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Abstract

Farinelli and Tibiletti (2008) propose a general risk-reward performance measurement ratio. Due to its simplicity and generality, the F-T ratios have gained much attentions. F-T ratios are ratios of average gains to average losses with respect to a target, each raised by some power index. Omega ratio and Upside Potential ratio are both special cases of F-T ratios. In this paper, we establish the consistency of F-T ratios with respect to first-order stochastic dominance. It is shown that second-order stochastic dominance is not consistent to the F-T ratios. This point is illustrated by a simple example.

KEYWORDS: Stochastic Dominance, Upside Potential Ratio, Farinelli and Tibiletti ratio.

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1 Introduction

Due to its simplicity and easy interpretation, the Sharpe ratio is widely used in the investment industry. However, the standard deviation, which is adopted in the Sharpe ratio, is not a good

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measure of risk because it penalizes upside deviation as well as downside deviation. In fact people view upside and downside deviation differently. The occurrence of negative returns over some target return is considered as risk and the occurrence of positive returns over the target return is what we would like to see. Downside risks thus become important components in the construction of performance measures. Classic measures of downside risk include semi-deviation and absolute semi-deviation (see Markowitz, 1959, 1987). Risk measures based on below-target returns are first proposed by Fishburn (1977) in the context of portfolio optimization. Farinelli and Tibiletti (2008) propose a general risk-reward ratio suitable to compare skewed returns with respect to a benchmark. The F-T ratios are essentially ratios of average above-benchmark returns (gains) to average below-benchmark returns (losses), each raised by some power index to proxy for the investor's degree of risk aversion. When the power index is equal to one for both numerator and denominator, the performance measure is the Omega ratio, first discussed by Keating and Shadwick (2002).

This paper focuses on the F-T ratio because of its intuitive simplicity and generality. It is clear that the higher is an investment's F-T ratio, the more attractive it is to an investor who cares about downside risk. On the other hand, stochastic dominance (SD) theory can be used to compare different investments without assuming specific form of utility function. Then people may ask the following question: if we find an investment is preferred compared with another one by stochastic dominance theory, can its F-T ratios always higher than those of the other one? In this paper, we show that the answer depends on the order of stochastic dominance. Specifically, it is proven that first-order stochastic dominance is consistent with the F-T ratio for pairwise comparisons. However, this is not true for higher-order stochastic dominance. We present a simple example to show that second-order stochastic dominance is not consistent with the F-T ratio.

The rest of this paper is organized as follows. Section 2 gives a brief introduction of SD theory. Section 3 contains our main result. Section 4 concludes the paper.

2 Definitions and Notations

We define the j-order integral, $F_Z^{(j)}$, of Z to be

$$F_Z^{(j)}(\eta) = \int_{-\infty}^{\eta} F_Z^{(j-1)}(\xi) d\xi \ , \tag{2.1}$$

and the j-order reverse integral, $F_Z^{(j)R}$, of Z to be

$$F_Z^{(j)R}(\eta) = \int_{\eta}^{\infty} F_Z^{(j-1)R}(\xi) d\xi \ , \tag{2.2}$$

with $F_Z^{(0)R} = F_Z^{(0)} = f_Z$ to be the probability density function (pdf) of Z for $Z = X, Y$ and for any integer j. When $j = 1$, $F_Z^{(1)} = F_Z$ is the cumulative distribution function (cdf) of Z.

Following the definition of stochastic dominance (SD), see, for example, Hanoch and Levy (1969), prospect X first-order stochastically dominates prospect Y, denoted by

$$X \succeq_{FSD} Y \quad \text{if and only if} \quad F_X^{(1)}(\eta) \leq F_Y^{(1)}(\eta) \quad \text{for any} \ \ \eta \in R, \tag{2.3}$$

and prospect X nth-order stochastically dominates prospect Y, denoted by

$$X \succeq_{nSD} Y \ \ \text{if and only if} \ \ F_X^{(n)}(\eta) \leq F_Y^{(n)}(\eta) \ \text{for any} \ \ \eta \in R, \ \text{and} \ \ F_X^{(k)}(\infty) \leq F_Y^{(k)}(\infty) \tag{2.4}$$

with $2 \leq k \leq n$. Here, FSD and nSD stands for first- and nth-order stochastic dominance. For $n = 2$, 2SD can also be written as SSD (Second-order Stochastic Dominance). We note that

$$\text{if} \ \ X \succeq_{nSD} Y \quad \text{for any} \ \ n \geq 1 \ , \quad \text{then} \quad \mu_X \geq \mu_Y \ . \tag{2.5}$$

We need this property in the proofs of the theorems we developed in our paper.

Now, we follow Li and Wong (1999), Levy (2015), Guo and Wong (2016), and others to define risk-seeking stochastic dominance (RSD)[1] for risk seekers. Prospect X second-order risk-seeking stochastically dominates prospect Y, denoted by

$$X \succeq_{SRSD} Y \quad \text{if and only if} \quad F_X^{(2)R}(\eta) \geq F_Y^{(2)R}(\eta) \quad \text{for any} \ \ \eta \in R. \tag{2.6}$$

Here, SRSD or 2RSD denotes second-order RSD.

We turn to define Farinelli and Tibiletti (FT) ratio. Formally, for any prospect X, its FT ratio $\phi_{FT,X}(\eta)$ is defined as:

$$\phi_{FT,X}(\eta) = \frac{(E[(X-\eta)_+^p])^{1/p}}{(E[(\eta - X)_+^q])^{1/q}}.$$

Here, $x_+ = \max\{0, x\}$ and η is called the return threshold. For any investor, returns below her return threshold are considered as losses and returns above as gains. Furthermore, p and q are positive values to present investor's degree of risk aversion. Thus, the F-T ratio is the ratio of

[1]Levy (2015) denotes it as RSSD while we denote it as RSD.

average gain to average loss, each raised by some power index to proxy for the investor's degree of risk aversion.

As an illustration, we first consider the Upside Potential Ratio (Sortino et al. 1999). In fact, if we take $p = 1$ and $q = 2$, the above defined F-T ratio reduces to the Upside Potential Ratio, which is defined as follows:

$$U_X(\eta) = \frac{E[(X-\eta)_+]}{\sqrt{E[(\eta - X)_+^2]}}.$$

Applying Proposition 1 in Ogryczak and Ruszczyński (2001), we obtain the following general FT ratio in our paper since the FT ratio can be rewritten as:

$$\phi_{FT,X}(\eta) = \frac{(E[(X-\eta)_+^p])^{1/p}}{(E[(\eta - X)_+^q])^{1/q}} = \frac{(E[(X-\eta)_+^p])^{1/p}}{(q!F_X^{(q+1)}(\eta))^{1/q}} \ . \tag{2.7}$$

We state the following definition for the FT ratio:

Definition 2.1 *For any two prospects X and Y with FT ratios, $\phi_{FT,X}$ and $\phi_{FT,Y}$, respectively, X is said to dominate Y by the FT ratio, denote by*

$$X \succeq_{FT} Y \quad \textit{if} \quad \phi_{FT,X}(\eta) \geq \phi_{FT,Y}(\eta), \ \ \textit{for any } \eta \in R. \tag{2.8}$$

3 The Theory

Is mean-risk model consistent with stochastic dominance rule? Markowitz (1952) defined a mean-variance rule for risk averters and Wong (2007) defined a mean-variance rule for risk seekers. Wong (2007) further established consistent of mean-variance rules with second-order SD (SSD) rules under some conditions. Ogryczak and Ruszczyński (1999) showed that under some conditions the standard semi-deviation and absolute semi-deviation make the mean-risk model consistent with the SSD, Ogryczak and Ruszczyński (2002) established the equivalence between TVaR and the second-order stochastic dominance. In addition, Leitner (2005) showed that AV@R as a profile of risk measures is equivalent to the SSD under certain conditions. Ma and Wong (2010) showed the equivalence between SSD and the C-VaR criteria. Thus, some academics believe that FT ratio is consistent with SSD.

In this paper, we first establish the following property to say the relationship between FT ratio and SSD:

Property 3.1 *FT ratio is not consistent with SSD in the sense that for any two prospects X and Y with FT ratios $\phi_{FT,X}$ and $\phi_{FT,Y}$, respectively, the following statement does not hold:*

$$X \succeq_{SSD} Y \Rightarrow X \succeq_{FT} Y \ . \tag{3.1}$$

We construct the following example to support the argument stated in Property 3.1.

Example 3.1 *Consider two prospects X and Y with the following distributions:*

$$X = 10 \quad \textit{with prob. } 1, \quad \textit{and} \quad Y = \begin{cases} 1 & \textit{with prob. } 2/3 \\ 11 & \textit{with prob. } 1/3 \end{cases} . \tag{3.2}$$

We have $\mu_X = 10$ and $\mu_Y = 13/3$ and obtain the following

$$F_X^{(2)}(\eta) = \begin{cases} 0 & \text{if } \eta < 10 \\ \eta - 10 & \text{if } \eta \geq 10 \end{cases} , \quad F_Y^{(2)}(\eta) = \begin{cases} 0 & \text{if } \eta < 1 \\ 2(\eta - 1)/3 & \text{if } 1 \leq \eta < 11 \\ \eta - 13/3 & \text{if } \eta \geq 11 \end{cases} ,$$

$$F_X^{(2)R}(\eta) = \begin{cases} 10 - \eta & \text{if } \eta < 10 \\ 0 & \text{if } \eta \geq 10 \end{cases} , \quad F_Y^{(2)R}(\eta) = \begin{cases} 13/3 - \eta & \text{if } \eta < 1 \\ (11 - \eta)/3 & \text{if } 1 \leq \eta < 11 \\ 0 & \text{if } \eta \geq 11 \end{cases} .$$

It is easy to observe that $F_X^{(2)}(\eta) \leq F_Y^{(2)}(\eta)$, for all $\eta \in R$; that is, $X \succeq_{SSD} Y$. However, for any $10 \leq \eta < 11$, we have $F_X^{(2)R}(\eta) \equiv 0 < F_Y^{(2)R}(\eta)$. Recall the definition of $U_X(\eta)$, we can conclude that $U_X(\eta) \equiv 0 < U_Y(\eta)$ for any $10 \leq x < 11$, and thus, $X \not\succeq_{FT} Y$.

Thus, Example 3.1 shows that SSD is not sufficient to imply $U_X(\eta) \geq U_Y(\eta)$ for any η. However, we find that FSD is consistent with the FT ratio as shown in the following theorem:

Theorem 3.1 *For any two returns X and Y with Farinelli and Tibiletti ratios $\phi_{FT,X}(\eta)$ and $\phi_{FT,Y}(\eta)$, respectively, if $X \succeq_{FSD} Y$, then*

$\phi_{FT,X}(\eta) \geq \phi_{FT,Y}(\eta)$ for any $\eta \in R$ and for any nonnegative values p and q.

We give a short proof in the following: Define $u(x) = (x - \eta)_+^p$. It is easy to know that this function is non-decreasing. It is known that FSD is equivalent to the expected-utility/wealth maximization for any investor with increasing utility functions. As a result, if $X \succeq_{FSD} Y$, we must have $E[(X - \eta)_+^p] \geq E[(Y - \eta)_+^p]$ for any positive p.

If $X \succeq_{FSD} Y$, then due to the hierarchy, $X \succeq_{(q+1)SD} Y$ for any $q \geq 0$ holds. Immediately, we can have $F_X^{(q+1)}(\eta) \leq F_Y^{(q+1)}(\eta)$ for any $\eta \in R$. Since $E[(X - \eta)_+^p]$ is always positive, then we can conclude that $\phi_{FT,X}(\eta) \geq \phi_{FT,Y}(\eta)$ for any $\eta \in R$. Thus, the assertions in Theorem 3.1 holds.

4 Conclusions

In practice, investors care about losses more than gains of similar magnitude. The gains and losses are relative to specified benchmarks. Returns below the benchmarks are considered as losses and returns above as gains. The F-T ratio encodes both of these features in a simple way. We have shown that the simplicity of the F-T ratio belies its intimate connection with expected utility theory for all non-satiated investors (first-order stochastic dominance). However, the second-order stochastic dominance is in general not consistent with F-T ratio. We illustrate this point by a simple example.

We note that there are many studies, for example, Fong, et al. (2005, 2008), Egozcue and Wong (2010), Chan, et al. (2012), Qiao, et al. (2012), Vieito, et al. (2015), and Clark, et al. (2016), and others develop and/or apply SD to study some important issues in economics and finance. On the other hand, there are many studies, see, for example, Broll, et al. (2006, 2015), Leung and Wong (2008), Wong and Ma (2008), Ma and Wong (2010) and Bai, et al. (2009, 2012, 2013) study mean-risk models or apply the models to study some important issues in economics and finance. But as far as we know, there is no study applying both Farinelli and Tibiletti ratio and SD to address any important issue in economics and finance. With the theory developed in this paper, academics and practitioners would be able to apply both FT ratio and SD to study some important issues in economics and finance. For example, Qiao and Wong (2015), Tsang et al. (2016) and others apply SD to study the Hong Kong Housing market. Tsang et al. (2016) find that there are FSD relationship among smaller house and bigger house while Qiao and Wong (2015) find that there is not.

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