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Akamatsu, Takashi and Fujishima, Shota and Takayama, Yuki

23 October 2016

Discrete-Space Agglomeration Model with Social Interactions: Multiplicity, Stability, and Continuous Limit of Equilibria*

Takashi Akamatsu† Shota Fujishima‡ Yuki Takayama§

October 23, 2016

Abstract

This study examines the properties of equilibrium, including the stability, of discrete-space agglomeration models with social interactions. The findings reveal that while the corresponding continuous-space model has a unique equilibrium, the equilibrium in discrete space can be non-unique for any finite degree of discretization by characterizing the discrete-space model as a potential game. Furthermore, it indicates that despite the above result, any sequence of discrete-space models' equilibria converges to the continuous-space model's unique equilibrium as the discretization of space is refined.

Keywords: Social interaction; Agglomeration; Discrete space; Potential game; Stability; Evolutionary game theory

JEL Codes: C62; C72; C73; D62; R12

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*We are grateful to Daisuke Oyama, Chin-Sheng Chen, the editor Atsushi Kajii, and anonymous referees for helpful comments and discussions. We also would like to thank audiences at the Third Asian Seminar in Regional Science and the 27th Annual Meetings of the Applied Regional Science Conference for useful comments. Takashi Akamatsu acknowledges financial support from the Japan Society for the Promotion of Science (Grant-in-Aid for Scientific Research (B) 21360240 and 24360202). Shota Fujishima acknowledges financial support from the Japan Society for the Promotion of Science (Grant-in-Aid for Young Scientists (B) 15K20924). Any remaining errors are our own.

†Graduate School of Information Sciences, Tohoku University, 6-6-6 Aoba, Sendai, Miyagi 980-8579, Japan. Email: `akamatsu@plan.civil.tohoku.ac.jp`

‡School of Management, Tokyo University of Science, 1-11-2, Fujimi, Chiyoda, Tokyo 102-0071, Japan. Email: `sfujishima@rs.tus.ac.jp`

§Institute of Science and Engineering, Kanazawa University, Kakuma-machi, Kanazawa 920-1192, Japan. Email: `ytakayama@se.kanazawa-u.ac.jp`

1 Introduction

Beckmann's (1976) social interaction model has been an important benchmark for the study on spatial agglomeration. Considering the fact that face-to-face communications are important for understanding the mechanisms behind spatial distributions of economic activities, Beckmann (1976) presented a model in which people aiming to interact with others choose their area of location, referred to as a cell in this study. Although people can reduce the cost of interactions by locating close to one another, agglomeration can cause congestion, such as increases in housing prices. Equilibrium population distributions, which are of interest to this study, emerge as a result of the trade-off between the positive and negative effects of agglomeration. This type of model has been of particular interest for urban economists because the cell of an urban center is not specified a priori, unlike classical urban models such as the monocentric city model.[1]

Beckmann (1976) also considered social interactions among households for a linear city represented by a real line. After Beckmann's (1976) study, Tabuchi (1986) and Mossay and Picard (2011) considered social interactions among a single type of agent on the real line.[2] All these studies attained symmetric unimodal population distributions as unique equilibria. This uniqueness result is compelling, and the shape of the equilibrium distribution is intuitively reasonable. Moreover, this is good news for policymakers since they do not have to worry about multiple equilibria when internalizing externalities.

Although the results attained in continuous-space models serve as important theoretical benchmarks, examining whether these results are robust in terms of the discretization of space is also essential. In particular, if we would like to empirically test the model, we would have to discretize it. For example, it is virtually impossible to collect population data for each *point*. Whatever micro the data is, it is still aggregated over some geographical areas.[3] Thus, empirical studies cannot invoke the uniqueness result of the continuous-space model unless the properties of equilibria of the continuous-space model are transferred to those of the discrete-space model.

To address this issue, we consider social interactions among consumers in the discrete space in which a finite number of cells are distributed on a line segment, and we study the

[1] See, for example, Section 3.3 of Fujita and Thisse (2013).

[2] Mossay and Picard (2011) considered consumers, whereas Tabuchi (1986) considered firms. Besides models on the real line, O'Hara (1977) considered the social interactions of firms in a square city, and Borukhov and Hochman (1977) considered the social interactions of consumers in a circular city. They also obtained a symmetric unimodal distribution as a unique equilibrium. In Borukhov and Hochman (1977), though, the cost of social interaction was not weighted by population density, so social interactions did not cause any externality.

[3] In fact, Allen and Arkolakis (2014), who studied the relationship between economic activity and geography with data, "approximate the continuous space with a discrete number of locations (p. 1113)."

properties of equilibria accordingly. To this end, this study begins by creating a model for a general quasi-linear utility function, invoking the fact that this model of location choice can be described as a *potential game* (Monderer and Shapley, 1996).[4] One important consequence of being a potential game is that the equilibrium can be characterized with a finite-dimensional optimization problem. Indeed, by assuming that the pairwise interaction cost between cells is symmetric, we can identify a function, called a *potential function*, so that the set of equilibria exactly coincides with the set of Kurash–Kuhn–Tucker (KKT) points for the maximization problem of the function. Moreover, even if multiple equilibria arise, we can conduct the stability analysis with the potential function. In fact, every local maximizer of the potential function is a stable equilibrium under a broad class of myopic individualistic evolutionary dynamics. Note that the stability of equilibria has not been addressed in continuous-space models.[5] The discretization of space reduces the dimension of the stability analysis, which enables the properties of equilibria to be scrutinized more closely.

We provide both positive and negative results for the issue raised above. Regarding the negative result, we present cases in which the equilibrium in the discrete space is essentially non-unique as long as the interaction cost is not too small, meaning that equilibria having different numbers of populated cells coexist. In particular, we can pin down a range of interaction costs in which multiple equilibria arise for *any* finite number of cells. This result holds, for example, when the equilibrium condition is described as a system of linear equations, which is particularly relevant to empirical analysis. This also suggests that contrary to the case of the unique equilibrium, it is important to be cautious about interpreting implications from the analysis focusing on a particular equilibrium because it is possible that the equilibrium is unstable and another stable equilibrium with an essentially different population distribution exists.

Conversely, we explore the connection between continuous-space and discrete-space models by focusing on the linear interaction cost. In particular, we make each cell eventually non-atomic by increasing the number of cells while the total size of location space remains fixed, and study the limiting properties of equilibria. We show that *any* sequence of the discrete-space model's equilibria converges to the equilibrium of the continuous-space model as the number of cells goes to infinity or as the distance between adjacent cells vanishes. Since the equilibrium of the continuous-space model is unique, this means that the set of equilibria is continuous in the number of cells at their limit. This is a positive result since we may

[4]The potential function approach has been recognized as a promising analytical tool for regional science (Fujita and Thisse, 2013). See Oyama (2009a,b) and Fujishima (2013) for applications of the potential game approach to geography models.

[5]Naturally, continuous-space models are not always free from the problem of multiple equilibria, as we will discuss in the concluding remarks.

think that as long as the number of geographical zones is sufficiently large, any equilibrium of a discrete-space model is close to the equilibrium of a limiting continuous-space model. In other words, a continuous-space model can be viewed as the limit of a discrete-space model with regard to the size of geographical zones.

To the best of our knowledge, few papers on spatial social interactions have utilized a discrete-space model. Anas and Xu (1999) presented a multi-regional general equilibrium model in which every region employs labor and produces goods. Although the technology exhibited a constant returns to scale, the goods were differentiated over regions and the consumers traveled to each region to purchase them, which yielded an agglomeration force in the central region.[6] Although their model was useful for evaluating urban policies, they entirely relied on numerical simulations, thus forcing us to consider a particular equilibrium that might be unstable in the case of multiple equilibria. Turner (2005) and Caruso et al. (2009) considered one-dimensional discrete-space location models with neighborhood externalities in the sense that utility at a particular location depends on the population distribution of the neighborhood.[7] Caruso et al. (2009) relied on numerical simulations, while Turner (2005) generically attained a unique equilibrium outcome by considering an extreme type of neighborhood externalities in which an individual located between vacant neighborhoods receives a bonus. However, because they focused on the effects of residential locations on open spaces, they abstracted away from the endogenous determination of an urban center, although this remains an important feature of the model in which we are interested.[8] Moreover, we emphasize that none of the aforementioned works focused on the relation between continuous- and discrete-space models.

The remainder of this study is as follows. Section 2 introduces a general class of social interaction models and characterizes this class as a potential game. Section 3 examines the stability of equilibria and Section 4 investigates the uniqueness of equilibria. Section 5 studies the connections between discrete- and continuous-space models by increasing the number of cells. Finally, Section 6 concludes the study. Note that the proofs omitted in the main text are provided in the Appendix.

[6] Braid (1988) considered a five-town model having a similar structure, although he abstracted away from general equilibrium effects. He showed that, depending on the degree of product differentiation, the equilibrium firm distribution can be bimodal.

[7] Caruso et al. (2007) considered a two-dimensional discrete space.

[8] Moreover, they made the so-called open-city assumption in which the equilibrium utility level was exogenous, whereas the total city population was endogenous.

2 The model

We start with a general class of discrete-space social interaction models, which includes the discrete-space analogue of Beckmann's (1976) and Mossay and Picard's (2011) models as special cases. This description enables us to illustrate how the potential function approach generally works for the equilibrium characterization and the stability analysis of discrete-space social interaction models.

2.1 Basic assumptions

We consider a region represented by the unit interval $[0,1]$ divided into K cells. The cells are labeled by $i \in S \equiv \{1, 2, \cdots, K\}$ in order of distance from 0, where the width of cell i is denoted by δ_i. The boundary between cells i and $i+1$ for $i \in \{1, 2, ..., K-1\}$ is then $b_i = \delta_i + b_{i-1}$, where $b_0 = 0$ and $b_K = 1$. The center of cell i, which is $\frac{\delta_i}{2} + b_{i-1}$, is denoted by x_i.[9] We assume that the land is uniformly distributed, with a density of 1 over the region. As is common in the literature, the land is owned by absentee landlords. In addition, the opportunity cost of the land is normalized to zero.

There is a unit mass of identical consumers in this region. Let $n_i \in [0,1]$ be the mass of consumers in cell i, and let $\Delta \equiv \left\{ \boldsymbol{n} = (n_1, \cdots, n_K) \in \mathbb{R}_+^K \,:\, \sum_{i=1}^K n_i = 1 \right\}$ denote the set of consumers' spatial distributions. Each consumer travels to every other consumer for social interaction. In each cell, they obtain the same utility $u(z_i, y_i)$ for residential land y_i and for the composite good z_i, which is chosen as the numéraire. Given land rent r_i and population distribution $\boldsymbol{n} \in \Delta$, the utility maximization problem of consumers in cell i is expressed as

$$\max_{z_i, y_i} \left\{ u(z_i, y_i) \,|\, z_i + r_i y_i + T_i(\boldsymbol{n}) \leq Y,\, i \in S \right\}, \tag{1}$$

where r_i denotes the land rent in cell i, and Y is the fixed income. $T_i(\boldsymbol{n})$ is the total cost to consumers from cell i for traveling to other consumers, which is defined as

$$T_i(\boldsymbol{n}) \equiv \tau \sum_{j=1}^{K} d_{ij} n_j, \tag{2}$$

where τd_{ij} denotes the travel cost from cell i to j. We make the following assumption regarding the properties of $\boldsymbol{D} = (d_{ij})$:

Assumption 1 $\boldsymbol{D} = (d_{ij})$ *fulfills the following four conditions:*

[9] The geometric structure stated here is necessary only in Sections 4.1 and 5. For other places, it is sufficient to state "there are K cells where the area of cell i is δ_i and the interaction cost between cells i and j, to be defined below, is τd_{ij}."

(i) $d_{ii} = 0$ *for all* $i \in S$*;*

(ii) $d_{ij} = d_{ji}$ *for any* $i, j \in S$*;*

(iii) $\boldsymbol{D}$ *is negative definite on* $T\Delta \equiv \left\{\boldsymbol{z} = (z_1, \cdots, z_K) \in \mathbb{R}^K : \sum_{i=1}^K z_i = 0\right\}$*;*

In the terminology of spatial statistics, the first three conditions imply that d_{ij} is an isotropic *variogram*. This class of travel costs includes the exponential cost ($d_{ij} = e^{|x_i - x_j|} - 1$) and the linear cost ($d_{ij} = |x_i - x_j|$), both of which are commonly assumed in the literature of spatial interaction. From an economic point of view, condition (iii), which states that $\boldsymbol{D}$ is negative definite on the tangent space of Δ, can be interpreted as *self-defeating externalities*. Suppose that some players change their cells. Then, under this condition, the improvements in the interaction costs of cells to which they switch are dominated by the improvements in the interaction costs of cells they abandon.[10]

The utility function $u(z_i, y_i)$ is assumed to be quasi-linear:

$$u(z_i, y_i) = z_i + f(y_i), \tag{3}$$

where we make the following assumptions on the utility of land consumption f:

Assumption 2 f *is strictly increasing, concave, and twice continuously differentiable. Moreover,* $\lim_{x\to\infty} xf'(x) < \infty$*.*

If $f(x) = \alpha \ln x$ [resp. $f(x) = -\frac{\alpha}{2x}$] where $\alpha > 0$ is a constant, then we obtain the discrete-space analogue of Beckmann's (1976)[resp. Mossay and Picard's (2011)] model.

Given a population distribution, let us derive the maximum utilities attainable in each cell. Let population distribution $\boldsymbol{n} \in \Delta$ be given, and pick $i \in S$ such that $n_i > 0$. Then, consumers in cell i solve the utility maximization problem (1). Under the quasi-linear utility function specified in (3), the first-order condition with respect to y_i is

$$f'(y_i) \le r_i, \tag{4}$$

where the equality holds whenever $y_i > 0$. However, since the land market clears, we have $y_i = \delta_i / n_i > 0$. Hence, $r_i = f'(\delta_i / n_i)$. Then, we define

$$h(x) = f\left(x^{-1}\right) - x^{-1} f'\left(x^{-1}\right). \tag{5}$$

[10]Hofbauer et al. (2009) argued that the self-defeating externalities characterize *stable games*. In fact, if we consider a game in which the payoff of strategy $i \in S$ is $T_i(\boldsymbol{n})$, condition (iii) is the necessary and sufficient condition for the game to be a stable game.

This represents the net utility from the land consumption in cell i when $x = n_i/\delta_i$, which is the population density in cell i. Since $\lim_{x\to\infty} xf'(x) < \infty$, h is bounded below, but we allow $\lim_{x\to 0} h(x) = \infty$. Given $\boldsymbol{n} \in \Delta$, the maximum utility attained in cell i is then

$$v_i(\boldsymbol{n}) \equiv u_i\left(Y - T_i(\boldsymbol{n}) - \frac{\delta_i}{n_i}f'\left(\frac{\delta_i}{n_i}\right), \frac{\delta_i}{n_i}\right) = Y - T_i(\boldsymbol{n}) + h(n_i/\delta_i). \tag{6}$$

If $n_j = 0$ for some $j \in S$, then we assume that the utility attained is

$$v_j(\boldsymbol{n}) \equiv \lim_{\hat{n}\to 0}[Y - T_j(\boldsymbol{n}_{-j}, \hat{n}) + h\left(\hat{n}/\delta_j\right)], \tag{7}$$

where $(\boldsymbol{n}_{-j}, \hat{n}) = (n_1, ..., n_{j-1}, \hat{n}, n_{j+1}, ..., n_K)$. Note that we allow $v_j(\boldsymbol{n}) = \infty$ for $j \in S$ such that $n_j = 0$, which is actually the case in Beckmann's (1976) model.

2.2 Spatial equilibrium and potential game

We will now define the equilibrium. We consider a two-stage game in which each consumer first settles in a cell and chooses consumptions of the composite good and land in his/her cell. Since we impose the subgame perfection, we argue backwards to characterize equilibria. However, we have aleady specified each cell's utility levels given $\boldsymbol{n} \in \Delta$ (i.e., $\{v_i(\boldsymbol{n})\}_{i\in S}$). Then, in the first stage, each consumer chooses a cell that provides the highest utility, given the location decisions of other consumers. Formally, the equilibrium is defined as follows[11]:

Definition 1 A *spatial equilibrium* is a population distribution $\boldsymbol{n}^* \in \Delta$ such that given $\boldsymbol{n}^* \in \Delta$, no one has the incentive to change the cell. That is, there exists $u^* \in \mathbb{R}$ such that

$$\begin{cases} u^* = v_i(\boldsymbol{n}^*) & \text{if} \quad n_i^* > 0, \\ u^* \geq v_i(\boldsymbol{n}^*) & \text{if} \quad n_i^* = 0. \end{cases} \qquad \forall i \in S. \tag{8}$$

If $v_i(\boldsymbol{n}) = \infty$ when $n_i = 0$ for any $i \in S$, then only interior distribution can be an equilibrium. This is the case for Beckmann's (1976) model.

By writing the indirect utilities in a vector form, we obtain

$$\boldsymbol{v}(\boldsymbol{n}) \equiv (v_i(\boldsymbol{n}))_{i=1}^K = Y\mathbf{1} - \boldsymbol{T}(\boldsymbol{n}) + \boldsymbol{h}(\boldsymbol{n}) \tag{9}$$

where $\boldsymbol{T}(\boldsymbol{n}) = (T_i(\boldsymbol{n}))_{i=1}^K (= \tau\boldsymbol{D}\boldsymbol{n})$, $\boldsymbol{h}(\boldsymbol{n}) = (h(n_i/\delta_i))_{i=1}^K$, and $\mathbf{1}$ is a vector of ones with an appropriate dimension. People prefer to agglomerate to reduce the social interaction

[11] Strictly speaking, the definition of the equilibrium should state how people choose their allocations at the second stage. However, we make it implicit here because, in what follows, we focus on equilibrium population distributions.

costs, which are summarized by $\boldsymbol{T}(\boldsymbol{n})$. However, people also prefer to disperse and avoid the congestion from the land consumption that is summarized by $\boldsymbol{h}(\boldsymbol{n})$ since $h'(n_i/\delta_i) = (\delta_i/n_i)^3 f''(\delta_i/n_i) < 0$. As we will see, spatial equilibrium is attained as a result of tradeoffs between the agglomeration force, represented by $\boldsymbol{T}(\boldsymbol{n})$, and the dispersion force, represented by $\boldsymbol{h}(\boldsymbol{n})$.

In what follows, to characterize spatial equilibria and their stability, we invoke the properties of a *potential game* introduced by Monderer and Shapley (1996). Note that our model may be viewed as a game in which the set of players is $[0, 1]$, the (common) action set is S, and the payoff vector is $(v_i)_{i=1}^K$.[12] Moreover, as is evident from the definition, a spatial equilibrium is actually a Nash equilibrium of the game. Thus, let us denote our game by $G = (v_i)_{i=1}^K$. Then, we define that G is a potential game if $(v_i)_{i=1}^K$ allows for a continuously differentiable function W such that

$$\frac{\partial W(\boldsymbol{n})}{\partial n_i} - \frac{\partial W(\boldsymbol{n})}{\partial n_j} = v_i(\boldsymbol{n}) - v_j(\boldsymbol{n}) \quad \forall \boldsymbol{n} \in \Delta, \forall i, j \in S \tag{10}$$

where W is defined on an open set that contains Δ so that its partial derivative is well-defined on Δ. If the condition above holds, then W is referred to as a *potential function*.

For the moment, let us suppose that G is a potential game with the potential function W. As mentioned in the introduction, the equilibria of a potential game are characterized with the optimization problem of an associated potential function. Indeed, let us consider the following problem:

$$\max_{\boldsymbol{n} \in \Delta} W(\boldsymbol{n}). \tag{11}$$

Let γ be a Lagrange multiplier for the constraint $\sum_{i=1}^K n_i = 1$. Then, the first-order condition is $\frac{\partial W(\boldsymbol{n})}{\partial n_i} \leq \gamma$ in which the equality holds whenever $n_i > 0$. Then, by (10), we have $v_i(\boldsymbol{n}) = v_j(\boldsymbol{n})$ for any populated cells i and j, and $v_k(\boldsymbol{n}) \leq v_i(\boldsymbol{n})$ if $n_k = 0$ and $n_i > 0$. Thus, $\boldsymbol{n}$ is a spatial equilibrium. By similar reasoning, it follows that the converse is also true.[13] That is, if $\boldsymbol{n}$ is a spatial equilibrium, then it satisfies the necessary condition for problem (11). As a result, *the equilibrium set of G exactly coincides with the set of KKT points of problem* (11).

The necessary and sufficient condition for the existence of a potential function is given by, for example, Hofbauer and Sigmund (1988), who referred to the condition as *triangular integrability*. In our model, the agglomeration force $\boldsymbol{T}(\boldsymbol{n})$ is linear and the dispersion force

[12] A game with a continuum of anonymous players is called a *population game* (Sandholm, 2001). In our game, players are anonymous in that the payoff depends on only strategy distributions.

[13] See Proposition 3.1 of Sandholm (2001).

$\boldsymbol{h}(\boldsymbol{n})$ is separable. As a result, the condition is stated as

$$d_{ij} + d_{jk} + d_{ki} = d_{ik} + d_{kj} + d_{ji} \quad \text{for any } i, j, k \in S. \tag{12}$$

Recall that our travel costs are pairwise symmetric (i.e., $d_{ij} = d_{ji}$ for any $i, j \in S$). Hence, the above condition necessarily holds, and our game is, in fact, a potential game. Indeed, the following lemma explicitly constructs a potential function for $(v_i)_{i=1}^{K}$.

Lemma 1 *G is a potential game with the potential function*

$$W(\boldsymbol{n}) \equiv \tau W_1(\boldsymbol{n}) + W_2(\boldsymbol{n}) \tag{13}$$

where

$$W_1(\boldsymbol{n}) = -\oint \boldsymbol{T}(\boldsymbol{n}')\mathrm{d}\boldsymbol{n}' = -\frac{1}{2}\sum_{i=1}^{K}\sum_{j=1}^{K} d_{ij} n_i n_j, \tag{14}$$

$$W_2(\boldsymbol{n}) = \oint \boldsymbol{h}(\boldsymbol{n}')\mathrm{d}\boldsymbol{n}' = \sum_{i=1}^{K} n_i f\left(\frac{\delta_i}{n_i}\right). \tag{15}$$

Here, $\oint$ denotes the line integral over a path in Δ connecting $\mathbf{0}$ to $\boldsymbol{n}$. Since $d_{ij} = d_{ji}$ for any $i, j \in S$, it is guaranteed that the line integrals are path-independent.

Let us observe that in our potential game, we can capture the tradeoff between centrifugal and centripetal forces as the tradeoff between the concavity and convexity of the potential function. Indeed, W_2 is strictly concave since f_i's are strictly concave, whereas W_1 is quasi-convex since $\boldsymbol{D}$ is non-negative and negative definite on the tangent space of Δ.[14] Moreover, if the concavity of W_2 dominates so that W is strictly concave, then a dispersed population distribution (i.e., an interior point in Δ) is attained as a unique equilibrium. Conversely, if the convexity of W_1 dominates, then the equilibrium population distributions would be more agglomerated. Therefore, W_1 represents the centripetal force, whereas W_2 represents the centrifugal force.[15]

[14] See, for example, Theorem 4.4.6 of Bapat and Raghavan (1997).

[15] Blanchet et al. (2016) generalize the analysis of Mossay and Picard (2011) by taking the potential function(al) approach to characterize the equilibria of a continuous-space spatial interaction model. Takayama and Akamatsu (2010), Akamatsu et al. (2014), and this study examine the properties of discrete-space models by using the potential function, which is a discrete analogue of their potential functional.

3 Stability of equilibrium

3.1 Adjustment dynamics

We are interested in the stability of equilibria, especially since our model generally includes multiple equilibria, as shown in the next section. More specifically, we examine whether we can justify an equilibrium through the existence of a learning process that makes players settle down in their equilibrium strategies. In this study, we describe players' learning processes with *evolutionary dynamics*, or a (set-valued) dynamical system V that maps population distribution $\boldsymbol{n}^0 \in \Delta$ to a set of Lipschitz paths in Δ, which starts from $\boldsymbol{n}^0$.[16] Although we usually consider a specific evolutionary dynamics for stability analysis, we will see that a more general analysis is possible owing to the existence of a potential function; that is, the stability of equilibria can be characterized under a broad class of dynamics. In particular, we consider the class of *admissible* dynamics, which is defined as follows:

Definition 2 *An evolutionary dynamics V is* admissible *for $G = (v_i)_{i=1}^K$, if, for almost all $t \geq 0$ and for all $\boldsymbol{n}^0 \in \Delta$, it satisfies the following conditions:*

(PC) $\dot{\boldsymbol{n}}(t) \neq 0 \Rightarrow \dot{\boldsymbol{n}}(t) \cdot v(\boldsymbol{n}(t)) > 0$ *for all* $\boldsymbol{n}(\,\cdot\,) \in V(\boldsymbol{n}^0)$,

(NS) $\dot{\boldsymbol{n}}(t) = 0 \Rightarrow \boldsymbol{n}(t)$ *is a Nash equilibrium of G for all* $\boldsymbol{n}(\,\cdot\,) \in V(\boldsymbol{n}^0)$.

In order to interpret condition (PC), which is called *positive correlation*, we rewrite it as

$$\dot{\boldsymbol{n}}(t) \cdot v(\boldsymbol{n}(t)) = \sum_{i=1}^{K} \dot{n}_i(t) \left(v_i(\boldsymbol{n}(t)) - \frac{1}{K} \sum_{j=1}^{K} v_j(\boldsymbol{n}(t)) \right). \tag{16}$$

In general, it would be reasonable to expect that each term in the summation over i is positive: if the payoff from city i is higher than the average payoff (i.e., $v_i(\boldsymbol{n}(t)) - \frac{1}{K}\sum_{j=1}^{K} v_j(\boldsymbol{n}(t)) > 0$), t hen the mass of consumers choosing city i should increase (i.e., $\dot{n}_i(t) > 0$), and vice versa. Condition (PC) only requires that this be true in the aggregate. Thus, in learning periods, it is possible that the mass of consumers choosing city i increases even though it yields a less-than-average payoff. Condition (NS), which is called *Nash stationary*, states that if there is a profitable deviation, some consumers change their cells. Under condition (PC), the converse is also true.[17] Therefore, under conditions (PC) and (NS), $\dot{\boldsymbol{n}}(t) = 0$ if and only if $\boldsymbol{n}(t)$ is a Nash equilibrium of G.

Specific examples of admissible dynamics include *best response dynamics* (Gilboa and Matsui, 1991), *Brown-von Neumann-Nash (BNN) dynamics* (Brown and von Neumann,

[16] Considering a general dynamical system allows us to include set-valued dynamics such as the best-response dynamics which is important from the game-theoretic point of view.

[17] See Proposition 4.3 of Sandholm (2001).

1950), and *projection dynamics* (Dupuis and Nagurney, 1993).[18] One important remark is that *replicator dynamics* (Taylor and Jonker, 1978), which is often used in spatial economic models (e.g., Fujita et al., 1999), is *not* admissible. Under replicator dynamics, a rest point is always attained on the boundary, but the boundary points are not always Nash equilibria. Therefore, condition (NS) does not hold under replicator dynamics.[19]

3.2 Stability condition of equilibrium

Admissible dynamics are closely related to the potential function and thereby to the stability of Nash equilibria. Given such dynamics, we say that a population distribution $\boldsymbol{n} \in \Delta$ is *stable* if there exists a neighborhood $U \subseteq \Delta$ of $\boldsymbol{n}$ such that $\boldsymbol{n}(t) \to \boldsymbol{n}$ for any trajectory $\boldsymbol{n}(\,\cdot\,)$ of the dynamics with $\boldsymbol{n}(0) \in U$. In particular, if we can consider Δ for U, then $\boldsymbol{n}$ is *globally stable*. $\boldsymbol{n} \in \Delta$ is *unstable* if it is not stable.

To understand how admissible dynamics are related to potential function, let us consider our game $G = (v_i)_{i=1}^K$, with the potential function W given by (15). Note that, by conditions (PC) and (NS), any trajectory $\boldsymbol{n}(\cdot)$ of an admissible dynamic monotonically ascends the potential function until it reaches a Nash equilibrium since

$$\dot{W}(\boldsymbol{n}(t)) = \sum_{i=1}^{K} \frac{\partial W(\boldsymbol{n}(t))}{\partial n_i} \dot{n}_i(t) = \sum_{i=1}^{K} v_i(\boldsymbol{n}(t)) \dot{n}_i(t) > 0 \tag{17}$$

whenever $\dot{\boldsymbol{n}}(t) \neq 0$.[20] Hence, if Nash equilibrium $\boldsymbol{n}^*$ does not locally maximize W, then we can perturb $\boldsymbol{n}^*$ so that the trajectory ascends W and moves away from the equilibrium. In other words, assuming that each Nash equilibrium is isolated, *a Nash equilibrium is stable under any admissible dynamics if and only if it locally maximizes an associated potential function.*[21] Therefore, if a game has a potential function, we can characterize the stability of equilibria under admissible dynamics by examining the shape of the potential function.

3.3 Instability of population distributions

In light of the stability condition stated above, we start with investigating the relation between the interaction cost τ and the *instability* of spatial equilibria. Given a population

[18] See Sandholm (2005) for more examples.

[19] Any non-Nash rest point is never stable, where the stability is defined below, under the replicator dynamics, though (Sandholm, 2010, Proposition 8.1.1). The replicator dynamics belongs to the class of *strict myopic adjustment dynamics* due to Swinkels (1993) where Nash stationary is not imposed.

[20] Recall that $\dot{\boldsymbol{n}}(t) = 0$ if and only if $\boldsymbol{n}(t)$ is a Nash equilibrium.

[21] See Sandholm (2001) for a formal argument about this.

distribution, we obtain a sufficient condition under which the population distribution could not be stable even if it were a spatial equilibrium.

Given a population distribution $\boldsymbol{n} \in \Delta$, let $\operatorname{supp}(\boldsymbol{n})$ be the support of $\boldsymbol{n}$ (i.e., $\operatorname{supp}(\boldsymbol{n}) = \{i \in S : n_i > 0\}$). Let us denote the cardinality of a vector $\boldsymbol{x}$ by $|\boldsymbol{x}|$. Since a stable spatial equilibrium locally maximizes potential function W, we may investigate its Hessian while we have to respect the constraint $\boldsymbol{n} \in \Delta$. To this end, let us consider the projection of Δ onto $\mathbb{R}^{K-1}$, which is given by $\Pi\Delta = \{\boldsymbol{\nu} \in \mathbb{R}_+^{K-1} : \sum_{i=1}^{K-1} \nu_i \leq 1\}$, and represent the constraint $\boldsymbol{\nu} \in \Pi\Delta$ with the following inequalities:

$$q_i(\boldsymbol{\nu}) = -\nu_i \leq 0 \quad \text{for } i = 1, 2, ..., K-1, \tag{18}$$

$$q_K(\boldsymbol{\nu}) = \sum_{i=1}^{K-1} \nu_i - 1 \leq 0, \tag{19}$$

for $\boldsymbol{\nu} \in \mathbb{R}^{K-1}$. Then, the maximization problem of the potential function is written as

$$\max_{\boldsymbol{\nu} \in \mathbb{R}^{K-1}} W(-q(\boldsymbol{\nu})) \quad \text{s.t. } q_i(\boldsymbol{\nu}) \leq 0 \text{ for all } i \in S, \tag{20}$$

where $q = (q_i)_{i=1}^K$. Let $\gamma_i(\boldsymbol{\nu}) \geq 0$ be the Lagrange multiplier for the constraint $q_i(\boldsymbol{\nu}) \leq 0$. We assume that the problem (20) satisfies the *strict complementary* condition; that is, $q_i(\boldsymbol{\nu}) = 0 \Rightarrow \gamma_i(\boldsymbol{\nu}) > 0$ for all $i \in S$. In our context, this means that the spatial equilibrium is *quasi-strict* (i.e., the payoff of each unpopulated cell is strictly smaller than the (common) payoff of populated cells).

Assumption 3 *Every spatial equilibrium is quasi-strict.*

Let

$$Q(\boldsymbol{\nu}) = \left[\nabla q_i(\boldsymbol{\nu}) : i \notin \operatorname{supp}(-q(\boldsymbol{\nu}))\right], \tag{21}$$

where $\nabla q_i(\boldsymbol{\nu})$ is the gradient of $q_i(\boldsymbol{\nu})$ which is a $(K-1)$-dimensional vector. $Q(\boldsymbol{\nu})$ is the matrix comprising the gradients of constraints that are active at $\boldsymbol{\nu}$. Let us denote the matrix of the orthogonal basis of the null space of $Q(\boldsymbol{\nu})$ by $Z(\boldsymbol{\nu})$.

Since $q_i(\boldsymbol{\nu})$ is linear in $\boldsymbol{\nu}$ for all $i \in S$, $[\nabla q_i(\boldsymbol{\nu}) : i \in S] = \begin{pmatrix} -1 & 0 & \cdots & 0 \\ 0 & -1 & \cdots & 0 \\ & & \ddots & \\ 0 & 0 & \cdots & -1 \\ 1 & 1 & \cdots & 1 \end{pmatrix}$ is independent of $\boldsymbol{\nu} \in \Pi\Delta$. Hence, we denote it by Q. Let $\nabla^2 W(\boldsymbol{n})$ be the Hessian of $W(\boldsymbol{n})$ at $\boldsymbol{n} \in \Delta$. Then, it follows that, if $\boldsymbol{\nu} \in \Pi\Delta$ locally maximizes W,

$$H(\boldsymbol{\nu}) \equiv Z(\boldsymbol{\nu})' Q' \nabla^2 W(-q(\boldsymbol{\nu})) Q Z(\boldsymbol{\nu}), \tag{22}$$

is negative semi-definite.[22] This is the Hessian that needs to be examined.[23] In other words, $-q(\boldsymbol{\nu}) \in \Delta$ does not locally maximize W and, thus, is not a stable equilibrium if $H(\boldsymbol{\nu})$ is not negative semi-definite, which also indicates that the largest eigenvalue of $H(\boldsymbol{\nu})$ is positive.

Example 1 Let $K = 4$ and consider a population distribution $\boldsymbol{\nu} \in \Pi\Delta = \{\boldsymbol{\nu}' \in \mathbb{R}^3_+ : \sum_{i=1}^3 \nu_i' \leq 1\}$ such that $\nu_1 = 0, \nu_2 > 0, \nu_3 > 0$, and $\nu_2 + \nu_3 < 1$. Then, $Q(\boldsymbol{\nu}) = \begin{pmatrix} -1 & 0 & 0 \end{pmatrix}$. The null space of $Q(\boldsymbol{\nu})$ is two dimensional, and we can take $Z(\boldsymbol{\nu}) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}'$ as its orthogonal basis. Let $\boldsymbol{n} = -q(\boldsymbol{\nu})$. Then, since $QZ(\boldsymbol{\nu}) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ -1 & 0 \\ 0 & -1 \\ 1 & 1 \end{pmatrix}$,

$$
\begin{aligned}
H(\boldsymbol{\nu}) &= \begin{pmatrix} 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{pmatrix} J_{\boldsymbol{v}}(\boldsymbol{n}) \begin{pmatrix} 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{pmatrix}' & (23) \\
&= \begin{pmatrix} \frac{\partial(v_2(\boldsymbol{n})-v_4(\boldsymbol{n}))}{\partial n_2} - \frac{\partial(v_2(\boldsymbol{n})-v_4(\boldsymbol{n}))}{\partial n_4} & \frac{\partial(v_2(\boldsymbol{n})-v_4(\boldsymbol{n}))}{\partial n_3} - \frac{\partial(v_2(\boldsymbol{n})-v_4(\boldsymbol{n}))}{\partial n_4} \\ \frac{\partial(v_3(\boldsymbol{n})-v_4(\boldsymbol{n}))}{\partial n_2} - \frac{\partial(v_3(\boldsymbol{n})-v_4(\boldsymbol{n}))}{\partial n_4} & \frac{\partial(v_3(\boldsymbol{n})-v_4(\boldsymbol{n}))}{\partial n_3} - \frac{\partial(v_3(\boldsymbol{n})-v_4(\boldsymbol{n}))}{\partial n_4} \end{pmatrix} & (24) \\
&= \tau \left\{ \begin{pmatrix} d_{42} & d_{43} \\ d_{42} & d_{43} \end{pmatrix} + \begin{pmatrix} d_{24} & d_{24} \\ d_{34} & d_{34} \end{pmatrix} - \begin{pmatrix} d_{22} & d_{23} \\ d_{32} & d_{33} \end{pmatrix} \right\} \\
&\qquad + \begin{pmatrix} \delta_2^{-1} h'(n_2/\delta_2) + \delta_4^{-1} h'(n_4/\delta_4) & \delta_4^{-1} h'(n_4/\delta_4) \\ \delta_4^{-1} h'(n_4/\delta_4) & \delta_3^{-1} h'(n_3/\delta_3) + \delta_4^{-1} h'(n_4/\delta_4) \end{pmatrix}, & (25)
\end{aligned}
$$

where $J_{\boldsymbol{v}}(\boldsymbol{n})$ is the Jacobian of $\boldsymbol{v}$ at $\boldsymbol{n}$. ■

As shown in the example above, there is room for discretion regarding how to take $Z(\boldsymbol{\nu})$ even though it does not affect the stability. If $\operatorname{supp}(-q(\boldsymbol{\nu})) = \{i\}$ for some $i \in S$, then it follows that $H(\boldsymbol{\nu}) \propto \frac{\partial v_i(-q(\boldsymbol{\nu}))}{\partial n_i}$ whatever $Z(\boldsymbol{\nu})$ we take. Hence, $-q(\boldsymbol{\nu})$ cannot be a stable equilibrium if $\frac{\partial v_i(-q(\boldsymbol{\nu}))}{\partial n_i} > 0$. If more than one cell is populated, then room for discretion arises since only the payoff difference matters for the equilibrium. Hence, we may take any cell in the support as a "reference cell" from which the payoff difference is computed. In the following, we take the cell having the highest index in the support as the reference cell. More specifically, given $\boldsymbol{\nu} \in \Pi\Delta$, let $\boldsymbol{n} = -q(\boldsymbol{\nu})$ and $\operatorname{supp}(\boldsymbol{n}) = \{i_1, i_2, ..., i_L\}$. Then, let us take $Z(\boldsymbol{\nu})$ so that

$$H(\boldsymbol{\nu})_{k\ell} = \frac{\partial(v_{i_k}(\boldsymbol{n}) - v_{i_L}(\boldsymbol{n}))}{\partial n_{i_\ell}} - \frac{\partial(v_{i_k}(\boldsymbol{n}) - v_{i_L}(\boldsymbol{n}))}{\partial n_{i_L}}. \tag{26}$$

In matrix form, we have

$$H(\boldsymbol{\nu}) = \tau \tilde{D}_{\operatorname{supp}(\boldsymbol{n})} + \tilde{H}(\boldsymbol{n}), \tag{27}$$

[22] See the Appendix.

[23] $H(\boldsymbol{\nu})$ is called the *reduced* Hessian. See, for example, Griva et al. (2009).

where

$$\tilde{D}_{\mathrm{supp}\,(\boldsymbol{n})} = \mathbf{1} \otimes (d_{ii_L})_{i \in \mathrm{supp}\,(\boldsymbol{n}) \setminus \{i_L\}} + (d_{ii_L})'_{i \in \mathrm{supp}\,(\boldsymbol{n}) \setminus \{i_L\}} \otimes \mathbf{1}' - D_{\mathrm{supp}\,(\boldsymbol{n})} \tag{28}$$

$$= \begin{pmatrix} d_{i_1 i_L} & d_{i_1 i_L} & \cdots & d_{i_1 i_L} \\ d_{i_2 i_L} & d_{i_2 i_L} & \cdots & d_{i_2 i_L} \\ \vdots & \vdots & \ddots & \vdots \\ d_{i_{L-1} i_L} & d_{i_{L-1} i_L} & \cdots & d_{i_{L-1} i_L} \end{pmatrix} + \begin{pmatrix} d_{i_1 i_L} & d_{i_1 i_L} & \cdots & d_{i_1 i_L} \\ d_{i_2 i_L} & d_{i_2 i_L} & \cdots & d_{i_2 i_L} \\ \vdots & \vdots & \ddots & \vdots \\ d_{i_{L-1} i_L} & d_{i_{L-1} i_L} & \cdots & d_{i_{L-1} i_L} \end{pmatrix}'$$

$$- \begin{pmatrix} d_{i_1 i_1} & d_{i_1 i_2} & \cdots & d_{i_1 i_{L-1}} \\ d_{i_2 i_1} & d_{i_2 i_2} & \cdots & d_{i_2 i_{L-1}} \\ \vdots & \vdots & \ddots & \vdots \\ d_{i_{L-1} i_1} & d_{i_{L-1} i_2} & \cdots & d_{i_{L-1} i_{L-1}} \end{pmatrix}, \tag{29}$$

$$\tilde{H}(\boldsymbol{n}) = \frac{1}{\delta_{i_L}} h'(\tfrac{n_{i_L}}{\delta_{i_L}}) \mathbf{1}\mathbf{1}' + \mathrm{diag}\left(\frac{1}{\delta_{i_1}} h'(\tfrac{n_{i_1}}{\delta_{i_1}}),\ \frac{1}{\delta_{i_2}} h'(\tfrac{n_{i_2}}{\delta_{i_2}}),\ \cdots,\ \frac{1}{\delta_{i_{L-1}}} h'(\tfrac{n_{i_{L-1}}}{\delta_{i_{L-1}}}) \right). \tag{30}$$

In (28), $D_{\mathrm{supp}\,(\boldsymbol{n})}$ is the submatrix of D that corresponds to the indices in $\mathrm{supp}\,(\boldsymbol{n})$ and $\otimes$ denotes the Kronecker product, whereas, in (30), $\mathrm{diag}(\boldsymbol{x})$ is the diagonal matrix having $\boldsymbol{x}$ as its diagonal elements.

To attain a threshold value of τ above which the largest eigenvalue of $H(\boldsymbol{\nu})$ is positive, we invoke Weyl's inequality, which states that

$$\mu_{\max}(H(\boldsymbol{\nu})) \equiv \mu_{L-1}(H(\boldsymbol{\nu})) \geq \tau \mu_{L-j}(\tilde{D}_{\mathrm{supp}\,(\boldsymbol{n})}) + \mu_j(\tilde{H}(\boldsymbol{n})) \tag{31}$$

for $2 \leq j \leq L-1$, where $\mu_i(M)$ is the i-th smallest eigenvalue of matrix M.[24] Although we made some adjustments to account for feasibility constraints, we can see that $\tilde{D}_{\mathrm{supp}\,(\boldsymbol{n})}$ corresponds to the agglomeration force $W_1(\boldsymbol{n})$, whereas $\tilde{H}(\boldsymbol{n})$ corresponds to the dispersion force $W_2(\boldsymbol{n})$. Indeed, as shown in the proof of Proposition 1 below, $\tilde{D}_{\mathrm{supp}\,(\boldsymbol{n})}$ is positive definite and all its eigenvalues are positive. Thus, $\tilde{D}_{\mathrm{supp}\,(\boldsymbol{n})}$ acts as the destabilizing force against interior distributions. Conversely, since h is decreasing, all of $\tilde{H}(\boldsymbol{n})$'s eigenvalues, except for one zero eigenvalue, are negative. Hence, $\tilde{H}(\boldsymbol{n})$ acts as the stabilizing force. Furthermore, the threshold value is attained when these two forces are balanced:

Proposition 1 *Under Assumptions 1-3, a population distribution* $\boldsymbol{n} \in \Delta$ *such that* $\mathrm{supp}\,(\boldsymbol{n}) =$

[24] Weyl's inequality states that $\mu_p(B+C) \leq \mu_{p+q}(B) + \mu_{n-q}(C)$ for $q \in \{0, 1, 2, ..., n-p\}$ and $\mu_p(B+C) \geq \mu_{p-q+1}(B) + \mu_q(C)$ for $q \in \{1, 2, ..., p\}$ where B and C are $n \times n$ symmetric matrices. See Theorem 4.3.1 and Corollary 4.3.3 of Horn and Johnson (2013).

$\{i_1, i_2, ..., i_L\}$ *where* $L \geq 2$ *cannot be a stable spatial equilibrium if*

$$\tau > \min_{2 \leq j \leq L-1} \frac{\mu_{j-1}(\mathrm{diag}[(\delta_i^{-1}|h'(n_i/\delta_i)|)_{i \in \{i_1, i_2, ..., i_{L-1}\}}])}{\mu_{L-j}(\tilde{D}_{\mathrm{supp}\,(\boldsymbol{n})})}.$$

Proof. See the Appendix. ■

As an illustrating example, let us consider a discrete-analogue of Mossay and Picard's (2011) model in which $d_{ij} = |x_i - x_j|$, $f(x) = -\frac{\alpha}{2x}$, and $\delta_i = 1/K$. Observe that the uniform discretization, where the same amount of land is allocated to each cell, necessarily implies $\delta_i = 1/K$ for all $i \in S$ since the total area of the region is normalized to one.

Assumption 4 (Uniform discretization) $\delta_i = 1/K$ *for all* $i \in S$*.*

Let $\mathrm{supp}\,(\boldsymbol{n}) = \{i_1, i_2, ..., i_L\}$. The specifications above imply $\mu_j(\mathrm{diag}[(\delta_i^{-1}|h'(n_i/\delta_i)|)_{i \in \{i_1, i_2, ..., i_{L-1}\}}]) = \alpha K$ for any $j \in \{1, 2, ..., L-1\}$. We aim to derive explicit expressions for the threshold values of τ.

To this end, we exploit the fact that if the interaction cost is linear, then the support of a spatial equilibrium can be considered to be a downsized replica of the entire region. More specifically, any populated cells in a spatial equilibrium are congregated (i.e., there is no vacant cell between any populated cells), as shown in the following lemma[25]:

Lemma 2 *Suppose Assumption 2 and* $d_{ij} = |x_i - x_j|$*. Then, the support of spatial equilibrium is given by* $\{i_1, i_2 ..., i_L\} \subseteq S$*, where* $i_{k+1} = i_k + 1$ *for any* $k = 1, ..., L-1$*.*

Proof. See the Appendix. ■

Let $\boldsymbol{n} \in \Delta$ be a spatial equilibrium such that $\mathrm{supp}\,(\boldsymbol{n}) = \{i_1, i_2, ..., i_L\}$, where $L \geq 2$. By Lemma 2, $\tilde{D}_{\mathrm{supp}\,(\boldsymbol{n})}$ is written as

$$\tilde{D}_{\mathrm{supp}\,(\boldsymbol{n})} = \frac{2}{K} \begin{pmatrix} L-1 & L-2 & L-3 & \cdots & 1 \\ L-2 & L-2 & L-3 & \cdots & 1 \\ L-3 & L-3 & L-3 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 1 \end{pmatrix}. \tag{32}$$

Note that if two equilbria have the same number of populated cells, then the Hessian is the same in the model. This will be a key observation when addressing the multiplicity of equilibria in the following section. When deriving eigenvalues of $\tilde{D}_{\mathrm{supp}\,(\boldsymbol{n})}$, it turns out to be

[25] Mossay and Picard (2011) invoke an analogue observation for their continuous space model.

more convenient to examine its inverse, which is given by

$$\tilde{D}^{-1}_{\mathrm{supp}\,(\boldsymbol{n})} = \frac{K}{2}\begin{pmatrix} 1 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & -1 & 2 & -1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & -1 & 2 & -1 \\ & & & & -1 & 2 \end{pmatrix}. \tag{33}$$

This is an $(L-1)\times(L-1)$-dimensional tridiagonal Toeplitz matrix in which the upper-left corner is perturbed and for which explicit expressions of eigenvalues are known. In particular, we have $\mu_j(\tilde{D}^{-1}_{\mathrm{supp}\,(\boldsymbol{n})}) = K\left(1-\cos\frac{(2j-1)\pi}{2L-1}\right)$, and hence $\mu_j(\tilde{D}_{\mathrm{supp}\,(\boldsymbol{n})}) = \frac{1}{K}\left(1-\cos\frac{(2(L-j)-1)\pi}{2L-1}\right)^{-1}$. Note that these eigenvalues depend on the number of populated cells but not on the distribution over the support of equilibrium. Hence, the following result is obtained:

Corollary 1 *Suppose Assumptions 3-4, $d_{ij} = |x_i - x_j|$, and $f(x) = -\frac{\alpha}{2x}$. Then, a population distribution $\boldsymbol{n} \in \Delta$ having $L\,(\geq 2)$ populated cells cannot be a stable spatial equilibrium if*

$$\tau > \tau^l(L) \equiv \left(1-\cos\frac{3\pi}{2L-1}\right)\alpha K^2. \tag{34}$$

There are two remarks here. First, since $\tau^l(2) > \tau^l(3) > \tau^l(4) > \cdots$, *the maximum possible number of populated cells that might constitute a stable spatial equilibrium is non-increasing in τ.* Second, since $K \mapsto \tau^l(K)$ is continuous on $\mathbb{R}_+$ and $\lim_{K\to\infty}\tau^l(K) = \frac{9}{8}\alpha\pi^2 < \infty$, it is bounded. Thus, *if τ is sufficiently large, then a population distribution with full support cannot be a stable spatial equilibrium for any finite K.*

4 Characterization of equilibria

This section characterizes the equilibrium of the discrete-space model to compare its properties with those of the continuous-space model. In the continuous-space model, it follows that a unimodal population distribution is attained as the unique spatial equilibrium up to translation (Mossay and Picard, 2011).[26] Section 4.1 shows that, as in the continuous-space model, the equilibrium population density of the discrete-space model is unimodal. However, in Section 4.2, we see that the equilibrium is essentially non-unique in the sense that equilibria with different population distributions over the support coexist.

[26]In the next section, we will show that the spatial equilibrium of a general class of continuous-space model is unimodal (Lemma 5)

4.1 Equilibrium population distribution

Suppose that the interaction cost is linear (i.e., $d_{ij} = |x_i - x_j|$). Then, by Lemma 2, populated cells in a spatial equilibrium are congregated, i.e., $\mathrm{supp}\,(\boldsymbol{n}) = \{i_-, i_- + 1, ..., i_+ - 1, i_+\}$ for some $i_-, i_+ \in S$. Then, since $v_i(\boldsymbol{n}) = v_{i-1}(\boldsymbol{n})$ for all $i \in \mathrm{supp}\,(\boldsymbol{n})\backslash\{i_-\}$, the equilibrium condition is written as

$$h\left(\frac{n_i}{\delta_i}\right) - h\left(\frac{n_{i-1}}{\delta_{i-1}}\right) = \tau\epsilon_i \left\{2\sum_{k=i_-}^{i-1} n_k - 1\right\} \qquad \forall i \in \mathrm{supp}\,(\boldsymbol{n})\backslash\{i_-\}, \tag{35}$$

where $\epsilon_i = x_i - x_{i-1} > 0$. Since $\sum_{k\in\mathrm{supp}\,(\boldsymbol{n})} n_k = 1$, (35) implies that there exist $i^* \in S$ such that $h(n_i/\delta_i)$ is decreasing [resp. increasing] in i for $i \leq i^*$ [resp. $i \geq i^*$], as long as $\sum_{k=i_-}^{i-1} n_k \neq \frac{1}{2}$ for any $i \in \mathrm{supp}\,(\boldsymbol{n}) \setminus \{i_-\}$. If $\sum_{k=i_-}^{i-1} n_k = \frac{1}{2}$ for some $i \in \mathrm{supp}\,(\boldsymbol{n}) \setminus \{i_-\}$, then there are two cells at which h attains its bottom. In any case, $\{-h(n_i/\delta_i)\}_{i\in S}$ is unimodal. Since h is strictly decreasing, this is also true for $(n_i/\delta_i)_{i\in S}$.

Proposition 2 *Suppose Assumption 2 and $d_{ij} = |x_i - x_j|$. Then, the equilibrium population density distribution $(n_i/\delta_i)_{i\in S}$ of the discrete-space model is unimodal.*

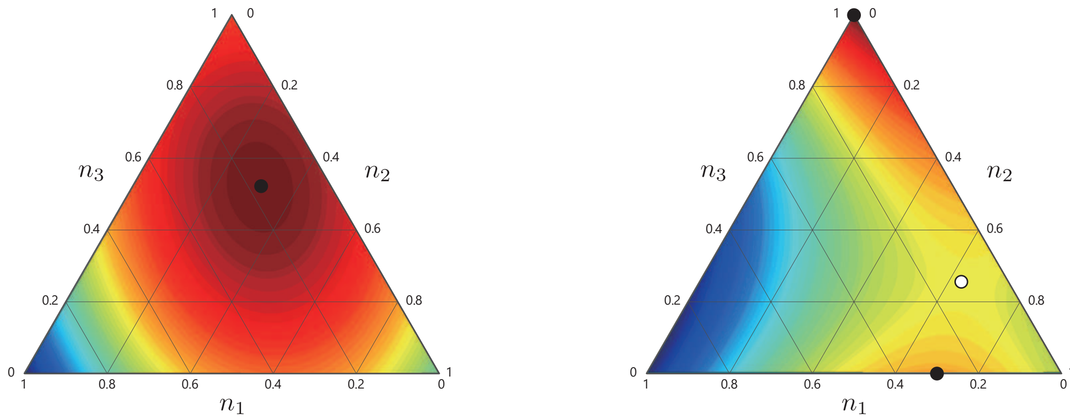
This proposition shows that the equilibrium population density distribution $(n_i/\delta_i)_{i\in S}$ of the discrete-space model exhibits a property similar to that of the continuous-space model. Furthermore, the equilibrium population distribution $\boldsymbol{n}$ is also unimodal, especially if we consider the uniform discretization of space (i.e., $\delta_i = \frac{1}{K}$ for all $i \in S$).

4.2 Multiplicity of spatial equilibria

4.2.1 Non-uniform discretizations

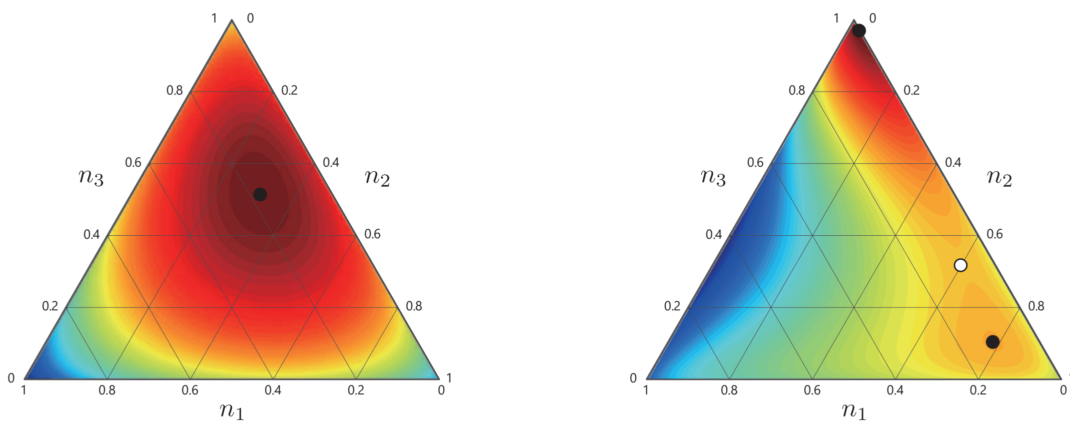
In this subsection, we examine the uniqueness of equilibrium in discrete space. We first consider the case of the non-uniform discretization of space (i.e., there exists $i, j \in S$ such that $\delta_i \neq \delta_j$).[27] We show through examples that different population distributions can be KKT points of problem (11) (and thus spatial equilibria) in this case. We consider two models: Beckmann's (1976) model in which $f(x) = \alpha \ln x$ and Mossay and Picard's (2011) model in which $f(x) = -\frac{\alpha}{2x}$. In either model, we consider the linear interaction cost in which $d_{ij} = |x_i - x_j|$. We assume that $K = 3$, $(b_1, b_2) = (0.2, 0.5)$, and $\alpha = 1.0$. Under these parameters, Figures 1 and 2 depict the contour plots of each model's potential function,

[27] Note that when we perform an empirical analysis, we often need to discretize a space non-uniformly since social and economic data is aggregated over some geographical areas and, in general, these areas are not uniformly sized.



(a) $\tau = 1.0$ (b) $\tau = 8.0$

Figure 1: Contour plot of the potential function of Mossay and Picard's (2011) model (•: stable, ∘: unstable)



(a) $\tau = 1.0$ (b) $\tau = 8.0$

Figure 2: Contour plot of the potential function of Beckmann's (1976) model (•: stable, ∘: unstable)

respectively. In these figures, the background color represents the value of potential function: the regions in which the value is the largest are red, while the regions in which the value is the smallest are blue. To characterize equilibria with these figures, we invoke the fact that local maximizers of potential function are stable equilibria, whereas any other KKT points are unstable equilibria.

According to Figure 1, we can see that when $\tau = 1.0$, the potential function is strictly concave, and thus, there exists a unique equilibrium that is stable. However, when $\tau = 8.0$, the potential function fails to concave, and three equilibria arise: 1) full agglomeration in cell 3, the largest cell; 2) the population is agglomerated in cells 1 and 2, the two smaller cells; and 3) full support in which all cells are populated. As shown in Figure 2, Beckmann's (1976) model has qualitatively similar properties to those of Mossay and Picard's (2011)

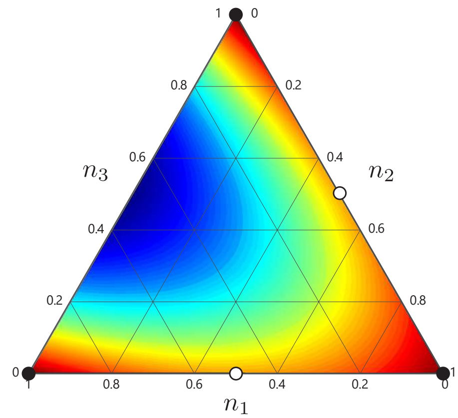


Figure 3: Contour plot of the potential function of uniformly discretized Mossay and Picard's (2011) model (•: stable equilibrium, ∘: unstable equilibrium)

model. These results show that if we consider a non-uniform discretization, then equilibria with different population distributions can coexist.

4.2.2 Uniform discretizations

Based on the results above, one might think that they stem from the exogenous asymmetry in space. Hence, we next consider uniform discretizations. In this case, the label of the cell should not matter when discussing the multiplicity of equilibria. That is, we do not distinguish between two equilibria such that one equilibrium is obtained by horizontally shifting the other one, which can arise in this case. For example, let us look at two unstable equilibria in Figure 3, which depicts a contour plot of the potential function of Mossay and Picard's (2011) model with $K = 3$, $[b_1, b_2] = [\frac{1}{3}, \frac{2}{3}]$, $\alpha = 1.0$, and $\tau = 15.0$. One equilibrium is $(\frac{1}{2}, \frac{1}{2}, 0)$ whereas the other equilibrium is $(0, \frac{1}{2}, \frac{1}{2})$. However the two population distributions can be merged through translation.

In what follows, we assume that the equilibrium, if any, is unique for each possible support.

Assumption 5 *The number of equilibria is, at most, one for each possible support.*

This is true for Mossay and Picard's (2011) model in which the equilibrium solves the system of linear equations. In this case, since the interaction cost is symmetric, we can regard any two equilibria as qualitatively identical in the above sense whenever they have the same number of populated cells. In other words, two equilibria are indistinguishable up to translation unless they have different numbers of populated cells. Therefore, under Assumption 5, we say that the spatial equilibrium is *essentially non-unique* if equilibria with different numbers of populated cells simultaneously exist, and we focus on this essential multiplicity.

We show that the equilibrium can be essentially non-unique, even if the space is uniformly discretized. To this end, we refer to the index theorem of Simsek et al. (2007), which is applicable to the set of KKT points (See Proposition 5.2 of the paper). This is relevant for us since our equilibrium problem is reduced to finding KKT points of the optimization problem of the potential function. In our context, their result is stated as follows.

Recall that the problem (20) is the optimization problem that characterizes spatial equilibria. Denote the map $\boldsymbol{\nu} \mapsto W(-q(\boldsymbol{\nu}))$ by $W \circ (-q)$. Given the problem (20), let

$$\Gamma(\boldsymbol{\nu}) = Z(\boldsymbol{\nu})' \left(H_{W\circ(-q)}(\boldsymbol{\nu}) + \sum_{i \notin \mathrm{supp}\,(-q(\boldsymbol{\nu}))} \gamma_i(\boldsymbol{\nu}) H_{q_i}(\boldsymbol{\nu}) \right) Z(\boldsymbol{\nu}), \tag{36}$$

where $H_{W\circ(-q)}(\boldsymbol{\nu})$ [resp. $H_{q_i}(\boldsymbol{\nu})$] is the Hessian of $W \circ (-q)$ [resp. q_i] at $\boldsymbol{\nu}$. Let $\mathrm{KKT}(W, q)$ be the set of KKT points of the problem (20). Then, for each $\boldsymbol{\nu} \in \mathrm{KKT}(W, q)$, we define the index by

$$\mathrm{ind}_\Gamma(\boldsymbol{\nu}) = \begin{cases} 1 & \text{if} \quad \det(\Gamma(\boldsymbol{\nu})) > 0, \\ 0 & \text{if} \quad \det(\Gamma(\boldsymbol{\nu})) = 0, \\ -1 & \text{if} \quad \det(\Gamma(\boldsymbol{\nu})) < 0, \end{cases} \tag{37}$$

where $\det(\Gamma(\boldsymbol{\nu}))$ is the determinant of $\Gamma(\boldsymbol{\nu})$. Then, under the assumptions mentioned later, the index theorem states that

$$\sum_{\boldsymbol{\nu} \in \mathrm{KKT}(W,q)} \mathrm{ind}_\Gamma(\boldsymbol{\nu}) = 1. \tag{38}$$

Note that all of the constraints $(q_i)_{i=1}^K$ are linear. Hence, H_{q_i} is the zero matrix for any $i \in S$. Then, since $H_{W\circ(-q)}(\boldsymbol{\nu}) = Q'\nabla^2 W(-q(\boldsymbol{\nu}))Q$, it turns out that $\Gamma(\boldsymbol{\nu}) = H(\boldsymbol{\nu})$. Observe that this is exactly the Hessian that we have used for the stability analysis. This illustrates how useful the potential function is, not only for the stability analysis, but also for analysis of the multiplicity of equilibria.

The index theorem holds under the following three assumptions. The first one is that $W \circ (-q)$ is twice continuously differentiable, which holds under Assumption 2. The second one is that the problem (20) satisfies the strict complementary condition, which holds under Assumption 3. The final one is that $\Gamma(\boldsymbol{\nu})$ is non-singular at any $\boldsymbol{\nu} \in \mathrm{KKT}(W, q)$. Since $\Gamma(\boldsymbol{\nu}) = H(\boldsymbol{\nu})$ and the KKT points of the problem (20) correspond to spatial equilibria, the following assumption is necessary:

Assumption 6 *For any $\boldsymbol{\nu} \in \Pi\Delta$ such that $-q(\boldsymbol{\nu}) \in \Delta$ is a spatial equilibrium, $H(\boldsymbol{\nu})$ is nonsingular.*

In order to apply the index theorem to the issue of the multiplicity of equilibria, we assume the linear interaction cost in order to invoke Lemma 2. Lemma 2, together with Assumption 5, enables us to determine the number of equilibria for each number of populated cells. More specifically, if an equilibrium with L populated cells exists, then there are $K - L + 1$ such equilibria. Moreover, since these equilibria have the same distribution over the support (i.e., these equilibria are essentially indistinguishable), all their indices have the same value. These observations lead us to the following result:

Lemma 3 *Suppose Assumptions 2-6 and* $d_{ij} = |x_i - x_j|$*. If there is a spatial equilibrium* $\boldsymbol{n}$ *such that* $|\mathrm{supp}\,(\boldsymbol{n})| < K$*, then there is another spatial equilibrium* $\boldsymbol{n}'$ *such that* $|\mathrm{supp}\,(\boldsymbol{n}')| \neq |\mathrm{supp}\,(\boldsymbol{n})|$.

Proof. Suppose, to the contrary, that for some $L < K$, $|\mathrm{supp}\,(\boldsymbol{n})| = L$ for any equilibrium $\boldsymbol{n}$. By Lemma 2, the number of equilibria is then $K-L+1$. Since the population distribution over the support is identical for all equilibria, the index given by (37) is the same for all equilibria. Hence, the total sum of the indices is either $K - L + 1, -(K - L + 1)$, or 0. Any of them contradicts the index theorem. ■

Thus, if a spatial equilibrium having some unpopulated cells exists, then there is necessarily another spatial equilibrium that is essentially different from the equilibrium. Therefore, if the spatial equilibrium exists uniquely, then all the cells in the region must be populated in the unique equilibrium.

Recall that, given a full-support distribution, we have obtained a sufficient condition under which it is not an equilibrium or it is an unstable equilibrium (Proposition 1). Observe that, since the potential function is continuous and Δ is compact, a stable equilibrium, which is a maximizer of the potential function, exists. Hence, if a full-support distribution is not an equilibrium, then Lemma 3 immediately implies the essential multiplicity of equilibria. Furthermore, if a full-support distribution is an equilibrium but unstable, a maximizer of the potential function, which is a stable equilibrium, does not have the full support. Thus, even if a full-support distribution is an equilibrium, Lemma 3 applies as long as it is not stable. Therefore, by Proposition 1, we conclude the following result.

Proposition 3 *Suppose Assumptions 2-6 and* $d_{ij} = |x_i - x_j|$*. If*

$$\tau > \sup_{\{\boldsymbol{n}\in\Delta:\mathrm{supp}\,(\boldsymbol{n})=S\}} \min_{2\leq j\leq K-1} \frac{\mu_{j-1}(\mathrm{diag}[(\delta_i^{-1}|h'(n_i/\delta_i)|)_{i\in S}])}{\mu_{K-j}(\tilde{D}_S)},$$

then the equilibrium is essentially non-unique.

For a corollary of the above result, let us consider Mossay and Picard's (2011) model in which $f(x) = -\frac{\alpha}{2x}$. Then, the threshold value of τ is independent of the population distribution, and the equilibrium is essentially non-unique when $\tau > \tau^l(K)$, where τ^l is given by (34). Moreover, since $\max_{K\in\mathbb{N}} \tau^l(K)$ exists, the spatial equilibrium is essentially non-unique for *any* finite K if τ is larger than the maximal value.

Overall, the (essential) multiplicity of equilibria in a discrete space suggests that implications derived from (theoretical/empirical) analyses focusing on a particular equilibrium might be questionable since its equilibrium might be unstable and another stable equilibrium with an essentially different population distribution can exist.

5 The limit of discrete-space models

5.1 Continuous-space models

This section examines the continuous limit of discrete-space model. However, we begin by presenting the structure of a continuous-space model and consider the real line for the region, even though we subsequently focus on the unit interval, as in the discrete-space model. Let $\lambda \in \{\lambda \in L^1(\mathbb{R}) : \lambda \geq 0, \int \lambda(x)\mathrm{d}x = 1\}$ be an (integrable) population density over $\mathbb{R}$. In the continuous-space model, the indirect utility at location x_i is given by

$$v_\lambda(x_i) \equiv Y - \tau \int |x - x_i|\lambda(x)\mathrm{d}x + h(\lambda(x_i)), \tag{39}$$

where we focus on the linear interaction cost. Let $\mathrm{supp}\,(\lambda) \subseteq \mathbb{R}$ be the support of population density λ. For a continuous-space model, the spatial equilibrium is defined in relation to the population density. That is, λ^* is a spatial equilibrium if there exists $u^* \in \mathbb{R}$ such that $v_{\lambda^*}(x) = u^*$ for any $x \in \mathrm{supp}\,(\lambda^*)$ whereas $v_{\lambda^*}(x) \leq u^*$ for any $x \notin \mathrm{supp}\,(\lambda^*)$. In addition, we make the following assumption on the support of the spatial equilibrium.

Assumption 7 $\mathrm{supp}\,(\lambda^*)$ *is finite for every spatial equilibrium* λ^**.*

For example, this is the case for Mossay and Picard's (2011) model.

In a discrete-space model, we have seen that the population distribution is congregated at any spatial equilibrium when the interaction cost is linear (Lemma 2). Naturally, this also holds for the continuous-space model:

Lemma 4 *Suppose Assumptions 2 and 7, and the linear interaction cost. Then, the support of the spatial equilibrium of the continuous-space model is a finite open interval.*

Proof. See the Appendix. ■

In Section 4.1, we represented the equilibrium condition as a difference equation in order to characterize equilibria of the discrete-space model. Likewise, we represent the equilibrium condition as a differential equation to characterize equilibria of the continuous-space model. To this end, let $\mathcal{H}(x) = h(\lambda(x))$ be the net utility from the land consumption at location x. As it turns out, looking at $\mathcal{H}$, instead of directly looking at λ, is more convenient for characterizing equilibria. In fact, at the equilibrium, $\mathcal{H}(x)$ satisfies the following condition:[28]

Lemma 5 *Suppose Assumptions 2 and 7, and the linear interaction cost. Then, $\mathcal{H}(x)$ is the equilibrium net utility from the land consumption of the continuous-space model if and only if it solves the following equations:*

$$\frac{\mathrm{d}\mathcal{H}(x)}{\mathrm{d}x} = \tau \left\{ 2 \int_{x_-}^{x} g(\mathcal{H}(z))\mathrm{d}z - 1 \right\} \qquad \forall x \in [x_-, x_+], \tag{40a}$$

$$\int_{x_-}^{x_+} g(\mathcal{H}(x))\mathrm{d}x = 1, \quad g(\mathcal{H}(x)) \geq 0 \qquad \forall x \in [x_-, x_+], \tag{40b}$$

$$\mathcal{H}(x_-) = \mathcal{H}(x_+) = h(0), \tag{40c}$$

for some $x_-, x_+ \in \mathbb{R}$ with $x_- \leq x_+$, where g is the inverse function of h.

Proof. See the Appendix. ■

(40a) and (40b) imply that there exists $x^0 \in [x_-, x_+]$ such that $\frac{\mathrm{d}\mathcal{H}(x)}{\mathrm{d}x} < 0$ for $x \in [x_-, x^0)$ and $\frac{\mathrm{d}\mathcal{H}(x)}{\mathrm{d}x} > 0$ for $x \in (x^0, x_+]$. That is, $\mathcal{H}(x)$ is U-shaped. This shows that as in the discrete-space model (Proposition 2), *the spatial equilibrium $\lambda(x) = h^{-1}(\mathcal{H}(x))$ of the continuous-space model is unimodal.*

5.2 Continuous limit of discrete-space models

The continuous-space model has been rigorously studied by Blanchet et al. (2016), and we can invoke some of their results if we additionally assume that $\lim_{x \to 0} h(x) = 0$.[29] In particular, since the utility of land consumption f is strictly concave and increasing, and the interaction cost is symmetric and linear, *the spatial equilibrium of the continuous-space model is essentially unique* by Theorem 3 in Blanchet et al. (2016). This is in sharp contrast to the results of the discrete-space model obtained thus far. Nevertheless, we show in this

[28] Lemma 5 implies that for a given x_- (or x_+), the equilibrium net utility from the land consumption and x_+ (or x_-) of the continuous-space model is obtained by solving (40).

[29] In this case, Beckmann's (1976) model is excluded.

subsection that the continuous-space model can be viewed as the continuous limit of the discrete-space model.

For discrete-space models, we consider a finite region, whereas the region of continuous-space models is infinite. One might worry that the boundaries of this region will affect the analysis. However, the following lemma shows that, as long as we consider a region that is larger than the length of the continuous-space model's equilibrium support, a full-support distribution cannot be an equilibrium of the discrete-space model when K is large. Observe that, since the spatial equilibrium is unique up to translation in the continuous-space model, the length of the continuous-space model's equilibrium support is uniquely determined.

Lemma 6 *Suppose Assumption 2 and* $d_{ij} = |x_i - x_j|$. *Consider a discrete-space model for which the region is given by* $[0, L]$ *where* L *is larger than the length of the equilibrium support of the continuous-space model over* $\mathbb{R}$. *Then, a full-support distribution cannot be an equilibrium for sufficiently large* K*s.*

Proof. See the Appendix. ■

Hence, as long as the region can contain the equilibrium of the continuous-space model, we do not have to worry about the exogenous boundaries. Therefore, without loss of generality, we assume that the length of the continuous-space model's equilibrium support is less than one so that we can keep the original setting in which the region is given by the unit interval.

Now let us consider a sequence $\{\boldsymbol{n}^K\}$ of equilibria of discrete-space models, where $\boldsymbol{n}^K$ is an equilibrium of the discrete-space model with K cells. Given $\{\boldsymbol{n}^K\}$, let $\mathcal{H}_i^K = h(\lambda_i^K)$, where $\lambda_i^K = n_i^K/\delta_i$, denote the equilibrium net utility from the land consumption at cell i in the discrete-space model with K cells. By Lemma 2, the support of $\boldsymbol{n}^K$ is represented by $\{i_-^K, i_-^K+1, ..., i_+^K-1, i_+^K\}$ for some i_-^K and i_+^K where $1 \leq i_-^K \leq i_+^K \leq K$. Then, it follows from (35) that, for each K, $\mathcal{H}^K = \{\mathcal{H}_i^K\}_{i \in \text{supp}\,(\boldsymbol{n}^K)}$ solves the following equations:

$$\frac{\mathcal{H}_i^K - \mathcal{H}_{i-1}^K}{\epsilon_i} = \tau \left\{ 2 \sum_{j=i_-^K}^{i-1} g(\mathcal{H}_j^K)\delta_j - 1 \right\} \qquad \forall i \in \{i_-^K, i_-^K+1, ..., i_+^K-1, i_+^K\}, \tag{41a}$$

$$\sum_{j=i_-^K}^{i_+^K} g(\mathcal{H}_j^K)\delta_k = 1, \; g(\mathcal{H}_i^K) \geq 0 \qquad \forall i \in \{i_-^K, i_-^K+1, ..., i_+^K-1, i_+^K\}, \tag{41b}$$

which converge to (40a) and (40b) respectively as $K \to \infty$. Let $x_- = \lim_{K\to\infty} x_{i_-^K}$ and $x_+ = \lim_{K\to\infty} x_{i_+^K}$. Since $x_{i_-^K}, x_{i_+^K} \in (0,1)$ for all K, $x_- \geq 0$ and $x_+ \leq 1$. Furthermore, since the length of the continuous-space model's equilibrium support is unique and less than one, we have either $x_- > 0, x_+ < 1$, or both by Lemma 6. Without loss of generality, suppose

$x_+ < 1$. Then, for sufficiently large K, $x_{i_+^K+1} \leq 1$ for which $v_{i_+^K} \geq v_{i_+^K+1}$ and $\lambda^K_{i_+^K+1} = 0$. Hence,

$$h(0) - h(\lambda(x_{i_+^K})) = \mathcal{H}^K_{i_+^K+1} - \mathcal{H}^K_{i_+^K} \leq \tau\epsilon_{i_+^K+1} \Rightarrow h(0) - h(\lambda(x_+)) \leq 0. \tag{42}$$

Since h is decreasing, $\lambda(x_+) = 0$. The symmetry of the equilibrium then implies $\lambda(x_-) = 0$. Thus, the equilibrium condition of the discrete-space model eventually coincides with that of the continuous-space model, as $K \to \infty$.

Unfortunately, the finite difference method, which studies the relationships between difference and differential equations, shows that the convergence of the equilibrium condition does not imply that of the equilibrium.[30] Therefore, in the following, we show the convergence of the equilibrium.

Let $\boldsymbol{n}^K$ be an equilibrium of the discrete-space model with K cells. Our aim is to show that, for any sequence of equilibria $\{\boldsymbol{n}^K\}$, there exists an equilibrium λ of the continuous-space model such that

$$\lim_{K\to\infty} \left\{ \max_{i \in \operatorname{supp}(\boldsymbol{n}^K)} |\lambda(x_i) - \lambda_i^K| \right\} = 0, \tag{43}$$

where $\lambda_i^K = n_i^K/\delta_i^K$. However, as the preceding arguments have demonstrated, considering the net utility from the land consumption is more convenient than directly considering the population density. More specifically, let $\mathcal{H}(x) = h(\lambda(x))$ and $\mathcal{H}^K = \{h(\lambda_i^K)\}_{i \in \operatorname{supp}(\boldsymbol{n}^K)}$. The following proposition shows that

$$\lim_{K\to\infty} \left\{ \max_{i \in \operatorname{supp}(\boldsymbol{n}^K)} \left|\mathcal{H}(x_i) - \mathcal{H}_i^K\right| \right\} = 0. \tag{44}$$

Under Assumption 2, g, the inverse of h, is Lipschitz continuous. As a result, for some $C > 0$,

$$|\lambda(x_i) - \lambda_i^K| = |g(\mathcal{H}(x_i)) - g(\mathcal{H}_i^K)| \leq C|\mathcal{H}(x_i) - \mathcal{H}_i^K|. \tag{45}$$

Therefore, (44) implies (43).

In order to obtain the result, we impose a restriction on how space is discretized, by assuming that $\lim_{K\to\infty} \max_{i\in S} K\delta_i^K < \infty$. Obviously, this includes the case of the uniform discretization since $\delta_i^K = 1/K$ for all $i \in S$ implies $\max_{i\in S} K\delta_i^K = 1$ for all K.

Proposition 4 *Suppose Assumptions 2 and 7, and the linear interaction cost. Moreover, suppose* $\lim_{K\to\infty} \max_{i\in S} K\delta_i^K < \infty$. *Let* $\{\boldsymbol{n}^K\}$ *be a sequence of the discrete-space model's*

[30] See LeVeque (2007).

equilibria with support $\{i_-^K, i_-^K+1, \cdots, i_+^K-1, i_+^K\}$ *for each* K*. Then, there exists a continuous-space model's equilibrium* λ *with support* $(x_-, x_+) \subset [0,1]$ *where* $x_- = \lim_{K\to\infty} x_{i_-^K}$ *and* $x_+ = \lim_{K\to\infty} x_{i_+^K}$ *such that* (44) *holds.*

Proof. See the Appendix. ■

Observe that the sequence of spatial equilibria is arbitrary. Thus, any sequence of spatial equilibria converges on the unique equilibrium of the continuous-space model. In addition, recall that the spatial equilibrium in a discrete space is generally not unique. Nevertheless, every equilibrium converges on the single equilibrium as $K \to \infty$. This means that the set of spatial equilibria parametrized by K is upper hemi-continuous at the limit. Furthermore, since the spatial equilibrium in the continuous space is unique, the lower hemi-continuity is implied by the upper hemi-continuity. Therefore, *the set of spatial equilibria is continuous in* K *at the limit.*[31] This is a positive result for the continuous-space model. As long as K is sufficiently large, the continuous-space model can be viewed as a good approximation of the discrete-space model, which is relevant for real economies.

6 Conclusion

We studied the discrete-space agglomeration model with social interactions and its connection to the corresponding continuous-space model. We showed that any sequence of the discrete-space model's equilibria converges on the unique equilibrium of the continuous-space model, as the distance between adjacent cells vanishes. However, by appealing to the properties of the potential game, we found that, contrary to the continuous-space model, the spatial equilibrium can be essentially non-unique for any finite number of cells. Thus, while all equilibria should be close to one another when the cell size is sufficiently small, the problem of multiple equilibria is not negligible.

In this paper, we considered social interactions among a single type of agents. Hence, the natural extension is to consider multiple types of agents. There is rich literature on (continuous-space) social interaction models that include both consumers and firms.[32] Owing to general equilibrium effects, the properties of equilibrium are more complex than the class of models considered here. In particular, equilibrium is generally not unique even in the continuous-space model, although the stability of equilibria has not been explored. It is

[31] Recall that the equilibrium of the continuous-space model is only unique up to translation. Thus, strictly speaking, what we are actually considering here is the set of population distributions over their supports that are attained at equilibria, rather than the set of equilibria itself.

[32] See Chapter 6 of Fujita and Thisse (2013) and references therein.

difficult to determine the stability of equilibria in the continuous-space model, but we may be able to address this by approximating the model with a discrete-space model.[33]

Finally, although we did not engage in policy discussions, the spatial equilibrium of our model is generally not efficient since social interactions cause externalities. Indeed, the population distribution is more concentrated at the social optimum than at the market equilibrium. This is a consequence of positive externalities in social interactions, which yields under-agglomeration.[34] Thus, in order to achieve the social optimum, it is necessary that planners internalize these externalities. However, since the equilibrium under such an intervention is not necessarily unique as in a laissez-faire case, there may exist a stable equilibrium besides social optima. Therefore, in contrast to the continuous world, the policy design to achieve a social optimum in the discrete world is not straightforward, owing to the multiplicity of equilibria. This is an important subject for future research.[35]

Appendix

Derivation of the Hessian (22)

Suppose that the first-order conditions of the problem (20) hold at $\boldsymbol{\nu} \in \mathbb{R}^{K-1}$. Then, we are interested in whether $\boldsymbol{\nu}$ (locally) maximizes the potential function. A feasible direction $\boldsymbol{p}$ from $\boldsymbol{\nu}$ satisfies $Q(\boldsymbol{\nu})\boldsymbol{p} \leq \mathbf{0}$. Let $\hat{\boldsymbol{\gamma}}(\boldsymbol{\nu})$ be the vector of Lagrange multipliers for active constraints (i.e., $\hat{\boldsymbol{\gamma}}(\boldsymbol{\nu}) = \{\gamma_i(\boldsymbol{\nu})\}_{i \notin \operatorname{supp}(-q(\boldsymbol{\nu}))}$). If $Q(\boldsymbol{\nu})\boldsymbol{p} < \mathbf{0}$, then we have

$$\boldsymbol{p}'\nabla W(-q(\boldsymbol{\nu}))' = \boldsymbol{p}'Q(\boldsymbol{\nu})'\hat{\boldsymbol{\gamma}}(\boldsymbol{\nu}) < 0$$

under the strict complementarity. Thus, we may focus on direction $\boldsymbol{p}$ such that $Q(\boldsymbol{\nu})\boldsymbol{p} = \mathbf{0}$. Note that any feasible point is written as $\boldsymbol{\nu}' = \boldsymbol{\nu} + \boldsymbol{p}$. Because $\boldsymbol{p}$ belongs to the null space of $Q(\boldsymbol{\nu})$, the set of feasible points is then $\{\boldsymbol{\nu}' : \boldsymbol{\nu}' = \boldsymbol{\nu} + Z(\boldsymbol{\nu})\boldsymbol{x}, \boldsymbol{x} \in \mathbb{R}^{K-1}\}$.

Hence, we may study the second-order condition of the problem (20) at $\boldsymbol{\nu} \in \Pi\Delta$ by examining the following unconstrained problem:

$$\max_{\boldsymbol{x} \in R^{K-1}} W(-q(\boldsymbol{\nu} + Z(\boldsymbol{\nu})\boldsymbol{x})). \tag{46}$$

The condition that the Hessian (22) is negative-semidefinite corresponds to the second-order

[33] Blanchet et al. (2016) pave the way for using a potential function(al) to characterize the equilibria of a continuous-space model. However, they still abstract away from the stability analysis.

[34] Observe that the dispersion force due to the housing congestion is a pecuniary externality.

[35] Sandholm (2007) and Fujishima (2013) consider Pigouvian tax policies in the presence of multiple equilibria.

necessary condition of the problem (46) at $\boldsymbol{x} = \mathbf{0}$.

Proof of Lemma 2. Suppose, on the contrary, that there exists an equilibrium $\boldsymbol{n} \in \Delta$ in which, for some $i, j \in \operatorname{supp}(\boldsymbol{n})$ with $j - i \geq 2$, $n_\ell = 0$ for all $i < \ell < j$. Let $k \in \{i+1, .., j-1\}$. Since $n_k = 0$ at equilibrium, $h(0) < \infty$. Without loss of generality, suppose $\sum_{\ell=1}^{i} n_\ell \geq \sum_{\ell=j}^{K} n_\ell$. Then, because $d_{\ell k} + d_{kj} = d_{\ell j}$ for $\ell \leq i$ whereas $d_{kj} + d_{j\ell} = d_{k\ell}$ for $\ell \geq j$,

$$
\begin{aligned}
v_i(\boldsymbol{n}) - v_j(\boldsymbol{n}) \geq v_k(\boldsymbol{n}) - v_j(\boldsymbol{n}) \\
&= \tau \sum_\ell (d_{j\ell} - d_{k\ell}) n_\ell + h(0) - h(n_j/\delta_j) \\
&> \tau \sum_\ell (d_{j\ell} - d_{k\ell}) n_\ell \quad \because h \text{ is decreasing} \\
&= \tau d_{jk} \left(\sum_{\ell=1}^{i} n_\ell - \sum_{\ell=j}^{K} n_\ell \right) \geq 0,
\end{aligned}
$$

which contradicts the supposition $i, j \in \operatorname{supp}(\boldsymbol{n})$. ■

Proof of Proposition 1. Given $\boldsymbol{\nu} \in \Pi\Delta$, let $\boldsymbol{n} = -q(\boldsymbol{\nu})$ and $\operatorname{supp}(\boldsymbol{n}) = \{i_1, i_2, ..., i_L\} \subseteq S$ where $L \geq 2$ (and hence $|\operatorname{supp}(\boldsymbol{n})| = L$). At first, we show that $\tilde{D}_{\operatorname{supp}(\boldsymbol{n})}$ is positive definite. Let $\boldsymbol{x} \in \mathbb{R}^K$ be a vector such that $x_i = 0$ for all $i \notin \{i_1, i_2, ..., i_{L-1}\}$, and let $\tilde{\boldsymbol{x}} \in \mathbb{R}^K$ be a vector such that $\tilde{x}_i = x_i$ for all $i \neq i_L$ and $\tilde{x}_{i_L} = -\sum_{i=i_1}^{i_{L-1}} x_i$. Note that $\tilde{\boldsymbol{x}} \in T\Delta$. Since $[\tilde{D}_{\operatorname{supp}(\boldsymbol{n})}]_{ij} = d_{ii_L} + d_{i_L j} - d_{ij}$,

$$
\begin{aligned}
\sum_{i,j=i_1}^{i_{L-1}} (d_{ii_L} + d_{i_L j} - d_{ij}) x_i x_j &= -\sum_{i=i_1}^{i_{L-1}} d_{ii_L} x_i \tilde{x}_{i_L} - \sum_{j=i_1}^{i_{L-1}} d_{i_L j} \tilde{x}_{i_L} x_j - \sum_{i,j=i_1}^{i_{L-1}} d_{ij} x_i x_j \\
&= -\sum_{i,j=i_1}^{i_L} d_{ij} \tilde{x}_i \tilde{x}_j = -\sum_{i,j=1}^{K} d_{ij} \tilde{x}_i \tilde{x}_j > 0,
\end{aligned}
$$

where the last inequality follows from the assumption that D is negative definite on $T\Delta$. Then, since $(x_i)_{i \in \{i_1, i_2, ..., i_{L-1}\}}$ is arbitrary, $\tilde{D}_{\operatorname{supp}(\boldsymbol{n})}$ is positive definite. Hence, all of $\tilde{D}_{\operatorname{supp}(\boldsymbol{n})}$'s eigenvalues are positive. Second, we have

$$
\tilde{H}(\boldsymbol{n}) = \operatorname{diag}[(\delta_i^{-1} h'(n_i/\delta_i))_{i \in \{i_1, i_2, ..., i_{L-1}\}}] + \delta_{i_L}^{-1} h'(n_{i_L}/\delta_{i_L}) \mathbf{1}\mathbf{1}', \tag{47}
$$

where $\operatorname{diag}(\boldsymbol{x})$ is a diagonal matrix having x_i for its (i,i)-th element, and $\mathbf{1}$ is a row vector of ones with an appropriate dimension. The eigenvalues of $\delta_{i_L}^{-1} h'(n_{i_L}/\delta_{i_L}) \mathbf{1}\mathbf{1}'$ are $(L-1)\delta_{i_L}^{-1} h'(n_{i_L}/\delta_{i_L})$ and 0. Accordingly, the matrix has exactly one negative eigenvalue, since h is decreasing. Thus, by Weyl's inequality, $\mu_j(\tilde{H}(\boldsymbol{n})) \geq \mu_{j-1}(\operatorname{diag}[(\delta_i^{-1} h'(n_i/\delta_i))_{i \in \{i_1, i_2, ..., i_{L-1}\}}]) + \mu_2(\delta_{i_L}^{-1} h'(n_{i_L}/\delta_{i_L}) \mathbf{1}\mathbf{1}') = \mu_{j-1}(\operatorname{diag}[(\delta_i^{-1} h'(n_i/\delta_i))_{i \in \{i_1, i_2, ..., i_{L-1}\}}])$ for each $j = 2, 3, ..., L$.

Then, by invoking Weyl's inequality for $\tau\tilde{D}_{\mathrm{supp}\,(\boldsymbol{n})}+\tilde{H}(\boldsymbol{n})$, we obtain

$$\mu_{\max}(H(\boldsymbol{\nu}))\equiv\mu_{L-1}(H(\boldsymbol{\nu}))\geq\tau\mu_{L-j}(\tilde{D}_{\mathrm{supp}\,(\boldsymbol{n})})+\mu_j(\tilde{H}(\boldsymbol{n})) \tag{48}$$

$$\geq\tau\mu_{L-j}(\tilde{D}_{\mathrm{supp}\,(\boldsymbol{n})})+\mu_{j-1}(\mathrm{diag}[(\delta_i^{-1}h'(n_i/\delta_i))_{i\in\{i_1,i_2,\dots,i_{L-1}\}}]) \tag{49}$$

where $2\leq j\leq L-1$. Since $n_i>0$ for all $i\in\{i_1,i_2,...,i_{L-1}\}$, $\mu_{j-1}(\mathrm{diag}[(\delta_i^{-1}h'(n_i/\delta_i))_{i\in\{i_1,i_2,\dots,i_{L-1}\}}])\in(-\infty,0)$ for any $j\in\{2,3,...,L-1\}$. Therefore, we obtain the stated result. ■

Proof of Corollaries 1. We invoke the results of Yueh and Cheng (2008), which are restated as follows:[36]

Theorem 2 (Yueh and Cheng, 2008) *Consider the $n\times n$ real matrix of the form*

$$\begin{pmatrix} m_2+m_4 & m_3 & & & & \\ m_1 & m_2 & m_3 & & & \\ & m_1 & m_2 & m_3 & & \\ & & \ddots & \ddots & \ddots & \\ & & & m_1 & m_2 & m_3 \\ & & & & m_1 & m_2+m_5 \end{pmatrix}. \tag{50}$$

(i) Let $0<\theta<\pi$. Then, the eigenvalues of the matrix above are given by

$$\mu=m_2+2\sqrt{m_1m_3}\cos\theta, \tag{51}$$

where θ solves

$$m_1m_3\sin(n+1)\theta+m_4m_5\sin(n-1)\theta-\sqrt{m_1m_3}(m_4+m_5)\sin n\theta=0. \tag{52}$$

(ii) If $\tilde{\rho}=\sqrt{m_1/m_3}$ [resp. $\tilde{\rho}=-\sqrt{m_1/m_3}$] solves

$$m_1m_3(n+1)+m_4m_5(n-1)-m_3\tilde{\rho}(m_4+m_5)n=0, \tag{53}$$

then $m_2+2\sqrt{m_1m_3}$ [resp. $m_2-2\sqrt{m_1m_3}$] is an eigenvalue of the matrix above (these correspond to the case in which $\theta\in\{0,\pi\}$).

Our matrix (33) corresponds to the case in which $m_1=m_3=-\frac{K}{2},m_2=K,m_4=$

[36] They consider more general matrices than (50), where upper right and lower left corners can also be nonzero. They also allow complex matrices.

$-\frac{K}{2}, m_5 = 0$, and $n = L-1$. Then, (52) becomes

$$\frac{K^4}{4}(\sin L\theta - \sin(L-1)\theta) = \frac{K^4}{4}\sin\frac{\theta}{2}\cos\frac{(2L-1)\theta}{2} = 0. \tag{54}$$

Because $0 < \theta < \pi$, $\sin\frac{\theta}{2} \neq 0$ and hence $\cos\frac{(2L-1)\theta}{2} = 0 \Rightarrow \theta = \frac{(2j-1)\pi}{2L-1}$ where $j = 1, 2, ..., L-1$. Moreover, (53) is given by

$$\frac{K^2}{4}(L - \tilde{\rho}(L-1)) = 0, \tag{55}$$

and neither $\tilde{\rho} = 1$ nor $\tilde{\rho} = -1$ solves the equation above. Hence, the eigenvalues of matrix (33) are given by $K(1 - \cos\frac{(2j-1)\pi}{2L-1})$ where $j = 1, 2, ..., L-1$. In particular, since $K(1 - \cos\frac{(2j-1)\pi}{2L-1})$ is its j-th smallest eigenvalue, $\mu_j(\tilde{D}_{\mathrm{supp}\,(\boldsymbol{n})}) = \frac{1}{K}(1 - \cos\frac{(2(L-j)-1)\pi}{2L-1})^{-1}$. Thus,

$$\begin{aligned}
\min_{2\leq j\leq L-1} \frac{\mu_{j-1}(\mathrm{diag}[(|h_i'(n_i)|)_{i\in\{i_1,i_2,...,i_{L-1}\}}])}{\mu_{L-j}(\tilde{D}_{\mathrm{supp}\,(\boldsymbol{n})})} &= \frac{\alpha K}{\max_{2\leq j\leq L-1}\mu_{L-j}(\tilde{D}_{\mathrm{supp}\,(\boldsymbol{n})})} && (56)\\
&= \alpha K \mu_{L-2}(\tilde{D}_{\mathrm{supp}\,(\boldsymbol{n})})^{-1} && (57)\\
&= \left(1 - \cos\frac{3\pi}{2L-1}\right)\alpha K^2. && (58)
\end{aligned}$$

■

Proof of Lemma 4. By Corollary 1 of Blanchet et al. (2016), the spatial equilibrium is continuous. Hence, the support of the equilibrium is open. We argue in an analogue manner as the proof of Lemma 2. Suppose, to the contrary, that there exists an equilibrium λ for which $\mathrm{supp}\,(\lambda) = (x_-^1, x_+^1) \cup (x_-^2, x_+^2)$ with $x_+^1 < x_-^2$.

Let $x^1 \in (x_-^1, x_+^1), x^2 \in (x_-^2, x_+^2)$, and $\hat{x} \in (x_+^1, x_-^2)$, respectively. Without loss of generality, suppose $\int_{x_-^1}^{x_+^1}\lambda(z)\mathrm{d}z \geq \int_{x_-^2}^{x_+^2}\lambda(z)\mathrm{d}z$. Then, we have

$$\begin{aligned}
v_\lambda(x^1) - v_\lambda(x^2) &\geq v_\lambda(\hat{x}) - v_\lambda(x^2)\\
&> -\tau\int|\hat{x} - z|\lambda(z)\mathrm{d}z + \tau\int|x^2 - z|\lambda(z)\mathrm{d}z \quad \because h \text{ is decreasing}\\
&= \tau(x^2 - \hat{x})\left(\int_{x_-^1}^{x_+^1}\lambda(z)\mathrm{d}z - \int_{x_-^2}^{x_+^2}\lambda(z)\mathrm{d}z\right) \geq 0,
\end{aligned}$$

which contradicts the supposition $x^1, x^2 \in \mathrm{supp}\,(\lambda)$. ■

Proof of Lemma 5. Suppose that λ is an equilibrium. By Lemma 4, $\mathrm{supp}\,(\lambda) = (x_-, x_+)$ for some $x_-, x_+ \in \mathbb{R}$. Then, $v_\lambda(x) = v^*$ for all $x \in (x_-, x_+)$ where v^* is the equilibrium utility level. Moreover, since $v_\lambda(x)$ is continuous in x, $v_\lambda(x_-) = v_\lambda(x_+) = v^*$. Hence,

$$y - \tau \int_{x_-}^{x_+} |x-y|\lambda(y)\mathrm{d}y + h(\lambda(x)) = v^* \qquad \forall x \in [x_-, x_+]. \tag{59}$$

Differentiating this equation with respect to x yields

$$\frac{\mathrm{d}h(\lambda(x))}{\mathrm{d}x} = \tau \left\{ \int_{x_-}^{x} \lambda(z)\mathrm{d}z - \int_{x}^{x_+} \lambda(z)\mathrm{d}z \right\} \qquad \forall x \in [x_-, x_+]. \tag{60}$$

Substituting the population constraint $\int_{x_-}^{x_+} \lambda(z)\mathrm{d}z = 1$ and $\lambda(x) = g(H(x))$ into this,[37] we have (40a). Meanwhile, (40b) is obtained from the population constraint, while (40c) is obtained from $\lambda(x_-) = \lambda(x_+) = 0$.

Conversely, suppose that λ solves the system (40a)-(40c) for some $x_-, x_+ \in \mathbb{R}$. Let $x^1, x^2 \in [x_-, x_+]$ with $x^2 > x^1$. Then, integrating (60) over $[x^1, x^2]$,

$$h(\lambda(x^2)) - h(\lambda(x^1)) = \tau \left\{ \int_{x^1}^{x^2} \int_{x_-}^{x} \lambda(z)\mathrm{d}z\mathrm{d}x - \int_{x^1}^{x^2} \int_{x}^{x_+} \lambda(z)\mathrm{d}z\mathrm{d}x \right\}. \tag{61}$$

By integration by parts,

$$\int_{x^1}^{x^2} \int_{x_-}^{x} \lambda(z)\mathrm{d}z\mathrm{d}x = x^2 \int_{x_-}^{x^2} \lambda(z)\mathrm{d}z - x^1 \int_{x_-}^{x^1} \lambda(z)\mathrm{d}z - \int_{x^1}^{x^2} z\lambda(z)\mathrm{d}z, \tag{62}$$

$$\int_{x^1}^{x^2} \int_{x}^{x_+} \lambda(z)\mathrm{d}z\mathrm{d}x = x^2 \int_{x^2}^{x_+} \lambda(z)\mathrm{d}z - x^1 \int_{x^1}^{x_+} \lambda(z)\mathrm{d}z + \int_{x^1}^{x^2} z\lambda(z)\mathrm{d}z. \tag{63}$$

However, observe that

$$\begin{aligned} &\int_{x_-}^{x_+} |x^2 - z|\lambda(z)\mathrm{d}z - \int_{x_-}^{x_+} |x^1 - z|\lambda(z)\mathrm{d}z \\ &= x^2 \left\{ \int_{x_-}^{x^2} \lambda(z)\mathrm{d}z - \int_{x^2}^{x_+} \lambda(z)\mathrm{d}z \right\} - x^1 \left\{ \int_{x_-}^{x^1} \lambda(z)\mathrm{d}z - \int_{x^1}^{x_+} \lambda(z)\mathrm{d}z \right\} - 2\int_{x^1}^{x^2} z\lambda(z)\mathrm{d}z. \end{aligned} \tag{64}$$

Therefore, $v_\lambda(x^1) = v_\lambda(x^2)$. This implies that there exists $v^* \in \mathbb{R}$ such that $v_\lambda(x) = v^*$ for

[37] Since $h(\cdot)$ is a strictly decreasing function, the inverse function $g(\cdot)$ exists.

all $x \in [x_-, x_+]$. Now, let $x < x_-$. Because $\lambda(x_-) = \lambda(x) = 0$ and $\int_{x_-}^{x_+} \lambda(z)\mathrm{d}z = 1$,

$$v_\lambda(x) - v_\lambda(x_-) = \tau \left\{ \int_{x_-}^{x_+} (z - x_-)\lambda(z)\mathrm{d}z - \int_{x_-}^{x_+} (z - x_- + x_- - x)\lambda(z)\mathrm{d}z \right\} \tag{65}$$

$$= \tau(x - x_-) < 0. \tag{66}$$

Similarly, we have $v_\lambda(x) < v_\lambda(x_+)$ for $x > x_+$. Hence, λ is an equilibrium. ■

Proof of Lemma 6. Suppose that the region is given by $[0, L]$ where L is larger than the length of the equilibrium support of the continuous-space model over $\mathbb{R}$. Let $\boldsymbol{n}^K$ be a full-support distribution in the discrete-space model with K cells. Then, let

$$\lambda^K(x) = \frac{n_i^K}{\delta_i^K} \quad \text{for } x \in [b_{i-1}^K, b_i^K) \tag{67}$$

where $i \in S$ and $\lambda^K(x) = 0$ otherwise. For each $j \in S$, let x_j^K be the middle point of $[b_{j-1}^K, b_j^K]$ at which the discrete model's payoff in cell j is defined. Then,

$$v_{\lambda^K}(x_i^K) = Y - \tau \int |x_i^K - y|\lambda^K(y)\mathrm{d}y + h(\lambda^K(x_i^K)) \tag{68}$$

$$= Y - \tau \sum_{j=1}^{K} \int_{b_{j-1}^K}^{b_j^K} |x_i^K - y| \frac{n_j^K}{\delta_j^K}\mathrm{d}y + h(n_i^K/\delta_i^K) \tag{69}$$

$$= Y - \tau \sum_{j=1}^{i-1} \int_{b_{j-1}^K}^{b_j^K} (x_i^K - y) \frac{n_j^K}{\delta_j^K}\mathrm{d}y + \tau \sum_{j=i+1}^{K} \int_{b_{j-1}^K}^{b_j^K} (x_i^K - y) \frac{n_j^K}{\delta_j^K}\mathrm{d}y \tag{70}$$

$$- \tau \int_{b_{i-1}^K}^{b_i^K} |x_i^K - y| \frac{n_i^K}{\delta_i^K}\mathrm{d}y + h(n_i^K/\delta_i^K). \tag{71}$$

We have

$$\int_{b_{j-1}^K}^{b_j^K} (x_i^K - y) \frac{n_j^K}{\delta_j^K}\mathrm{d}y = n_j^K x_i^K - \frac{n_j^K}{\delta_j^K} \frac{(b_j^K)^2 - (b_{j-1}^K)^2}{2} \tag{72}$$

$$= n_j^K \left(x_i^K - \frac{b_j^K + b_{j-1}^K}{2} \right) \tag{73}$$

$$= n_j^K \left(x_i^K - x_j^K \right) \quad \because b_j^K = x_j^K + \frac{\delta_j^K}{2}, b_{j-1}^K = x_j^K - \frac{\delta_j^K}{2}, \tag{74}$$

for $j \in S \setminus \{i\}$. Moreover, since x_i^K is the middle point of $[b_{i-1}^K, b_i^K]$,

$$\begin{aligned}\int_{b_{i-1}^K}^{b_i^K} |x_i^K - y| \frac{n_i^K}{\delta_i^K} \mathrm{d}y &= 2 \int_{b_{i-1}^K}^{x_i^K} (x_i^K - y) \frac{n_i^K}{\delta_i^K} \mathrm{d}y \\ &= 2n_i^K \left(\frac{1}{2} x_i^K - \frac{x_i^K + b_{i-1}^K}{4} \right) \quad \because b_{i-1}^K = x_i^K - \frac{\delta_i^K}{2} \\ &= \frac{1}{2} n_i^K (x_i^K - b_{i-1}^K) = \frac{1}{4} n_i^K \delta_i^K\end{aligned}$$

Hence,

$$\sum_{j=1}^{K} \int_{b_{j-1}^K}^{b_j^K} |x_i^K - y| \frac{n_j^K}{\delta_j^K} \mathrm{d}y = \sum_{j=1}^{K} |x_i^K - x_j^K| n_j^K + \frac{1}{4} n_i^K \delta_i^K. \tag{75}$$

Therefore,

$$v_{\lambda^K}(x_i^K) = v_i(\boldsymbol{n}^K) - \frac{\tau}{4} n_i^K \delta_i^K. \tag{76}$$

Let $z \in [0, L]$. For each K, there exists $i_z^K \in \{1, 2, ..., K\}$ such that $z \in [b_{i_z^K - 1}^K, b_{i_z^K}^K)$. Then, we have

$$v_{\lambda^K}(x_{i_z^K}^K) - \tau |x_{i_z^K}^K - z| \leq v_{\lambda^K}(z) \leq v_{\lambda^K}(x_{i_z^K}^K) + \tau |x_{i_z^K}^K - z| \tag{77}$$

$$\Leftrightarrow v_{i_z^K}(\boldsymbol{n}^K) - \frac{\tau}{4} n_{i_z^K}^K \delta_{i_z^K}^K - \tau |x_{i_z^K}^K - z| \leq v_{\lambda^K}(z) \leq v_{i_z^K}(\boldsymbol{n}^K) - \frac{\tau}{4} n_{i_z^K}^K \delta_{i_z^K}^K + \tau |x_{i_z^K}^K - z|, \tag{78}$$

where we use the triangle inequality for (77); and (76) for (78).

Suppose that, for any $\bar{K} \geq 1$, there exists $K \geq \bar{K}$ such that $\boldsymbol{n}^K$ is an equilibrium. Then, we can take a sequence $\{K_\ell\}_{\ell \in \mathbb{N}}$ such that $\boldsymbol{n}^{K_\ell}$ is an equilibrium for all ℓ.[38] Then, in light of (78), we have

$$v_1(\boldsymbol{n}^{K_\ell}) - \frac{\tau}{4} n_{i_z^{K_\ell}}^{K_\ell} \delta_{i_z^{K_\ell}}^{K_\ell} - \tau |x_{i_z^{K_\ell}}^{K_\ell} - z| \leq v_{\lambda^{K_\ell}}(z) \leq v_1(\boldsymbol{n}^{K_\ell}) - \frac{\tau}{4} n_{i_z^{K_\ell}}^{K_\ell} \delta_{i_z^{K_\ell}}^{K_\ell} + \tau |x_{i_z^{K_\ell}}^{K_\ell} - z| \tag{79}$$

$$\begin{aligned}&\Leftrightarrow v_{\lambda^{K_\ell}}(x_1^{K_\ell}) + \frac{\tau}{4} (\delta_1^{K_\ell} n_1^{K_\ell} - n_{i_z^{K_\ell}}^{K_\ell} \delta_{i_z^{K_\ell}}^{K_\ell}) - \tau |x_{i_z^{K_\ell}}^{K_\ell} - z| \\ &\qquad \leq v_{\lambda^{K_\ell}}(z) \leq v_{\lambda^{K_\ell}}(x_1^{K_\ell}) + \frac{\tau}{4} (\delta_1^{K_\ell} n_1^{K_\ell} - n_{i_z^{K_\ell}}^{K_\ell} \delta_{i_z^{K_\ell}}^{K_\ell}) + \tau |x_{i_z^{K_\ell}}^{K_\ell} - z|,\end{aligned} \tag{80}$$

where we use the equilibrium condition of $\boldsymbol{n}^{K_\ell}$ for (79); and (76) for (80). Since $\delta_i^{K_\ell}$ vanishes for all i when $\ell \to \infty$, whereas $\boldsymbol{n}^{K_\ell}$ is finite for any ℓ, $\delta_1^{K_\ell} n_1^{K_\ell} - n_{i_z^{K_\ell}}^{K_\ell} \delta_{i_z^{K_\ell}}^{K_\ell} \to 0$ as $\ell \to \infty$. Moreover, since $x_1^{K_\ell} = \delta_1^{K_\ell}/2$, $\lim_{\ell \to \infty} v_{\lambda^{K_\ell}}(x_1^{K_\ell}) = v_\lambda(0)$ where $\lambda = \lim_{\ell \to \infty} \lambda^{K_\ell}$. Then, since $x_{i_z^{K_\ell}}^{K_\ell} \to z$ as $\ell \to \infty$, (80) implies $v_\lambda(z) = v_\lambda(0)$. Since $z \in [0, L]$ is arbitrary, we have obtained a spatial equilibrium λ of the continuous-space model with support $[0, L]$, but this

[38] There exists $K_1 \geq 1$ such that $\boldsymbol{n}^{K_1}$ is an equilibrium. This, in turn, enables us to take $K_2 \geq K_1 + 1$ such that $\boldsymbol{n}^{K_2}$ is an equilibrium. Continuing in this way, we obtain the desired sequence.

is a contradiction. ■

Proof of Proposition 4. Let $\{\boldsymbol{n}^K\}$ be a sequence of discrete-space models' equilibria with $\operatorname{supp}(\boldsymbol{n}^K) = \{i_-^K, i_-^K+1, \cdots, i_+^K-1, i_+^K\}$. Since the length of the equilibrium support of the continuous-space model is assumed to be less than one, we have either $\lim_{K\to\infty} x_{i_-^K} > 0$ or $\lim_{K\to\infty} x_{i_+^K} < 1$ (or both) by Lemma 6. Without loss of generality, suppose $\lim_{K\to\infty} x_{i_-^K} > 0$. Let λ be the solution to the system (40a)-(40c) where $x_- = \lim_{K\to\infty} x_{i_-^K}$. By Taylor's theorem, if $i \in \operatorname{supp}(\boldsymbol{n}^K)$ and $\lambda(x_i) > 0$ so that λ is differentiable, $\mathcal{H}(x_i)$ is expressed as

$$\mathcal{H}(x_i) = \mathcal{H}(x_{i-1}) + \epsilon_i^K \frac{\mathrm{d}\mathcal{H}(x_{i-1})}{\mathrm{d}x} + \frac{(\epsilon_i^K)^2}{2}\frac{\mathrm{d}^2\mathcal{H}(x_{i-1}+\theta_i\epsilon_i^K)}{\mathrm{d}x^2}, \tag{81}$$

where $\theta_i \in (0,1)$. Thus, by (41a) and Lemma 5, $\mathcal{H}(x_i) - \mathcal{H}_i^K$ is given by

$$\mathcal{H}(x_i) - \mathcal{H}_i^K = \mathcal{H}(x_{i-1}) - \mathcal{H}_{i-1}^K + \tau\epsilon_i^K\Upsilon_i^K + \frac{(\epsilon_i^K)^2}{2}\frac{\mathrm{d}^2\mathcal{H}(x_{i-1}+\theta_i\epsilon_i^K)}{\mathrm{d}x^2}, \tag{82}$$

where $\Upsilon_i^K = 2\left\{\int_{x_-}^{x_{i-1}} g(\mathcal{H}(z))\mathrm{d}z - \sum_{j=i_-^K}^{i-1} g(\mathcal{H}_j^K)\delta_j^K\right\}$. Furthermore, Taylor's theorem yields

$$\begin{aligned}
\int_{b_{i-1}^K}^{b_i^K} g(\mathcal{H}(z))\mathrm{d}z &= \int_{x_-}^{b_i^K} g(\mathcal{H}(y))\mathrm{d}y - \int_{x_-}^{b_{i-1}^K} g(\mathcal{H}(y))\mathrm{d}y \\
&= \int_{x_-}^{x_i} g(\mathcal{H}(y))\mathrm{d}y + \frac{\delta_i^K}{2} g(\mathcal{H}(x_i)) + \frac{(\frac{\delta_i^K}{2})^2}{2}\frac{\mathrm{d}g(\mathcal{H}(x_i+\eta_i\frac{\delta_i^K}{2}))}{\mathrm{d}x} \\
&\quad - \int_{x_-}^{x_i} g(\mathcal{H}(y))\mathrm{d}y + \frac{\delta_i^K}{2} g(\mathcal{H}(x_i)) - \frac{(-\frac{\delta_i^K}{2})^2}{2}\frac{\mathrm{d}g(\mathcal{H}(x_i-\psi_i\frac{\delta_i^K}{2}))}{\mathrm{d}x} \\
&= \delta_i^K g(\mathcal{H}(x_i)) + \frac{(\delta_i^K)^2}{8}\left\{\frac{\mathrm{d}g(\mathcal{H}(x_i+\eta_i\frac{\delta_i^K}{2}))}{\mathrm{d}x} - \frac{\mathrm{d}g(\mathcal{H}(x_i-\psi_i\frac{\delta_i^K}{2}))}{\mathrm{d}x}\right\},
\end{aligned} \tag{83}$$

where $\eta_i \in (0,1)$ and $\psi_i \in (0,1)$. Therefore, we have

$$\Upsilon_i^K = 2\left[\sum_{j=i_-^K}^{i-1}\left\{g(\mathcal{H}(x_j)) - g(\mathcal{H}_j^K)\right\}\delta_j^K - \int_{x_{i-1}}^{b_{i-1}^K} g(\mathcal{H}(z))\mathrm{d}z - \int_{b_{i_-^K-1}^K}^{x_-} g(\mathcal{H}(z))\mathrm{d}z + m_{i-1}^K\right], \tag{84}$$

where

$$m_i^K = \sum_{j=i_-^K}^{i}\frac{(\delta_j^K)^2}{8}\left\{\frac{\mathrm{d}g(\mathcal{H}(x_j+\eta_j\delta_j^K/2))}{\mathrm{d}x} - \frac{\mathrm{d}g(\mathcal{H}(x_j-\psi_j\delta_j^K/2))}{\mathrm{d}x}\right\}. \tag{85}$$

By Lemma A1, $\left|\frac{\mathrm{d}^2\mathcal{H}(x_{i-1}+\theta_i\epsilon_i^K)}{\mathrm{d}x^2}\right| \leq 2\tau\bar{g}_0$ where $\bar{g}_0$ is a bound of $g(\mathcal{H}(x))$. Hence,

$$|\mathcal{H}(x_i) - \mathcal{H}_i^K| \leq |\mathcal{H}(x_{i-1}) - \mathcal{H}_{i-1}^K| + \tau\epsilon_i^K|\Upsilon_i^K| + \tau\bar{g}_0(\epsilon_i^K)^2. \tag{86}$$

We also have $\left|\frac{\mathrm{d}g(\mathcal{H}(x))}{\mathrm{d}x}\right| \leq \bar{g}_1$ for some $\bar{g}_1 > 0$ by Lemma A1. Hence,

$$|m_i^K| \leq \frac{\bar{g}_1}{4}\sum_{j=i_-^K}^{i}(\delta_j^K)^2 \leq \frac{\bar{g}_1}{4}\bar{\delta}^K\sum_{j=i_-^K}^{i}\delta_j^K \leq \frac{\bar{g}_1}{4}\bar{\delta}^K,$$

where $\bar{\delta}^K = \max_{i\in S}\delta_i^K$. Then,

$$\begin{aligned}\frac{1}{2}|\Upsilon_i^K| &\leq \sum_{j=i_-^K}^{i-1}|g(\mathcal{H}(x_j)) - g(\mathcal{H}_j^K)|\delta_j^K + \frac{\bar{g}_0\delta_{i-1}^K}{2} + \bar{g}_0|x_- - b_{i_-^K-1}^K| + \frac{\bar{g}_1}{4}\bar{\delta}^K \\ &\leq \sum_{j=i_-^K}^{i-1}|g(\mathcal{H}(x_j)) - g(\mathcal{H}_j^K)|\delta_j^K + \bar{g}_0|x_- - x_{i_-^K}| + \left(\bar{g}_0 + \frac{\bar{g}_1}{4}\right)\bar{\delta}^K\end{aligned}$$

where the last inequality follows from

$$|x_- - b_{i_-^K-1}^K| \leq |x_- - x_{i_-^K}| + |x_{i_-^K} - b_{i_-^K-1}^K| = |x_- - x_{i_-^K}| + \frac{\delta_{i_-^K}^K}{2},$$

and $\delta_{i_-^K}^K, \delta_{i_-^K-1}^K \leq \bar{\delta}^K$. Therefore,

$$\begin{aligned}\left|\mathcal{H}(x_i) - \mathcal{H}_i^K\right| \leq \left|\mathcal{H}(x_{i-1}) - \mathcal{H}_{i-1}^K\right| &+ 2\tau\epsilon_i^K\sum_{j=i_-^K}^{i-1}|g(\mathcal{H}(x_j)) - g(\mathcal{H}_j^K)|\delta_j^K \\ &+ 2\tau\epsilon_i^K\left\{\bar{g}_0|x_- - x_{i_-^K}| + \left(\bar{g}_0 + \frac{\bar{g}_1}{4}\right)\bar{\delta}^K + \frac{\bar{g}_0\epsilon_i^K}{2}\right\}.\end{aligned} \tag{87}$$

In addition, since g is continuously differentiable (and hence Lipschitz continuous), there exists $C \in (0,\infty)$ such that $|g(\mathcal{H}(x_i)) - g(\mathcal{H}_i^K)| \leq C|\mathcal{H}(x_i) - \mathcal{H}_i^K|$. Thus, we obtain

$$\begin{aligned}\left|\mathcal{H}(x_i) - \mathcal{H}_i^K\right| \leq \left|\mathcal{H}(x_{i-1}) - \mathcal{H}_{i-1}^K\right| &+ 2\tau C\epsilon_i^K\sum_{j=i_-^K}^{i-1}|\mathcal{H}(x_j) - \mathcal{H}_j^K|\delta_j^K \\ &+ 2\tau\epsilon_i^K\left\{\bar{g}_0|x_- - x_{i_-^K}| + \left(\bar{g}_0 + \frac{\bar{g}_1}{4}\right)\bar{\delta}^K + \frac{\bar{g}_0\epsilon_i^K}{2}\right\}.\end{aligned} \tag{88}$$

Then, we consider the following difference equation: for $i \in \operatorname{supp}(\boldsymbol{n}^K) \setminus \{i_-^K\}$,

$$X_i^K = X_{i-1}^K + 2\tau C\bar{\epsilon}^K \bar{\delta}^K \sum_{j=i_-^K}^{i-1} X_j^K + 2\tau\bar{\epsilon}^K \left\{\bar{g}_0|x_- - x_{i_-^K}| + \left(\bar{g}_0 + \frac{\bar{g}_1}{4}\right)\bar{\delta}^K + \frac{\bar{g}_0\bar{\epsilon}^K}{2}\right\}, \tag{89}$$

and $X_{i_-^K}^K = |\mathcal{H}(x_{i_-^K}) - \mathcal{H}_{i_-^K}^K|$ where $\bar{\epsilon}^K = \max_{i\in S}\epsilon_i^K$. By induction, $X_i \geq |\mathcal{H}(x_i) - \mathcal{H}_i^K|$ for all $i \in \operatorname{supp}(\boldsymbol{n}^K)$. Hence, it suffices to show that, for all $i \in \operatorname{supp}(\boldsymbol{n}^K)$, X_i^K converges to zero as $K \to \infty$.

Since $\lim_{K\to\infty} K\bar{\delta}^K < \infty$, $K\bar{\epsilon}^K \leq K\bar{\delta}^K \leq \bar{\delta}$ for some $\bar{\delta} \in (0,\infty)$ (c.f., $\epsilon_i^K = \frac{\delta_i^K + \delta_{i-1}^K}{2}$). Then, by $X_{i+1}^K > X_i^K$, we have

$$\begin{aligned}
X_i^K &\leq (1 + 2\tau C\bar{\epsilon}^K)X_{i-1}^K + 2\tau\bar{\epsilon}^K\left\{\bar{g}_0|x_- - x_{i_-^K}| + \left(\bar{g}_0 + \frac{\bar{g}_1}{4}\right)\bar{\delta}^K + \frac{\bar{g}_0\bar{\epsilon}^K}{2}\right\} \\
&\leq (1 + 2\tau C\bar{\epsilon}^K)^{i-i_-^K} X_{i_-^K}^K - \frac{1 - (1 + 2\tau C\bar{\epsilon}^K)^{i-i_-^K}}{C}\left\{\bar{g}_0|x_- - x_{i_-^K}| + \left(\bar{g}_0 + \frac{\bar{g}_1}{4}\right)\bar{\delta}^K + \frac{\bar{g}_0\bar{\epsilon}^K}{2}\right\} \\
&\leq \left(1 + \frac{2\tau C\bar{\delta}}{K}\right)^K \left\{X_{i_-^K}^K + \frac{1}{C}\left(\bar{g}_0|x_- - x_{i_-^K}| + \left(\bar{g}_0 + \frac{\bar{g}_1}{4}\right)\bar{\delta}^K + \frac{\bar{g}_0\bar{\epsilon}^K}{2}\right)\right\}.
\end{aligned} \tag{90}$$

Then, because $\bar{\delta}^K \to 0, \bar{\varepsilon}^K \to 0, |x_- - x_{i_-^K}| \to 0$, and $\left(1 + \frac{2\tau C\bar{\delta}}{K}\right)^K \to e^{2\tau C\bar{\delta}} < \infty$ as $K \to \infty$, we obtain

$$\lim_{K\to\infty} X_i^K = 0 \quad \forall i \in \operatorname{supp}(\boldsymbol{n}^K) \qquad \text{if} \quad \lim_{K\to\infty} X_{i_-^K}^K = 0. \tag{91}$$

Because $x_- > 0$ by supposition, for sufficiently large K, $x_{i_-^K - 1} > 0$ for which $\lambda_{i_-^K-1}^K = 0$. Since $\lambda(x_-) = 0$, we have $\mathcal{H}_{i_-^K-1}^K = \mathcal{H}(x_-)$, and thus

$$\lim_{K\to\infty} X_{i_-^K}^K \leq \lim_{K\to\infty}\left[|\mathcal{H}(x_{i_-^K}) - \mathcal{H}(x_-)| + |\mathcal{H}_{i_-^K-1}^K - \mathcal{H}_{i_-^K}^K|\right] \leq \lim_{K\to\infty} \tau\epsilon_{i_-^K-1}^K = 0. \tag{92}$$

This and (91) yield the desired conclusion. ■

Lemma A1 *Let $\mathcal{H}(x) = h(\lambda(x))$ where λ is an equilibrium such that* $\operatorname{supp}(\lambda) = (x_-, x_+)$. *Then, under Assumption 2 and the linear interaction cost, $g(\mathcal{H}(x))$, $\frac{\mathrm{d}g(\mathcal{H}(x))}{\mathrm{d}x}$, and $\frac{\mathrm{d}^2\mathcal{H}(x)}{\mathrm{d}x^2}$ are bounded over $[x_-, x_+]$.*

Proof. It follows from Corollary 1 of Blanchet et al. (2016) that $g(\mathcal{H}(x)) = \lambda(x)$ is continuous in x at the equilibrium. In addition, $g(\mathcal{H}(x))$ satisfies the population constraint. Thus, $g(\mathcal{H}(x))$ is bounded. Moreover, since $\frac{\mathrm{d}h(\lambda(x))}{\mathrm{d}x} = h'(\lambda(x))\frac{\mathrm{d}\lambda(x)}{\mathrm{d}x}$ and it follows from (60) [resp. Assumption 2] that $\frac{\mathrm{d}h(\lambda(x))}{\mathrm{d}x}$ [resp. $h'(\lambda(x))$] is continuous, $\frac{\mathrm{d}\lambda(x)}{\mathrm{d}x}$ is continuous. Hence, $\frac{\mathrm{d}g(\mathcal{H}(x))}{\mathrm{d}x}$

is bounded over $[x_-, x_+]$. Furthermore, differentiating (40a) with respect to x, we obtain

$$\frac{\mathrm{d}^2\mathcal{H}(x)}{\mathrm{d}x^2} = 2\tau g(\mathcal{H}(x)) \qquad \forall x \in [x_-, x_+]. \tag{93}$$

This shows that $\frac{\mathrm{d}^2\mathcal{H}(x)}{\mathrm{d}x^2}$ is bound over $[x_-, x_+]$. ■

References

Akamatsu, T., Fujishima, S., Takayama, Y., 2014. On stable equilibria in discrete-space social interaction models. MPRA Paper, University Library of Munich, Germany, 55938.

Allen, T., Arkolakis, C., 2014. Trade and the topography of the spatial economy. The Quarterly Journal of Economics 129 (3), 1085–1140.

Anas, A., Xu, R., 1999. Congestion, land use, and job dispersion: A general equilibrium model. Journal of Urban Economics 45 (3), 451–473.

Bapat, R. B., Raghavan, T. E. S., 1997. Nonnegative Matrices and Applications. Cambridge University Press.

Beckmann, M. J., 1976. Spatial equilibrium in the dispersed city. In: Papageorgiou, G. J. (Ed.), Mathematical Land Use Theory. Lexington Books, pp. 117–125.

Blanchet, A., Mossay, P., Santambrogio, F., 2016. Existence and uniqueness of equilibrium for a spatial model of social interactions. International Economic Review 57 (1), 31–60.

Borukhov, E., Hochman, O., 1977. Optimum and market equilibrium in a model of a city without a predetemined center. Environment and Planning A 9 (8), 849–856.

Braid, R. M., 1988. Heterogeneous preferences and non-central agglomeration of firms. Regional Science and Urban Economics 18 (1), 57–68.

Brown, G. W., von Neumann, J., 1950. Solutions of games by differential equations. In: Kuhn, H. W., Tucker, A. W. (Eds.), Contributions to the Theory of Games I. Princeton University Press.

Caruso, G., Peeters, D., Cavailhès, J., Rounsevell, M., 2007. Spatial configurations in a periurban city. A cellular automata-based microeconomic model. Regional Science and Urban Economics 37 (5), 542–567.

Caruso, G., Peeters, D., Cavailhès, J., Rounsevell, M., 2009. Space-time patterns of urban sprawl, a 1D cellular automata and microeconomic approach. Environment and Planning B 36 (6), 968–988.

Dupuis, P., Nagurney, A., 1993. Dynamical systems and variational inequalities. Annals of Operations Research 44 (1), 9–42.

Fujishima, S., 2013. Evolutionary implementation of optimal city size distributions. Regional Science and Urban Economics 43 (2), 404–410.

Fujita, M., Krugman, P. R., Mori, T., 1999. On the evolution of hierarchical urban systems. European Economic Review 43 (2), 209–251.

Fujita, M., Thisse, J.-F., 2013. Economics of Agglomeration: Cities, Industrial Location, and Globalization, 2nd Edition. Cambridge University Press.

Gilboa, I., Matsui, A., 1991. Social stability and equilibrium. Econometrica 59 (3), 859–867.

Griva, I., Nash, S. G., Sofer, A., 2009. Linear and Nonlinear Optimization. Society for Industrial and Applied Mathematics.

Hofbauer, J., Oechssler, J., Riedel, F., 2009. Brown-von Neumann-Nash dynamics: the continuous strategy case. Games and Economic Behavior 65 (2), 329–373.

Hofbauer, J., Sigmund, K., 1988. The Theory of Evolution and Dynamical Systems. Cambridge University Press.

Horn, R. A., Johnson, C. R., 2013. Matrix Analysis, 2nd Edition. Cambridge University Press.

LeVeque, R. J., 2007. Finite Difference Methods for Ordinary and Partial Differential Equations. Society for Industrial and Applied Mathematics.

Monderer, D., Shapley, L. S., 1996. Potential games. Games and Economic Behavior 14 (1), 124–143.

Mossay, P., Picard, P. M., 2011. On spatial equilibria in a social interaction model. Journal of Economic Theory 146 (6), 2455–2477.

O'Hara, D. J., 1977. Location of firms within a square central business district. The Journal of Political Economy 85 (6), 1189–1207.

Oyama, D., 2009a. Agglomeration under forward-looking expectations: Potentials and global stability. Regional Science and Urban Economics 39 (6), 696–713.

Oyama, D., 2009b. History versus expectations in economic geography reconsidered. Journal of Economic Dynamics and Control 33 (2), 394–408.

Sandholm, W. H., 2001. Potential games with continuous player sets. Journal of Economic Theory 97 (1), 81–108.

Sandholm, W. H., 2005. Negative externalities and evolutionary implementation. The Review of Economic Studies 72 (3), 885–915.

Sandholm, W. H., 2007. Pigouvian pricing and stochastic evolutionary implementation. Journal of Economic Theory 132 (1), 367–382.

Sandholm, W. H., 2010. Population Games and Evolutionary Dynamics. MIT Press.

Simsek, A., Ozdaglar, A., Acemoglu, D., 2007. Generalized Poincaré-Hopf theorem for compact nonsmooth regions. Mathematics of Operations Research 32 (1), 193–214.

Swinkels, J. M., 1993. Adjustment dynamics and rational play in games. Games and Economic Behavior 5 (3), 455–484.

Tabuchi, T., 1986. Urban agglomeration economies in a linear city. Regional Science and Urban Economics 16 (3), 421–436.

Takayama, Y., Akamatsu, T., 2010. Non-uniqueness and stability of equilibrium urban configuration for Beckmann's spatial interaction model. Journal of Japan Society of Civil Engineers, Series D 66 (2), 232–245.

Taylor, P. D., Jonker, L. B., 1978. Evolutionary stable strategies and game dynamics. Mathematical Biosciences 40 (1-2), 145–156.

Turner, M. A., 2005. Landscape preferences and patterns of residential development. Journal of Urban Economics 57 (1), 19–54.

Yueh, W.-C., Cheng, S. S., 2008. Explicit eigenvalues and inverses of tridiagonal Toeplitz matrices with four perturbed corners. The ANZIAM Journal 49 (03), 361.