Competitive equilibrium with indivisible objects

Yang, Yi-You

Department of Economics, Aletheia University

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Yi-You Yang†

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Abstract

We study exchange economies in which objects are heterogeneous and indivisible, and may not be substitutes for each other. We give new equilibrium existence results with the p-substitutability condition, under which a certain degree of complementarity among objects is permitted according to the parameter vector p. Moreover, we introduce conditions under which the contributions of objects to the social welfare are equilibrium prices.

Keywords: Indivisibility, competitive equilibrium, gross substitutability, p-substitutability.

1 Introduction

We study the equilibrium existence problem for exchange markets with heterogeneous indivisible objects and preferences that are quasi-linear in money. The *gross substitutability* (GS) condition on agents' preferences is a sufficient condition for the existence

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†Department of Economics, Aletheia University, New Taipei City 251, Taiwan. E-mail address: yyyang@mail.au.edu.tw

of a competitive equilibrium and has been extensively studied in the literature.[1] Kelso and Crawford [8] prove that when all agents view objects as substitutes in the sense that their preferences satisfy GS, a price adjustment procedure will end up at a competitive equilibrium. In this paper, we try to extend their analysis to incorporate markets in which objects may not be considered as substitutes by all agents with a weaker condition called *p-substitutability*, where a parameter vector p is employed to permit a certain degree of complementarity among objects.

Suppose that some agent j promises to purchase any set of objects from other agents at price level $p = (p_a)$, where p_a is the minimal marginal value of object a for j. We say that agent i's preferences are *p-substitutable* if, taking into account j's promise, i would view objects as substitutes for each other. We prove that there exists a competitive equilibrium if each agent's preferences are p-substitutable.

It should be noted that since the parameter vector p is derived from the preferences of a certain agent j in the market, p-substitutability is an endogenous condition, and thus, in general, cannot guarantee the existence of an equilibrium for another market. Hence, our existence result does not contradict to the maximal domain theorem by Gul and Stacchetti [5], which shows that if any agent's preferences fail GS, then all other agents having GS preferences does not guarantee an equilibrium to exist. Moreover, since agent j's preferences satisfy GS whenever j has p-substitutable preferences, our result complements Gul and Stacchetti's theorem in the sense that the a single agent j having GS preferences is helpful for sustaining an equilibrium by relaxing the GS restriction on other agents' preferences.

Based on the foregoing observations, we try to further extend our analysis and give

[1] Related literature includes Gul and Stacchetti [5], Beviá et al.[3], Reijnierse et al. [11], Fujishige and Yang [4], Lien and Yan [9], Milgrom and Strulovici [10], Hatfield et al. [6, 7], Baldwin and Klemperer [1, 2] and Shioura and Tamura [12], among many others.

an existence result that can be applied to markets in which no agent has GS preferences. Suppose that the social welfare function of the market has decreasing marginal returns and let $p = (p_a)$ be the vector consisting of objects' contributions to the social welfare. We prove that if all agents' preferences are p-substitutable, then (i) there exists a competitive equilibrium; and (ii) p_a is the largest competitive price of object a. It is well-known that the contribution of object a to the social welfare is greater than or equal to any competitive price of a.[2] We prove that under p-substitutability, this bound itself is a competitive price of a.

This paper is organized as follows. We present the model and some fundamental results in Section 2. In Section 3, we introduce the notion of p-substitutability with an illustrative example and give an existence result. We then relate the existence problem to social welfare function and study equilibrium prices in Section 4, and conclude in Section 5.

2 Preliminaries

Consider an economy with a finite set $N = \{1, \ldots, n\}$ of agents and a finite set $\Omega = \{a_1, \ldots, a_m\}$ of heterogeneous indivisible objects. Let $p = (p_a) \in \mathbb{R}^{|\Omega|}$ be a price vector, where p_a denotes the price of object $a \in \Omega$. We assume that agents' net utility functions are quasilinear in prices: each agent i's utility of consuming bundle $A \subseteq \Omega$ at price level p is

$$u_i(A, p) \equiv v_i(A) - p(A),$$

where $v_i : 2^\Omega \to \mathbb{R}$ is a valuation function satisfying $v_i(\emptyset) = 0$ and $p(A)$ is a shorthand for $\sum_{a \in A} p_a$. We also assume that agents are not subject to any budget constraints. Hence

[2] See, for example, Beviá et al. [3] and Gul and Stacchetti [5].

such a trading economy can be simply represented by $\mathcal{E} = \langle \Omega; (v_i, i \in N) \rangle$.

A *competitive equilibrium* for economy $\mathcal{E}$ is a pair $\langle p; \mathbf{X} \rangle$, where $\mathbf{X} = (X_1, \ldots, X_n)$ is an allocation of objects among agents and $p \in \mathbb{R}^{|\Omega|}$ is a price vector such that for all agent $i \in N$,

$$X_i \in D_{v_i}(p) \equiv \arg\max_{A \subseteq \Omega} u_i(A, p).$$

In Proposition 1, we recall the standard theorem of welfare economics and include a proof for completeness.

Proposition 1. *Let $\langle p; \mathbf{X} \rangle$ be a competitive equilibrium for $\mathcal{E} = \langle \Omega; (v_i, i \in N) \rangle$. Then*

(a) $\mathbf{X}$ *is efficient;*[3] *and*

(b) *for any efficient allocation $\mathbf{Y}$, $\langle p; \mathbf{Y} \rangle$ is also a competitive equilibrium for $\mathcal{E}$.*

Proof. Let $\mathbf{Y} = (Y_1, \ldots, Y_n)$ be an arbitrary allocation of objects among agents. Since $X_i \in D_{v_i}(p)$ for each $i \in N$, we have

$$\begin{aligned}\sum_{i=1}^{n} v_i(X_i) &= \sum_{i=1}^{n} [v_i(X_i) - p(X_i)] + p(\Omega) \\ &\geq \sum_{i=1}^{n} [v_i(Y_i) - p(Y_i)] + p(\Omega) = \sum_{i=1}^{n} v_i(Y_i).\end{aligned}$$

Hence $\mathbf{X}$ is efficient.

In case $\mathbf{Y}$ is efficient, the above inequality implies that for all $i \in N$, $v_i(X_i) - p(X_i) = v_i(Y_i) - p(Y_i)$ and hence $Y_i \in D_{v_i}(p)$. □

The gross substitutability introduced by Kelso and Crawford [8] is an essential condition for the analysis of equilibrium. A valuation function $v_i : 2^{\Omega} \to \mathbb{R}$ satisfies *gross*

[3] An allocation $\mathbf{X} = (X_1, \ldots, X_n)$ is *efficient* if it maximizes the sum $\sum_{i \in N} v_i(X_i)$.

substitutability (GS) if for any vector $p \in \mathbb{R}^{|\Omega|}$, the following condition holds:

$$A \in D_{v_i}(p), p' \geq p \Rightarrow \exists B \in D_{v_i}(p') \text{ such that } \{a \in A : p'_a = p_a\} \subseteq B. \tag{1}$$

It is well-known that each GS valuation function $v_i : 2^{\Omega} \to \mathbb{R}$ has *decreasing marginal returns* [4] i.e., for each object $a \in \Omega$,

$$A \subseteq B \subseteq \Omega \backslash \{a\} \Rightarrow v_i(B \cup \{a\}) - v_i(B) \leq v_i(A \cup \{a\}) - v_i(A).$$

Theorem 2 of Kelso and Crawford [8] implies that a competitive equilibrium exists whenever all agnets' preferences satisfy GS. A natural question is how to extend their analysis to incorporate markets with non-GS preferences.

Gul and Stacchetti [5] address the issue and give a negative result: if any agent's preferences violate GS, then GS preferences can be found for other agents such that no equilibrium exists. In contrast to Gul and Stacchetti's approach, we focus on the question of whether the GS preferences of a single agent or a group of agents can help to sustain a competitive equilibrium. In what follows, we will first introduce the notion of p-substitutability to generalize GS, and then study economies in which agents' preferences may fail GS.

3 The p-substitutability condition

Our analysis begins with an illustrative example. Consider the three-agent economy $\mathcal{E}$ with $\Omega = \{a, b, c\}$ given in Table I. Although only agent 1's preferences satisfy GS, the efficient allocation $X_1 = \emptyset, X_2 = \{a\}, X_3 = \{b, c\}$ could be supported by prices

[4] See Gul and Stacchetti [5] and Reijnierse et al. [**?**] for details.

$p_a = 16, p_b = p_c = 9$ as a competitive equilibrium. The reason for this is that agent 1's preferences can complement other agents' preferences such that objects are viewed as substitutes by all agents in a certain context.

Table I

Agents' valuations

	$\{a\}$	$\{b\}$	$\{c\}$	$\{a,b\}$	$\{b,c\}$	$\{a,c\}$	$\{a,b,c\}$
v_1	7	7	7	13	13	13	19
v_2	16	4	4	20	7	21	25
v_3	5	11	11	16	20	17	26

Suppose that agent 1 promises to buy any set of objects from other agents at the price level $p^{v_1} = (p^{v_1}_\alpha) \in \mathbb{R}^{|\Omega|}$, where

$$p^{v_1}_\alpha \equiv v_1(\Omega) - v_1(\Omega \backslash \{\alpha\})$$

is the *minimal marginal value* of object α for agent 1.[5] In this case, agent i $(i = 2, 3)$ would act the same as an agent with the valuation function $v_i[p^{v_1}]$ given by

$$v_i[p^{v_1}](A) = \max\left\{v_i(B) + p^{v_1}(A \backslash B) : B \subseteq A\right\} \text{ for } A \subseteq \Omega,$$

[5]Since v_1 satisfies GS, it has decreasing marginal returns, and hence $v_1(\Omega) - v_1(\Omega\backslash\{\alpha\}) \leq v_1(A) - v_1(A\backslash\{\alpha\})$ for all objects α and all bundles A for which $\alpha \in A \subseteq \Omega$.

and thus leading to the economy $\mathcal{E}' = \langle \Omega; v_1, v_2[p^{v_1}], v_3[p^{v_1}] \rangle$ given in Table II.

Table II

Agents' valuations

	$\{a\}$	$\{b\}$	$\{c\}$	$\{a,b\}$	$\{b,c\}$	$\{a,c\}$	$\{a,b,c\}$
v_1	7	7	7	13	13	13	19
$v_2[p^{v_1}]$	16	6	6	22	12	22	28
$v_3[p^{v_1}]$	6	11	11	17	20	17	26

We first note that all agents in enonomy $\mathcal{E}'$ have GS preferences, it follows that there exists a competitive equilibrium $\langle p; X_1, X_2, X_3 \rangle$ for $\mathcal{E}'$. Then, by definition, we can choose $Y_i \subseteq X_i$ such that $v_i[p^{v_1}](X_i) = v_i(Y_i) + p^{v_1}(X_i \backslash Y_i)$ and verify that $Y_i \in D_{v_i}(p)$ for $i = 2, 3$. It is not difficult to check $X_1 \cup (X_2 \backslash Y_2) \cup (X_3 \backslash Y_3) \in D_{v_1}(p)$. Hence, we obtain that there is a competitive equilibrium $\langle p; X_1 \cup (X_2 \backslash Y_2) \cup (X_3 \backslash Y_3), Y_2, Y_3 \rangle$ for $\mathcal{E}$.

We now introduce the notion of p-substitutability, and study its relation to GS. The *marginal vector* of a valuation function $v_i : 2^\Omega \to \mathbb{R}$ is the vector $p^{v_i} = (p_a^{v_i}) \in \mathbb{R}^{|\Omega|}$ given by

$$p_a^{v_i} = v_i(\Omega) - v_i(\Omega \backslash \{a\}) \text{ for } a \in \Omega.$$

For any vector $p \in \mathbb{R}^{|\Omega|}$, the valuation function v_i is called *p-substitutable* if the function $v_i[p] : 2^\Omega \to \mathbb{R}$ given by

$$v_i[p](A) = \max\{v_i(B) + p(A \backslash B) : B \subseteq A\} \text{ for } A \subseteq \Omega$$

satisfies GS. By definition, it is clear that $v_i[p^{v_i}](A) = v_i(A)$ for all $A \subseteq \Omega$. Hence, v_i is p^{v_i}-substitutable if and only if v_i satisfies GS.

Proposition 2. *Let $p \in \mathbb{R}^{|\Omega|}$ and let $v_i : 2^\Omega \to \mathbb{R}$ be a valuation function.*

(a) *If v_i satisfies GS, then v_i is p-substitutable.*

(b) *Let $p' \in \mathbb{R}^{|\Omega|}$ be a vector such that $p' \geq p$. If v_i is p-substitutable, then v_i is also p'-substitutable.*

Proof. (a) Assume that v_i satisfies GS. Let $v_j : 2^\Omega \to \mathbb{R}$ be the function given by $v_j(A) = p(A)$ for $A \subseteq \Omega$ and let $C = \{i, j\}$. Suppose, to the contrary, that $v_i[p]$ fails GS. Theorem 2 of Gul and Stacchetti [5] implies that there exists GS valuation functions $v_2, \ldots, v_r$ such that the economy $\mathcal{E} = \langle \Omega; v_i[p], v_2, \ldots, v_r \rangle$ has no equilibrium. However, we are going to show that there exists an equilibrium for $\mathcal{E}$, yielding a contradiction.

Note that each agent's preferences in the economy $\mathcal{E}' = \langle \Omega; v_i, v_j, v_2, \ldots, v_r \rangle$ satisfy GS. Hence there exist an allocation $(X_i, X_j, X_2, \ldots, X_r)$ and an equilibrium price vector $q \in \mathbb{R}^{|\Omega|}$ such that $X_i \in D_{v_i}(q), X_j \in D_{v_j}(q)$ and $X_k \in D_{v_k}(q)$ for $k = 2, \ldots, r$. For any $Y \subseteq \Omega$, there exists $A \subseteq Y$ such that $v_i[p] = v_i(A) + p(Y \backslash A)$ and hence

$$\begin{aligned} v_i[p](X_i \cup X_j) - q(X_i \cup X_j) &= [v_i(X_i) - q(X_i)] + [v_j(X_j) - q(X_j)] \\ &\geq [v_i(A) - q(A)] + [v_j(Y \backslash A) - q(Y \backslash A)] \\ &= v_i[p](Y) - q(Y). \end{aligned}$$

This implies that $\langle q; X_i \cup X_j, X_2, \ldots, X_r \rangle$ is a competitive equilibrium for $\mathcal{E}$.

(b) Assume that v_i is p-substitutable. This implies that $v_i[p]$ is GS, and so is $(v_i[p])[p']$. It suffices to prove that $v_i[p']$ coincides with $(v_i[p])[p']$. Let $A \subseteq \Omega$ be an arbitrary bundle of objects. By definition, there exist two subsets B and B' of A such that $v_i[p'](A) = v_i(B) + p'(A \backslash B)$ and $(v_i[p])[p'](A) = v_i[p](B') + p'(A \backslash B')$. Similarly, there exists $C' \subseteq B'$ such that $v_i[p](B') = v_i(C') + p(B' \backslash C')$. Since $p' \geq p$, we have

$$\begin{aligned} v_i[p'](A) &= v_i(B) + p'(A \backslash B) \leq v_i[p](B) + p'(A \backslash B) \leq (v_i[p])[p'](A) \\ &= v_i[p](B') + p'(A \backslash B') = v_i(C') + p(B' \backslash C') + p'(A \backslash B') \\ &\leq v_i(C') + p'(A \backslash C') \leq v_i[p'](A). \end{aligned}$$

This implies $v_i[p'](A) = (v_i[p])[p'](A)$ and completes the proof.

□

The following result shows that p^{v_1}-substitutability is sufficient for the existence of a competitive equilibrium whenever v_1 satisfies GS.

Theorem 1. *Let $\mathcal{E} = \langle \Omega; (v_i, i \in N) \rangle$ be an economy. Assume that v_1 satisfies GS. If each agent's valuation function v_i satisfies p^{v_1}-substitutability, then $\mathcal{E}$ has a competitive equilibrium.*

The proof of Theorem 1 requires the following lemma.

Lemma 1. *Let $\mathcal{E} = \langle \Omega; v_1, \ldots, v_n \rangle$ be an economy. Assume that v_1 has decreasing marginal returns and that there exists a competitive equilibrium $\langle p; \mathbf{X} \rangle$ for the economy $\mathcal{E}' = \langle \Omega; v_1, v_2[p^{v_1}], \ldots, v_n[p^{v_1}] \rangle$. Then there exists $Y_i \subseteq X_i$ for $i = 2, \ldots, n$ such that $Y_1 \cup (\bigcup_{i=2}^{n} (X_i \backslash Y_i)) \in D_{v_1}(p)$, and $Y_i \in D_{v_i}(p)$ for $i = 2, \ldots, n$.*

Proof. We first note that $p_a^{v_1} \leq p_a$ for all $a \in \Omega \backslash X_1$. In case $p_a^{v_1} > p_a$ for some $a \in \Omega \backslash X_1$, since v_1 has decreasing marginal returns, it follows that

$$\begin{aligned} v_1(X_1 \cup \{a\}) - p(X_1 \cup \{a\}) = {} & [v_1(X_1 \cup \{a\}) - v_1(X_1) - p_a^{v_1}] + [p_a^{v_1} - p_a] \\ & + [v_1(X_1) - p(X_1)] > v_1(X_1) - p(X_1), \end{aligned}$$

contradicting to the fact $X_1 \in D_{v_1}(p)$.

For $i = 2, \ldots, n$, there exists $Y_i \subseteq X_i$ such that $v_i[p^{v_1}](X_i) = v_i(Y_i) + p^{v_1}(X_i \backslash Y_i)$, and hence

$$\begin{aligned}
v_i[p^{v_1}](Y_i) - p(Y_i) &\leq v_i[p^{v_1}](X_i) - p(X_i) \\
&\leq [v_i(Y_i) - p(Y_i)] + [p^{v_1}(X_i \backslash Y_i) - p(X_i \backslash Y_i)] \\
&\leq v_i(Y_i) - p(Y_i) \leq v_i[p^{v_1}](Y_i) - p(Y_i).
\end{aligned}$$

This implies $\sum_{i=1}^{n} [p^{v_1}(X_i \backslash Y_i) - p(X_i \backslash Y_i)] = 0$ and $v_i(Y_i) - p(Y_i) = v_i[p^{v_1}](X_i) - p(X_i) \geq v_i[p^{v_1}](A) - p(A) \geq v_i(A) - p(A)$ for $i = 2, \ldots, n$ and for all $A \subseteq \Omega$. Moreover, since

$$\begin{aligned}
v_1(Y_1) - p(Y_1) &\geq v_1(Y_1 \cup (\bigcup_{i=2}^{n} (X_i \backslash Y_i))) - p(Y_1 \cup (\bigcup_{i=2}^{n} (X_i \backslash Y_i))) \\
&\geq v_1(Y_1) - p(Y_1) + \sum_{i=1}^{n} [p^{v_1}(X_i \backslash Y_i) - p(X_i \backslash Y_i)] \geq v_1(Y_1) - p(Y_1),
\end{aligned}$$

we have $Y_1 \cup (\bigcup_{i=2}^{n} (X_i \backslash Y_i)) \in D_{v_1}(p)$. $\square$

We are now ready to prove Theorem 1.

Proof of Theorem 1. Assume that v_i satisfies p^{v_1}-substitutability for $i = 1, \ldots, n$. This implies each agent in $\mathcal{E}'$ has GS valuation function, and hence there exists an equilibrium for $\mathcal{E}'$. Moreover, since each GS valuation function has decreasing marginal returns and so does v_1, it follows that $\mathcal{E}$ has a competitive equilibrium by Lemma 1. $\square$

4 Markets with non-GS preferences

In this section, we will extend our analysis to study economies in which no agent has GS preferences with the notion of aggregate valuation function. For each coalition of agents $C \subseteq N$, the corresponding *aggregate valuation function* $v_C : 2^\Omega \to \mathbb{R}$ is defined by

$$v_C(A) = \max\left\{\sum_{i\in C} v_i(A_i) : \bigcup_{i\in C} A_i = A \text{ and } A_i \cap A_j = \emptyset \text{ for } i \neq j\right\} \text{ for } A \subseteq \Omega.$$

In particular, we call v_N the *social welfare function* of the economy $\mathcal{E} = \langle \Omega; (v_i, i \in N)\rangle$.

The following result shows that when the aggregate valuation function of some coalition C has decreasing marginal returns, p^{v_C}-substitutability is sufficient for an equilibrium to exist.

Theorem 2. *Let $\mathcal{E} = \langle \Omega; (v_i, i \in N)\rangle$ be an economy. Assume that there exists a coalition $C \subseteq N$ such that v_C has decreasing marginal returns. If each agent's valuation function v_i satisfies p^{v_C}-substitutability, then*

(a) *there exists a competitive equilibrium; and*

(b) *the social welfare function v_N satisfies GS.*

To illustrate the impact of Theorem 2, we consider the three-agent economy given in Table III. Note that each agent's preferences violate GS but satisfy p^{v_N}-substitutability. Since the social welfare function v_N has decreasing marginal returns, it follows that the market has an equilibrium by Theorem 2.

Table III

Agents' valuations

	$\{a\}$	$\{b\}$	$\{c\}$	$\{a,b\}$	$\{b,c\}$	$\{a,c\}$	$\{a,b,c\}$
v_1	7	3	3	8	7	8	13
v_2	7	7	7	8	8	8	12
v_3	3	3	7	7	8	8	13
v_N	7	7	7	14	14	14	21

Let $p \in \mathbb{R}^{|\Omega|}$ be the vector given by $p_a = p_b = p_c = 4$ and let $X_1 = \{a\}, X_2 = \{b\}, X_3 = \{c\}$. It can be verified that $\langle p; \mathbf{X} \rangle$ is a competitive equilibrium.

Proof of Theorem 2. Assume that v_i satisfies p^{v_C}-substitutability for $i = 1, \ldots, n$.

(a) Let $\mathcal{E}' = \langle \Omega; v_0, v_1, \ldots, v_n \rangle$ be the economy constructed from $\mathcal{E}$ by adding an agent 0 with the valuation function v_0 given by $v_0(A) = p^{v_C}(A)$ for $A \subseteq \Omega$. Since v_0 satisfies GS and $p^{v_0} = p^{v_C}$, the result of Theorem 1 implies that there exists a competitive equilibrium $\langle p; X_0, X_1, \ldots, X_n \rangle$ for $\mathcal{E}'$.

Note that in case $X_0 = \emptyset$, $\langle p; X_1, \ldots, X_n \rangle$ is a competitive equilibrium for $\mathcal{E}$ and we have done. Suppose $X_0 = \{a_1, \ldots, a_r\} \neq \emptyset$. Let $A_0 = \cup_{i \in C} X_i$ and let $A_j = A_{j-1} \cup \{a_j\}$ for $j = 1, \ldots, r$. Since v_C has decreasing marginal returns, we have

$$v_C(A_j) - v_C(A_{j-1}) \geq p^{v_C}_{a_j} \text{ for } j = 1, \ldots, r,$$

and $v_C(A_r) - v_C(A_0) \geq p^{v_C}(X_0) = v_0(X_0)$. Let $X_0' = \emptyset, X_i' = X_i$ for $i \in N \backslash C$, and let

$\{X_i'\}_{i\in C}$ be a partition of A_r such that $v_C(A_r) = \sum_{i\in C} v_i(X_i')$. It follows that

$$\begin{aligned}\sum_{i=0}^{n} v_i(X_i) &\geq \sum_{i=0}^{n} v_i(X_i') = v_C(A_r) + \sum_{i\in N\setminus C} v_i(X_i)\\ &\geq v_0(X_0) + v_C(A_0) + \sum_{i\in N\setminus C} v_i(X_i) \geq \sum_{i=0}^{n} v_i(X_i).\end{aligned}$$

This implies $X_0', X_1', \ldots, X_n'\rangle$ is an efficient allocation for $\mathcal{E}'$, i.e., $\sum_{i=0}^{n} v_i(X_i) = \sum_{i=0}^{n} v_i(X_i')$. By Proposition 1, $\langle p; X_0', X_1', \ldots, X_n'\rangle$ is a competitive equilibrium for $\mathcal{E}'$, and hence $\langle p; X_1', \ldots, X_n'\rangle$ is a competitive equilibrium for $\mathcal{E}$.

(b) Suppose, to the contrary, that v_N violates gross substitutability. By Theorem 2 of Gul and Stacchetti (1999), there exists a GS valuation function $v_{n+1}, \ldots, v_{n+r}$ such that there is the economy $\langle \Omega; v_N, v_{n+1}, \ldots, v_{n+r}\rangle$ has no competitive equilibrium. However, the result of (a) implies that there exists an equilibrium $\langle p; X_1, \ldots, X_{n+r}\rangle$ for the economy $\langle \Omega; v_1, \ldots, v_n, v_{n+r}\rangle$. Let $X_N = \cup_{i\in N} X_i$. It is not difficult to check that $\langle p; X_N, X_{n+1}, \ldots, X_{n+r}\rangle$ is an equilibrium for $\langle \Omega; v_N, v_{n+1}, \ldots, v_{n+r}\rangle$, yielding a contradiction. $\square$

It is well-known that in equilibrium, the competitive price of object $a \in \Omega$ is less than or equal to $p_a^{v_N} \equiv v_N(\Omega) - v_N(\Omega\setminus\{a\})$, i.e., the contribution of a to the social welfare. A proof by Beviá et al. [3] is included for completeness.

Proposition 3 (See Beviá et al. [3]). *Let $\langle p; \mathbf{X}\rangle$ be a competitive equilibrium for $\mathcal{E} = \langle \Omega; (v_i, i \in N)\rangle$. Then $p^{v_N} \geq p$.*

Proof. Let $a \in \Omega$ and let $(Y_1, \ldots, Y_n)$ be a partition of $\Omega\setminus\{a\}$ such that $\sum_{i=1}^{n} v_i(Y_i) =$

$v_N(\Omega\backslash\{a\})$. Since $\mathbf{X}$ is efficient and $X_i \in D_{v_i}(p)$ for $i = 1, \ldots, n$, it follows that

$$\begin{aligned} v_N(\Omega) - p(\Omega) &= \sum_{i=1}^{n}[v_i(X_i) - p(X_i)] \\ &\geq \sum_{i=1}^{n}[v_i(Y_i) - p(X_i)] = v_N(\Omega\backslash\{a\}) - p(\Omega\backslash\{a\}). \end{aligned}$$

This implies $p_a^{v_N} = v_N(\Omega) - v_N(\Omega\backslash\{a\}) \geq p_a$. $\square$

Following the above observation, Beviá et al. [3] and Gul and Stacchetti [5] study the question of under which conditions an efficient allocation can be supported by p^{v_N} as an equilibrium. In the following result, we try to generalize their results with p^{v_N}-substitutability.

Theorem 3. *Let $\mathcal{E} = \langle \Omega; (v_i, i \in N)\rangle$ be an economy. Assume that the social welfare function v_N has decreasing marginal returns. If each agent's valuation function v_i satisfies p^{v_N}-substitutability, then for any efficient allocation $\mathbf{X}$, $\langle p^{v_N}; \mathbf{X}\rangle$ is a competitive equilibrium.*

Proof. Consider the economy $\mathcal{E}' = \langle \Omega; v_0, v_1, \ldots, v_n\rangle$ where v_0 is the valuation function given by $v_0(A) = p^{v_N}(A)$ for $A \subseteq \Omega$ and let $N' = \{0, 1, \ldots, n\}$. The result of Theorem 1 implies that there is a competitive equilibrium $\langle p; Y_0, Y_1, \ldots, Y_n\rangle$ for $\mathcal{E}'$. Without loss of generality, we may assume that $Y_0 = \{a_1, \ldots, a_r\}$. Let $A_0 = \cup_{i=1}^{n} Y_i$ and let $A_j = A_{j-1} \cup \{a_j\}$ for $j = 1, \ldots, r$. Morover, since v_N has decreasing marginal returns, we have

$$\begin{aligned} v_N(\Omega) - v_N(A_0) &= \sum_{j=1}^{r}[v_N(A_j) - v_N(A_{j-1})] \\ &\geq p^{v_N}(Y_0) = v_0(Y_0). \end{aligned}$$

This implies $v_N(\Omega) \geq v_0(Y_0)+v_N(A_0) \geq v_0(Y_0)+\sum_{i=1}^n v_i(Y_i) = v_{N'}(\Omega) \geq v_0(\emptyset)+v_N(\Omega) = v_N(\Omega)$, and we have $v_N(\Omega) = v_{N'}(\Omega)$. Let $(X_1, \dots, X_n)$ be an arbitrary partition of Ω such that $\sum_{i=1}^n v_i(X_i) = v_N(\Omega) = v_{N'}(\Omega)$ and let $X_0 = \emptyset$. By Proposition 1, we have that $\langle p; X_0, X_1, \dots, X_n \rangle$ is also a competitive equilibrium for $\mathcal{E}'$. This implies that $\langle p; X_1, \dots, X_N \rangle$ is a competitive equilibrium for $\mathcal{E}$ and for each $a \in \Omega$,

$$v_0(\emptyset) - p(\emptyset) = 0 \geq v_0(\{a\}) - p_a = p_a^{v_N} - p_a.$$

Together with the fact $p_a^{v_N} \geq p_a$ by Proposition 3, we obtain that $p^{v_N} = p$. □

5 Concluding remarks

In contrast to our approach, Sun and Yang [13] and Teytelboym [14] extend the GS framework of Kelso and Crawford, and study the effect of complementarity on equilibrium results under the assumption that objects can be partitioned into different groups and agents' preferences are *alike* in the way that they all view objects in the same group as substitutes and objects across different groups as complements. In this paper, we introduce the notion of p-substitutability to permit complex types of complementarity, and give equilibrium results which can be applied to markets with agents having *divergent* preferences.

Hatfield et al. [6] address a model of trading networks which incorporates economies with indivisible objects as special cases, and prove that a number results from the exchange economy model continue to hold in their network model under the *full substitutability* condition. The question of generalizing the notion of p-substitutability to the network model might bring significant contribution to the matching literature, and is left for further work.

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