Vertical differentiation and collusion: pruning or proliferation?

Jean Gabszewicz, Jean and Marini, Marco A. and Tarola, Ornella

CORE, Université Catholique de Louvain, Sapienza Università di Roma, Sapienza Università di Roma

15 October 2016

Vertical Differentiation and Collusion: Pruning or Proliferation?*

Jean J. Gabszewicz,[a] Marco A. Marini[b]†and Ornella Tarola[c]
[a]CORE, Université Catholique de Louvain
[b, c] Università di Roma La Sapienza, Roma

August 2016

Abstract

In this paper, we tackle the dilemma of pruning versus proliferation in a vertically differentiated oligopoly under the assumption that some firms collude and control both the range of variants for sale and their corresponding prices, likewise a multiproduct firm. We analyse whether pruning emerges and, if so, a fighting brand is marketed. We find that it is always more profitable for colluding firms to adopt a pricing strategy such that some variants are withdrawn from the market. Under pruning, these firms commercialize a fighting brand only when facing competitors in a low-end market.

Keywords: *Vertically Differentiated Markets*, *Cannibalization*, *Market Pruning*, *Price Collusion.*

JEL Classification: D42, D43, L1, L12, L13, L41.

1 Introduction

What happens to the product lines of oligopolistic firms when they decide to collude in prices? Combining about prices should also be the occasion to

*We are grateful to Rabah Amir, Luca Benvenuti, Sergio Currarini, Alberto De Santis, Michael Kopel, Sébastien Mitraille, Michele Polo, Giorgio Rodano and the participants of Oligo Workshop 2016 in Paris Dauphine and EARIE Conference 2016 in Nova Universidade de Lisboa for useful comments and suggestions.

†Corresponding author: Marco A. Marini, Department of Social and Economic Sciences, Sapienza Università di Roma, Italy and CREI. Address: Piazzale Aldo Moro, 5, 00184, Roma.(Italy). Tel. +39-06-77274044. E-mail: marini@dis.uniroma1.it.

combine about product line policies: it provides the opportunity to revisit the existing competition among products sold by the *ex-ante* competitors. In particular, colluding firms can possibly *prune* some existing products appearing in their product lines. When firms decide to *proliferate* the number of their products, it reinforces competition, while pruning reduces their number and thereby reduces competition in the market. In both scenarios, one should evaluate the effect of price collusion on the decision of not only how many product variants, but also which product variants should be kept after collusion. We analyze in this paper the effect of pruning decisions on the product lines of a collusive entity when quality is a key competitive factor of the market.

Examples of pruning and proliferation can be found almost everywhere. While at first sight, these phenomena seem to be randomly widespread, proliferation prevail in horizontally differentiated markets, such as automobile industry, insurance markets, and the food industry, while pruning is frequently observed in industries where products are mainly differentiated along a quality dimension (Siebert, 2015). The high-tech industries provides nice examples of pruning: Apple withdraws from the mobile industry the iPhone 5 when marketing the higher quality iPhone 6, smart TVs launched by Samsung and LG make traditional apparels obsolete and likely to be pulled out of the market soon. Second, in the sectors where pruning prevails, fighting brands are sometimes marketed: beyond a high quality variant, a firm sells also a lower quality good designed to fight low-price competitors and possibly make them inactive. This strategy was adopted by IBM against its competitor Hewlett-Packard. For a long time, in the printer market IBM confined its production to a high quality product, the so called LaserPrinter. However, as a reaction to low-end competition coming from Hewlett-Packard, IBM introduced a defensive brand, the LaserPrinter E. This product was identical to the originally marketed LaserPrinter except for the fact that its software limited its printing to five rather than ten pages per minute.[1]

In this paper, we examine whether pruning emerges in post-merger competition when firms compete along a quality dimension and, if so, whether a kind of fighting brand is kept by the cartel in its portfolio.

The dilemma between pruning and proliferating products has been initially faced by the literature on monopoly price discrimination. In the pioneering contribution by Mussa and Rosen (1978), a price-discriminating

[1] See Ritson (2009),"Should You Launch a Fighter Brand" for a more detailed account of these case studies.

monopolist defines its optimal product line when products are of different qualities. It shows that the optimal solution differs crucially according to how the cost increases with respect to quality. Intuitively, if this increase is slow, the monopolist chooses to sell only the top quality to half of the whole population of consumers. On the contrary, if costs with respect to quality increase faster, the quality level offered to lower-value consumers is distorted downward: such a distortion is optimal since it prevents higher-value customers to buy the low quality good instead of the good targeted to them.

Of course, the trade-off between proliferation and pruning is made more intricate in the case of competition. When firms face rivals, the benefit of discriminating among consumers through proliferation has to be put in balance with the gain of moving product qualities apart from each other, thereby softening price competition, along with the benefit from escaping intra-firm cannibalization. When embracing this perspective, the most part of the theoretical analysis on proliferation versus pruning tends to solve this tension in terms of entry-deterring device (Schmalensee 1978, Bonanno 1987, Tirole, 1988): an incumbent firm decides to adjust its products line as a reaction to (potential) entrant(s), expanding its own product variety or rather withdrawing some goods depending on its cost function, marginal revenue and market size, *inter alia*.[2] More recent investigations have shown that proliferation strategies enable firms to match products to heterogeneous consumers (Kekre and Srinivasan 1990, Bayus and Putsis 1999, Siebert 2015).[3]

Our paper extends the above literature examining whether the formation of cartels into a market populated by an arbitrary number of firms has an impact on the range of products sold by these firms. Does cartel price optimization implies that some variants are pruned from the product portfolio of the cartel? And, if this is true, which among the variants remain for sale? To this end, we first determine the conditions characterizing the pre-merger noncooperative price equilibrium when all firms act independently and can each produce a single variant. Then, we assume that some k, with $k \leq n$, among these firms *collude* and, as a consequence, control both the range

[2] In Johnson and Myatt (2003), these drivers are considered when duopolists selling multiple quality-differentiated products and facing a potential entrant compete in quantity. The authors find that an incumbent never responds to the entrant by expanding its product line when marginal revenue is everywhere decreasing. Rather, under entry, the incumbent prunes lower-quality products from the basket of its sales, thereby choosing to "focus on quality."

[3] Empirical analysis also contribute to this issue. See Berry, Levinsohn and Pakes (1993), Berry and Waldvogel (1999), Davis (2002) and Petrin (2002), among many others.

of variants for sale and their corresponding prices, likewise a *multiproduct firm*.[4] In the case when $k = n$, a *full price collusion* occurs and the market is monopolized by an *a priori* multi-product monopolist. When $k < n$, the colluding firms can compete against single-product firms or against other groups of colluding firms. The former scenario resembles a multiproduct firm against a fringe of single-product competitors, while the latter mimics price competition among multiproduct firms. We describe the pricing behavior of firms in either scenario and examine how the quality gap(s) among products are consequently affected.

We find that *it is always more profitable for the cartel under either full or partial collusion to adopt a pricing strategy such that some existing variants are withdrawn from the market*. On the one hand, a reduction of product variety reduces the number of goods competing in the market with an upward movement of prices and a possible gain in profits. On the other hand, reducing the range of products in the market prevents firms from discriminating among consumers, thus entailing a loss in profits. The former gain from pruning is larger than the corresponding losses from missing the demand of some consumers. Thus, pruning always prevails, regardless of the number of firms deciding to collude and the quality of the variants that these firms initially produce in the market.[5]

Moreover, we show that, *under pruning, the colluding firms keep on sale in their portfolio a kind of fighting brand when facing competitors in the lower quality segment of the market.* Indeed, when $k = n$ (*full collusion*), only the top variant is kept for sale. When $k < n$, the variants for sale chosen by the colluding firms only consist *at most* of the top quality variant and the bottom quality one, among those initially existing in the bundle of variants owned by them. The bottom quality can thus be viewed as a *fighting brand* for the cartel. Further, under partial collusion, the prices are for all firms always higher than at the noncooperative Nash equilibrium without collusion, but lower than under full collusion.

[4] The dilemma between pruning and proliferation under mergers has received scarce attention. Some noticeable exceptions are the analysis by Lommerud and Sorgard (1997), Gandhi *et al.* (2008) and Chen and Schwartz (2013).

[5] Our analysis is somehow related to the one developed by Martinez-Giralt and Neven (1988), later discussed by Tabuchi (2012). In a horizontal differentiated market, Martinez-Giralt and Neven (1988) find, for both circle and line models, that duopolists with two stores each at equilibrium bunch their two stores at the same point. In our perspective, their result implies that the incentive to relax price competition dominates the incentive to segment the market. See on this also Chamorro-Rivas (2000), Giraud-Héraud et al. (2003), Bae and Choi (2007).

2 Pruning in a vertically differentiated market

2.1 The model

Let a set N of firms, $i = 1, 2, ..., n$, offer product variants $v_1, v_2, ..., v_n$, respectively, with $v_i \in (0, \infty)$ and $v_n > v_{n-1} > ... > v_1$ to a population of consumers in a vertically differentiated market.[6] Consumers are assumed uniformly distributed in the interval $[0, \beta]$, with $\beta < \infty$ and indexed by scalar θ. The parameter θ captures consumers' willingness to pay for quality. Our instantaneous demand set-up is directly inspired by the traditional model of vertical product differentiation (see Mussa and Rosen 1978; Gabszewicz and Thisse 1979). Accordingly, the utility consumer θ derives from buying at price p_i variant i, is given by

$$U(\theta) = \begin{cases} \theta v_i - p_i \text{ if she/he buys variant } i \\ 0 \text{ if she/he refrains from buying.} \end{cases} \tag{1}$$

With reservation prices defined for consumers included in the interval $[0, \beta]$, the market is endogenously *uncovered*.

Finally, since we assume that the qualities are exogenously given, we disregard costs. In general, our implicit assumption is that, for all firms $i = 1, 2, ..., n$, the quality levels $v_1, v_2, ..., v_n$ are such that all firms' equilibrium profits are non negative.

2.2 Ex-ante price equilibrium

We first consider the case in which all firms behave noncooperatively. The equilibrium behaviour of firms can be characterized by looking at the behaviour of three types of firms competing in the quality spectrum: *top*, *intermediate* and *bottom* quality firm. The top quality firm. i.e. the one selling the top quality variant and indexed with $i = n$, maximizes its profit

$$\pi_n = \left(\beta - \frac{p_n - p_{n-1}}{v_n - v_{n-1}}\right) p_n, \tag{2}$$

therefore setting the price according to its best-reply function

$$p_n(p_{n-1}) = \frac{1}{2}\left(p_{n-1} + \beta(v_n - v_{n-1})\right). \tag{3}$$

[6] In our model all firm variants $v_1, v_2, ..., v_n$ are given exogenously and the firms can only affect the variants remaining on sale by setting their prices. For a (three-firm) model looking at the firm strategic choice of qualities and prices, see Gabszewicz, Marini and Tarola (2015) and Marini (2016).

An intermediate quality firm, i.e. a firm selling an intermediate variant $i = 2, 3, ..., (n-1)$, maximizes its payoff

$$\pi_i = \left(\frac{p_{i+1} - p_i}{v_{i+1} - v_i} - \frac{p_i - p_{i-1}}{v_i - v_{i-1}} \right) p_i \tag{4}$$

and imposes a price respecting its best-reply function, namely

$$p_i(p_{i-1}, p_{+1}) = \frac{1}{2} \frac{p_{i-1}(v_{i+1} - v_i) + p_{i+1}(v_i - v_{i-1})}{(v_{i+1} - v_{i-1})}. \tag{5}$$

Finally, the bottom quality firm selling the bottom quality variant $(i = 1)$ maximizes the profit function

$$\pi_1 = \left(\frac{p_2 - p_1}{v_2 - v_1} - \frac{p_1}{v_1} \right) p_1, \tag{6}$$

sets its price according to its best reply function, namely

$$p_1(p_2) = \frac{1}{2} \frac{p_2 v_1}{v_2}. \tag{7}$$

Notice, from (2)-(6), that all firms profit functions are concave in their own prices. Moreover, for all firms, prices and qualities are strategic complements ($\frac{\partial_i^2 \pi_i}{\partial p_i \partial v_i} > 0$), so that firm best-reply shifts outward as a result of an increase in its quality. On the other hand, for every firm i, the effect of an increase in the quality of its direct rivals' variants v_j, for $j = (i+1)$ and $(i-1)$ is negative ($\frac{\partial_i^2 \pi_i}{\partial p_i \partial v_j} < 0$) and, therefore, price-competition becomes tougher as a result. Notice also that, since best-replies are *contractions*,[7] the existence of a unique (noncooperative) Nash equilibrium price vector p^* is guaranteed in the model for any (finite) number of firms competing in the market.[8]

[7] A sufficient condition for the contraction property to hold is (see, for instance, Vives 2000, p.47):

$$\frac{\partial^2 \pi_i}{\partial (p_i)^2} + \sum_{j \neq i} \left| \frac{\partial^2 \pi_i}{\partial p_i \partial p_j} \right| < 0,$$

which, using (4) for all intermediate firms $i = 2, ..., n-1$, becomes

$$-\frac{2(v_{i+1} - v_{i-1})}{(v_{i+1} - v_i)(v_i - v_{i-1})} + \frac{v_{i+1} - v_{i-1}}{(v_{i+1} - v_i)(v_i - v_{i-1})} = \frac{v_{i-1} - v_{i+1}}{(v_{i+1} - v_i)(v_i - v_{i-1})} < 0 \tag{8}$$

which is satisfied for $v_n > v_{n-1} > ... > v_1$. The same applies for top and bottom quality firms.

[8] See, for instance Friedman (1991), p.84.

2.3 Collusive agreements

A collusive agreement is viewed as a an agreement passed among a subset of individual firms to choose the variants kept for sale and the price charged for each of these variants.

2.3.1 Full collusion

We denote by *full collusion* the case when all n firms collude in prices. In this scenario, the cartel maximizes the sum of all firms' payoffs:

$$\Pi_N = \sum_{i \in N} \pi_i = \pi_1 + ... + \pi_{i-1} + \pi_i + \pi_{i+1} + ... + \pi_n.$$

For every colluding firm i, $i = 1, ..., n$, the first-order condition writes as[9]

$$\frac{\partial \Pi_N}{\partial p_i} = \frac{\partial \pi_{i-1}}{\partial p_i} + \frac{\partial \pi_i}{\partial p_i} + \frac{\partial \pi_{i+1}}{\partial p_i} = 0. \tag{9}$$

Since the top quality-firm $i = n$ in the cartel internalizes only the payoff of its lower-quality neighbour, its optimal reply writes as

$$p_n^c(p_{n-1}) = p_{n-1} + \frac{\beta}{2}(v_n - v_{n-1}). \tag{10}$$

Along the same rationale, for all intermediate firms $i = 2, 3, ..., (n-1)$ which are members of the cartel, the optimal reply writes as

$$p_i^c(p_{i-1}, p_{i+1}) = \frac{p_{i-1}(v_{i+1} - v_i) + p_{i+1}(v_i - v_{i-1})}{(v_{i+1} - v_{i-1})}, \tag{11}$$

since they internalize the payoff of their adjacent neighbour members of the cartel. Finally, the optimal reply of the bottom quality firm $i = 1$ is given by

$$p_1^c(p_2) = \frac{v_1}{v_2} p_2. \tag{12}$$

In the next proposition, we characterize the equilibrium prices set by the firms under full price collusion and the number of variants remaining on sale.

---

[9] Note that $\frac{\partial^2 \Pi_N}{\partial p_i^2} = -\frac{2(v_{i+1} - v_{i-1})}{(v_{i+1} - v_i)(v_i - v_{i-1})} < 0$ for $i = 2, 3, ..., n-1$, and, therefore, the joint profit Π_N is concave in every firm's price p_i. The same condition holds for the two extreme firms along the quality spectrum, i.e. $i = 1$ and $i = n$.

Proposition 1 *Under full price collusion: (i) every firm $i = 1, 2, ..., n$ sets a price p_i^c, with*

$$p_i^c = \frac{1}{2}\beta \sum_{j=1}^{i} (v_j - v_{j-1}) > p_i^*,$$

where p_i^* stands for firm i' s ex-ante noncooperative price; (ii) the demand $D_i(p_i^c)$ of the bottom $(i = 1)$ and of all intermediate quality firms $i = 2, ..., n-1$ is nil, while for the top-quality firm $i = n$, $D_n(p_n^c)$ is positive and corresponds to the demand of the upper part $\left[\frac{1}{2}, \beta\right]$ of the consumers' population.

Proof. See the Appendix. ■

The above result does not come as a surprise, and simply duplicates the well known result occurring under monopoly and quasi-linear preferences of consumers (see Mussa and Rosen, 1978).[10] Conversely, what appears as relatively unexplored is the case of partial price collusion in a vertical differentiated market, which we consider in detail below.

2.3.2 Partial price collusion

A vertically differentiated market is a market of local interaction, with every firm $i = 1, 2, ..., n$, having its demand depending on at most three prices: its own price and the two prices of its neighbours. There are two exceptions to this statement: first, the case of the top quality firm, since the demand faced by this firm depends only on its own price and the price of its sole adjacent firm; second, the case of the bottom quality firm, whose demand is depending on its own price and the price of its sole adjacent firm. In all other cases, whatever its quality, a firm interacts with its two adjacent rivals, a lower-quality competitor and a higher quality one**.**

Thus, when using the first-order conditions for characterizing an equilibrium with a cartel $S \subset N$, we need to consider separately the case of (i) any firm member of the cartel whose cartel neighbours are selling lower as well as higher quality variants than its own (call them *interior* cartel members and the corresponding cartel an *interior* cartel), (ii) the firm member

[10] In Gabszewicz, Shaked, Sutton and Thisse (1986) it is proved that if the unit cost is (zero or) constant, or rises only slowly with quality, the result of Proposition 1 holds for any multi-product monopoly. Extending the model to a duopoly, Champsaur and Rochet (1989, 1990) and Bonnisseau and Lahmandi-Ayed (2006) show that each firm produces a single quality rather than a range of qualities under a similar set-up to the one used here: quasilinear utility, uniform distribution of consumer taste, and quadratic cost of quality improvement.

of the cartel whose neighbour's variant is selling lower quality than its own (call him the *low boundary* cartel member), and (iii) the firm member of the cartel whose neighbour's variant is selling a higher quality than its own (call him the *upper boundary* cartel member). To each of these members, the first order condition of profit maximization writes as:

(i) in the case of *interior* cartel members:

$$\frac{\partial \Pi_S}{\partial p_i} = \frac{\partial \sum_{i \in S} \pi_i}{\partial p_i} = \frac{\partial \pi_{i-1}}{\partial p_i} + \frac{\partial \pi_i}{\partial p_i} + \frac{\partial \pi_{i+1}}{\partial p_i} = 0,$$

leading to the optimal reply function

$$p_i^{pc}(p_{i-1}, p_{i+1}) = \frac{p_{i-1}(v_{i+1} - v_i) + p_{i+1}(v_i - v_{i-1})}{(v_{i+1} - v_{i-1})}, \tag{13}$$

where the superscript pc stands for *partial collusion.*

(ii) In the case of *lower boundary* cartel member:

$$\frac{\partial \Pi_S}{\partial p_i} = \frac{\partial \sum_{i \in S} \pi_i}{\partial p_i} = \frac{\partial \pi_i}{\partial p_i} + \frac{\partial \pi_{i+1}}{\partial p_i} = 0,$$

leading to the best-reply function

$$p_i^{pc}(p_{i-1}, p_{i+1}) = \frac{\frac{1}{2} p_{i-1}(v_{i+1} - v_i) + p_{i+1}(v_i - v_{i-1})}{(v_{i+1} - v_{i-1})}. \tag{14}$$

(iii) Finally, in the case of u*pper boundary* cartel member:

$$\frac{\partial \Pi_S}{\partial p_i} = \frac{\partial \sum_{i \in S} \pi_i}{\partial p_i} = \frac{\partial \pi_i}{\partial p_i} + \frac{\partial \pi_{i-1}}{\partial p_i} = 0,$$

leading to the best-reply function

$$p_i^{pc}(p_{i-1}, p_{i+1}) = \frac{p_{i-1}(v_{i+1} - v_i) + \frac{1}{2} p_{i+1}(v_i - v_{i-1})}{(v_{i+1} - v_{i-1})}. \tag{15}$$

Notice that, for the firms $i = 1$ or $i = n$, the prices p_{i-1} and p_{i+1} in the above expressions have no meaning so that their best reply functions are, for $i = 1$,

$$p_1^{pc}(p_2) = \frac{v_1}{v_2} p_2$$

and, for $i = n$,

$$p_n^{pc}(p_{n-1}) = p_{n-1} + \frac{\beta}{2}(v_n - v_{n-1}).$$

We can now prove that

Proposition 2 *Under partial price collusion leading to an interior cartel, only two variants remain on sale, the top and the bottom quality good produced into the cartel portfolio. On the other hand, if the cartel includes the top quality variant $i = n$, only two variants remain on sale, the highest and lowest quality variants in the cartel portfolio. Finally, if the cartel includes the bottom quality $i = 1$, only one variant remains for sale, namely the top quality variant in the cartel portfolio.*

Proof. See the Appendix. ■

The above statements provide a characterization of equilibrium market configurations. Proposition 2 shows that it is optimal for any cartel to drop from the market as many variants as possible, except the top and the bottom quality ones among those previously on sale. Furthermore, when the cartel includes the firm selling the bottom quality among all existing variants, the cartel does not need to keep on sale the lowest variant.

Example 1 *Let an arbitrary number of firms compete in the markets, e. g. $N = 14$, therefore initially branding fourteen different variants, $v_1, v_2, ..., v_{14}$. (i) Suppose now that an interior cartel $S_I = \{5, 6, 7, 8\}$ forms. In this case, only variants v_5 and v_8 remain on sale by the cartel and, overall, only $14 - 4 + 2 = 12$ variants remain on sale in the market (see Figure 1). (ii) Alternatively, let the top cartel $S_T = \{10, 11, 12, 13, 14\}$ form. In this case, only variants v_{10} and v_{14} are on sale from the cartel and, overall, only $14 - 5 + 2 = 11$ variants are left for sale (Figure 2). Finally, when the bottom cartel $S_B = \{1, 2, 3, 4, 5, 6\}$ is formed only its highest variant v_6 remain on sale and a total of $14 - 6 + 1 = 9$ variants remain on sale in the market (Figure 3).*

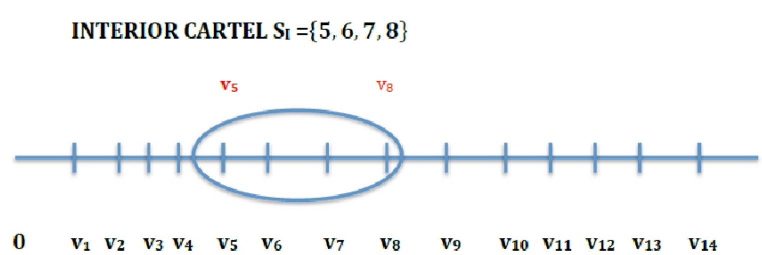

Figure 1

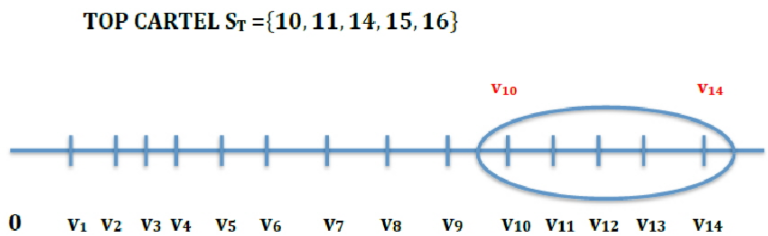

Figure 2

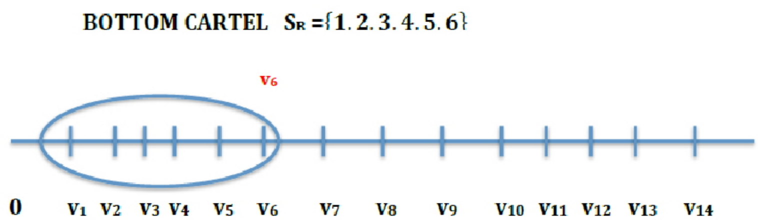

Figure 3

It is worth remarking that, although pruning *always* prevails, whatever the type of colluding entity occurring in the market (top/interior/bottom cartel), the set of variants on sale at equilibrium changes with the type of collusive agreement. In particular, we observe that a low quality variant is sold by a top cartel and an intermediate cartel, but never by a bottom cartel. In brief, the rationale underlying this finding can be described as follows. By (2)-(6) we know that, for every i-th firm, $\frac{\partial_i^2 \pi_i}{\partial p_i \partial v_j} < 0$, and, therefore, an increase in adjacent rivals' quality v_j (with $j = i - 1$ and $j = i + 1$) has a detrimental effect on its profit. Accordingly, cartel members possess a direct incentive to reduce the quality of their neighbours in order to increase the quality gap between the existing variants and, thus, their profit. Since the highest incentive among the colluding firms is for the top quality firm in the cartel, the members of the cartel keep only their top quality on sale and start pruning all existing variants in the cartel until their lowest quality one. However, when in presence of lower quality competitor(s), if the cartel decides to shut down also its bottom quality firm, it would directly compete with a firm external to the cartel whose pricing behaviour would definitely be more aggressive than that of its lowest quality member. So, it is optimal in this case to keep a *defensive* or *fighting brand*.[11]

[11] It would be interesting to consider these incentives in a horizontally differentiated

Casual observations show that a fighting brand usually appear when a firm competes against lower quality rival(s) in the market.[12] Through the lens of our result, the practice of *un–brand management* can, thus, be seen as a means to avoid intra-firm cannibalization and to widen as much as possible the existing quality gap between variants. As far as the bottom cartel, a fighting brand would not play any role since this cartel does not face a low-end competitor. Further, the un–brand management would not prevent the lowest quality variant produced by this cartel from cannibalizing the market share of the adjacent variant, namely the top one in this bottom cartel. Accordingly, the bottom cartel restricts its sales to the highest quality variant it can produce.

Moreover, the next result directly follows from Proposition 3. It allows to easily compute the number of variants on sale in any possible partial cartelization.

Proposition 3 *In a generic partition of the* n *firms* $P = (S_1, S_2, ..., S_m)$ *organized in* $m < n$ *cartels, a total of* $2m + (n - z) - 1$ *(resp.* $2m + (n - z)$*) variants are put on sale in the market when the partition includes (resp. does not include) the bottom cartel, for* $z = s_1 + s_2 + ... + s_m$*, where* s_j*, for* $j = 1, 2, ..., m$*, denotes the cardinality of every cartel.*

The next example clarify how Proposition 3 works.

Example 2 *Suppose again that* $N = 14$ *and some subsets of firms form two cartels, a bottom cartel* $S_B = (1, 2, 3, 4)$ *and a top cartel* $S_T = (9, 10, 11, 12, 13, 14)$*. In this case firms give rise to partition*

$$C = (\{1, 2, 3, 4\}, 5, 6, 7, 8, 9, \{10, 11, 12, 13, 14\})$$

and, according to Proposition 3 and 4, the number of variants remaining on sale are $2m + (n - z) - 1 = 2(2) + (14 - 10) - 1 = 7$*, i.e. one variant from the bottom cartel, two variants from the top cartel and four from the remaining firms in the fringe (see Figure 4).*

---

market. In a Hotelling model, Hinloopen and Martin (2016) provide the general conditions on the cost-of-location function such that a price-location Nash equilibrium in pure strategy exists. Their analysis could be a useful entry point for examining the incentive of colluding firms to prune or proliferate their product portfolio when located along a line. For a general discussion on horizontal differentiated markets and a comparison with vertical differentiated markets, see also Gabszewicz and Tarola (2016).

[12] This is in line with the evidence gathered in the introduction.

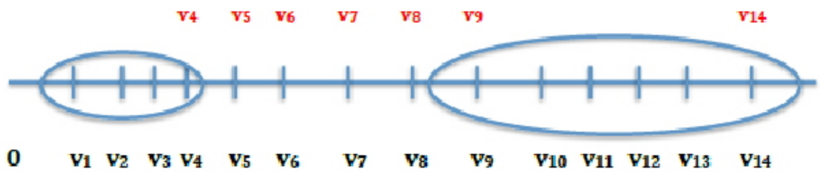

Figure 4

Finally, we conclude the characterization of partial collusion by introducing a price comparison with both the noncooperative and the fully collusive case.

Proposition 4 *Under partial price collusion all firms $i = 1, 2, ..., n$ set prices p_i^{pc} higher than the corresponding prices set at the noncooperative equilibrium p_i^* and lower than the ones occurring under full price collusion p_i^c.*

Proof. See the Appendix. ■

Notice that the above result holds for any arbitrary quality gap existing among firms' variants and its proof does not require to derive the equilibrium prices in closed form (see the Appendix). The latter task can be quite awkward without assuming some regularity conditions on quality gaps. By following the approach taken by Gabszewicz and Thisse (1980) of equidistant quality gaps, the next proposition characterizes the equilibrium prices for the noncooperative case.[13] A full taxonomy of partial collusive prices under this assumption is not presented here and it is left to interested readers.

Proposition 5 *Let market variants $v_1, v_2, ..., v_n$ be equispaced and such that $v_i - v_{i-1} = \delta$ for every $i = 1, 2, ..., n$, with $v_0 = 0$. Thus, the noncooperative (Nash) equilibrium price vector of all firms $i = 1, 2, ..., n$ is given by:*

$$p_i^* = \frac{\delta\beta(\left(2+\sqrt{3}\right)^i - \left(2-\sqrt{3}\right)^i)}{\sqrt{3}\left(2+\sqrt{3}\right)^n + \sqrt{3}\left(2-\sqrt{3}\right)^n},$$

Proof. See the Appendix. ■

[13] In Gabszewicz *et al.* (2016) it is shown that the equidistance bewteen variants is a sufficient condition for the whole market cartel to be coalitionally stable (*core-stable*) against any possible deviation of partial cartels.

3 Concluding Remarks

In this analysis, we have considered the dilemma between pruning and proliferation in a vertically differentiated market with more than two firms. We have shown that, the cannibalization effect inducing pruning is so significant that proliferation never occurs.

Our paper provides a further dimension of analysis on the effects determined by collusive agreements. The standard understanding of collusion is that firms producing homogeneous goods collude in order to mimic the behaviour of a monopoly. Based on this, cartels are typically viewed as a means to reduce competition. In the current paper, it is not clear *a priori* whether a cartel is detrimental to the market, since it yields a quality shift, in addition to the outcomes typically described in the literature on collusion. Similar considerations can be applied to the analysis of horizontal mergers. The traditional approach to mergers is mainly linked to industry-concentration measures whose value can determine presumptions of illegality. In our work, the effects of a merger are not only depending on the number of merging firms but also (and primarily) on the qualities initially produced by the firms.

While disentagling these issues goes beyond the aim of this paper, it opens the door to further research in the field of competition policy.

References

[1] Bae S. and J. P. Choi 2007. The Optimal Number of Firms With an Application to Professional Sports Leagues, *Journal of Sports Economics*, 8, 199-108.

[2] Bayus, B. and W Putsis 1999. Product proliferation: An empirical analysis of product line determinants and market outcomes, *Marketing Science* 18, 137-153.

[3] Berry, S., J. Levinsohn, and A. Pakes 1993. Applications and Limitations of Some Recent Advances in Empirical Industrial Organization: Price Indexes and the Analysis of Environmental Change, *American Economic Review, Papers and Proceedings* 83, 241-246.

[4] Berry, S. and J. Waldfogel 1999. Free Entry and Social Inefficiency in Radio Broadcasting, *Rand Journal of Economics*, 30, 397-420.

[5] Bonanno, G. 1987. Location Choice, Product Proliferation and Entry Deterrence, *Review of Economic Studies,* 54, 37-46.

[6] Bonnisseau J.M. and R. Lahmandi-Ayed 2006. Vertical Differentiation: Multiproduct Strategy to Face Entry?, *The B.E. Journal of Theoretical Economics*, 6 (1), 1-14.

[7] Champsaur, P. and J. C. Rochet 1989. Multiproduct Duopolists. *Econometrica*, 57 (3), 533-557.

[8] Chamorro-Rivas J.M 2000. Plant proliferation in a spatial model of Cournot competition, *Regional Science and Urban Economics*, 30 (5), 507–518.

[9] Chen, Y. and M. Schwartz 2013. Product Innovation Incentives: Monopoly versus Competition, *Journal of Economics and Management Science*, 22, 513-528.

[10] Davis, P. 2002. Fine Young Cannibals in the U.S. Motion Picture Exhibition Market, *mimeo.*

[11] Friedman, J.W. 1989. *Game Theory with Applications to Economics.* Oxford University Press, Oxford.

[12] Gabszewicz, J. J. and J-F. Thisse 1979. Price Competition, Quality and Income Disparities, *Journal of Economic Theory*, 20, 340-359.

[13] Gabszewicz, J. J., J-F. Thisse 1980 Entry (and Exit) in a Differentiated Industry, *Journal of Economic Theory*, 22, 327-338.

[14] Gabszewicz J.J., A. Shaked, J. Sutton and J.-F. Thisse 1986. Segmenting the Market: the Monopolist's Optimal Product Mix, *Journal of Economic Theory*, 9, 273-289.

[15] Gabszewicz J.J., Marini, M. A. and O. Tarola 2015. Alliance Formation in a Vertically Differentiated Market, *CORE Discussion Papers* 30/2015, Université Catholique de Louvain, Louvain-la-Neuve, Belgium.

[16] Gabszewicz J.J., Marini, M. A. and O. Tarola 2016. Core Existence in Vertically Differentiated Markets, *Economics Letters*, forthcoming.

[17] Gabszewicz J.J. and O. Tarola 2016. Oligopoly and Product Differentiation, in Corchon L., Marini M. A. (eds.), *Handbook of Game Theory and*

Industrial Organization, Edward Elgar, Chelthenam and Northampton. Mass., forthcoming.

[18] Gandhi, A., Froeb, L., Tschantz, S. and G. J. Werden 2008. Post-Merger Product Repositiong, *Journal of Industrial Economics*, 56: 49–67.

[19] Giraud-Héraud, E., Hammoudi, H. and M. Mokrane 2003. Multiproduct firm behaviour in a differentiated market. *Canadian Journal of Economics*, 36: 41–61

[20] Hinloopen J. and S. Martin 2016. Costly Location in Hotelling Duopoly, *Research in Economics,* forthcoming.

[21] Johnson J. P and D. Myatt 2003. Multiproduct Quality Competition: Fighting Brands and Product Line Pruning, *American Economic Review*, 93, 748-774.

[22] Kekre, S. and K. Srinivasan 1990. Broader product line: A necessity to achieve success?, *Management Science* 36, 1216–1231.

[23] Lommerud, K. and L. Sørgard 1997. Merger and Product Range Rivalry, *International Journal of Industrial Organization* 16, 21-42.

[24] Marini, M. A. 2016. Collusive Agreements in Vertically Differentiated Markets, in Corchon L., Marini M. A. (eds.), *Handbook of Game Theory and Industrial Organization*, Edward Elgar, Chelthenam and Northampton. Mass., forthcoming.

[25] Martinez-Giralt, X. and D.J. Neven 1988. Can price competition dominate market segmentation?, *Journal of Industrial Economics,* 36, 431-442.

[26] Mussa, M. and S. Rosen 1978. Monopoly and product quality, *Journal of Economic Theory,* 18, 301–317.

[27] Petrin, A. 2002. Quantifying the Benefits of New Products: The Case of the Minivan, *Journal of Political Economy*, 110 (4), 705-729.

[28] Ritson, M. 2009. Should You Launch a Fighter Brand, *Harvard Business Review,* October.

[29] Schmalensee, R. 1978. Entry Deterrence in the Ready-to-eat Breakfast Cereal Industry, *Bell Journal of Economics,* 9, 305–327.

[30] Siebert, R. 2015. Entering New Markets in the Presence of Competition: Price Discrimination versus Cannibalization, *Journal of Economics & Management Strategy*, 24, 369-389.

[31] Tabuchi, T. 2012. Multiproduct firms in Hotelling's spatial competition, *Journal of Economics & Management Strategy*, 21 (2), 445-467.

[32] Tirole, J. 1988. *The Theory of Industrial Organization.* MIT Press, Cambridge, Mass.

[33] Vives, X. 2000. *Oligopoly Pricing. Old Ideas and New Tools.* MIT Press, Cambridge, Mass.

4 Appendix

Proof of Proposition 1. (i) After some manipulations the price of every firm $i = 1, 2, ..., n$ under full collusion can be easily obtained in closed form as

$$p_i^c = \frac{1}{2}\beta \sum_{j \leq i} \delta_j, \tag{16}$$

where $\delta_i = (v_i - v_{i-1})$ denotes the quality gap between every firm $i = 1, 2, ..., n$ and its lower quality-neighbour $(i-1)$, whereas $\delta_1 = (v_1 - v_0) = v_1$. The fact that $p_i^c > p_i^*$ for every $i = 1, 2, ..., n$, can be proved through the following steps: (a) Start with the profile of Nash equilibrium prices, $p^* = (p_1^*, p_2^*, ..., p_n^*)$ and assume, with no loss of generality, that firms $i = 2, 3, ..., n$ respond, instead of noncooperatively, by setting prices according to their fully collusive optimal replies (10)-(11). Comparing (3)-(7) with (10)-(12) we see that all firms' replies are positively sloped and additionally that the collusive optimal replies are twice as steep as the noncooperative best-replies. Thus, it follows that after step (a), all firms $i = 2, 3, ..., n$ increase their prices. (b) Let now also firm 1 respond cooperatively, hence increasing its price as well. (c) Let such adjustment process continue for all firms and, given that all firms' optimal replies are contractions, a new (fully collusive) price profile $p^c = (p_1^c, p_2^c, ..., p_n^c)$ will be finally reached, with the property that, for every $i = 1, 2, 3, ..., n$, $p_i^* < p_i^c$. (ii) Using (6) and (16) the demand of the bottom-quality firm $i = 1$ under full collusion is

$$D_1(p_1^c, p_2^c) = \left(\frac{p_2^c - p_1^c}{v_2 - v_1} - \frac{p_1^c}{v_1}\right) = \left(\frac{\frac{1}{2}\beta\left(\delta_1 + \delta_2\right) - \frac{1}{2}\beta\delta_1}{\delta_2} - \frac{\frac{1}{2}\beta\delta_1}{\delta_1}\right) = 0$$

and, analogously, that of every intermediate firm $i = 2, 3, ..., n-1$ writes as

$$\begin{aligned} D_i(p^c_{i-1}, p^c_i, p^c_{i+1}) &= \left(\frac{p^c_{i+1} - p^c_i}{\delta_{i+1}} - \frac{p^c_i - p^c_{i-1}}{\delta_{i-1}} \right) = \\ &= \left(\frac{\frac{1}{2}\beta \sum_{j \le i+1} \delta_j - \frac{1}{2}\beta \sum_{j \le i} \delta_j}{\delta_{i+1}} - \frac{\frac{1}{2}\beta \sum_{j \le i} \delta_j - \frac{1}{2}\beta \sum_{j \le i-1} \delta_j}{\delta_{i-1}} \right) = \\ &= \left(\frac{\frac{1}{2}\beta\delta_{i+1}}{\delta_{i+1}} - \frac{\frac{1}{2}\beta\delta_i}{\delta_i} \right) = 0. \end{aligned}$$

Finally, for the top quality firm $i = n$, the demand under full collusion is:

$$\begin{aligned} D_i(p^c_{n-1}, p^c_n) &= \left(\beta - \frac{p^c_n - p^c_{n-1}}{v_n - v_{n-1}} \right) = \\ &= \left(\beta - \frac{\frac{1}{2}\beta \sum_{j \le n} \delta_j - \frac{1}{2}\beta \sum_{j \le n-1} \delta_j}{\delta_n} \right) = \left(\beta - \frac{\frac{1}{2}\beta\delta_n}{\delta_n} \right) = \frac{1}{2}\beta. \end{aligned}$$

Therefore, when the whole industry cartel forms and signs a binding agreement on prices, it behaves as a single monopolist by offering uniquely its top variant at a marketable price and covering only one-half of the whole population of consumers. **Q.E.D.**

Proof of Proposition 2. Take a generic *intermediate* cartel of $k \le n-2$ firms initially selling variants

$$v_i, v_{i+1}, v_{i+2}, ..., v_{i+k}$$

and competing, both with a left-hand fringe of independent firms selling lower quality variants $v_1, v_2, ..., v_{i-1}$ and with a righ-hand fringe selling, alternatively, higher quality variants $v_{i+k+1}, v_{i+k+2}, ..., v_n$. Using expressions (13)-(15) the optimal-replies of the firms in the cartel are

$$\begin{aligned} p^{pc}_i(p_{i-1}, p_{i+1}) &= \frac{\frac{1}{2}p_{i-1}(v_{i+1} - v_i) + p_{i+1}(v_i - v_{i-1})}{(v_{i+1} - v_{i-1})} \\ p^{pc}_{i+1}(p_i, p_{i+2}) &= \frac{p_i(v_{i+2} - v_{i+1}) + p_{i+2}(v_{i+1} - v_i)}{(v_{i+2} - v_i)} \\ p^{pc}_{i+2}(p_{i+1}, p_{i+3}) &= \frac{p_i(v_{i+3} - v_{i+2}) + p_{i+3}(v_{i+2} - v_{i+1})}{(v_{i+3} - v_{i+1})} \\ &\quad \cdots\cdots\cdots\cdots\cdots\cdots \\ p^{pc}_{i+k}(p_{i+k-1}, p_{i+k+1}) &= \frac{p_{i+k-1}(v_{i+k+1} - v_{i+k}) + \frac{1}{2}p_{i+k+1}(v_{i+k} - v_{i+k-1})}{v_{i+k+1} - v_{i+k-1}}. \end{aligned}$$

where only the two extreme firms i and $i+k$ in the cartel are directly competing with the firms outside. Without loss of generality, take a generic firm inside the cartel producing an intermediate variant (i.e neither the bottom nor the top quality within the cartel), say firm $i+1$. Using both the optimal reply of firm $i+1$ and those of the firms connected to it (i.e. firms i and $i+2$) and re-arranging, we obtain the optimal replies of these three firms as functions of p_{i-1} and p_{i+3} only.

$$\begin{aligned}
\tilde{p}_i &= p_i^{pc}(p_{i-1}, p_{i+3}) = \frac{1}{2}\frac{p_{i-1}\left(v_{i+3}-v_i\right) + 2p_{i+3}(v_i - v_{i-1})}{v_{i+3}-v_{i-1}}, \\
\tilde{p}_{i+1} &= p_{i+1}^{pc}(p_{i-1}, p_{i+3}) = \frac{1}{2}\frac{p_{i-1}(v_{i+3}-v_{i+1}) + 2p_{i+3}(v_{i+1} - v_{i-1})}{v_{i+3}-v_{i-1}}, \\
\tilde{p}_{i+2} &= p_{i+2}^{pc}(p_{i-1}, p_{i+3}) = \frac{1}{2}\frac{p_{i-1}(v_{i+3}-v_{i+2}) + 2p_{i+3}(v_{i+2} - v_{i-1})}{v_{i+3}-v_{i-1}}.
\end{aligned}$$

Using the above, we can easily compute the optimal market share of firm $(i+1)$ as

$$D_{i+1}(\tilde{p}_i, \tilde{p}_{i+1}, \tilde{p}_{i+2}) = \frac{\tilde{p}_{i+2}-\tilde{p}_{i+1}}{v_{i+2}-v_{i+1}} - \frac{\tilde{p}_{i+1}-\tilde{p}_i}{v_{i+1}-v_i} = 0$$

which proves that under partial collusion every intermediate firm of an *intermediate* cartel obtains zero market share. Repeating now the same procedure for the firm producing the lowest quality in the cartel (here firm i), we obtain instead that

$$D_i(\tilde{p}_i, \tilde{p}_{i+1}, \tilde{p}_{i-1}) = \frac{\tilde{p}_{i+1}-\tilde{p}_i}{v_{i+1}-v_i} - \frac{\tilde{p}_i-\tilde{p}_{i-1}}{v_i-v_{i-1}} = \frac{1}{2}\frac{\tilde{p}_{i-1}}{(v_i - v_{i-1})} > 0$$

for $\tilde{p}_{i-1} > 0$. Finally, computing the optimal replies of the highest quality firm in the cartel, i.e. firm $(i+k)$, and of the firms directly connected to it, we obtain

$$\begin{aligned}
\tilde{p}_{i+k-1}(p_{i+k-2}, p_{i+k}) &= \frac{p_{i+k-2}(v_{i+k-1}-v_{i+k-2}) + p_{i+k}\left(v_{i+k-1}-v_{i+k-2}\right)}{v_{i+k}-v_{i+k-2}} \\
\tilde{p}_{i+k}(p_{i+k-1}, p_{i+k+1}) &= \frac{p_{i+k-1}(v_{i+k+1}-v_{i+k}) + \frac{1}{2}p_{i+k+1}\left(v_{i+k}-v_{i+k-1}\right)}{v_{i+k+1}-v_{i+k-1}} \\
\tilde{p}_{i+k+1}(p_{i+k}, p_{i+k+2}) &= \frac{1}{2}\frac{p_{i+k}(v_{i+k+2}-v_{i+k+1}) + p_{i+k+2}\left(v_{i+k+1}-v_{i+k}\right)}{v_{i+k+2}-v_{i+k}}.
\end{aligned}$$

Using the above,

$$\begin{aligned}
D_{i+k}(\tilde{p}_{i+k-1}, \tilde{p}_{i+k}, \tilde{p}_{i+k+1}) &= \frac{\tilde{p}_{i+k+1}-\tilde{p}_{i+k}}{v_{i+k+1}-v_{i+k}} - \frac{\tilde{p}_{i+k}-\tilde{p}_{i+k-1}}{v_{i+k}-v_{i+k-1}} = \\
&= \frac{1}{2}\frac{\tilde{p}_{i+k+1}}{(v_{i+k}-v_{i+k-1})} > 0.
\end{aligned}$$

showing that only the variants produced by the two firms at the extremes of this (generic) intermediate cartel are sold at prices implying *positive* market shares. Exactly the same procedure proves that, in a *top cartel,* only the highest and the lowest quality variants initially sold by the cartel remain on sale.

Finally, let us consider a *bottom cartel,* i.e. a cartel formed by firms $1, 2, ..., k$ initially selling k variants $v_1, v_2, ...v_k$ and competing with $(n-k)$ independent firms selling the higher quality variants $v_{k+1}, v_{k+2}, ..., v_n$. Again, we can apply the same argument used above to show that every firm in the *interior* of the cartel (i.e neither selling the lowest quality nor the highest quality variant in the cartel) obtains zero market share. Also, for the top quality firm in the cartel (here firm k), we obtain that $D_k(\tilde{p}_k, \tilde{p}_{k-1}, \tilde{p}_{k+1}) > 0$. Finally, when considering a firm selling the lowest quality variant in any *bottom* cartel, its market share simply writes as:

$$D_1(p_2, p_1) = \frac{p_2 - p_1}{v_2 - v_1} - \frac{p_1}{v_1},$$

that, using firm' s 1 optimal collusive reply $p_1^{pc}(p_2) = \frac{v_1}{v_2} p_2$, becomes

$$D_1(p_2, \tilde{p}_1) = \frac{p_2 - \frac{v_1}{v_2} p_2}{v_2 - v_1} - \frac{\frac{v_1}{v_2} p_2}{v_1} = 0,$$

showing that, differently from other cartels, a *bottom cartel* optimally produces only its top-quality variant v_k. **Q.E.D.**

Proof of Proposition 4. Let us assume here, for simplicity, that only one cartel $S \subset N$ has formed, and that the remaining firms play as singletons. However, the same reasoning would apply to the case with more than one cartel. It can be easily checked that the joint profit of an arbitrary cartel $\Pi_S = \sum_{i \in S} \pi_i$ is concave with respect to the price p_i of every firm $i \in S$. Moreover, optimal reply functions of partially collusive firms $i \in S$ are *contractions* (cf. footnote 5) and, hence, a unique partially collusive price profile p^{pc} exists for any given profile of qualities $v_1, v_2, ...v_n$. Furthermore, as for the proof of proposition 1, we can: (a) start with a profile $p^*$of Nash equilibrium prices. (b) Let firms in $S \subset N$ reply using their partially collusive optimal replies. A quick comparison of the optimal replies under partial collusion (13)-(15) and those of their noncooperative counterparts (3)-(7) shows that the former are more reactive to prices than the latter and positively sloped, so that the firms in the cartel will set now higher prices than in the noncooperative scenario. (c) The same occur to all firms in the fringe playing noncooperatively: given the higher prices of the cartel, they will now respond in accordance to their best-replies by increasing their prices as well.(d) The described adjustment process, given the contraction property of firms' optimal reply functions, converges to a new profile of prices p^{pc}such that $p_i^{pc} > p_i^*$ for every $i = 1, 2, ..., n$. The inequality $p_i^c > p_i^{pc}$ for all $i = 1, 2, ..., n$ can be proved along similar lines. **Q.E.D.**

Proof of Proposition 5. From the n first order conditions of noncooperative firms the following system of second order difference equations is obtained:

$$\begin{aligned} \delta_1 p_2 - 2(\delta_1 + \delta_2)p_1 &= 0 \\ \delta_i p_{i+1} - 2(\delta_i + \delta_{i+1})p_i + \delta_{i+1}p_{i-1} &= 0, \text{ for } i = 2, ..., n-1 \\ 2p_n - p_{n-1} - \beta\delta_n &= 0, \end{aligned}$$

where $\delta_i = v_i - v_{i-1}$ for $i = 1, 2, ..., n$ indicates the existing quality gap between firm variants, and $v_0 = 0$. Under equispaced variants $\delta_i = \delta$ for every $i = 1, 2, ..., n$ the system of equations becomes

$$\begin{aligned} p_2 - 4p_1 &= 0 \\ p_{i+1} - 4p_i + p_{i-1} &= 0 \text{ for } i = 2, ..., n-1 \\ 2p_n - p_{n-1} &= \beta\delta. \end{aligned}$$

Using a standard technique, the equation of every firm's $i = 1, 2, ..., n-1$ can be written as

$$Ab^{i+1} - 4Ab^i + Ab^{i-1} = 0.$$

whose characteristic equation possesses the following two distinct real roots

$$b_1 = 2 + \sqrt{3},\ b_2 = 2 - \sqrt{3}.$$

Therefore, for every $i = 1, 2, ...n$ we can write

$$p_i = A_1 b_1^i + A_2 b_2^i. \tag{17}$$

Now, using the fact that for the bottom quality firm,

$$p_2 - 4p_1 = 0$$

we can set

$$p_0 = A_1 b_1^0 + A_2 b_2^0 = A_1 + A_2 = 0$$

implying

$$A_2 = -A_1. \tag{18}$$

Moroever, since for the top quality firm

$$2p_n - p_{n-1} = \beta\delta$$

from which

$$p_{n-1} = A_1 b_1^{n-1} + A_2 b_2^{n-1} = A_1 \left(b_1^{n-1} - b_2^{n-1}\right) = 2A_1 \left(b_1^n - b_2^n\right) - \beta\delta,$$

we obtain

$$A_1 \left(b_1^{n-1} - b_2^{n-1}\right) - 2A_1 \left(b_1^n - b_2^n\right) + \beta\delta = 0$$

and, then,

$$A_1 = \frac{\beta\delta}{\left(2b_1^n - 2b_2^n - b_1^{n-1} + b_2^{n-1}\right)}.$$

As a final step, by inserting coefficients A_1 and A_2 in (17), we obtain

$$p_i^* = A_1 (b_1)^i + A_2 (b_2)^i = A_1 (b_1)^i - A_1 (b_2)^i = \frac{\beta\delta \left(b_1^i - b_2^i\right)}{\sqrt{3}b_1^n + \sqrt{3}b_2^n},$$

for every $i = 1, 2, ..., n$, with $b_1 = \left(2 + \sqrt{3}\right)$ and $b_2 = \left(2 - \sqrt{3}\right)$, which concludes the proof. **Q.E.D.**