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11 November 2016

Online at https://mpra.ub.uni-muenchen.de/75030/
MPRA Paper No. 75030, posted 12 November 2016 07:06 UTC

ORDINAL SPACE, UTILITY, AND CONSUMER DEMAND: A CLARIFYING NOTE

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SUMMARY: Concepts such as marginal utility, expected-utility, etc. are severely criticized in some quarters where economists are accused of performing mathematical operations in ordinal spaces. Haplessly, economists' counter-claims are far from being substantive. This note shows that there exists an order-isomorphism relating preference ordering to a substantive set of real numbers and thus obviates the need for a utility index.

KEYWORDS: Ordinal Spaces, Binary Relation, Poset, Total Pre-ordering, Isomorphisms.

JEL Classification: D10, D11.

INTRODUCTION:

In a few recent papers, Barzilai (2010, 2011, 2013) reminds us that on ordinal spaces (non Euclidean) only *relations* are defined and, therefore, all elementary mathematical operations are inapplicable, including concepts of algebra and calculus. Thus, if norms, metrics, derivatives, convexity concepts, etc. are undefined in ordinal spaces, then ordinal utility functions are not differentiable. Consequently, works on decision theory by von Neumann and Morgenstern (1944), Pareto (1971), Hicks (1946), Samuelson (1948), and Debreu (1954) are based on fundamental errors. So stated, Barzilai then accuses these economists and their followers of misidentifying mathematical spaces, and of drawing erroneous conclusions such as the derivation of individual demand functions and their properties from constrained utility maximization.

Katzner (2014) took issue with these claims, arguing in return that Barzilai's criticism is founded on the general notion of ordinality arising out of the mathematical theory of measurement. The theory of consumer demand in economic theory, he argues, is based on a *different notion of ordinal utility*, which is independent of the mathematical theory of measurement. Thus according to Katzner, whether it is possible to measure utility ordinally or cardinally has to do with the kind of scale upon which the elements of the function values of the utility functions are measured. In other words, the ordering of baskets of commodities is used to construct a scale on which utility is measured in utils, say. Katzner next assumes that under certain technical restrictions (not specified), there exists an ordinal scale to measure utility. In Katzner's words (2014, 2):

> "Assume that some ordering relation orders the objects of [the set] A by pleasure. Under certain technical restrictions on that ordering, there exists an ordinal scale for measuring pleasure. When a utility function is present and when its function values are taken to be ordinally measured in this sense, the functions map baskets of commodities into measured pleasure recorded as quantities of utils."

The nebulosity of that argument stretches the confusion. The economist uses the *observed* preference ordering to construct a utility scale. Then on p. 3, Katzner states that the utility function values have no intrinsic meaning other than the information they provide concerning preferences *already observed*. One finds the same reasoning in Hicks (1946, 4-19) and in Varian who writes (1978, 81):

> "In classical economic theory, one often summarizes a consumer's behavior by means of a [n unobservable] utility function …."

And a few lines below:

> "A utility function is often a very convenient way to describe preferences, but it should not be given any psychological interpretation. The only relevant feature of a utility function is its ordinal character."

Confusion abounds. Utility functions summarize consumer behavior. Utility functions describe preferences but their relevant feature is their ordinal character. Yet we perform mathematical operations on them despite their ordinal character.

Barzilai's argument focuses on the fact that von Neumann and Morgenstern have built a scale to measure preference and next seem to equate preference and utility. Thus they failed to present an order-monomorphism from preference to something real on which mathematical operations could be performed. In that connection, notions such as expected utility and constrained maximization via the Lagrangian method are devoid of substance. Katzner too failed to touch on this crucial element. Instead, he refers the reader to another source for technical restrictions.

The purpose of this note is first clarify both the claims of Barzilai and the counterclaims of Katzner, and next to show that the needed isomorphism exists; therefore, the whole notion of utility is vacuous.

PRELIMINARIES

Terminology and symbols may vary from author to author, but concepts and properties do not give rise to controversies. For tractability, therefore, we begin by defining our main terms. And since the properties of the relation of order are standard, we will state them without proof.

Poset: is a partially ordered consumption set, say, X whose elements are x, y ,z, ….

Binary relation: A binary relation R on a set X is a subset of the Cartesian product, denoted: $(X, R) \subseteq X \times X$. A binary relation R on a set X is a *partial order* if it satisfies: $\Delta_x \subseteq R$; $R \cap R^{-1} \subseteq \Delta_x$, and $R \circ R \subseteq R$. And it is a *quasi-order* if it satisfies: $\Delta_x \cap R = \emptyset$; $R \cap R^{-1} \subseteq \Delta_x$, and $R \circ R \subseteq R$.

Diagonal of X: The diagonal of X is denoted: $\Delta_x = \{(x, x) | x \in X\}$.

Inverse of R: The inverse of R is denoted: $R^{-1} = \{(y, x) \mid (x, y) \in R$.

Total Preorder: A total preorder on X, denoted (X, $\precsim$), is a preorder such that if (x, y) $\in$ X, then (x $\precsim$ y) $\vee$ (y $\precsim$ x) $\vee$ (x ~ y). If (X, $\precsim$) is a preorder, then (x $\prec$ y) $\Leftrightarrow$ (x $\precsim$ y) $\wedge$ $\neg$ (y $\precsim$ x); if (x ~ y) $\Leftrightarrow$ (x $\precsim$ y) $\vee$ (y $\precsim$ x). If a preorder is antisymmetric, we have: x $\precsim$ y $\vee$ y $\precsim$ x $\Leftrightarrow$ x = y, then it is a poset.

Ordinal Space: An ordinal space is a set such as X on which only the relations of order and equivalence are defined.

We will now focus on the properties of the relations of order and equivalence.

The *Quasi-order* Relation satisfies[1]:

i) $\Delta_x \cap R = \emptyset$;

ii) $R \cap R^{-1} \subseteq \Delta_{x}$;

iii) $R \circ R \subseteq R$.

That is, the Quasi-order relation is irreflexive, antisymmetric, and transitive. If it is reflexive, then it is a poset. The equivalence relation, on the other hand, satisfies:

iv) $\Delta_x \subseteq R$;

v) $R = R^{-1}$;

vi) $R \circ R \subseteq R$.

Thus, the main difference between the two relations is that the quasi-order relation is irreflexive, and antisymmetric; the poset is reflexive and antisymmetric, while the equivalence relation is reflexive, symmetric, and transitive.

Remark 1: In a poset on a set X, the following relations are defined as follows: $\preccurlyeq$ is the inverse of $\succcurlyeq$; $\prec$ is the inverse of $\succ$. Further, if x $\preccurlyeq$ y, then x $\prec$ y $\vee$ y = x; if x $\prec$ y, then x $\preccurlyeq$ y $\wedge$ x $\neq$ y; if x $\gtrsim$ y, then y $\preccurlyeq$ x; and if x $\succ$ y, then y $\prec$ x.

THE RESULT

It follows that if the pair (X, $\lesssim$) is a total preorder on X that is antisymmetric[2]), it is then said to be representable if there is a function φ: X $\rightarrow$ $\Re$, also denoted *order isomorphism* if φ is a bijection and order preserving. Put differently, (x ≤ y) if and only if φ(x) ≤ φ(y). Since φ is one-to-one, if follows that (x < y) if and only if φ(x) < φ (y). However,

[1] See Stanat and McAllister (1977), Rosen (1999). See also: www.cs.odu/-nerzic/content/relation/eq-relation/eq-relation/eq-relation.rtml. A quasi-order is a poset if the relation ≤ is antisymmetric. Pequignot (2015) defies two elements x and y of a quasi-order, which is reflexive and transitive, as equivalent if both x ≤ y and y ≤ x hold. He then declares that x and y are equivalent while y ≠ x. Moreover, x and y are not comparable when both x $\not\geq$y and y $\not\geq$ x. Then an embedding between two such quasi-orders becomes an equivalence.

[2] That is $\Delta_x \cap R = \emptyset$.

such a φ exists almost surely if $(R \cap R^{-1}) \subseteq \Delta_x$; in such a case φ is an order isomorphism from X into $\Re$, where $\Re$ is the real line.

Put differently, for every element $x \varepsilon X$, $\exists$ $a \varepsilon \Xi$ such that $(x, a) \varepsilon R$. Then φ is a function. Let $\preccurlyeq_x$ and $\preccurlyeq_\Xi$ be partial orders on the sets X and Ξ, respectively, $\exists$ $\varphi : X \rightarrow \Xi$, then φ is increasing if and only if $x \preccurlyeq_{x\,y} \Leftrightarrow \varphi(x) \preccurlyeq_\Xi \varphi(y)$. The two partially ordered sets $(X, \preccurlyeq_x)$ and $(\Xi, \preccurlyeq_\Xi)$ are said to be isomorphic if there exists a one-to-one φ from X to Ξ such that $x \preccurlyeq_{X\,y} \varphi(\Xi)$ if and only if $\varphi(x) \preccurlyeq_\Xi \varphi(y)$, $\forall x, y \varepsilon X$. If $x \prec_{x\,y}$, then by definition $\varphi(x) \preccurlyeq_\Xi \varphi(y)$. But if $\varphi(x) = \varphi(y)$, then $x = y$ since φ is one-to-one; this is a contradiction because by antisymmetry $x R y \vee y R x$, but $x \neq y$. Hence, $\varphi(x) \prec_\Xi \varphi(y)$.

We are then claiming the following:

Claim 1: Let X and Ξ be sets satisfying ii), then $\exists$ an isomorphic relation between them[3].

Claim 2: If X and Ξ are sets that are relationally isomorphic, then the following statements are equivalent:

$\varphi: X \rightarrow \Xi$ is an injective function.

And $\varphi^{-1}: \Xi \rightarrow X$ is a surjective function[4,5].

This result can now be expressed more formally as:

$$\{\bigwedge_X \bigwedge_R ((X, \precsim) \text{ satisfies i) – vi)} \bigvee_\varphi \bigvee_\Re \varphi: X \rightarrow \Xi\ \varepsilon\ (0, 1) \mid (R \cap R^{-1}) \subseteq \Delta_x \},$$

where Ξ is the set of the shares of the budget of the consumer as revealed in equilibrium. To my knowledge and up to now, the economics profession has only alluded to the existence of φ. Yet it is easily seen that φ maps the preference ordering into the budget share's set Ξ, observable in equilibrium.

The construction of a utility index presents other complications too. As utility functions must be differentiable with a convex hypograph, then continuity and strict convexity are arbitrarily imposed on consumer preference. However, this result obviates the need to impose conditions on a concept whose underlining mechanism is not even known. As far as we know, the origin of a consumer's preference may be genetical, cultural, psychological, or dependent on the consumers' information sets to some extent, or a combination of all of these. The reason why the consumer prefers x to y, say, is of little importance next to the information revealed in the distribution of his or her budget.

This result also obviates the need to do mathematical operations where they are undefined, and even obviates the need to appeal to an unobservable utility function.

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3 For a proof see Warner (1965)
4 For a proof, see Karolyi (2016).
5 See, Remark 2 below.

Remark 2: The pair ($X_{\varepsilon \Re}$, ≤) satisfies: h) $\Delta_x \subseteq R$; i) $R \cap R^{-1} \subseteq \Delta_x$; and j) $R \circ R \subseteq R$. The pair ($X_{\varepsilon \Re}$, <) satisfies: k) $R \cap R^{-1} \subseteq \Delta_x$; l) $R \circ R \subseteq R$; and l) $\Delta_x \cap R = \emptyset$. The pair ($X_{\varepsilon \Re}$, =) satisfies: m) $\Delta_x \subseteq R$; n) $R = R^{-1}$; o) $R \cap R^{-1} \subseteq \Delta_x$, and p) $R \circ R \subseteq R$.

For more on these, the reader is referred to Campion *et al*. (2012, p.2-3). It should also be noted that the relation ≤ on X can be reversed. For example, if $(x \le y) \vee (x < y)$, one can write $(y \ge x) \vee (y > x)$.

Finally, our result underlines the difference in claims by different authors. For example, for Varian, (1978, 80-81), the order relation is complete, reflexive and transitive. In Takayama (1986, 176-183), the order relation is reflexive and transitive. Layard and Walters (1978, 124-125) simply assume that goods are infinitely divisible so that they can assign real numbers to the ordering. I do not know whether or not Varian realizes the enormity of the demand he is putting on consumers to *linearly order their preference*. Layard and Walters, on the other hand, exclude most goods from exchange without making a dent in the problem at hand. The property of antisymmetry is nowhere mentioned, yet it a fundamental property for R to be a functional. Even the pair ($X_{\varepsilon \Re}$, ≤) is reflexive, antisymmetric and transitive. Hopefully the above result will contribute to a needed clarification of the issue.

INDIVIDUAL DEMAND FUNCTION

Consider now a market with m consumers, n goods, where each consumer possesses n endowments (ω). The best approach is to fix supply and examine the demand side. In equilibrium, consumer i devotes a share α of his or her budget (B) to good j. Then:

$$x^{i*}_j = \alpha^i_j B^i / p_j = \alpha^i_j \omega^i_j + (\alpha^i_j \sum^n_{k \neq j} p_k \omega_k) / p_j, \quad p_j > 0, \forall j \varepsilon n.$$

The instantaneous price elasticity (η) at the equilibrium point, is:

$$\eta = \partial x^{i*} p_j / \partial p_j x^{i*} = - (\alpha^i_j \sum_k p_k \omega^i_k) / p_j x^{i*}_j < 0;$$

and for any small increment e around x^{i*} ($\eta < 0$). Hence, one can infer that there is an inverse relationship between price and quantity and that, at the observed equilibrium quantity demanded, price elasticity is negative. *A demand function exists almost surely, but only a point on the demand curve is observable*. Yet, it can easily be seen that at the equilibrium point income elasticity and cross price elasticity are both positive. Further, the price elasticity of expensive goods (that absorb a significant share of the budget) tend to be relatively low (inelastic), while that of cheaper goods tend to be elastic. As income elasticity is positive, this simple mechanism of exchange also shows that real growth is constrained by resource availability. Thus, the demand side of the market process reduces to consumers' effort to map their preference orderings to real-valued sets, under the constraint of their budget. Therefore, nothing can be said about continuity and marginal rate of substitution in this set-up.

At the equilibrium point: $Q_j = \sum_i^m x_j^{*i}$ is possible, and per force Walras's Law holds as a zero excess demand is observed. But elsewhere, excess demand for any good j is undefined. Hence, a community excess demand function is also unobservable.

Obviously, this result is valid in a linear world, i. e. in pure exchange, the equilibrium is unique and stable. In real non-linear (due to feedbacks), non-ergodic, and uncertain settings, multiple equilibria are to be expected (Debreu, 1970; Sonnenschein (1972), among others); however, this is left for further research. In the meantime, it is easily seen that revealed preference is just that; hence to assume that it reconciles demand theory and unobservable utility function is not justified.

CONCLUSION

Barzilai's criticism is very useful in the sense that it forces economists to be more specific about a real-value set (Ξ) on which mathematical operations are defined, and it obviates the need for a utility index as shown in McKensie (1957) and Dominique (2008).

In this regard, it is instructive to return to the correspondence between Léon Walras and Henri Poincaré to show that the problem with a utility index is a recurrent one. According to Ingrao and Isreael (1990), after receiving many criticisms from scientists for doing mathematical operations on an unmeasurable satisfaction function, Walras sent a copy of his *Elements* to Poincaré asking for an endorsement. Poincaré in essence told Walras that he saw nothing wrong with Walras' attempt to measure an unmeasurable magnitude as long as there exists a measurable magnitude that is positively correlated with satisfaction, perhaps as the quantity of heat contained in a body can be mapped into the height of a column of mercury. But, once a proxy function is found, he (Walras) should no longer mention the unmeasurable magnitude. However, Walras never provide the needed isomorphism in question. On the contrary, as Jaffé (1965) reports, Walras wrote to his followers that he had the support of a great scientist to perform mathematical operations on the concept of satisfaction (utility).

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