Distributional Neutral Welfare Ranking-Extending Pareto Principle

Marjit, Sugata and Sarkar, Sandip

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Distribution-neutral Welfare Ranking-Extending Pareto Principle

Sugata Marjit[1] **and Sandip Sarkar**[2]

[1,2] *Centre for Studies in Social Sciences, Calcutta, India*

Abstract

The well known Pareto criterion used in the context of efficiency and welfare has to do with absolute changes whereas in every domain of economic behavior inequality or relative changes has become a major concern. We propose an inequality-preserving or distribution neutral Pareto criterion-the strong Pareto superior or SPS allocation which preserves the initial distribution and makes everyone better off. Our main result is that whenever there is a gain in the aggregate value of the relevant attribute, there exists a unique counterfactual allocation which is SPS.

Keywords Pareto Superiority, Strong Pareto Superiority, Inequality, Distribution

1 Introduction

Pareto ranking or Pareto efficiency is a topic economists are exposed to very early in their career. In particular the basic welfare comparison between two social situations starts with the ranking in terms of a principle Pareto have talked about in the nineteenth century. If we compare two social situations A and B, we say B is Pareto superior to A iff everyone is as well of in B as in A and at least one is strictly better off in situation B compared to A. This comparison is done in terms of utility or welfare levels individuals enjoy in A and B. In the theory of social welfare this has been a widely discussed topic with seminal contributions from De Scitovszky (1941), Samuelson (1958), Arrow (1963), Stiglitz (1987), Sen (1970) and others to make recent treatments such as Mandler (1999), Cornes and Sandler (2000) and many others.

Pareto's principle provides a nice way to compare situations when some gain and some lose by considering whether transfer from gainers to losers can lead to a new distribution in B such that B turns out to be Pareto superior to A, the initial welfare distribution. It is obvious that if sum of utilities increases in B relative to A, then whatever be the actual distribution in B, a transfer mechanism will always exist such that transfer-induced redistribution will make B Pareto superior to A. The great example is how gains from international trade can be redistributed in favor of those who lose from trade such that everyone gains due to trade. Overall gains from trade lead to a highly level of welfare, under ideal conditions and therefore one can show that under free trade eventually nobody may lose as gainers *'bribe'* the losers. But whatever it is Pareto ranking definitely does not address the inequality issue. There will be situations where B will be Pareto superior to A, but inequality in B can be much greater than A. The purpose of this short paper is to extend the basic principle of Pareto's welfare ranking subjecting it to a stricter condition that keeps the degree of inequality intact between A and B after transfer from gainers to losers, but at the same time guaranteeing that everyone gains in the end.

Thus we coin a Strong-Pareto criterion which not only insists that everyone must be better of in B compared to A, but also requires that degree of inequality must remain the same between A and B. Only then B will be Strongly Pareto Superior (SPS) to A. Concern for such a Strong principle stems from the fact that people do care about inequality and inequality has become a worldwide popular point of debate in public domain (see Stewart, 2004; Stiglitz, 2012; Piketty, 2014; Atkinson and Stiglitz, 2015, for further readings).

Pareto superior move as such may not contain agitation to change policies further because of rising inequality. We are also motivated by the query as to whether the basic condition that guarantees Pareto superiority of B to A, would also guarantee that B is SPS to A. Apparently it need not be since there can be transfer that make B PS A, but that aggravate inequality.

We show that if total utility in B is grater than total utility of A, we can always construct a counterfactual distribution C which is SPS to A. The counterfactual allocation is obtained by taxing a subset of individual and redistributing the collected tax to the rest of the individuals. In order to keep the inequality level same we redistribute the aggregate gains proportionate to that of individuals utility at the initial stage.

Section 2 describes the environment and the result. Last section concludes.

2 Model

Consider an n $(n > 1)$ agent society being observed for two time points. The initial time point is denoted by 0, whereas the final time point is denoted by 1. The utility profile for the set of individuals at time $t(t \in \{0, 1\})$ is defined in the following fashion:

$$U_t = \{u_{t1}, u_{t2}, ..., u_{tn}\}, \ \forall t \in \{0, 1\} \tag{1}$$

We assume that the individual utilities are cardinal. Furthermore, we also assume that the individual utilities are strictly positive, i.e. $u_{ti} > 0$, $\forall t \in \{0, 1\}$ and $\forall i \in \{1, 2, .., n\}$.[1] Let $\mathbb{D}^n$ be the set of all such n coordinated utility profiles.

Pareto superiority (PS) is defined as the situation where no one looses from the initial to the final period but at least one individual gains. However, PS allocation may aggravate inequality. We thus introduce "*Strong Pareto Superiority*" (SPS). By SPS we mean a situation where the utility of all the individuals increases and the inequality also remains same, compared to that of the initial distribution. In the present paper by inequality we restrict our attention to the family of relative inequality indices of the form $I : \mathbb{D}^n \longrightarrow \mathbb{R}$ which are homogeneous of degree 0, i.e.,

$$I(u_1, u_2, .., u_n) = I(\delta u_1, \delta u_2, .., \delta u_n) \ \forall \delta > 0 \text{ and } \delta \text{ is finite.} \tag{2}$$

In order to derive such a SPS allocation we first assume that their exists a social planner who taxes a subgroup of the population and distributes the collected tax to the rest of the population. Throughout this paper we refer this phenomenon as a tax-transfer mechanism[2]. However, we also impose an additional condition that such tax-transfer mechanism keeps the level of inequality same as that of the initial distribution where the class of inequality measures satisfies equation 2. We thus formally define the notion of SPS allocations in the following fashion:

Definition 1. ***SPS allocation*** *For all $U_0, U_1 \in \mathbb{D}^n$, and let $\hat{U} = \{\hat{u}_1, \hat{u}_2, .., \hat{u}_n\} \in \mathbb{D}^n$ be the counterfactual allocation which is obtained from U_1 following a tax-transfer mechanism. Then $\hat{U}$ is said to be a SPS allocation to U_0 which is denoted by $\hat{U} \succ_{SPS} U_0$, if and only if $\hat{u}_i > u_{0i}$ and $\frac{u_{0i}}{u_{0j}} = \frac{\hat{u}_i}{\hat{u}_j}$, $\forall i, j \in \{1, 2, .., n\}$.*

Note that if we scale up utilities of all the individual's of the initial allocation by any positive scalar greater than 1, we necessarily get a SPS allocation.

[1]We make this assumption for mathematical simplicity. We can always allow an utility function which takes negative values. However, in such cases we have to restrict the class of utility functions that are invariant to any change in the origin of the utilities of all the individuals belonging in that profile.

[2]Note that the tax-transfer mechanism is best understood if we consider income profiles. In our context one can assume utility is an increasing function of income. Hence, the gain(loss) of utility is also an increasing function of the amount of money received(paid) as a transfer (tax).

Nevertheless, such an allocation is not feasible if the aggregate utility of the counterfactual allocation exceeds that of the final allocation. Formally we define a feasible SPS allocation in the following fashion:

Definition 2. ***Feasible SPS allocation:*** *For all $U_0, U_1 \in \mathbb{D}^n$, and $\exists \hat{U} = \{\hat{u}_1, \hat{u}_2, .., \hat{u}_n\} \in \mathbb{D}^n$ such that $\hat{U} \succ_{SPS} U_0$, then $\hat{U}$ is said to be a feasible SPS allocation if and only if $\sum_{i=1}^{n} \hat{u}_i \leq \sum_{i=1}^{n} u_{1i}$.*

A feasible SPS allocation may not be the most efficient. This is particularly when there is some resource left as a residual which can be further redistributed amongst the agents to make every one better off. We define the most efficient SPS allocation, among the set of feasible SPS allocations in the following fashion:

Definition 3. ***Most efficient SPS allocation:*** *For all $U_0, U_1 \in \mathbb{D}^n$, and $\exists \hat{U} = \{\hat{u}_1, \hat{u}_2, .., \hat{u}_n\} \in \mathbb{D}^n$ such that $\hat{U} \succ_{SPS} U_0$, then $\hat{U}$ is said to be the most efficient SPS allocation if and only if $\hat{U}$ is a feasible SPS allocation and $\hat{U}$ is Pareto Superior to any other feasible SPS allocation.*

We now characterize one important property associated with the most efficient SPS allocation.

Lemma 1. *For all $U_0, U_1 \in \mathbb{D}^n$, and $\exists \hat{U} = \{\hat{u}_1, \hat{u}_2, .., \hat{u}_n\} \in \mathbb{D}^n$ such that $\hat{U} \succ_{SPS} U_0$, then $\hat{U}$ is the most efficient SPS allocation if and only if $\sum_{i=1}^{n} \hat{u}_i = \sum_{i=1}^{n} u_{1i}$.*

Proof:

Only if

Let $\sum_{i=1}^{n} \hat{u}_i \neq \sum_{i=1}^{n} u_{1i}$. Now if $\sum_{i=1}^{n} \hat{u}_i > \sum_{i=1}^{n} u_{1i}$ the SPS allocation is infeasible, hence $\hat{U}$ is not the most efficient. On the other hand if $\sum_{i=1}^{n} \hat{u}_i < \sum_{i=1}^{n} u_{1i}$, we can always construct another feasible SPS allocation $\hat{Z} = \{\hat{z}_1, \hat{z}_2, .., \hat{z}_n\} \in \mathbb{D}^n$ where $\hat{z}_i = \hat{u}_i + \left(\frac{\sum_{i=1}^{n} u_{1i} - \sum_{i=1}^{n} \hat{u}_i}{\sum_{i=1}^{n} u_{0i}} \right) u_{0i}$. Clearly $\hat{Z}$ is Pareto superior to $\hat{U}$ which implies that $\hat{U}$ is not a most efficient SPS allocation.

If

Suppose $\hat{V} = \{\hat{v}_1, \hat{v}_2, .., \hat{v}_n\} \in \mathbb{D}^n$ be the most efficient SPS allocation. Hence, $\hat{V}$ is Pareto superior to any other allocation including $\hat{U}$. Given that both $\hat{V}$ and $\hat{U}$ are SPS allocations they are related in the following fashion: $\hat{v}_i =$

$\hat{u}_i + \kappa.u_{0i}$ where $\kappa \in \mathbb{R}$. Now $\hat{V}$ being the most efficient SPS implies $\kappa > 0$. Furthermore, a necessary condition for a most efficient SPS allocation is feasibility which implies that $\sum_{i=1}^{n} \hat{v}_i \leq \sum_{i=1}^{n} u_{1i} \implies \sum_{i=1}^{n} \left(\hat{u}_i + \kappa.u_{0i}\right) \leq \sum_{i=1}^{n} u_{1i}$. Now given $\kappa > 0 \implies \sum_{i=1}^{n} \hat{u}_i < \sum_{i=1}^{n} u_{1i}$, which is a contradiction. **Q.E.D.**

We are now ready to introduce the main result of this paper. By SPS allocations we mean a counterfactual distribution which has same inequality as the initial distribution and also is Pareto superior to the initial distribution. Obviously such a distribution will never exist if there is aggregate loss in the society. This is because the net loss must make at least one individual worse off and eventually there does not exist any feasible Pareto Superior allocation. However, if there is a net gain any SPS allocation can be obtained by taxing a subgroup of individual and transferring the collected tax to the rest of the population. Our next result characterizes the most efficient allocation and the associated tax-transfer vector. Formally:

Proposition 1. *For all $U_0, U_1 \in \mathbb{D}^n$, $\exists \hat{U} = \{\hat{u}_1, \hat{u}_2, .., \hat{u}_n\} \in \mathbb{D}^n$ such that $\hat{U} \succ_{SPS} U_0$ and $\hat{U}$ is the most efficient SPS allocation, if and only if $\sum_{i=1}^{n} u_{1i} > \sum_{i=1}^{n} u_{0i}$ and the tax-transfer vector is $T = \{T_1, T_2, .., T_n\} = \{u_{11} - \hat{u}_1, u_{12} - \hat{u}_2, .., u_{1n} - \hat{u}_n\}$, and $\hat{u}_i = u_{0i}\left(\frac{\sum_{i=1}^{n} u_{1i}}{\sum_{i=1}^{n} u_{0i}}\right)$, $\forall i \in \{1, 2, .., n\}$.*

Proof: Only if

Given $\hat{U} \succ_{SPS} U_0 \implies \exists \theta$ such that $\hat{u}_i = \theta.u_{0i}$ $\forall i \in \{1, 2, .., n\}$ and $\theta > 1$. Now since $\hat{U}$ is the most efficient SPS allocation, following lemma 1 we can write $\sum_{i=1}^{n} \hat{u}_i = \sum_{i=1}^{n} u_{1i} \implies$.

$$\theta = \frac{\sum_{i=1}^{n} u_{1i}}{\sum_{i=1}^{n} u_{0i}} \tag{3}$$

Now $\theta > 1 \implies \sum_{i=1}^{n} u_{1i} > \sum_{i=1}^{n} u_{0i}$. Putting $\hat{u}_i = \theta u_{0i} = u_{0i} \frac{\sum_{i=1}^{n} u_{1i}}{\sum_{i=1}^{n} u_{0i}}$ in the elements of T, (i.e., $T_i = u_{1i} - \hat{u}_i$), we can write $T_i = u_{1i} - u_{0i}\left(\frac{\sum_{i=1}^{n} u_{1i}}{\sum_{i=1}^{n} u_{0i}}\right)$.

If

Given

$$\hat{u}_i = u_{0i}.\left(\frac{\sum_{i=1}^{n} u_{1i}}{\sum_{i=1}^{n} u_{0i}}\right) \tag{4}$$

Furthermore, it is also given that $\sum_{i=1}^{n} u_{1i} > \sum_{i=1}^{n} u_{0i} \implies \hat{u}_i > u_{0i}$. Since, 4 is satisfied it implies that the distributions $\hat{U}$ and U_0 have same inequality, following any relative inequality measure satisfying property 2. Furthermore, from equation 4 we have $\sum_{i=1}^{n} \hat{u}_i = \sum_{i=1}^{n} u_{1i}$ Combining these three arguments it is straightforward to write that $\hat{U} \succ_{SPS} U_0$ and $\hat{U}$ is also most efficient. **Q.E.D.**

For an illustration of the SPS allocations consider $U_0 = \{1, 2, 5, 12\}$ and $U_1 = \{1, 1, 27, 111\}$. Following the proposed approach the most efficient SPS allocation would be $\hat{U} = \{7, 14, 35, 84\}$, whereas the associated tax transfer vector is $\{6, 13, 8, -27\}$. In other words the first three individual receives transfers, whereas the last individual pays tax. Note that the tax payer must be a gainer from the initial to the final period. However, all the gainers are not tax payers (3rd individual).

It is quite straightforward to show that there exists infinitely many SPS allocations and associated tax transfer mechanisms. However, if we focus our attention only on the most efficient SPS allocation then the counterfactual allocation and eventually the associated tax transfer vector is unique. We illustrate this formally in the following fashion:

Proposition 2. *Given Proposition 1, the most efficient SPS allocation $\hat{U}$ and the associated Tax transfer vector T is unique.*

Proof: We begin with the assumption that there exists any arbitrary $\tilde{T} = \{\tilde{T}_1, \tilde{T}_2, .., \tilde{T}_n\} \in \mathbb{R}^n$ associated with a counterfactual SPS allocation $\hat{V} = \{\hat{v}_1, \hat{v}_2, ..., \hat{v}_n\} = \{u_{11} - \tilde{T}_1, u_{12} - \tilde{T}_2, .., u_{1n} - \tilde{T}_n\} \in \mathbb{D}^n$, such that $\tilde{T} \neq T$. Let us begin with the assumption that both $\hat{U}$ and $\hat{V}$ are most efficient SPS allocation. Now any two vectors of the same order are related in the following fashion: $\tilde{T} = T + \epsilon$ where $\epsilon = \{\epsilon_1, \epsilon_2, .., \epsilon_n\} \in \mathbb{R}^n$. Since $T \neq \tilde{T} \implies \exists i \in \{1, 2, .., n\}$ such that $\epsilon_i \neq 0$. Clearly, inequality in $\hat{V}$ is same as U_0 following 2 if and only if $\exists \alpha (\alpha \in \mathbb{R})$ such that $\epsilon_i = \alpha u_{0i}$, $\forall i \in \{1, 2, .., n\}$. Hence we can write $\hat{v}_i = u_{0i}.\left(\theta - \alpha\right)$, where $\theta = \frac{\sum_{i=1}^{n} u_{1i}}{\sum_{i=1}^{n} u_{0i}}$. Clearly if $\alpha > 0$ the allocation is a feasible SPS but is not the most efficient. On the other hand if $\alpha < 0$

then $\sum_{i=1}^{n} \hat{v}_i > \sum_{i=1}^{n} u_{1i} \implies$ the allocation is not feasible and eventually is not the most efficient. Hence, $\alpha = 0 \implies T \equiv \tilde{T}$ and $\hat{U} \equiv \hat{V}$. **Q.E.D.**

Note that unlike SPS, uniqueness is not guaranteed for PS allocations. In order to illustrate this we consider the following utility profiles: $U_0 = \{1, 2, 3\}$ and $U_1 = \{1.5, 2, 5.5\}$. Consider the following two counterfactual allocations: $\tilde{U} = \{1.5, 2.5, 5\}$, $\bar{U} = \{2.5, 3, 3.5\}$. Here both $\tilde{U}$ and $\bar{U}$ are PS to their initial distributions and both are most efficient.

In the formulation of the SPS we have considered the notion of relative inequality measures. However, these results can also be extended to obtain a unique most efficient SPS allocation which preserves absolute inequality i.e., family of measures which are translation invariant[3] (see Kolm, 1976a,b, for further details on absolute inequality measures).

3 Conclusion

We have extended the basic Pareto principle to focus on inequality-neutral or distribution neutral Pareto superior allocation which we call strongly Pareto superior or SPS allocation which guarantees higher individual welfare keeping the degree of inequality same as before. We have shown that whenever there is aggregate gain in the society we can compute a counterfactual distribution obtained by taxing a subgroup of population and redistributing the collected tax to the rest of the population such that the counterfactual allocation is a SPS allocation. In the counterfactual distribution the aggregate gains of utility has been redistributed among the individuals in the proportionate to their utilities of the initial distribution. This keeps the inequality level same and also ensures that the SPS is feasible and is the most efficient one.

Our approach retains the spirit of Pareto criterion but instead of keeping the absolute level intact, we keep the inequality level the same. Any Pareto superior allocation can be converted to a SPS allocation.

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[3] An inequality measure $I : \mathbb{D}^n \longrightarrow \mathbb{R}$ is translation invariant if and only if $I(u_1, u_2, .., u_n) = I(\delta + u_1, \delta + u_2, .., \delta + u_n)\ \forall \delta \in \mathbb{R}$.

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