "ceteris paribus laws". The latter are widespread in economic modeling. See Reutlinger et al. (2015) for a discussion.

Approach 3 corresponds to "ad hoc laws", i.e. regularities that are sometimes applicable and sometimes not. In particular, the applicability of an ad hoc "law" differs from country to country, while (in contrast to ceteris paribus laws) it is not clearly stated when the model is applicable and when not. From the methodological point of view, the models representing "general laws" or "ceteris paribus laws" seems preferable, since among others, such models are directly testable by empirical evidence, in contrast to ad hoc models.[8] Furthermore, in structural change modeling, "ad hoc laws/models" seem unnecessary, since there are many similarities in structural change patterns across countries, which can be modeled as (ceteris paribus) laws. In particular, it is, therefore, possible to replace "ad hoc laws" by "ceteris paribus laws", where the latter can account for cross-country differences in structural change patterns, while being testable and explicitly naming the parameters that are responsible for the observable differences across countries.

For these reasons, Approaches 1 and 2 ("general law" and "ceteris paribus law") seem to be preferable over Approach 3 (cf. Section 5.2).

5. ON DIFFERENT WAYS OF GENERATING (SELF-)INTERSECTING TRAJECTORIES IN MODELS DESCRIBED BY DIFFERENTIAL EQUATIONS

While Section 4 discusses the general/methodological aspects of the theoretical explanation of trajectory-related empirical evidence, Section 5 is more specific and discusses how a specific type of trajectory-related empirical evidence, namely the (self-)intersection of trajectories, can be explained by structural change models that are described by differential equation systems. Especially, Section 5.2.1 merges the (methodological) results from Section 4 with some lessons from the mathematical theory of differential equations to derive concrete approaches for generating intersecting trajectories in (structural change) models that are described by differential equation systems. Note that the focus on differential equation systems (as opposed to general dynamical systems) in this section is justified by three facts: (a) the most structural change models are representable by differential equations, since the typical long-run modeling assumptions rely on smooth (production and utility) functions; for example, all the models discussed in Section 8 are continuous and differentiable (with respect to time); (b) the most economists are familiar with the basic aspects of differential equations; and (c) we can rely on the many useful results of the mathematical literature on differential equations. In contrast, in Section 8, we do not rely on differential equations as descriptions of structural change but on a more topological approach based on homeomorphisms.

We start the discussion in Section 5.1 by recapitulating the well-known result from differential equation theory that smooth autonomous differential equation systems generate only non-(self-)intersecting trajectories for given/constant system parameters. Then, we discuss how deviations from this standard case can generate intersecting (cf. Section 5.2.1)

[8] It is difficult to test the validity of model assumptions, if the model is only valid for one or two countries. At least, cross-country and panel data cannot be used in this case.

and self-intersecting (cf. Section 5.2.2) trajectories. In each section, we discuss as well which of these deviations can be used for generating (self-)intersections in Section 7.

5.1 Autonomous Differential Equation Systems and Non-(Self-)Intersection

In this section, we recapitulate some standard differential equation theory, which is the basis for our discussion (e.g. in Section 5.2). For references on all the statements, see, e.g., Stijepic (2015b), p.84f.

Assume a model (cf. Section 7) that generates the following initial value problem associated with an autonomous n-dimensional differential equation system:

(16) $\forall t \in \mathbf{D}' \subseteq \mathbf{R}\ \ \forall X_0 \in \mathbf{U}' \subseteq \mathbf{R}^n\ \ \forall P \in \mathbf{P}'\ \ dX(t)/dt = \Phi(X(t), P),\ X(0) = X_0,\ 0 \in \mathbf{D}'$

where P is a parameter vector taking values in the set **P'**. It is well known from the mathematical literature on differential equations that there exists a *unique solution* of (16) (on a set $\mathbf{U} \subseteq \mathbf{U}'$, a set $\mathbf{P} \subseteq \mathbf{P}'$, and an open interval $\mathbf{D} \subseteq \mathbf{D}'$ containing 0) if the function Φ has certain (smoothness) characteristics[9] (for $P \in \mathbf{P}$). Such a *unique solution* of (16) is simply a family of functions ϕ^I : $\mathbf{D} \times \mathbf{P} \to \mathbf{U}$ (with the index $I \in \mathbf{I}$ and the parameter vector $P \in \mathbf{P}$) that has the following characteristics: (a) the corresponding family of trajectories ($\mathbf{T}^I(P) := \{ \phi^I(t, P) \in \mathbf{U}: t \in \mathbf{D}\}$, where $I \in \mathbf{I}$ and $P \in \mathbf{P}$) is continuous and non-intersecting (cf. Definitions 3 and 5), and (b) $\forall P \in \mathbf{P}$, $\forall I \in \mathbf{I}$, $\mathbf{T}^I(P)$ is non-self-intersecting (cf. Definition 4).

Overall, a unique solution of (16) generates a family of trajectories that are continuous, non-intersecting, and non-self-intersecting (cf. Definitions 3-5) for a given parameter setting (P). In other words, if the structural dynamics are representable by a model of smooth autonomous differential equations with a given parameter vector, (self-)intersections do not arise (cf. Approach 1, Section 4.2.1). Therefore, we discuss now how to generate (self-)intersections by deviating from this model.

5.2 Some Mathematical Models of (Self-)Intersection

In this section, we present several mathematical conditions under which self-intersection of a (country's structural) trajectory and intersection of the members of a family of trajectories (where each trajectory belonging to the family represents the structural dynamics of a country) can occur. The discussion is based on the standard results of the mathematical

[9] The mathematical literature discusses different sets of conditions that ensure the "uniqueness of solutions" (for a given parameter setting P). In general, these conditions require that the function Φ (cf. (16)) is smooth in some sense (for a given parameter setting P). For an overview of these conditions, see Stijepic (2015b), p.84f.

literature on differential equation systems. To demonstrate the applicability of the results of Section 5.2, we apply them in Section 7 for analyzing under which conditions can the Kongsamut et al. (2001) model generate (self-)intersecting trajectories.

5.2.1 Models of families of intersecting trajectories

Assume that we observe structural change in two countries. For each of the two countries, we construct a structural trajectory based on the empirical data. Furthermore, assume that the two trajectories intersect. (We will see in Section 6 that this is a common empirical observation.) We discuss now how this intersection can be modelled or explained by using the concepts introduced in Sections 4.2 and 5.1.

Intersections between two trajectories representing two different countries (country A and country B) can occur in the following five cases, which are implied by the standard mathematical theory of differential equation systems.

a) Non-autonomous differential equation systems (time-varying "law"). Assume that the structural changes in the two countries under consideration follow one and the same structural "(pseudo) law" and that the latter can be expressed as follows

(17) $\forall t \in \mathbf{D} \subseteq \mathbf{R}\ \ \forall X_0 \in \mathbf{U} \subseteq \mathbf{R}^n\ \ dX(t)/dt = \Gamma(X(t),t),\ \ X(0) = X_0,\ \ 0 \in \mathbf{D}$

Furthermore, assume that country A has the initial condition X(0) = A $\in \mathbf{U}$ and country B has the initial condition X(0) = B $\in \mathbf{U}$, where A $\neq$ B. Implicitly, we assume here that both countries have the same parameter vector; therefore, (17) does not display the parameter vector explicitly. All these assumptions imply that we rely here on *Approach 1* (cf. Section 4.2.1). We can see that Γ is not only dependent on X but also on time and, thus, the differential equation system (17) is non-autonomous. It is well known that the non-autonomous differential equation system (17) can generate trajectories that intersect each other (cf. Definition 5) even if Γ is smooth in the sense discussed in Section 5.1. Thus, the "(pseudo) law" (17) may imply that the trajectories (of the two countries following this law and having different initial conditions) intersect in the sense of Definition 5.[10] We do not use

[10] However, since Γ (cf. (17)) is dependent on time, the "(pseudo) law" (17) is not a law in common sense, where the latter is, in general (e.g. in natural sciences), defined as a regularity independent of time (cf. Section 4.2.4). Therefore, among others, it makes sense to find a representation of (17) that separates the autonomous component (representing the law) and the time-dependent component (representing exogenous impacts or parameter shocks). This can be done by, e.g., relying on alternatives d and e (which we discuss later in this section), or finding an autonomous transformation of (17), where the resulting autonomous differential equation system represents the law. This is often done in growth theory. For example, the versions of the Solow (1956) model and the Ramsey-(1928)/Cass-(1965)/Koopmans-(1967) model with technological progress generate non-

this approach (i.e. non-autonomous systems) in Section 7, since the model discussed there can be represented by an autonomous differential equation system.

b) Non-smooth vector fields. Following *Approach 1* (cf. Section 4.2.1), assume that: (i) the structural law (followed by the two countries) is described by the autonomous differential equation system (16); (ii) both countries have the same parameter values (P); (iii) the two countries have not the same initial conditions; and (iv) Φ does *not* satisfy the usual smoothness conditions discussed in Section 5.1, such that the solution of (16) is not unique in the sense used in Section 5.1. Such a solution can be associated with a family of intersecting trajectories. Thus, the trajectories of the two countries modelled by this system could intersect each other (if the initial conditions of the countries are not the same). As noted at the beginning of Section 5, the typical structural change models (and most of the long-run growth models) are continuous and assume smooth (utility and production) functions such that the resulting dynamical systems are smooth. Therefore, among others, we cannot rely on non-smooth vector fields as an explanation of intersection in Section 7.

c) "Law" differs across countries. Following *Approach 3* (cf. Section 4.2.3), assume that the structural "(pseudo) law" in country A can be described by (16) (with a fixed parameter setting P), and the structural "(pseudo) law" in country B can be described by

(18) $\forall t \in \mathbf{D} \subseteq \mathbf{R} \;\; \forall X_0 \in \mathbf{U} \subseteq \mathbf{R}^n \;\; dX(t)/dt = \Delta(X(t)), \quad X(0) = X_0, \quad 0 \in \mathbf{D}$

Furthermore, assume that Φ and Δ are sufficiently smooth such that unique solutions of (16) and (18) exist (and, thus, each system generates a family of continuous and non-intersecting trajectories (cf. Definition 5)). These assumptions state that each country follows its own "(pseudo) law" (i.e. the "laws" are ad hoc); thus, the trajectories of the countries could intersect despite the existence of unique solutions for each country. As discussed in Section 4.2.4, ad hoc models are not only inferior to the models that generate statements that are valid across several countries but also can be replaced by "ceteris paribus models/laws" (see point d) when modelling structural change.

autonomous differential equation systems explaining the dynamics of consumption and capital. The standard approach to analysis of these models is based on the autonomous transformation of these systems, where the variables "consumption" and "capital" are transformed into the variables "consumption in labor efficiency units" and "capital in labor efficiency units" and the (transformed) differential equation system describing the dynamics of these transformed variables is autonomous. See, e.g., Barro and Sala-i-Martin (2004).

d) Ceteris paribus laws. Assume that the structural law can be expressed by (16), where P is a parameter vector taking values in the set $\mathbf{P} \subseteq \mathbf{P'}$. Furthermore, assume that Φ is sufficiently smooth such that there exists a unique solution of (16) (for each $P \in \mathbf{P}$) corresponding to a family of non-intersecting trajectories. Let the two countries under consideration follow the law (16) and differ only by P, i.e. country A has the parameter value $P = A \in \mathbf{P}$ and country B has the parameter value $P = B \in \mathbf{P}$, where $A \neq B$. We can see immediately that these assumptions reflect Approach 2 (see Section 4.2.2) and that the law (16) is a ceteris paribus law (see Section 4.2.4). In this case, cross-country differences regarding the trajectory characteristics are generated by cross-country parameter variation.

e) Parameter perturbations. Assume that the two countries follow the law (16) and that Φ is sufficiently smooth such that there exists a unique solution of (16) (for each $P \in \mathbf{P} \subseteq \mathbf{P'}$ and for all $X(0) \in \mathbf{U} \subseteq \mathbf{U'}$) corresponding to a family of non-intersecting trajectories. Moreover, assume that both countries are characterized by the same P (i.e. country A has the parameter value $P = C \in \mathbf{P}$ and country B has the parameter value P = C) but differ by initial conditions (i.e. country A has the initial condition $X(0) = A \in \mathbf{U}$ and country B has the initial condition $X(0) = B \in \mathbf{U}$, where $A \neq B$). Assume now that a perturbation of P occurs at some point in time t > 0. In this case, the trajectories of the two countries may intersect. For example, assume that one country is a latecomer (i.e. moves slowly through the state space) and intersects after the perturbation the pre-perturbation segment of the fast-developing country. This can occur even in structurally stable systems, since structural stability does not prevent intersection of the perturbed and non-perturbed system. This effect can occur easily in systems with bifurcations. Finally, note that case e reflects Approach 1 (cf. Section 4.2.1) with parameter perturbations.

Note that the cases a to e are archetypes. It is possible to create intersections by combining these archetypes. For example, we could generate intersections by assuming that the two countries follow one and the same ceteris paribus law (case d) and are subject to (asymmetric) parameter perturbations (case e).
Overall, the discussion of the points a to e shows that we will rely on cross-country parameter differences (case d) and parameter perturbations (case e) (or some combination of them) when trying to generate (self-)intersections in the model example of Section 7.

5.2.2 Models of self-intersecting trajectories

Now, we turn to the question under which circumstances a self-intersection of a trajectory (representing the structural dynamics of a country) can occur, where, again, we use the concepts described in Section 5.1 to answer this question. A self-intersection of a country's structural trajectory can occur in the following cases.

i) Non-autonomous differential equation systems (time-varying "law"). Assume that the country follows the law implied by the non-autonomous differential equation system (17) and that the initial state of the country is given, i.e. $X(0) = X_0 \in \mathbf{U}$. It is well known that the solution of non-autonomous systems of type (17) for a given initial value can be associated with self-intersecting trajectories in the sense of Definition 3.

ii) Non-smooth vector fields. Assume that (I) the structural law (followed by the country) is described by the autonomous differential equation system (16), (II) the parameter vector (P) is fixed, and (III) Φ does not satisfy the usual smoothness conditions discussed in Section 5.1, such that the solution of (16) is not unique in the sense used in Section 5.1. For a given initial condition X(0) (representing the country's initial state), such a solution could be associated with a self-intersecting trajectory (describing the dynamics of the economy).

iii) Parameter perturbations. Assume that: (I) the country follows the law (16); (II) Φ is sufficiently smooth such that there exists a unique solution of (16) (for each $P \in \mathbf{P} \subseteq \mathbf{P'}$ and for all $X(0) \in \mathbf{U} \subseteq \mathbf{U'}$) corresponding to a family of non-intersecting trajectories; (III) the initial state of the economy is given, i.e. $X(0) = X_0 \in \mathbf{U}$; and (IV) initially, the parameter value for the country is given by $P = C \in \mathbf{P}$. Assume now that a perturbation of P occurs at some point in time $t = z > 0$, i.e. $P = C$ for $t < z$ and $P = C' \in \mathbf{P}$ for $t \geq z$, where $C \neq C'$. In this case, the post-perturbation ($t > z$) segment of the country's trajectory can intersect the pre-perturbation ($t < z$) segment of the country's trajectory, such that the overall trajectory (which is the union of the post- and pre-perturbation segment) intersects itself according to Definition 3. This can occur even in structurally stable systems, since structural stability does not prevent intersection of the perturbed and non-perturbed system.

For the reasons discussed in Section 5.2.1 (points a and b), we can exclude alternatives (i) and (ii). Thus, in Section 7, we will try to explain the empirically observable self-intersections by assuming that there are perturbations of the parameters (case iii) of the Kongsamut et al. (2001) model.

6. EVIDENCE ON THE TOPOLOGICAL PROPERTIES OF STRUCTURAL CHANGE TRAJECTORIES AND STYLIZED FACTS

Sections 6-7 can be regarded as an application of the method developed in Sections 2-5 to different topics covered by our definition of structural change. Since our definition of structural change (i.e. Definition 2) and method (i.e. topological approach) cover a wide range of topics (cf. Section 2.2), it is not possible to discuss the evidence on (self-)intersection of trajectories associated with all these topics. Therefore, in Section 6.2, we focus on a specific type of structural change covered by Definition 2, namely, labor re-allocation across agriculture, manufacturing, and services. Nevertheless, in Section 6.1, we discuss briefly evidence on (non-)self-intersection of trajectories associated with some other topics covered by Definition 2, since it is relatively easy to construct this evidence on the basis of well-known and well-available data.

6.1 On the Construction of Evidence on Non-Self-Intersection of Trajectories Associated with Definition 2

Following Stijepic (2015b), p.82f, it can be relatively easy to identify non-self-intersection in empirical data on structural change. Assume that we have data on the vector $X(t) \equiv (x_1(t), x_2(t), \ldots x_n(t))$ for the time points $t_0, t_1, t_2, \ldots t_m$. The corresponding trajectory $\mathbf{T} := \{X(t) : t \in \{t_0, t_1, t_2, \ldots t_m\}\}$ is non-self-intersecting (cf. Definition 4) if there exists an $i \in \{1,2,\ldots n\}$ with the property that $x_i(t)$ increases or decreases monotonously over the period $t_0, t_1, t_2, \ldots t_m$. By using this proposition, we can easily identify non-self-intersecting trajectories by relying on well-known data, as demonstrated in Examples 11 and 12.

Example 11. Kongsamut et al. (2001), p.873, provide data on the US consumption structure (cf. Example 8) in the three-sector framework (cf. Example 2). The data depicts the dynamics of the agricultural, manufacturing, and services consumption shares over the period 1940-2000. As this data reveals, besides some short-run fluctuations, the consumption share of services increases monotonously over this period. Thus, we can conclude that the trajectory representing the Kongsamut et al. (2001) data on the consumption structure in the three-sector model is non-self-intersecting.

Example 12. The same procedure can be applied to the US employment shares data presented by Kongsamut et al. (2001), p.873, for showing that the US labor allocation trajectory (cf. Example 2) over the period 1869-1998 is non-self-intersecting in the long run, since the services employment share increases monotonously (except for some short run fluctuations).

Much stronger statements can be made about two-dimensional structures, i.e. for n = 2 (cf. Definition 1). Assume that we have data on the vector $X(t) \equiv (x_1(t),x_2(t))$ for the time points t_0, t_1, t_2,…t_m. The trajectory **T**:= $\{X(t) : t \in \{t_0, t_1, t_2,\ldots t_m\}\}$ is self-intersecting (cf. Definition 4) if there exists an $i \in \{1,2\}$ with the property that $x_i(t)$ is non-monotonous over the period t_0, t_1, t_2,…t_m. (The proof is obvious.) This proposition can be easily used to quickly identify self-intersection in well-known data, as demonstrated in Examples 13 and 14.

Example 13. Antras (2001), p.28, provides evidence on the dynamics of the savings rate in the OECD countries for the period 1950-1990. As explained in Section 2.2 (Example 3), the savings rate dynamics are covered by our structural change definition (i.e. Definition 2) for n = 2. According to the Antras (2001) evidence, the savings rate dynamics display long-run cycles and are, therefore, non-monotonous (in the long run). Therefore, the savings-consumption trajectory (cf. Example 3) generated by this data is self-intersecting.

Example 14. OECD (2015), p.15, provides long-run data on the labor income share reflecting the functional income distribution for the period 1856-2009. As explained in Section 2.2 (Example 4), the functional income distribution is covered by our Definition 2 with n = 2. As indicated by the OECD (2015) data, the labor income share dynamics are characterized by long-run fluctuations and, thus, are non-monotonous in the long run. Thus, the trajectory of functional income distribution representing this data is self-intersecting.

This discussion implies a lot of new research topics related to the identification of self-intersection (and mutual intersection) of the trajectories associated with the topics (cf. Section 2.2) covered by our structural change definition and the explanation of it.

6.2 Evidence on the (Self-)Intersection of Long-run Labor Allocation Trajectories

The long-run dynamics of the labor allocation across the agricultural, manufacturing, and services sector is a classical topic of development and growth theory. See Schettkat and

Yocarini (2006), Krüger (2008), Silva and Teixeira (2008), and Herrendorf et al. (2014) for overviews of literature dealing with this topic. For an explanation of this topic and some references, see Section 2.2, Example 2.

As explained there, the allocation of labor across agriculture, manufacturing, and services can be represented by the vector $X(t):= (x_1(t),x_2(t),x_3(t))$, where $x_1(t)$, $x_2(t)$, and $x_3(t)$ represent the employment share of the primary sector (agriculture), secondary sector (manufacturing), and tertiary sector (services) at time t, respectively. Structural change refers to the changes in X(t) over time and, thus, labor re-allocation. As discussed in Section 3.1, the vector X(t) (which represents the labor allocation at time t) can be represented by a point on the standard 2-simplex, and structural change over the period [a,b] can be represented by a trajectory on the standard 2-simplex connecting the points X(a) and X(b).

In accordance with (9), we construct the labor allocation trajectory of each country in our sample as follows. Assume that we have data on labor allocation (X(t)) across agriculture, manufacturing, and services in county A for the time points t_0, t_1,…t_m. That is, we have the data points $X(t_0)$, $X(t_1)$,…$X(t_m)$ associated with country A. We construct the labor allocation trajectory of country A by depicting the points $X(t_0)$, $X(t_1)$,…$X(t_m)$ on the standard 2-simplex and connecting them (while preserving their timely order) by line segments. We indicate the direction of movement (i.e. the timely order of the points) along the trajectory by an arrow at the last observation point.

We do this procedure with all the countries from our samples and depict the trajectories of all countries from the respective sample on one and the same simplex. In this way, we can not only observe self-intersections but also mutual intersections between countries' trajectories.

In Figures 10, 11, and 12, we depict the data on the long-run labor allocation dynamics in the OECD countries on the standard 2-simplex, where the simplex refers to the employment shares of agriculture (x_1), manufacturing (x_2), and services (x_3) and the vertices (V_1, V_2, and V_3) are given by (4)-(6); cf. Figure 1 in Section 3.1. For better visibility, Figure 12 depicts the enlarged segment of Figure 11 containing all the trajectories depicted in Figure 11. In Figures 11 and 12, we omit the arrows indicating the direction of movement along the trajectories in the most cases for reasons of clarity. Furthermore, note that the direction of movement along the trajectories is not relevant for our discussion.

Figure 10 depicts the data on labor re-allocation over very long periods of time (ranging from 1820 to 1992). As we can see, the trajectories of the countries *intersect mutually*. We can observe intersections of the trajectories of the following countries: (a) Germany and UK, (b) US and France, (c) Netherlands and France, (d) US and France, (e) Netherlands and US, (f)

China and US, (g) Russia and France, (h) Russia and Netherlands, (i) Japan and France, (j) Japan and Netherlands, and (k) Japan and US. Moreover, we cannot identify any self-intersections in Figure 10.

Figures 11 and 12 present higher-frequency data. As we can see, this data reveals again numerous mutual intersections, thus, confirming the results derived from Figure 10. Moreover, the high-frequency data presented in Figures 11 and 12 shows many (short-run) self-intersections. For example, the trajectories of the following countries self-intersect: Australia, Belgium, Chile, Ireland, Island, Latvia, Luxemburg, New Zealand, Norway, Slovakia, Slovenia, Suisse, Sweden, and Turkey. Longer-run self-intersections, e.g. large loops (covering long time periods), seem not to occur.

Figure 10. *Labor allocation trajectories for USA, France, Germany, Netherlands, UK, Japan, China, and Russia.*

- insert Figure 10 here –

Notes: *Data source: Maddison (1995). The black dot represents the barycenter of the simplex. Abbreviations: C – China, F – France, G – Germany, J – Japan, N – Netherlands, R – Russia, US – United States, UK – United Kingdom. Data points (years in parentheses): USA (1820, 1870, 1913, 1950, 1992), France (1870, 1913, 1950, 1992), Germany (1870, 1913, 1950, 1992), Netherlands (1870, 1913, 1950, 1992), UK (1820, 1870, 1913, 1950, 1992), Japan (1913, 1950, 1992), China (1950, 1992), Russia (1950, 1992).*

Figure 11. *Labor allocation trajectories of OECD countries over the 1980ies, 1990ies, 2000s, and 2010s.*

- insert Figure 11 here -

Notes: *Data source: The Worldbank, World Databank. The black dot represents the barycenter of the simplex. Arrows indicating the direction of movement along the trajectories are omitted in the most cases for reasons of clarity of representation.*

Figure 12. *The labor allocation trajectories depicted in Figure 11 enlarged.*

- insert Figure 12 here –

Notes: *The black dot represents the barycenter of the simplex. The edges of the simplex are not visible in Figure 12. Arrows indicating the direction of movement along the trajectories are omitted in the most cases for reasons of clarity of representation.*

We can summarize the discussion in Section 6.2 by formulating the following two stylized facts.

Stylized Fact 1. *The labor allocation trajectories of different countries intersect mutually (in the long run).*

Stylized Fact 2. a) *Labor allocation trajectories self-intersect. The intersections are of short-run nature, i.e. there are no long-run loops.* ***b)*** *The long-run dynamics of labor allocation can be represented by non-self-intersecting trajectories.*

7. AN APPLICATION TO THE THEORETICAL LABOR RE-ALLOCATION LITERATURE

In this section, we demonstrate how to apply our topological approach (developed in Sections 2-5) for comparing standard labor re-allocation models (cf. Example 2) with the stylized facts derived in Section 6.2. Since this discussion tends to be lengthy as we will see, we discuss only the Kongsamut et al. (2001) model as a major example of the modern labor re-allocation modeling literature. Of course, this choice is arbitrary to some extent and we regard all the other models[11] as interesting and important contributions to structural change theory.

In Section 5, we have discussed different approaches to generate (self-)intersection in models representable by differential equations. Now, we apply these results. In particular, we show that the Kongsamut et al. (2001) model belongs to the autonomous differential equation class discussed in Section 5.1; thus, for given parameter values, the Kongsamut et al. (2001) model cannot generate (self-)intersections (cf. Section 5.1). Therefore, we try to generate (I) trajectory intersections in this model by assuming that there are cross-country differences (case d) and perturbations (case e) regarding the parameters of this model (cf. Section 5.2.1) and (II) self-intersections by assuming that there are parameter perturbations (cf. case iii in Section 5.2.2). Note that we discuss here self-intersections although the Section 6 results show that self-intersection is not a long-run phenomenon. We do this since self-intersections occur in the shorter run and, thus, it is interesting to see whether the Kongsamut et al. (2001) model can explain short-run self-intersections.

[11] See Section 6.2 for some literature overviews dealing with long-run labor re-allocation models.

Recall that we return now to Example 2, where $x_1(t)$, $x_2(t)$, and $x_3(t)$ stand for the employment shares of the agricultural, manufacturing, and services sector, respectively and, thus, $X(t) \equiv (x_1(t), x_2(t), x_3(t))$ represents the labor allocation at time t.

Kongsamut et al. (2001) focus on the discussion of their model in its dynamic equilibrium state, which is named "generalized balanced growth path" (henceforth: GBGP). They justify their focus on the GBGP by referring to the fact that the GBGP is consistent with the empirical evidence known as "Kaldor-facts", among others. The GBGP and similar types of dynamic equilibrium are widespread in the modern structural change analysis (cf. Example 7 in Section 2.2).

After some calculations based on the equations provided by Kongsamut et al. (2001), we derive the following equations describing the dynamics of labor allocation along the GBGP of the Kongsamut et al. (2001) model:

(19a) $x_1(t) = \beta\chi + \dfrac{B_M \overline{A}}{B_A Y_0 \exp(gt)}$

(19b) $x_2(t) = 1 - (1-\gamma)\chi$

(19c) $x_3(t) = \theta\chi - \dfrac{B_M \overline{S}}{B_S Y_0 \exp(gt)}$

The "parameters" of this differential equation system satisfy the following restrictions (along the GBGP), as assumed by Kongsamut et al. (2001):

(20a) $\beta + \gamma + \theta = 1$

(20b) $B_S \overline{A} = B_A \overline{S}$

(20c) $\beta, \gamma, \theta, g, B_A, B_M, B_S, \overline{A}, \overline{S}, Y_0 > 0$

Although we do not seek to economically interpret the equation system generated by the Kongsamut et al. (2001) model, note that (a) Y_0 represents the aggregate output (in manufacturing terms) at time t = 0, where aggregate output grows at the rate g along the GBGP, and (b) χ stands for the aggregate consumption-expenditures-to-output ratio, which is constant along the GBGP of the Kongsamut et al. (2001) model and obviously, satisfies the following condition

(20d) $0 < \chi < 1$

Furthermore, it makes sense to assume that the parameters of the model are such that

(21) $X(0) \in \mathbf{S}_2$

Otherwise, the employment shares would be negative, which does not make sense economically.

Note that the system (19)-(21) can be represented by the following differential equation system satisfying the parameter conditions (20) and (21):

(22a) $\forall t \;\; x_1'(t) = \beta\chi g - g x_1(t)$

(22b) $\forall t \;\; x_2'(t) = 0$

(22c) $\forall t \;\; x_3'(t) = -x_1'(t)$

Thus, the GBGP dynamics of the Kongsamut et al. (2001) model are representable by a linear autonomous differential equation system.

It is obvious that the system (19)-(21) generates a line segment on the simplex that is parallel to the V1-V3 edge of the simplex (cf. (4)-(6) and Figure 1). This is true for any parameterization of the model satisfying (20) and for all initial conditions satisfying (21). This fact implies that: (a) the system (19)-(21) belongs to the class of models discussed in Section 5.1, i.e. the system (19)-(21) does not generate (self-)intersections unless there is some sort of parameter variation; and (b) we cannot generate mutual trajectory intersection by using approach d (cf. Section 5.2.1), since the countries' trajectories are always parallel (even if the parameters differ across countries).[12] However, (self-)intersections can be generated by assuming parameter perturbations, i.e. by using (a combination of approach d and) approach e (cf. Section 5.2.1) and approach iii (cf. Section 5.2.2). For example, (self-)intersections can be generated by assuming parameter sequences that generate the dynamics depicted in Figure 13, where the (self-)intersection occurs implicitly when the country A jumps from trajectory segment 3 to trajectory segment 4. (In empirical data, such jumps are not distinguishable from "continuous" intersections, since the empirical data is non-continuous.)

Figure 13. *An implicit mutual intersection and an implicit self-intersection generated by parameter perturbations.*

- insert Figure 13 here -

In general, such parameter sequences seem relatively complex; models that can generate (self-)intersections by relying on simpler parameter sequences or on approach d seem

[12] Note that the countries' trajectories do not overlap completely if the parameters differ across countries, as assumed in approach d.

preferable. However, this hypothesis cannot be discussed without econometric tests, which are beyond the scope of our paper. In general, the question whether the complex parameter shock sequences required to generate (self-)intersections in the system (19)-(21) occur in reality when (self-)intersections are observed or whether other explanations (not consistent with the system (19)-(21)) are preferable, seems interesting and is left for further research. Moreover, recall that (19)-(21) represents the dynamics of the Kongsamut et al. (2001) model *along the GBGP*. If we studied the economy off the GBGP, χ would not be not constant and, thus, the trajectory not linear and intersections could be possible even without the assumption of complex parameter shock sequences. We omit a detailed study of this topic, since the discussion above seems to be sufficient to demonstrate the applicability of our topological approach.

8. TOWARDS A THEORETICAL EXPLANATION OF NON-SELF-INTERSECTION OF STRUCTURAL CHANGE TRAJECTORIES

In this section, we remove our focus from labor allocation and return to our general definition of structural change, i.e. Definition 2. Thus, in this section, the term "structural change" covers among others all the topics discussed in the Examples 1-8.

Furthermore, we focus here on the theoretical explanation of non-self-intersection, whereas we leave the theoretical explanation of mutual intersection for further research. We have three reasons for this decision. First, non-self-intersection seems to be a stylized fact of long-run labor allocation dynamics (cf. Section 6.2) and of other structural change types (cf. Section 6.1). Second, the explanation of mutual intersection by relying on exogenous cross-country parameter differences and exogenous parameter shocks, as suggested in Section 5.2.1, seems to be an acceptable explanation in (empirical) sciences. At least, it seems to make sense to assume that exogenous parameter shocks exist in the models/topics discussed in Section 2.2. In contrast, the explanation of long-run *non*-self-intersection cannot rely on exogenous shocks but requires more complex arguments (e.g. arguments based on the results of Section 5.1 and smoothness of structural change systems). Third, the models from the previous literature (endogenously) generate non-self-intersecting trajectories, but not necessarily mutually intersecting trajectories (cf. Sections 5.1 and 7). Thus, for explaining non-self-intersection we can built on the previous literature to some extent, whereas explaining mutual intersection (beyond stating that it occurs due to exogenous parameter variation, as done in Sections 5.2.1 and 7) requires a devoted model, which is permitted by the space restrictions that we have here.

As we show in Sections 7 and 8.1, each of the standard structural change models (discussed in Sections 2.2 and 7) represents an (endogenous) theoretical foundation of the non-self-intersection, since each model generates non-self-intersecting trajectories (for given parameter values). However, the previous literature does not attempt to explain non-self-intersection explicitly. At least, we are not aware of such an attempt. Furthermore, the economic topics covered by our structural change definition (Definition 2) differ strongly; for example, it seems obvious that the non-self-intersection of the labor-allocation trajectory (cf. Example 1) has not the same explanation as the non-self-intersection of the trajectory of wealth distribution (cf. Example 6). In particular, in each model, the non-self-intersection results from a large set of assumptions, where the assumption sets differ significantly across models. Therefore, the standard structural change models (cf. Examples 1-8) do not imply a simple and uniform explanation of non-self-intersection of structural change trajectories covered by Definition 2. In this section, we try to provide a set of "simple" (partial) explanations common to many models (and not to only one model). We choose two ways.

In Section 8.1, we study a set of structural change models (cf. Examples 1-8) dealing with quite heterogeneous topics covered by Definition 2 and isolate the common model-components/modules ("underlying systems") that are representable by non-self-intersecting trajectories. These components/modules represent the partial explanations of non-self-intersection common to all models (from the corresponding model class). The non-self-intersection in each of the models is then a result of this partial/common explanation and further assumptions specific to each topic/structural change type. Technically speaking, we show that some component ("underlying system") common to all models (of the respective model class) is representable by a non-self-intersecting trajectory. The trajectory of structural change in each of the models is then a homeomorphism of this trajectory (of the underlying system), where in each model, the homeomorphism results from an assumption set specific to the respective model/topic. Thus, despite the heterogeneity of the topics we can separate the common model components (which are represented by the non-self-intersecting trajectory of the underlying system) from the specific model components (which are represented by the homeomorphisms) and, thus, isolate the common partial explanation of non-self-intersection by using the methods of topology. As we will see, the common partial explanations for non-self-intersection (within the model sample) are the monotonicity of consumption-capital dynamics in the neoclassical framework and the monotonicity of technological progress and population growth (in the long run), where the latter can be explained in R&D-models and endogenous fertility models.

While this approach is rather mathematical, we choose an intuitive approach in Section 8.2, where we argue verbally that the typical assumptions of (neoclassical) long-run models (e.g. rationality, efficiency, perfect foresight, and utility and profit maximization) imply monotonous trajectories and that self-intersecting trajectories seem to be inefficient from some point of view. Thus, if the economy is efficient in the long run (as believed by many neoclassical economists) non-monotonicity and, in particular, self-intersection does not arise.

8.1 Partial Explanations of Non-Self-Intersection Derived by Using a Topological Approach

Now, we extract from a sample of quite heterogeneous structural change models two partial explanations of non-self-intersection. It makes sense to use topological methods, since non-self-intersection is a topological characteristic of trajectories.

We choose the following sample of topics/models (cf. Examples 1-8) covered by our definition of structural change (i.e. Definition 2):

(I) dynamics of functional income distribution in the Solow (1956) model,

(II) savings and consumption rate dynamics in the Ramsey-(1928)/Cass-(1965)/Koopmans-(1967) model,

(III) labor re-allocation across sectors in the Baumol (1967) model,

(IV) dynamics of the consumption structure in the Kongsamut et al. (2001) model,

(V) dynamics of the consumption and capital sector in the Ngai and Pissarides (2007) model,

(VI) dynamics of the personal wealth distribution in the Caselli and Ventura (2000) model.

For a proof that these topics are covered by Definition 2, see Section 2.2

A closer look at the models (I)-(VI) reveals that there are two model categories that represent two typical characteristics of the neoclassical growth literature: (1) reliance on exogenous variables ("first category", "exogenous structural change") and (2) the centrality of the consumption/capital dynamics ("second category"). As we will prove now, these two model categories represent two different (partial) explanations of non-self-intersection: monotonicity of exogenous variables (first category) and monotonicity of the capital-consumption system (second category).

8.1.1 First category models ("exogenous structural change") and monotonicity of exogenous variables

The first category encompasses the topics III and IV. The mathematical model structure is set up around the assumption that there is an m-dimensional vector of exogenous variables, say $A(t) \equiv (a_1(t), a_2(t), \ldots a_m(t)) \in \mathbf{R}^m$, where $a_i(t) = a_i^0 \exp(g_i t)$ and $a_i^0, g_i \in \mathbf{R}$ are given (and constant) for $i = 1, \ldots m$ and $t \in [0, \infty)$. As is typical for the greatest part of (neoclassical) growth theory, these exogenous variables refer to population and (sectoral) technology parameters. Then, assumptions are made about the values of g_i (and a_i^0) based on theoretical or empirical arguments (i.e. it is assumed that there are sectors with higher and lower productivity growth rates and that population grows).

We can already see that the curve A(t), $t \in [0, \infty)$, generates a continuous and non-self-intersecting trajectory ($\mathbf{T}_A := \{A(t) \in \mathbf{R}^m\text{: } t \in [0, \infty)\}$) in m-dimensional real space (cf. Definitions 3 and 5); the curve/trajectory starts in $A(0) = (a_1^0, a_2^0, \ldots a_m^0)$ and converges to infinity or zero (in some dimension) for $t \to \infty$. In other words, the trajectory $\mathbf{T}_A$ is homeomorphic to the [0,1) interval.

Finally, the models of the first category explain how the exogenous variables (A) and the structural variables (X)[13] are related by using the typical neoclassical assumptions (production/utility functions, perfect markets, and market clearing). In fact, these assumptions establish a relationship between X and A of the form $X(t) = \Psi(A(t))$ for $t \in [0, \infty)$, where Ψ is a homeomorphism (i.e. it is a bijective, invertible, and continuous function with a bijective inverse). In particular, these assumptions ensure that the structural trajectory ($\mathbf{T}_X := \{X(t) \in \mathbf{R}^n$: $t \in [0, \infty)\}$ is homeomorphic to $\mathbf{T}_A$. Thus, the structural trajectory is a homeomorphism of the [0,1) interval and, thus, *non-self-intersecting*. For proofs (referring to models III and IV), see Stijepic (2014).

The fact that there exists a homeomorphism between the underlying system ($\mathbf{T}_A$) and the structural system ($\mathbf{T}_X$) results from relatively large sets of assumptions. It could be argued that the homeomorphisms arise because the variables ($a_i(t)$) of the underlying system increase strictly monotonously over time (per assumption) and the functions relating the exogenous system (A) to the structural system (X) are monotonous due to typical neoclassical assumptions (e.g. concave utility and production functions). (In part, this is implied by the lines of arguments used by Stijepic (2014) to show the existence of the homeomorphisms.) However, such arguments seem to be too complicated (since they must be derived from

[13] Note that in topic III (IV), the vector of structural variables X refers to sectoral employment (consumption) shares.

relatively large sets of model assumptions) and to differ strongly across models. Therefore, we choose a simpler way of discussing the intuitive/economic explanation of the homeomorphism/non-self-intersection in Section 8.2.

8.1.2 Second category models and the monotonicity of the consumption-capital system

The models of the "second category", which encompasses the topics I, II, V, and VI, are set up around a differential equation system describing the dynamics of consumption and capital. While model I postulates this consumption-capital system almost per assumption, models II, V, and VI derive the consumption-capital differential equation system from the typical neoclassical theoretical microfoundation (intertemporal utility maximization problem).

It is shown (by the authors of the models I, II, V, and VI) that the solution of the consumption-capital differential equation system (or a transformation of it) generates a saddle path along which the economy converges to a fixed point ("steady state"). Economic arguments[14] are provided ensuring that the economy is always placed on one of the two stable arms of the saddle-path, which we name here $\mathbf{T}_{CK1}$ and $\mathbf{T}_{CK2}$. Thus, for all (empirically relevant) initial conditions, the economy is located on either $\mathbf{T}_{CK1}$ or $\mathbf{T}_{CK2}$ and converges along one of these stable arms to the fixed point. The stable arms are continuous and non-(self-)intersecting trajectories in the sense of Definitions 3-5 and are, thus, homeomorphisms of the [0,1) interval.

It can be shown that in the "second category models", the trajectories describing the dynamics of the structural vector (X) are simply homeomorphisms of $\mathbf{T}_{CK1}$ and $\mathbf{T}_{CK2}$ (see Stijepic (2014) for a detailed discussion and proofs). Thus, the structural trajectories are homeomorphisms of the [0,1) interval and, therefore, *non-self-intersecting*.

In each of the models I, II, V, and VI, the homeomorphism between the structural trajectories and the underlying system ($\mathbf{T}_{CK1}$/$\mathbf{T}_{CK2}$) results from a relatively large set of economic/mathematical assumptions that differs significantly across models and that, in general, refers to the properties of utility/production functions, markets, and market clearing. Therefore, it is difficult to explain directly/explicitly/uniformly the existence of this homeomorphism by referring to the assumption sets of the respective models. For this reason, we discuss the intuitive/economic explanation of the homeomorphism/non-self-intersection by relying on more simple/uniform principles in Section 8.2.

[14] See, e.g., Barro and Sala-i-Martin (2004) for an example of such arguments.

8.1.3 Summary: topological properties of neoclassical structural change models and partial explanations of non-self-intersection of structural change trajectories

Overall, the discussion in Sections 8.1.1 and 8.1.2 implies the following properties of neoclassical structural change models (topics I-VI).

1.) The *structural system* (i.e. the system of equations/curves describing the dynamics of the structure X (cf. Definition 1)) is a sort of covering of some underlying system, where the latter is either an equation system describing the dynamics of exogenous variables growing at constant rates or a differential equation system describing consumption and capital dynamics.

2.) The trajectory of the underlying system is either per assumption (in category one models) or per neoclassical microfoundation (in category two models) a homeomorphism of the [0,1) interval, i.e. (the image of) a continuous and non-self-intersecting curve. The trajectory of the underlying system is non-self-intersecting because the technology and population dynamics or the capital-consumption dynamics are monotonous. Note that the theoretical foundation of the former can be found in, e.g., R&D-theories (cf., e.g., Romer-(1990)-type models) and endogenous fertility theories. The theoretical foundation of the latter is provided by the Ramsey-(1928)/Cass-(1965)/Koopmans-(1967) model or the Solow (1956) model.

3.) The structural trajectory is simply a homeomorphism of the trajectory of the underlying system. The theoretical foundation of this homeomorphism rests on complex assumption sets and differs significantly across models.

4.) The structural change trajectories of neoclassical structural change models (I-VI) are continuous and non-self-intersecting, since they are homeomorphisms of the continuous and non-self-intersecting trajectories of the underlying systems.

Especially due to point 3, we do not discuss the numerous heterogeneous sets of theoretical assumptions generating the homeomorphism between the underlying system and the structural trajectory but discuss the intuitive/economic explanation of this homeomorphism from a rather more general perspective in Section 8.2.

Overall, this discussion shows that the theoretical explanations of non-self-intersection of the trajectories of the underlying systems are partial explanations of the non-self-intersection of the structural change trajectories. This implies that the models that explain the monotonous development of technology (e.g. Romer-(1990)-type models) and of the capital-consumption system (e.g. Ramsey-(1928)/Cass-(1965)/Koopmans-(1967)) are partial explanations of non-self-intersection.

8.2 An Intuitive/Theoretical Explanation of Non-Self-Intersection of Structural Trajectories based on Efficiency Arguments

While Section 8.1 extracts (partial) explanations of non-self-intersection from the literature by using mathematical methods, we discuss the intuitive/economic aspects of the non-self-intersection of structural change trajectories, where we focus on long-run labor re-allocation in (neoclassical) multi-sector models. Similar arguments can be derived for the other topics covered by our structural change definition (cf. Section 2.2).

A discussion of *intuitive*/economic arguments seems to be necessary here, since our previous discussion did not provide many arguments of this sort and the latter are still regarded as an important pillar of economic thinking (even in macroeconomic analyses).

It seems to be interesting to explain the non-self-intersection from the *long-run* perspective, since the empirical evidence (cf. Section 6) and the long-run growth models (cf. Section 7 and 8.1) imply that non-self-intersection is a long-run phenomenon. Note that the long-run horizon does not automatically imply that we consider only linear trends and trajectories and, thus, trajectory self-intersection is excluded per definition of the framework of analysis ("long run"). As is well known in mathematics[15] and economics[16], long-run dynamics cannot be always described by linear trend curves or trajectories and, therefore, the assumption of "long run" does not automatically imply linear dynamics and non-self-intersection.

In the context of (neoclassical) long-run labor re-allocation models, the non-self-intersection of trajectories can be interpreted as an *efficiency* characteristic of the economy, as explained in the following.

Assume that a trajectory intersects itself at the coordinate point S. The point S represents a certain allocation of labor as any other point on the trajectory (on the simplex). Self-intersection of the trajectory means that the economy is at two points of time in point S: the first time (say at t = 1) when it traverses S and the second time (say at t = 2) when it intersects itself. In other words: first, the economy realizes the labor allocation S at t = 1; then, it deviates from this allocation over the time interval (1,2), i.e. the economy re-allocates labor across sectors; finally (at t = 2), the economy returns to the allocation S again, i.e. finally, the

[15] The mathematical literature on dynamical systems shows that the limit dynamics (i.e. the dynamics of a system as time goes to infinity), which represent the long-run dynamics in mathematical growth models, cannot be always described by linear trajectories. This is particularly true when the omega limit set of a trajectory is not of dimension zero (as in the case of limit cycles), or when the dynamics are chaotic (as in the case of strange attractors).

[16] Many economic long-run phenomena are cyclical (e.g. Kondratiev waves) or non-linear (e.g., the results of the structural change literature imply that the trajectory of long-run labor re-allocation is non-linear, as discussed by Stijepic (2015b), p.75).

economy re-allocates the labor back to the old allocation. (Of course, later, i.e. for $t > 2$, the economy may leave S again.) In general, labor re-allocation across sectors is associated with costs (unemployment, change of skills, cost of geographical relocation, etc.). This is particularly true for the re-allocation in the three-sector framework, where the qualification patterns differ significantly across agriculture, manufacturing, and services. Thus, deviating from S over the time interval (1,2) and, thus, accumulating all the re-allocation costs and then, returning to S seems to be *inefficient*, since the same end-result can be achieved by staying in S over the time interval (1,2), which is not associated with any re-allocation costs. That is, with respect to re-allocation costs, self-intersection seems to be inferior to staying in S (where the latter is not defined as self-intersection according to Definition 4).

This *"inefficiency argument"* for excluding self-intersection applies almost directly in neoclassical growth theory, which assumes that the economy is governed by a rational representative household that plans the economic dynamics to infinity. Within such a framework (with static preferences and perfect foresight of technology dynamics) it is hard to explain why the household chooses a self-intersecting trajectory (which is associated with significant re-allocation costs) instead of staying in S (which is not associated with any re-allocation costs).

Now, we could provide arguments stating that (short run) supply side *shocks* affecting the production function/technology or demand side shocks affecting the preferences are common and can lead to temporary deviation (over the time period (1,2)) from the (long-run) technology and preferences structure such that the economy deviates from optimal allocation (over the period (1,2)) and finally returns to it (at $t = 2$). However, we can exclude such arguments by the fact that we analyze the long-run dynamics, which abstract from such short run fluctuations. Recall that the empirical evidence shows that self-intersections seem to be short-run phenomena.

Moreover, the "shock argument" is excluded in neoclassical growth models, which assume that exogenous variables (such as technological/productivity parameters and population) grow at constant rates and, thus, are characterized by monotonous dynamics, as discussed in Section 8.1. In this case, monotonicity of these variables in association with our "inefficiency argument" ensures that the household chooses a monotonous (labor re-allocation) path to its future destination. In other words, our "inefficiency argument" can be regarded as a theoretical foundation of the homeomorphisms in "category one" and "category two" models (cf. Section 8.1), where the latter assume monotonous dynamics of the underlying system (which represents e.g. the exogenous technology dynamics).

Moreover, even if the dynamics of the exogenous variables were non-monotonous, our inefficiency argument ("avoidance of re-allocation costs") implies that the representative household, which can see into the future and foresee the technology non-monotonicities, seeks to compensate for them in order to avoid the re-allocation costs. Furthermore, if technology were endogenous, self-intersection of technology paths would be inefficient due to re-allocation costs and seem not to arise in frameworks with rational households, as implied by, e.g., the R&D models (e.g. Romer-(1990)-type models).

All in all, our inefficiency argument seems to be acceptable if we believe that the economy works efficiently in the long run. This is a rather neoclassical way of thinking. It would be interesting to develop alternative arguments (e.g. by studying whether non-self-intersection can be derived/established as an evolutionary law in the branch of evolutionary economics).

9. CONCLUDING REMARKS

In general, the term "structural change", as it is used in the literature, covers a wide range of topics (cf. Section 2). Traditionally, the structural change literature relies on the mathematical branch of calculus/analysis (and differential equations). Our paper is devoted to the exploration of the applicability of topological concepts (such as self-intersection and mutual-intersection of trajectories as well as homeomorphisms) in the analysis of structural change. The first part of our paper (Sections 2-5) is devoted to the discussion of the conceptual, mathematical and methodological aspects of the topological approach to structural change analysis. In the second part of our paper (i.e. in Sections 6 and 7), we demonstrate briefly how these results can be applied for (a) studying the empirical evidence on labor re-allocation and deriving stylized facts (cf. Section 6) and (b) comparing the theoretical models with the these stylized facts (cf. Section 7). Since we are not aware of the existence of a theoretical/intuitive explanation of the empirically observable non-self-intersection, we elaborate and discuss such explanations in Section 8.

Overall, we have demonstrated how topological characteristics can be used to study empirical evidence, classify models, compare the models with the evidence, and derive new theories and research topics.

While we apply our approach to labor allocation dynamics, it can be applied to many other topics (cf. Section 2.2). Furthermore, we apply our method to only one labor re-allocation model (namely, the Kongsamut et al. (2001) model). Of course, all the other labor re-allocation models (e.g. the models listed in Examples 1 and 2) can be analyzed regarding their (self-)intersection properties and compared with the evidence. This analysis can go much further than the analysis in our paper, which was limited by space restrictions and the necessity to lay the foundations of our approach. For example, each structural change model from the previous literature can be analyzed (on the basis of the results of Section 5.2) upon two questions: (1.) which exogenous model parameters must be varied to generate (self-)intersection of the structural trajectory; (2.) did such parameter variations occur in the countries that experienced (self-)intersection. Depending on the answers to these questions, model critique can be formulated and new model classes may become necessary. Furthermore, it could be interesting to continue the discussion started in Section 8 and develop further explanations of non-self-intersection. Overall, it seems that our approach generates a huge set of new research topics. These are left for further research.

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Figure 1

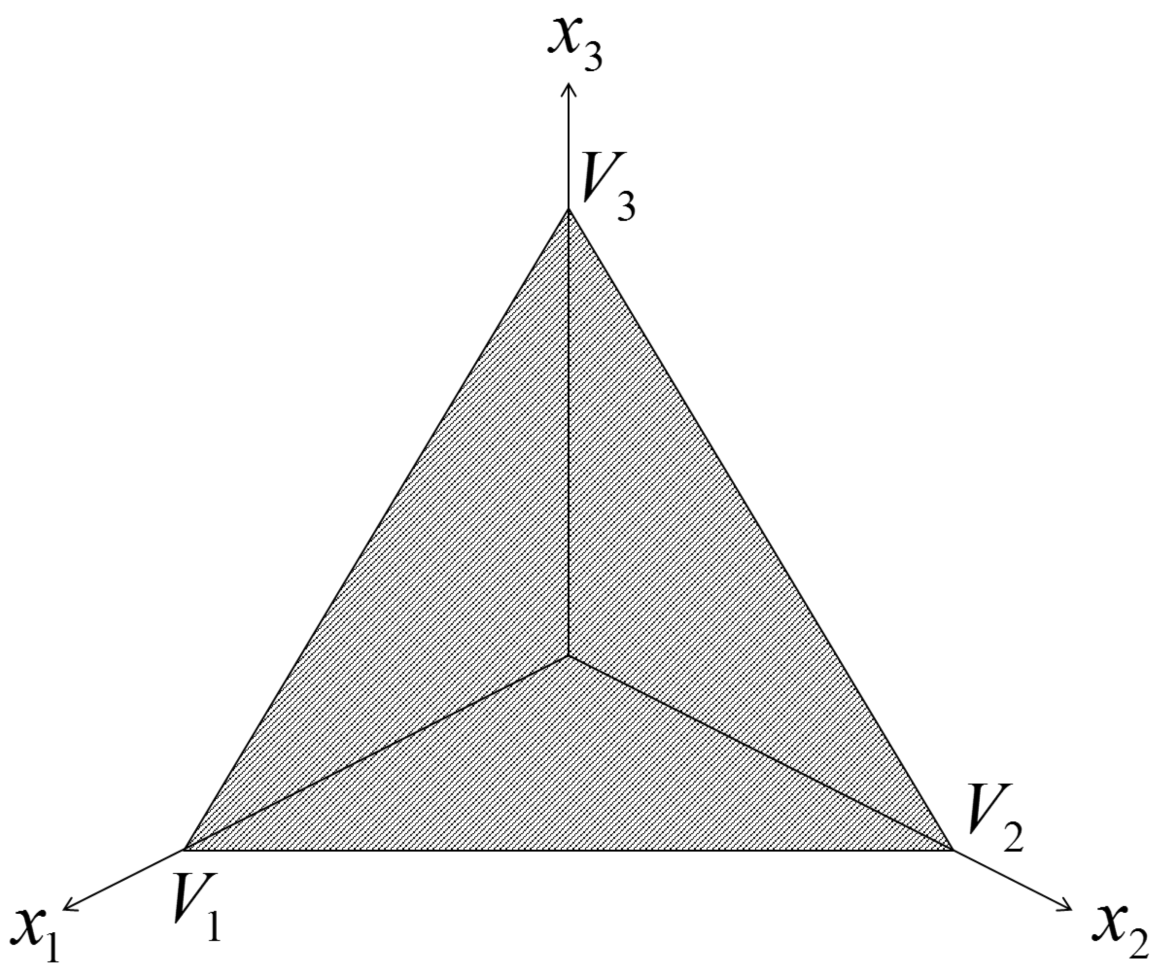


Figure 2

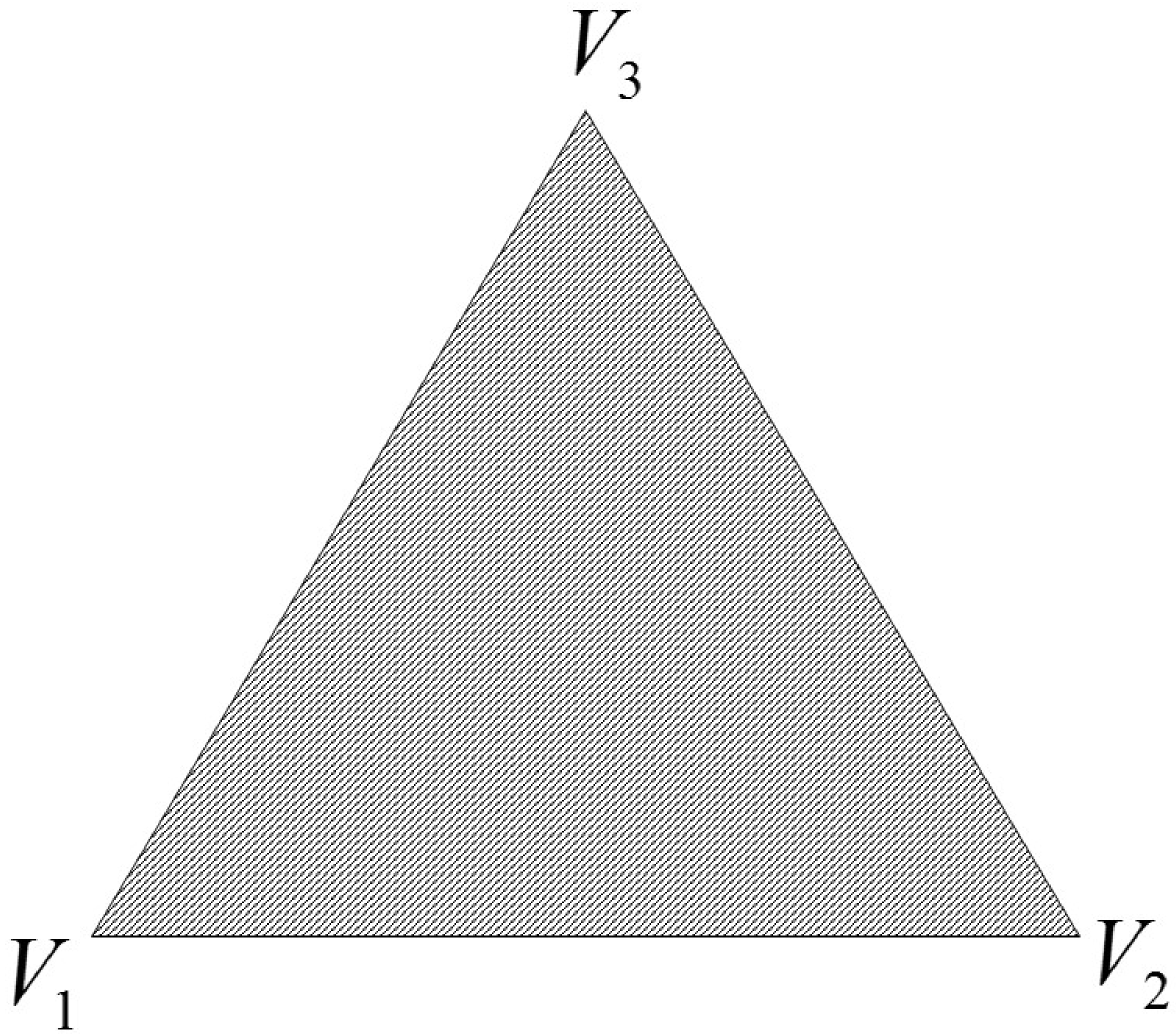
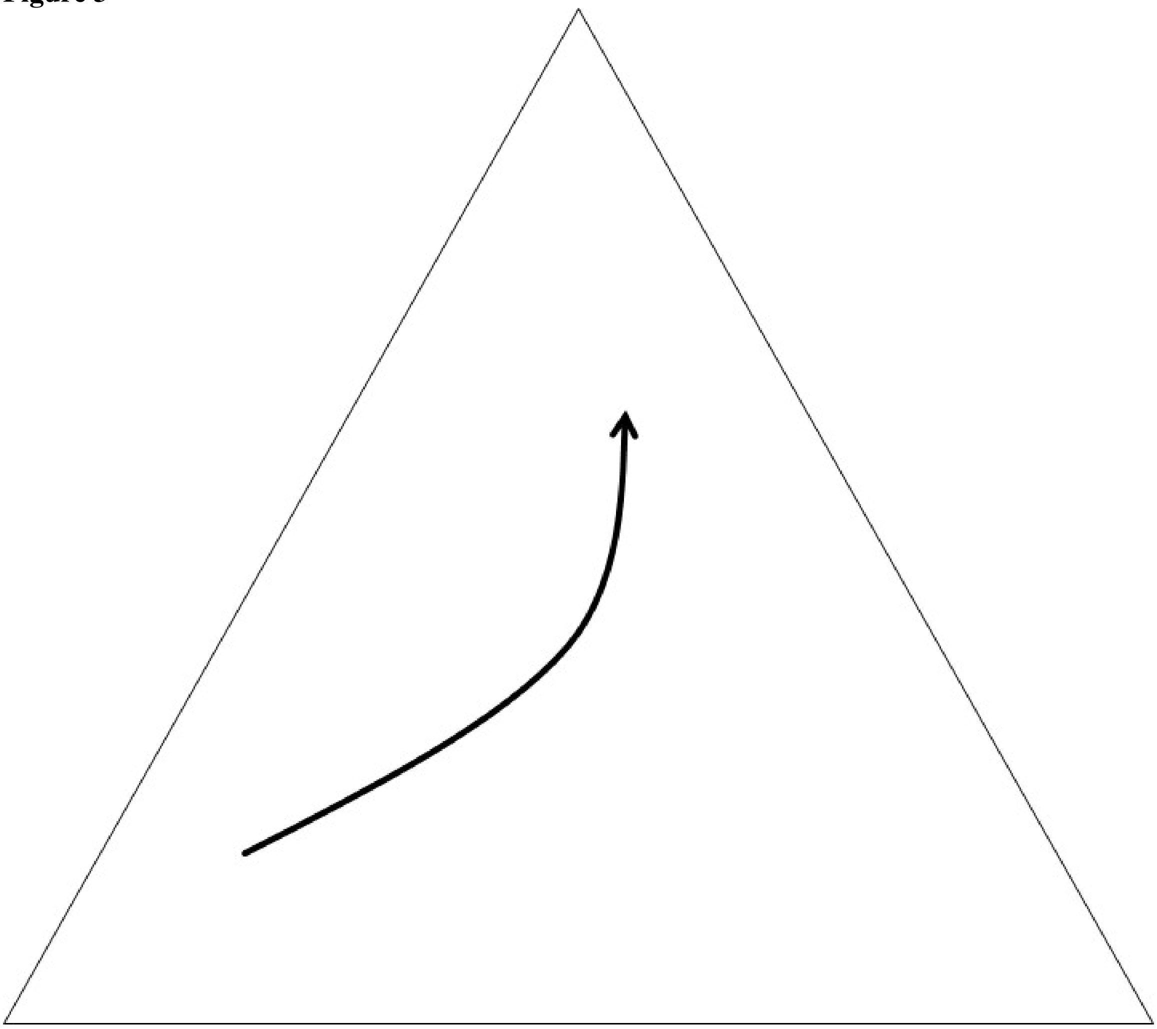


Figure 3



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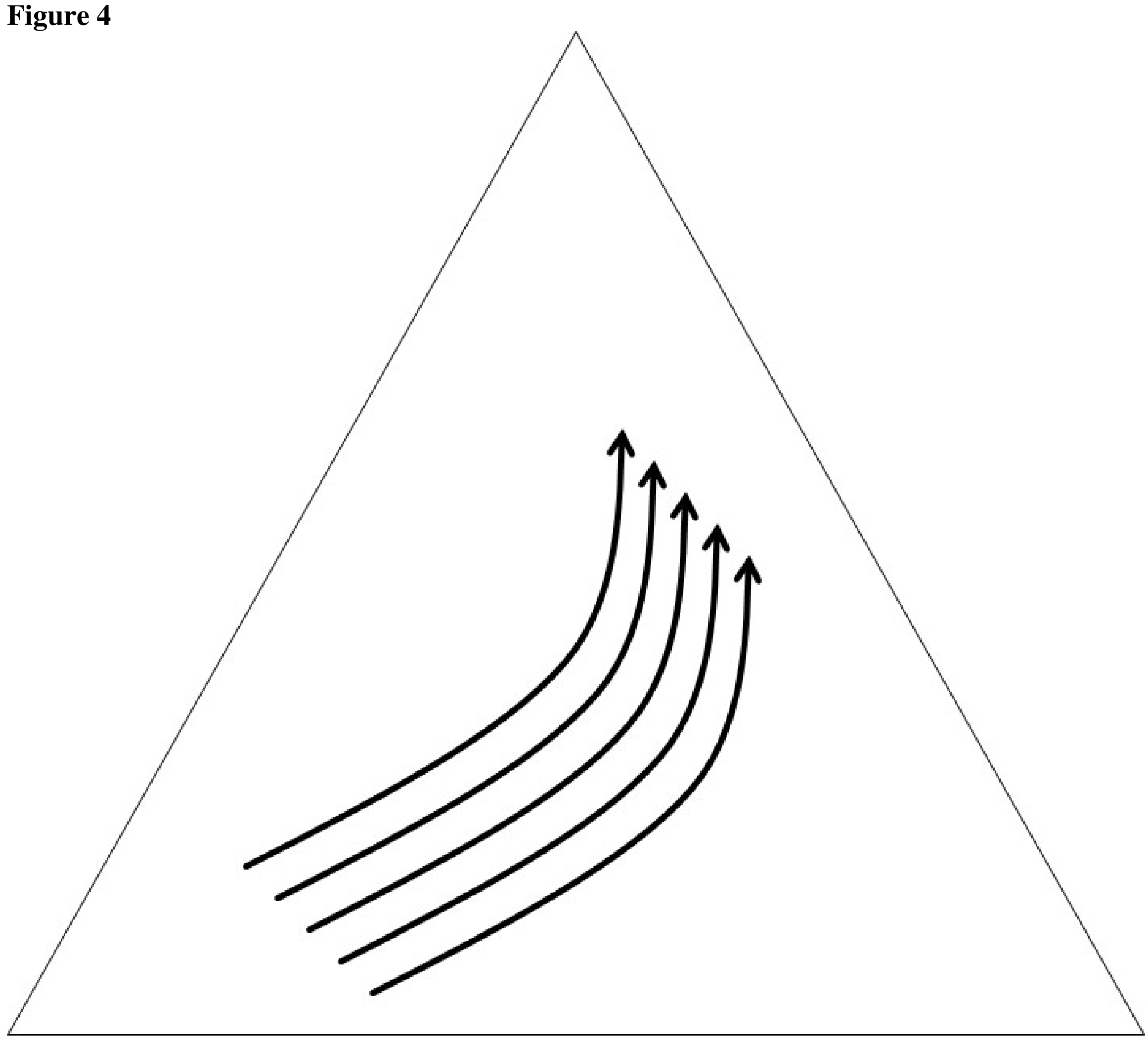

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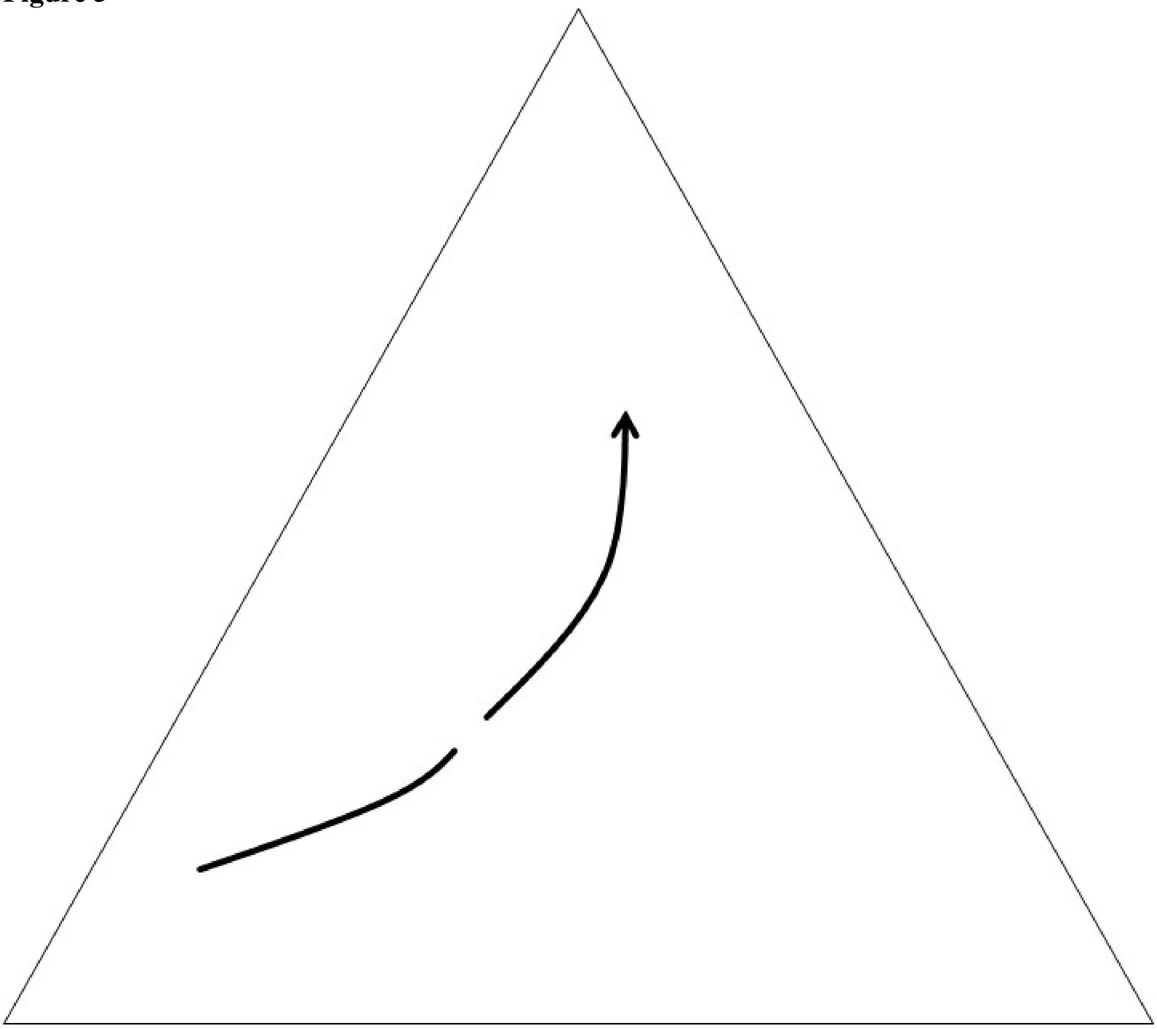

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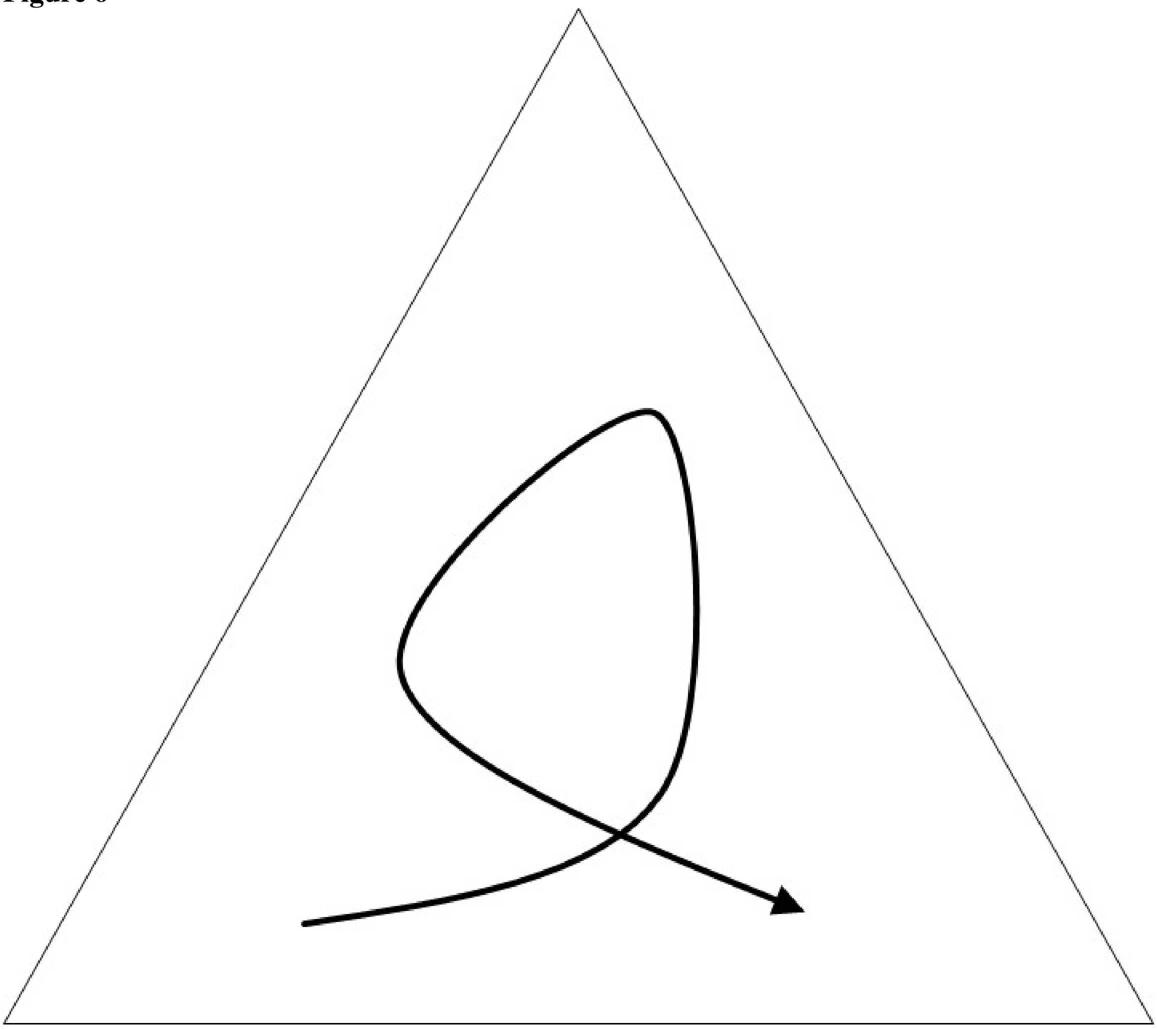

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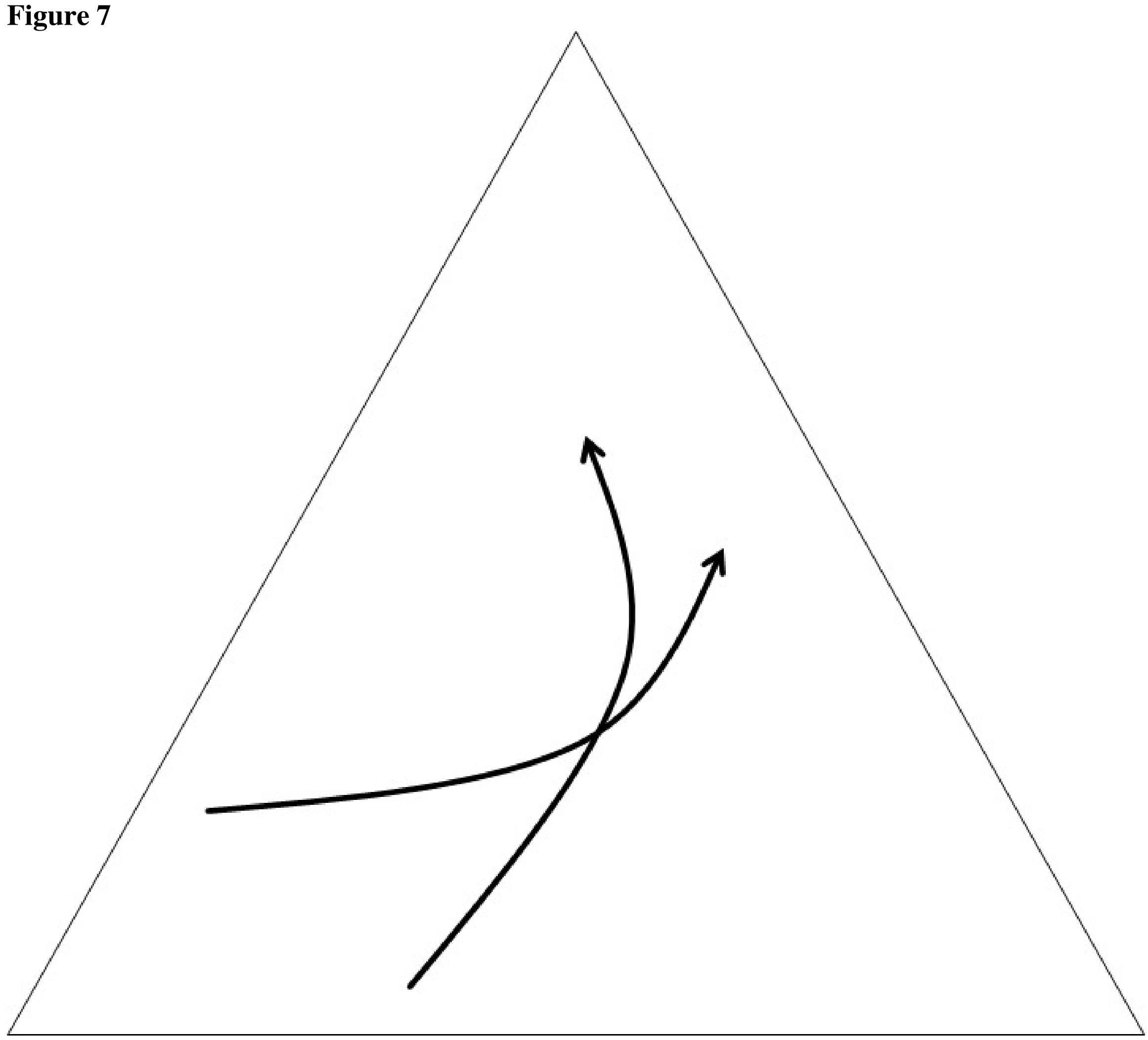


Figure 8

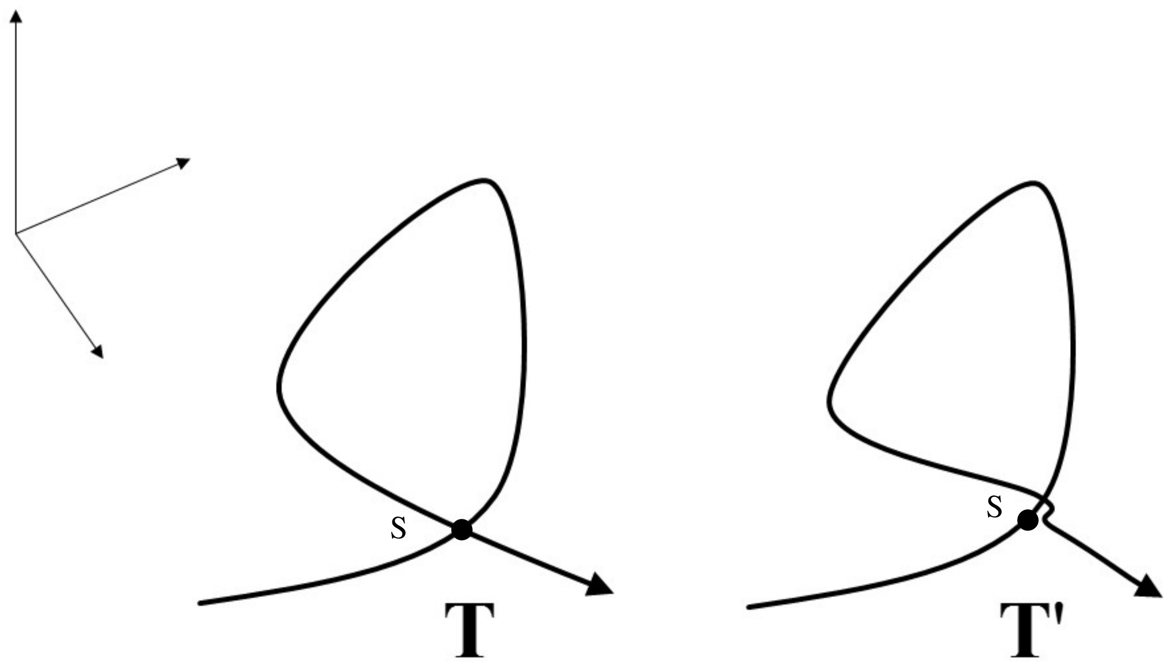


Figure 9

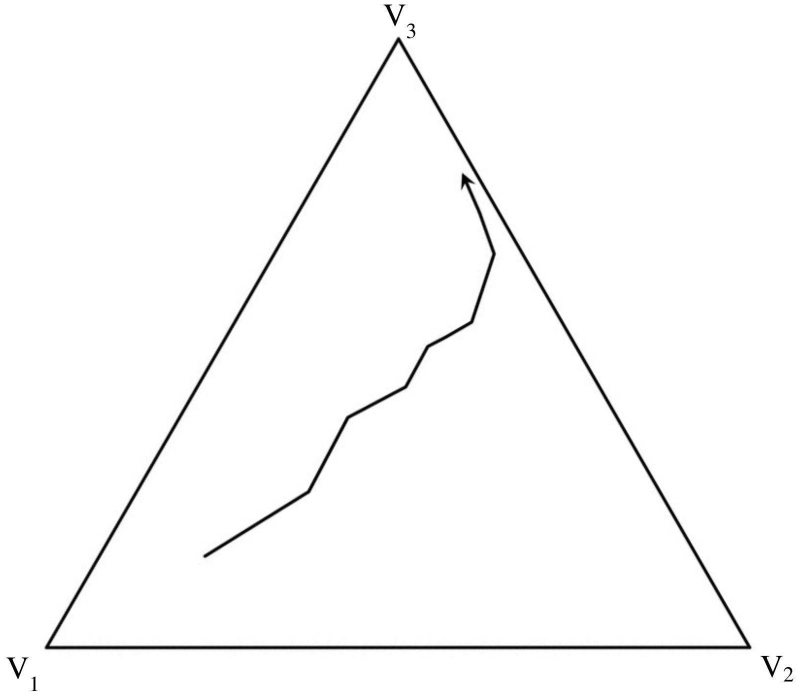
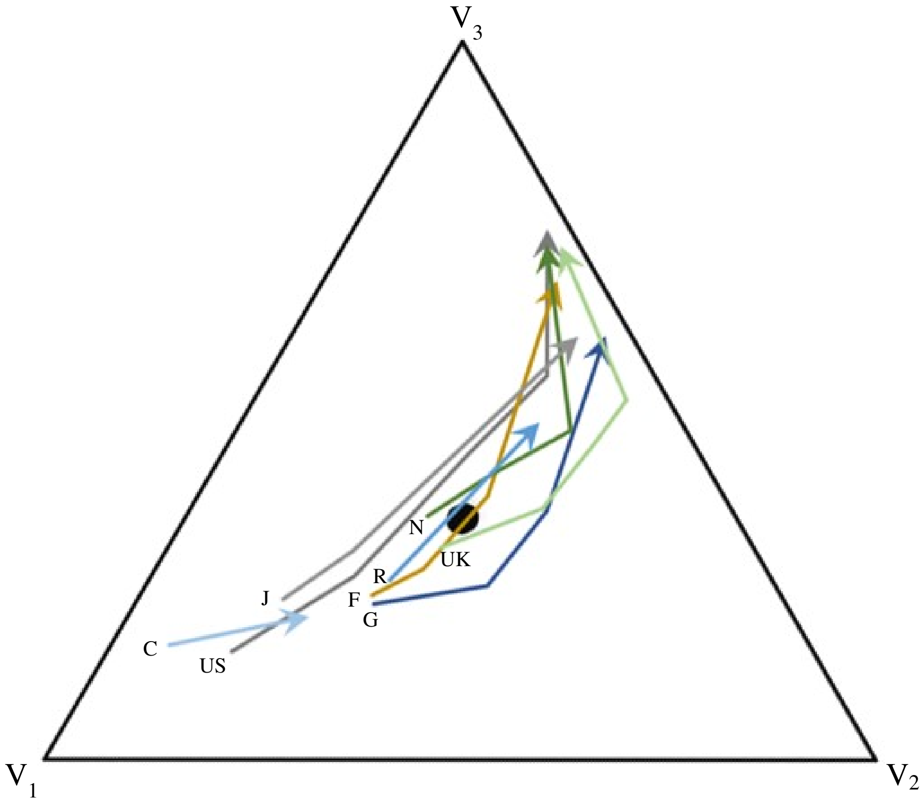


Figure 10



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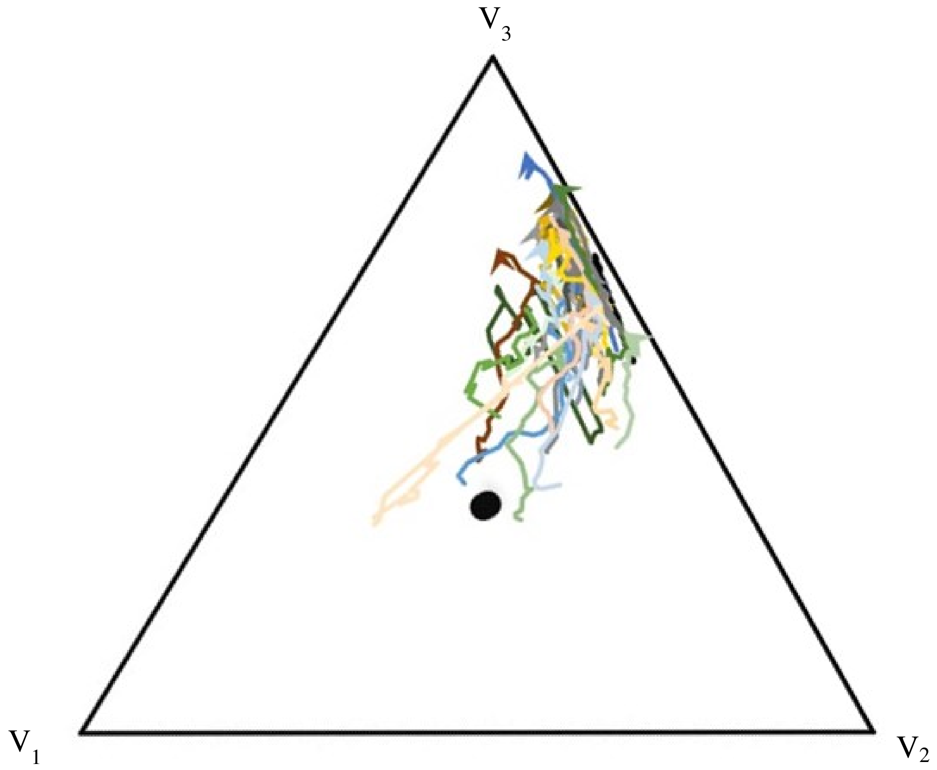

Figure 12

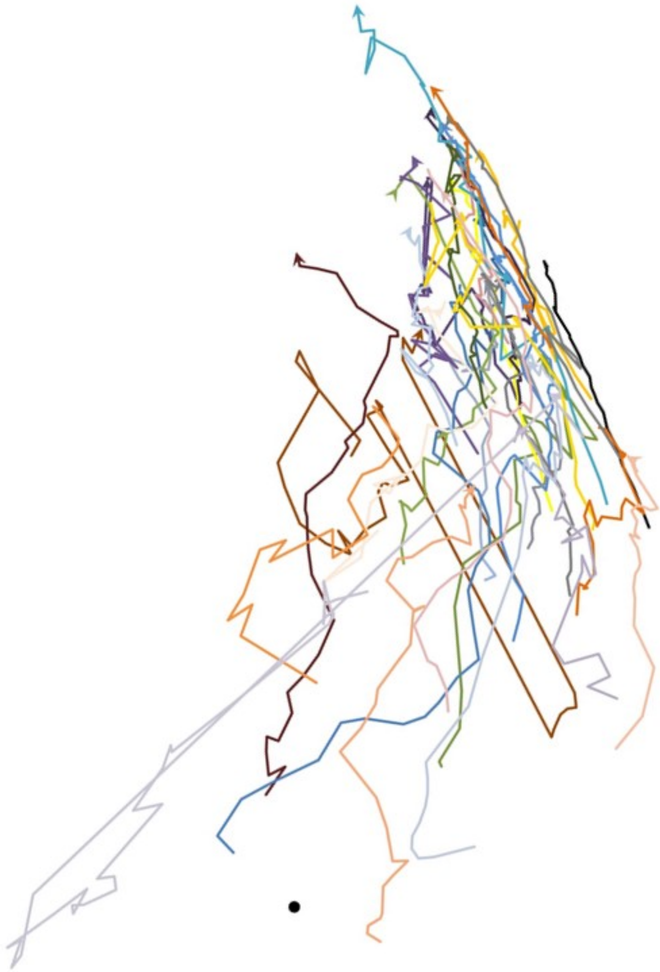

Figure 13