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Abstract

In this article, we consider illiquid life annuity contracts and show that they may be preferred to those illustrated by Yaari (1965). In an overlapping-generation economy, liquid life annuities are demanded only if the equilibrium is dynamically inefficient. Conversely, an equilibrium displaying a positive demand for illiquid life annuities is efficient. In this latter case, the welfare at steady-state is larger if illiquid life annuity contracts are available.

1. Introduction

In this article, we challenge the common thought that the life annuity contract proposed by Yaari in his seminal 1965 paper is optimal. We indeed show, in a standard neo-classical framework, that another contract, which actually resembles more the contracts offered by annuity providers, may be preferred by rational individuals.

The economic theory of annuities has been strongly influenced by Yaari (1965), a paper that has studiedoptimal demand for annuities in a life-cycle model with or without bequest motives. The financial asset that is named "annuity" by Yaari has positive returns if the bearer is alive and zero if he is not. Annuities are nevertheless demanded since their returns are larger than the one yielded by risk-free bonds. The difference between the two yields is the annuity premium, which is said to be fair when it equals the inverse of the survival probability. Importantly, as the individual ages, the premium increases. This characterization of an annuity has been quite influential and has lead to numerous studies (See among others Davidoff et al, 2005, and Sheshinski, 2008).

However, there are many types of annuity contracts (Cannon and Tonks, 2008). Their features are quite different from Yaari's annuities. For instance, the premium is age-independent. The individual purchases some annuities during youth and, after a given age -let's say post retirement- he periodically receives a fixed amount as long as he survives. Another feature is that the contract is irreversible. Once payments have begun, one can not recover the amount invested. An implicit assumption in Yaari is that agents, upon survival, receive the capital and the interests of their annuity. This means that they are in position to renegotiate their contract at each period, hence the premium increases as the individual ages.

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In this article, we propose a standard framework in which the individual has the choice between two types of life annuity contracts. The first one, which we name here "flexible", is the one proposed by Yaari (1965). The second one, which we name "illiquid", is irreversible and proposes age-independent returns. In both case, we suppose that the annuity premium is such that annuity providers make no profit. Illiquid annuities have been introduced in life-cycle models by Horneff et al. (2008) and Peijnenburg et al. (2011) in order to discuss the issue of low demand for annuities. Our purpose is to study analytically the equilibrium and welfare consequences of the existence of such contracts.

First, we analyze the consumer's optimal decisions over the life-cycle under uncertain lifetime. We consider a setting in which the individual ages, which more precisely means that survival probabilities decrease with age. We therefore depart from the two-period life-cycle setting, which would not have allowed us to make a clear distinction between increasing and fixed returns. We show that illiquid annuities are preferred to flexible ones if the expected returns of the first are sufficiently greater than those of the second. This is the consequence of an arbitrage between more flexibility and more returns. We are aware that partial equilibrium analysis may bias the evaluation of the efficiency of annuity markets (Heijdra and Mierau, 2012) and, therefore, we move to a general equilibrium analysis.

Second, we study an overlapping generation economy with neoclassical production (Diamond, 1965), in which returns of both contracts are determined at the equilibrium. We show that illiquid annuities are preferred when the equilibrium is dynamically efficient, while flexible annuities are preferred when it is inefficient. This result is based on the fact that illiquid annuities represent a transfer from one generation to the next generation. When the population growth rate is relatively low, which is the case when the equilibrium is efficient, this transfer is inexpensive and the investment is profitable. We then discuss the optimality of both annuity contracts. In particular, for the dynamically efficient equilibrium, the welfare at steady-state is larger if illiquid life annuity contracts are available.

The efficiency of transfer from the old to the young has been shown in overlapping generation models with accidental bequests by Pecchenino and Pollard (1997), Feigenbaum et al. (2013) and Heijdra et al. (2014). In equilibrium, partial annuitization may dominate full annuitization strategies as they imply an unintended transfer to the next generation and an increase in savings. Our results differ from theirs for two reasons. First, we assume full annuitization and compare annuity contracts. Second, we show that illiquid annuities are preferred at both the individual level and the long-run equilibrium.

Finally, to test the robustness of our results, we propose three extensions of our model by considering successively a background risk, a bequest motive and a subjective evaluation of survival probabilities. We also discuss our result by introducing an alternative annuity pricing where redistribution between cohorts is forbidden.

2. Individual behavior

2.1. Demographics

We consider an overlapping generations model in which agents live a finite yet uncertain length of time. They live for a maximum of three periods, also called ages, which are denoted $i = \{0, 1, 2\}$. The probability of being alive at age i, conditional on survival until age $i-1$, is denoted p_i. Survival probabilities at each age are constant over time, but decrease with age.

Let $N_{i,t}$ be the number of agents of age i at time t. At each time $t = 0, 1, 2...$, $N_{0,t}$, identical agents are born. Thus, the number of agents of age 1 born at time t is $N_{1,t+1} = p_1 N_{0,t}$ and the number of agents of age 2 born at time t is $N_{2,t+2} = p_1 p_2 N_{0,t}$. Finally, we assume that the number of agents of age 0 grows at a constant rate, denoted n, with $n > -1$:

$$N_{0,t} = (1+n)\, N_{0,t-1}. \tag{1}$$

2.2. Annuity markets

Agents can invest in two types of financial products: bonds and life annuities. The yield on bonds is risk free: each unit of consumption invested at time $t-1$ yields R_t units of consumption at time t. As for annuities, two types of contracts are offered by annuity providers. It is assumed that information on the probability of survival is perfect and that markets for each contract are competitive, which implies that the proposed contracts are fair. It is further assumed that a company cannot cross-subsidize the types of contracts it offers. All these assumptions imply that the profit of annuity providers is zero for each contract. The characteristics of the anuities are explained below.

The first annuity contract offered to agents is that found in most articles of the literature, dating back to the seminal article of Yaari (1965). This is an actuarially fair contract that can be renegotiated each time. If the agent survives, he recovers the capital plus interest and can consume or invest again. Because of this feature, we refer to it as a *flexible* annuity contract. Assuming zero profit as stated above, calculation of the annuity yield is well known; it results from sharing, among the survivors of a cohort, the capital plus the interest of the deceased agents. A unit of consumption invested at time $t-1$ by an agent of age i, $i = \{1, 2\}$, therefore yields R_t/p_{i+1} units of consumption at time t, upon survival. We denote $a_{0,t}$ and $a_{1,t+1}$ as the demand for flexible annuities at ages 0 and 1 by an agent born at time t. At age 2, the demand for annuities must be zero because the agent has reached, by assumption, the last period of life.

The second annuity contract proposed to agents has the following features: (1) the investment must be made at age 0; (2) the capital cannot be recovered before age 2 and; (3) the remuneration received is independent of age. This annuity is said to be *illiquid* because, at age 1, the agent receives only the interest of his investment. Equivalently, it can be said that the agent must invest at age 1 the same amount that he invested at age 0. We denote b_t as the demand for illiquid annuities by an agent of age 0 at time t. To calculate the annuity yield, the condition of zero profit for annuity providers is applied. The companies collect at time $t-1$ the agent's savings and invest them at the risk-free rate. At time t, the value of this investment, which is equal to

$$(N_{0,t-1} b_{t-1} + N_{1,t-1} b_{t-2})\, R_t, \tag{2}$$

is redistributed among the surviving agents. If we denote R_t/π_t as the yield at time t for each unit of consumption invested in $t-1$ or $t-2$, we conclude that the amount distributed must be equal to

$$(N_{1,t} b_{t-1} + N_{2,t} b_{t-2})\, \frac{R_t}{\pi_t}. \tag{3}$$

Consequently, by equalizing (2) and (3), the inverse of the premium solves for the following

$$\pi_t = \frac{p_1 (1+n) b_{t-1} + p_1 p_2 b_{t-2}}{(1+n) b_{t-1} + p_1 b_{t-2}}. \tag{4}$$

If the demand for illiquid annuities are positive, it is easy to show that $\pi_t \in [p_2, p_1]$. We conclude that the interest paid at age 1 is higher than that of the flexible annuities, while the interest paid at age 2 is lower. Hence, the flexible contract is more profitable the older the agent and illiquid annuities can be interpreted as an intergenerational transfer from agents age 2 to agents age 1. This explains why the yield R_t/π_t is a decreasing function of the population growth rate, n. Finally, we note that in the limit case $p_1 = p_2$, which is considered in Blanchard (1985), the yields of the two annuity contracts are equal.

In some countries, like the UK, the redistribution between cohorts of annuitants is forbidden in order to guarantee sufficient assets to pay for annuitants in the distant future. As we consider here a transfer from old generations to young ones, the sustainability of the system is ensured. But, to put our results in perspective, we discuss this issue in Section 5.

2.3. Consumer's choice along the life cycle

Each agent chooses a portfolio and a savings strategy to achieve an optimal consumption allocation between the different ages. The intertemporal expected utility of an agent of age 0 at time t is the following,

$$u(c_{0,t}) + \theta p_1 u(c_{1,t+1}) + \theta^2 p_1 p_2 u(c_{2,t+2}), \tag{5}$$

where $c_{i,t+i}$ is the consumption at age i, and $\theta > 0$ is a discount factor. The instantaneous utility function u, is increasing and concave, $u' > 0$ and $u'' < 0$, and is such that $\lim_{x\to 0} u'(x) = +\infty$ and $\lim_{x\to +\infty} u'(x) = 0$. The budget constraints are as follows: at time t, the agent of age 0 receives a wage, denoted w_t, which he allocates between consumption and savings. It may consist of flexible annuities, $a_{0,t}$, and illiquid annuity, b_t. The budget constraint at age 0 is:

$$c_{0,t} = w_t - a_{0,t} - b_t. \tag{6}$$

We note that investment in risk-free bonds is not modeled here because it is never an optimal strategy. Furthermore, short selling constraints are imposed on both investments, which together with positivity constraints on consumption allow us to eliminate the strategy that consists of borrowing an infinite amount of one asset to purchase an infinite amount of the other one.

$$a_{0,t} \geq 0, b_t \geq 0, c_{0,t} \geq 0, c_{1,t+1} \geq 0, c_{2,t+2} \geq 0. \tag{7}$$

At time $t+1$, the agent receives the capital and interest of his flexible annuity investment and the interest of his illiquid annuity investment. These financial revenues are used by the agent to finance his consumption and savings in the form of flexible annuities, for which the demand is denoted $a_{1,t+1}$. Note that no borrowing constraint is imposed on $a_{1,t+1}$ in this basic framework, an assumption that is discussed in Section 4. The budget constraint at age 1 is:

$$c_{1,t+1} = a_{0,t}\frac{R_{t+1}}{p_1} + b_t\left(\frac{R_{t+1}}{\pi_{t+1}} - 1\right) - a_{1,t+1}. \tag{8}$$

At time $t+2$, which corresponds to the last period of life of the agent, consumption is equal to the capital and interest of his flexible and illiquid annuity investments. The bounded lifespan hypothesis implies that the capital invested in illiquid life annuities is recovered at age 2. The budget constraint at age 2 is:

$$c_{2,t+2} = a_{1,t+1}\frac{R_{t+2}}{p_2} + b_t\frac{R_{t+2}}{\pi_{t+2}}. \tag{9}$$

Remark. The assumption of the recovery of capital at age 2 is necessary as death is certain at the end of the period 2. This assumption is the counterpart of a transversality condition that should be introduced in a more realistic model with a large number of periods of life where survival probabilities converge to 0 when age tends to infinity. In such a model the annuity income would be the expected present discounted value of the annuity payments, if the contract is fair (Inkmann et al., 2011).

The problem of the agent is to choose $\{c_{0,t}, c_{1,t+1}, c_{2,t+2}, a_{0,t}, b_t, a_{1,t+1}\}$ that maximizes (5) subject to (6), (7), (8) and (9).

Let us denote:

$$\overline{R}_{t+1} := \frac{p_1}{\pi_{t+1}}R_{t+1} - p_1\left(1 - \frac{p_2}{\pi_{t+2}}\right). \tag{10}$$

Our first result is the following.

Proposition 1. *The optimal portfolio satisfies:*

$$\begin{cases} b_t > 0 \quad \text{and} \quad a_{0,t} = 0 \quad \text{if} \quad \overline{R}_{t+1} > R_{t+1} \\ b_t = 0 \quad \text{and} \quad a_{0,t} > 0 \quad \text{if} \quad \overline{R}_{t+1} < R_{t+1} \end{cases} \tag{11}$$

A portfolio satisfying $b_t > 0$ and $a_{0,t} > 0$ can be optimal only if $\overline{R}_{t+1} = R_{t+1}$.
Proof. See Appendix.

Through π_{t+1} and π_{t+2} given in (4), we see that $\overline{R}_{t+1}$ is affected by the demands for annuities by past and future generations. In particular, $\overline{R}_{t+1}$ increases with b_{t-1} and decreases with b_{t+1}. Because the illiquid annuity contract acts as a transfer from the oldest to the youngest, the more it is demanded by the previous generation, the more the comparative advantage increases, but the more it is demanded by the next generation, the more the comparative advantage decreases. We also note that the yields of the two contracts are equal in the limit case $p_1 = p_2$.

With Proposition 1, we have seen that the portfolio is generically composed of a single type of contract. The optimal consumption allocation of the agent then depends on the chosen contract. If flexible annuities are chosen at age 0, the result is typical of that found in the literature: consumption dynamics are independent of survival probabilities and increase according to the ratio of the interest factor over the discount factor (Yaari, 1965). Conversely, if illiquid annuities are chosen, the optimal consumption dynamics can be characterized by the following proposition.

Proposition 2. *Suppose that* $\overline{R}_{t+1} > R_{t+1}$*. The optimal consumption allocation satisfies:*

$$\frac{u'(c_{0,t})}{u'(c_{1,t+1})} > \frac{u'(c_{1,t+1})}{u'(c_{2,t+2})} \; \textit{if} \, R_{t+1} \geq R_{t+2}. \tag{12}$$

Thus, if the utility function is homogenous, inequality (12) *can be rewritten as:*

$$\frac{c_{1,t+1}}{c_{0,t}} > \frac{c_{2,t+2}}{c_{1,t+1}} \; \textit{if} \, R_{t+1} \geq R_{t+2}. \tag{13}$$

Proof. See Appendix.

Provided that the interest rate is not increasing and that the utility function has standard properties, the holding of a portfolio composed of illiquid annuities implies that the consumption growth rate decreases with age. This is explained by the fact that the marginal rate of substitution (MRS) between ages 0 and 1 is given by $\overline{R}_{t+1}$, which is higher than R_{t+1} (as shown in Proposition 1), turns to be greater than the MRS between ages 1 and 2, which is given by R_{t+2}. Between ages 1 and 2, all additional savings are indeed invested in flexible annuities. The lower yield of investment opportunities when the agent ages can explain the decrease in the growth of consumption. Introducing illiquid annuities in a life-cycle model allows better reproduction of the stylized facts of the individual's consumption during his life cycle (see, e.g., Gourinchas and Parker, 2002 and Fernández-Villaverde and Krueger, 2007) even though annuities are fairly priced.[1] It should be noted, moreover, that the MRS between periods 0 and 1 depends on the survival probabilities even if the intertemporal utility function is additively separable. For a given and constant demand for annuities, it can be shown that the relationship is positive if the interest rate is higher than the population growth rate.

2.4. A numerical illustration

In this section we parametrize the model and solve it numerically. This serves to illustrate our theoretical results, but also leads to some new insights. Each period corresponds to ten years. Period 0 corresponds to ages 51-60, which is roughly the ages of annuity purchasers in the OECD countries. Periods 1 and 2 correspond to ages 61-70 and 71-80, respectively.

We compare a representative individual of the U.S. population and one of the European population. We assume that preferences are the same and that they differ according to their survival probabilities and the prevailing interest rates. Preferences are represented by a Constant Relative Risk Aversion utility function. We use fairly standard parameters in the literature: the Relative Risk Aversion is equal to 1.6; the annual discount factor is set to 0.996 (the ten-year discount factor is thus equal to 0.96).[2] To compute the survival probabilities, we used data from the United Nations World Population Prospects (the 2015 revision): p_1 is the probability of surviving till age 65 conditional on being alive at age 55 while p_2 is the probability of surviving till age 75 conditional on being alive at age 65. The long-term interest rates refer to government bonds maturing in ten years,

[1] Alternatively, a concave consumption can be obtained if annuities are not available (Davis, 1981) or not fairly priced (Hansen and Imrohoroglu, 2008), or if agents are not utility maximizers (Drouhin, 2015).

[2] Our result are robust to other relevant values of RRA.

we use OECD data.[3] The wage is normalized to 10 for each individual. Finally, if the illiquid annuity market is created, we assume that the premium for both periods are the same for all countries. We fixed their level to an intermediary one between the minimum value of p_1 and the maximum value of p_2, here $\pi_1 = \pi_2 = 0.9$. Table 1 presents the main parameter values.

θ	=	0.96	One-period discount factor
RRA	=	1.6	Relative Risk Aversion
w	=	10	Initial wage
R_1^{US}	=	1.0214	Risk-free yield at period 1 in the U.S.
R_2^{US}	=	1.032	Risk-free yield at period 2 in the U.S.
R_1^{EU}	=	1.0096	Risk-free yield at period 1 in the E.U.
R_2^{EU}	=	1.014	Risk-free yield at period 2 in the E.U.
π_1	=	0.94	Illiquid annuity premium at period 1
π_2	=	0.94	Illiquid annuity premium at period 2

Table 1: Parameters' values

From Figure 1, we can see that illiquid annuities are not demanded at the beginning. The yields are too low to be attractive compared to the ones for flexible annuities. Following Proposition 1, $\overline{R}_1$ is lower than R_1 for all countries. Considering the demographic scenarios used by UN, the probabilities of surviving will increase and the probabilities in the U.S. will be higher than the European ones. As a consequence, the demand for illiquid annuities appears earlier in U.S. than in the E.U. In our simulations, U.S. households have a positive demand for illiquid annuities in 2030-2035 while E.U. households have a positive demand in 2045-2050.

Insert Figure 1.

3. General equilibrium analysis

In the previous section, we showed that there exists a set of interest rate values for which illiquid annuities are purchased by agents. In this section, we analyze the choice of agents when prices are determined by the equilibrium conditions in all markets.

3.1. Annuities and the efficiency of the equilibrium

The production side of the model is standard. There exists a unique good that is produced by many firms operating in a perfectly competitive market. The production function displays constant returns-to-scale and satisfies the Inada conditions. We assume that only agents of age 0 are working and denote by k_t the capital stock per worker at time t. Assuming that capital depreciation rate is 100% per period, the optimality conditions of the firms can be written as:

$$\begin{aligned} w_t &= f(k_t) - k_t f'(k_t) \\ R_t &= f'(k_t). \end{aligned} \tag{14}$$

[3] OECD(2016), long-term interest rates (indicator) doi: 10.1787/2cc37d77-en (Accessed on 27 May 2016).

The equilibrium condition on the capital market is satisfied if the capital stock at time $t+1$ is equal to the sum of the savings of agents born at times t and $t-1$. This condition can be written as:

$$k_{t+1} = \frac{a_{0,t} + b_t}{1+n} + \frac{p_1 \left(a_{1,t} + b_{t-1}\right)}{\left(1+n\right)^2}. \tag{15}$$

At the intertemporal equilibrium, agents and firms behave optimally and markets clear. In what follows, we focus on a steady-state analysis, whose existence is proven below.

Lemma 1. *There exists a steady-state equilibrium.*
Proof. See Appendix.

Let a starred variable denote the steady-state equilibrium value of the considered variable. Depending on the model parameter values, the interest rate at steady-state may be higher or lower than the population growth rate. It is well known that a converging trajectory to such a steady-state is efficient when $f'(k^*) \geq 1+n$ and inefficient otherwise (Cass, 1972, De La Croix and Michel, 2002). In the remainder of this article, we will compare various steady-states that could emerge for various parameter configurations. In particular, we will compare the equilibrium outcomes when the steady-state is efficient $(f'(k^*) \geq 1+n)$ and when it is not $(f'(k^*) < 1+n)$.

The following proposition, which is the counterpart at equilibrium of Proposition 1, characterizes the portfolio choices of agents based on the efficiency of the steady state.

Proposition 3. *At the steady-state equilibrium, the optimal portfolio satisfies:*

$$\begin{cases} b^* > 0 \quad and \quad a_0^* = 0 \quad if \quad f'(k^*) > 1+n, \\ b^* = 0 \quad and \quad a_0^* > 0 \quad if \quad f'(k^*) < 1+n. \end{cases} \tag{16}$$

A portfolio satisfying $b^* > 0$ and $a_0^* > 0$ can be optimal only if $f'(k^*) = 1+n$.
Proof. See Appendix.

Proposition 3 states that if the equilibrium is dynamically efficient, agents hold illiquid annuities in the steady state. It is only when the equilibrium is inefficient that they are not held. The proof is simple and is based on the difference between the yields offered by flexible and illiquid annuities. Using equations (4), (10) and (15), written in steady state, we observe that $\overline{R}^*$ can be written as a linear function of the marginal productivity of capital:

$$\overline{R}^* = \frac{1+n+p_1}{1+n+p_2} f'(k^*) - \frac{(p_1 - p_2)(1+n+p_2)}{1+n+p_2} \tag{17}$$

The yield on illiquid annuities is greater than that on flexible life annuities if, and only if, it is greater than $1+n$, the population growth factor. The figure below shows the spread in yields as a function of the steady state interest factor R^*.

Insert Figure 2.

The intuition behind the result stated in Proposition 3 is based on the fact that illiquid annuity represents a transfer from one generation to the next. When the population growth rate is relatively low, which is the case when the equilibrium is efficient, this transfer is inexpensive and the investment is profitable. Conversely, when the growth rate is high, illiquid annuity investment is unprofitable. Finally, at the Golden Rule, flexible and illiquid annuities have exactly the same profitability. In fact, illiquid life annuities are the opposite of a Pay-As-You-Go pension system, which is a transfer to the older generation and a profitable investment when the equilibrium is inefficient. A Pay-As-You-Go pension system has been shown to reduce welfare (Bruce and Turn, 2013). In the next section, it is shown that illiquid life annuities may have a positive impact on welfare.

3.2. Annuities and welfare at steady-state

The next step concerns the welfare of an agent in the steady state. We have seen that when the equilibrium is efficient, illiquid annuity is preferred to flexible annuity. This has been established for an equilibrium interest rate associated to the equilibrium level of capital per worker. It does not, however, take into account the fact that the capital per worker may be different in an economy where illiquid annuities are available, versus one where they are not. Therefore, to evaluate the effect of the supply of illiquid annuity contracts on welfare, we proceed as follows: we compare the welfare obtained in an economy where the two types of contracts are offered to welfare obtained in an economy where only flexible annuities are available. The result of this comparison is presented in the following proposition.

Proposition 4. *Let $f'(k^*) > 1+n$. The welfare at the steady-state equilibrium is larger if illiquid annuity contracts are available.*
Proof. See Appendix.

In the proof of Proposition 4, we show that the introduction of illiquid annuity contracts increases the capital per worker in the steady state. The intuition for this result is the following: as it induces a shift to youth, illiquid annuities stimulate savings. This increase is conducive to steady-state welfare when the equilibrium is inefficient, as the utility increases with capital in that case. The proof of Proposition 4 is based on the assumption of the existence of a unique steady state. In the case of multiple equilibria, the same comparison can be made using the stability properties.

In the long run, agents benefit from the existence of an illiquid annuity market provided that the equilibrium is efficient. However, the existence of an illiquid annuity market in the steady state depends on the decisions made by agents along the transitory path. This is demonstrated in the following proposition.

Proposition 5. *Illiquid annuity contracts are offered in the steady state only if all previous generations have purchased illiquid annuity.*
Proof. See Appendix.

Proposition 5 shows that the Pareto optimality of illiquid annuity contracts at steady-state is not a sufficient condition for the existence of such a market. While the generation born in t chooses not to invest in illiquid annuities, we see, by using equation (4), that the preceding generation benefits at age 2 from a yield equal to R_{t+1}/p_2, which s equal to the one of flexible annuities, and the generation that follows would settle at age 1 for a yield equal to R_{t+2}/p_1. As this yield is equal to one of flexible annuities, the generation born at $t+1$ has no interest in investing in illiquid annuities, after which the contract is never requested. We conclude that if it exists, the illiquid annuity contract represents a Pareto improvement for all generations.

Equivalently, we note that illiquid annuities will never be demanded if the contract is not available at the beginning of time $t = 0$, that is to say, if agents born at $t = -1$ do not have illiquid annuities in their portfolio at $t = 0$. If the contract does not initially exist, it will not appear spontaneously in a market economy. This fact makes it necessary to intervene in order to possibly compensate for earlier generations, and in order to increase the welfare of future generations. Although it is not sufficient, this result may also help understand the low participation in annuity markets (for a recent survey on the annuity puzzle, see Benartzi et al. 2011).

In this section, we have presented the conditions for the existence of an illiquid annuity market and demonstrated the Pareto improvement that it generates. In the next section, we discuss the robustness of our results.

4. Robustness

The results presented above are not changed if we consider alternative assumptions about the agents' preferences and the environment in which they make their decisions. We consider, in particular, a non-borrowing constraint at age 1, possibly with a background risk that may affect consumption at ages 1 and 2, an assumption of bequest motivated by joy-of-giving, and finally, a subjective evaluation of the survival probabilities. We show that in all these cases, Proposition 1 is not changed.

The first extension we consider is a non-borrowing constraint at age 1. In our framework, this implies that selling annuities short, or equivalently purchasing life insurance contracts (Bernheim, 1991), is not allowed. We therefore add the following inequality to the optimization problem described above:

$$a_{1,t+1} \geq 0. \tag{18}$$

Proposition 1 is modified as follows.

Proposition 6. *Let the agent maximize (5) subject to (6), (7), (8), (9) and (18). The optimal portfolio satisfies:*

$$\begin{cases} b_t > 0 & \text{if} \quad \overline{R}_{t+1} > R_{t+1}, \\ b_t = 0, a_{0,t} > 0, a_{1,t} > 0 & \text{if} \quad \overline{R}_{t+1} < R_{t+1}. \end{cases} \tag{19}$$

A portfolio satisfying $b_t > 0$ and $a_{0,t} > 0$ can be optimal only if $\overline{R}_{t+1} \geq R_{t+1}$. In the case $\overline{R}_{t+1} > R_{t+1}$, $a_{0,t} > 0$ can be optimal only if $a_{1,t+1} = 0$.
Proof. See Appendix.

With proposition 6, we see that introducing a non-borrowing constraint at age 1 barely modifies the optimal portfolio. Constraint (18) is binding only if the demand for illiquid annuities is positive, as the consumption at age 2 would be otherwise zero. Provided that constraint (18) is binding, the MRS between ages 0 and 1 is still greater than R_{t+1} while remaining lower than $\overline{R}_{t+1}$, whereas between ages 1 and 2 is greater than R_{t+2}. In a nutshell, it is the dynamics of consumption that are modified by the non-borrowing constraint, not the optimal portfolio.

Let us now introduce a background risk that may reduce consumptions at ages 1 and 2. This risk can be interpreted as health shocks that require costly treatments and against with it is not possible to be insured.[4] Together with constraint (18), this shock makes the annuity contract non flexible (Direr, 2010). Consumption at ages 1 and 2 are then written as random variables, denoted $\tilde{c}_{1,t+1}$ and $\tilde{c}_{2,t+2}$, and the expected utility of the agent of age 0 at time t reads as:

$$u\left(c_{0,t}\right)+\theta p_1 Eu\left(\tilde{c}_{1,t+1}\right)+\theta^2 p_1 p_2 Eu\left(\tilde{c}_{2,t+2}\right). \tag{20}$$

The optimal behavior of the agent is given in the following.

Proposition 7. *Let the agent maximize (20) subject to (6), (7), (8), (9) and (18). The optimal portfolio satisfies the same conditions as those described in Proposition 6.*
Proof. See Appendix.

As the portfolio choice depends on a comparison of yields, it is not affected by considering random utilities.

The second extension we consider is a bequest motive. The investment in regular bonds can indeed be justified on the grounds of intergenerational altruism and, as shown by Hong and Rios-Rull (2007), Inkmann and Michaelides (2012), Lockwood (2012) and Heijdra et al. (2014), which may help explain the low demand for annuities. Following Yaari (1965), the bonds held in the portfolio at the age of death are bequeathed, and the utility of the agent increases with the amount that is bequeathed. As in Davidoff et al. (2005), we assume that the capitalized value of bequest enter the expected utility, which reads as:

$$\begin{aligned}&u\left(c_{0,t}\right)+\theta p_1 u\left(c_{1,t+1}\right)+\theta^2 p_1 p_2 u\left(c_{2,t+2}\right)+\left(1-p_1\right) v\left(R_{t+1}R_{t+2}h_{0,t}\right)\\&+p_1\left(1-p_2\right) v\left(R_{t+2}h_{1,t+1}\right)+p_1 p_2 v\left(h_{2,t+2}\right),\end{aligned} \tag{21}$$

where $h_{i,t+i}$ is the demand for bonds made by an agent of age i, $i=\{0,1,2\}$, as of time $t+i$. Function v is increasing and concave and we assume that $\lim_{x\to 0} v'(x)=+\infty$, which restrict our analysis to interior solutions. The yield of bonds is the risk-free rate. Thus, the budget constraints (6), (8), (9) are replaced by the following ones:

$$c_{0,t} = w_t-a_{0,t}-b_t-h_{0,t}, \tag{22}$$

$$c_{1,t+1} = a_{0,t}\frac{R_{t+1}}{p_1}+b_t\left(\frac{R_{t+1}}{\pi_{t+1}}-1\right)+h_{0,t}R_{t+1}-a_{1,t+1}-h_{1,t+1}, \tag{23}$$

$$c_{2,t+2} = a_{1,t+1}\frac{R_{t+2}}{p_2}+b_t\frac{R_{t+2}}{\pi_{t+2}}+h_{1,t+1}R_{t+2}-h_{2,t+2}. \tag{24}$$

The optimal behavior of the agent is given in the following.

Proposition 8. *Let the agent maximize (21) subject to (7), (22), (23), and (24). The optimal portfolio satisfies conditions (11). Moreover, the capitalized bequests are such that:*

$$\begin{cases} R_{t+1}R_{t+2}h_{0,t}=R_{t+2}h_{1,t+1}=h_{2,t+2}\ \text{if}\, b_t=0,\\ R_{t+1}R_{t+2}h_{0,t}<R_{t+2}h_{1,t+1}=h_{2,t+2}\ \text{if}\, a_{0,t}=0.\end{cases} \tag{25}$$

[4] Long-Term care can be, however, an example of this even though insurance contracts are offered in some countries. See Brown and Finkelstein (2011) or de Donder and Pestieau (2016).

Proof. See Appendix.

The introduction of a joy-of-giving altruistic motive modifies the optimal portfolio as regular bonds are demanded in order to be bequeathed. However, the remainder of the optimal portfolio is composed of flexible annuities for $\overline{R}_{t+1} < R_{t+1}$ and of illiquid annuities for $\overline{R}_{t+1} > R_{t+1}$. With flexible annuities, the optimal trade-off between consumption and bequest is the same as in Davidoff et al. (2005). The capitalized value of the bequest is constant and the consumption at age 2 equals the return of what was invested in annuities at age 1. With illiquid annuities, the capitalized value of the bequest increases with age (for the same reasons as those detailed for consumption in Proposition 2). However, at age 2, the agent still consumes the share of his portfolio invested in annuities.

The third extension considers a subjective evaluation of the survival probabilities. Many studies have indeed demonstrated the importance of probability distortion in risky choices, and notably when the risk at stake concerns health and longevity (Brewer et al., 2007). We consider an agent endowed with subjective survival probabilities, denoted $\hat{p}_1$ and $\hat{p}_2$, which are such that $\hat{p}_1 \neq p_1$ and $\hat{p}_2 \neq p_2$. His preferences are represented by the following subjective expected utility[5]:

$$u(c_{0,t}) + \hat{p}_1 \theta u(c_{1,t+1}) + \hat{p}_2 \theta^2 u(c_{2,t+2}). \tag{26}$$

The rest of the model is the same as in section 2.3, which implies that agent's beliefs differ from the survival probabilities estimated by insurers.To simplify, we therefore do not take into account the possibility for insurers to use this information and modify annuity yields. The optimal behavior of the agent is given in the following.

Proposition 9. *Let the agent maximize (26) subject to (6), (7), (8), and (9). The optimal portfolio satisfies conditions (11).*
Proof. See Appendix.

Once again, our main results are robust. Introducing a subjective evaluation of longevity risk does not modify the preference for illiquid annuity as long as their *objective* yield is sufficiently large.

5. Discussion

In our model, we assume that insurance companies can collect premia and distribute yields between two generations, which permits the existence of illiquid contracts as we defined them. It is obvious that our results strongly rely on this possible transfer from the old to the young.

However, in some countries, the redistribution between cohorts is forbidden. We now discuss the consequences of this regulation in our framework. If transfers are not allowed, the pricing of the illiquid annuity premium is different. In that case, when an agent chooses to invest in illiquid annuities, she also 'chooses' a constant return since it no longer depends on the next generation. Let us denote by $1/\tilde{\pi}_t$ the constant premium offered to cohort t at date $t+1$ and $t+2$. Thus, a representative agent of age 0 at time t who is investing the quantity b_t in illiquid annuities will receive $\left(\frac{R_{t+1}}{\tilde{\pi}_t} - 1\right) b_t$ at age 1

[5]Although different models of representation of preferences under uncertainty have been proposed, in the case of two states of nature, the main models reach to one, namely the subjective model (Savage, 1954). For an application of non-expected utility models for annuities, see d'Albis and Thibault (2012).

and $\frac{R_{t+2}}{\tilde{\pi}_t} b_t$ at age 2. To compute the premium, we use the fact that insurance companies make no profit on the contracts they offer to a given generation and only redistribute savings within the same cohort. This restriction can be written as follows:

$$N_{0,t} b_t R_{t+1} R_{t+2} + p_1 N_{0,t} b_t R_{t+2} = p_1 N_{0,t} \left(\frac{R_{t+1}}{\tilde{\pi}_t} - 1 \right) b_t R_{t+2} + p_1 p_2 N_{0,t} \frac{R_{t+2}}{\tilde{\pi}_t} b_t. \quad (27)$$

The latter condition says that the capitalized value of savings that were collected from the cohort born at date t is equal to the capitalized value of the annuity returns that will be paid to the same cohort. Using (27), we deduce the annuity premium:

$$\tilde{\pi}_t = \frac{R_{t+1} + p_2}{\frac{R_{t+1}}{p_1} + 2}. \quad (28)$$

Interestingly, by replacing (28) in (10), we obtain $\bar{R}_{t+1} = R_{t+1} + p_1$ and conclude, using Proposition 1, that agents never purchase flexible annuity contracts. This is due to the fact that the return of illiquid annuities is always higher.

However, condition (27) does not imply that the profit that an insurance company makes at each date t is zero. To see that, we compute the profit at date t of all insurances companies made with contracts concluded with cohorts who were born in $t-1$ and $t-2$. Let Π_t denotes this profit. It satisfies:

$$\Pi_t = N_{0,t-1} \left\{ \left(b_{t-1} + \frac{p_1}{1+n} b_{t-2} \right) R_t - \left(p_1 b_{t-1} \left(\frac{R_t}{\tilde{\pi}_{t-1}} - 1 \right) + \frac{p_1 p_2}{1+n} b_{t-2} \frac{R_t}{\tilde{\pi}_{t-2}} \right) \right\}. \quad (29)$$

By replacing (28) in (29), we obtain:

$$\Pi_t = \left[\frac{(p_2 - p_1) R_t + p_1 p_2}{(R_t + p_2)} \right] b_{t-1} - \frac{R_t}{1+n} \left[\frac{(p_2 - p_1) R_{t-1} + p_1 p_2}{R_{t-1} + p_2} \right] b_{t-2}. \quad (30)$$

At steady-state, such that $R_t = R_{t-1}$ and $b_{t-1} = b_{t-2}$, we conclude that $\Pi_t \geq 0 \Longleftrightarrow R_t \leq 1 + n$. Hence, in a perfectly competitive economy, the only possible equilibrium is the Golden Rule such that $R_t = 1 + n$. Since the existence of this specific equilibrium is non-generic, we conclude that such a annuity contract cannot be considered in a competitive equilibrium framework. If one supposes that the equilibrium satisfies the Golden Rule, we note that $\tilde{\pi} = p_1 (1 + n + p_2) / (1 + n + 2p_1)$, which is lower than the inverse of the annuity premium (4) computed at the Golden Rule and which is equal to $\pi = p_1 (1 + n + p_2) / (1 + n + p_1)$. Conditional on its availability, illiquid annuities without intergenerational transfers induce higher welfare than those considered above. But this contract does not generically exist in our framework. An interesting extension could consider non-competitive insurance markets.

6. Conclusion

In this paper, we showed that illiquid annuity is preferred to flexible ones provided that the equilibrium is dynamically efficient. Moreover, the availability of illiquid annuities permit a welfare improvement in the long run. Nevertheless, they are offered in the steady-state only if all generations have purchased them in the past. Consequently, policy intervention can be justified even if the equilibrium is efficient.

This study can be extended in several directions. First a multi-period setting can be analyzed in order to investigate the issue of the optimal timing of annuity purchase

(Brugiavini, 1993) and discuss the opportunity represented by deferred annuities. Second, heterogeneous agents could be introduced in order to focus on adverse selection (Heijdra and Reijnders, 2013) and redistribution issues (Cremer et al., 2010). Finally, aggregate risk on mortality (Schulze and Post, 2010) as well as other aggregate risks could be introduced in order to discuss the risk sharing properties (Gollier, 2008) of the illiquid annuities we considered.

7. Appendix

Proof of Proposition 1. We denote μ_t as the Kuhn-Tucker multiplier associated with the non-negativity constraint: $a_{0,t} \geq 0$, and λ_t as the one associated with: $b_t \geq 0$. The first order conditions of the optimization problem can be written as:

$$\begin{cases} u'(c_{0,t}) - \theta R_{t+1} u'(c_{1,t+1}) = \mu_t, \\ u'(c_{0,t}) - \theta p_1 \left(\frac{R_{t+1}}{\pi_{t+1}} - 1 \right) u'(c_{1,t+1}) - \theta^2 p_1 p_2 \frac{R_{t+2}}{\pi_{t+2}} u'(c_{2,t+2}) = \lambda_t, \\ u'(c_{1,t+1}) - \theta R_{t+2} u'(c_{2,t+2}) = 0, \end{cases} \tag{31}$$

while the complementary slackness conditions are:

$$\mu_t a_{0,t} = 0 \text{ and } \lambda_t b_t = 0. \tag{32}$$

By rearranging equations in system (31), we obtain:

$$\left(\tilde{R}_{t+1} - R_{t+1} \right) \theta u'(c_{1,t+1}) + \lambda_t - \mu_t = 0, \tag{33}$$

where $\overline{R}_{t+1}$ is defined in (10).

Let us first note that having both $\lambda_t > 0$ and $\mu_t > 0$ is not possible as we can see, using the complementary slackness conditions (32) and the budget constraints (8) and (9), that this would imply:

$$c_{1,t+1} = -a_{1,t+1} \text{ and } c_{2,t+2} = a_{1,t+1} \frac{R_{t+2}}{p_2}, \tag{34}$$

which contradicts the fact that optimal consumption should be positive. Consequently, we use (33) to state that:

$$\begin{cases} \lambda_t = 0, \ \mu_t > 0 \text{ if } \overline{R}_{t+1} - R_{t+1} > 0, \\ \lambda_t > 0, \ \mu_t = 0 \text{ if } \overline{R}_{t+1} - R_{t+1} < 0, \\ \lambda_t = \mu_t = 0 \text{ if } \overline{R}_{t+1} - R_{t+1} = 0, \end{cases} \tag{35}$$

which, using the complementary slackness conditions (32), allow us to conclude the proof. □

Proof of Proposition 2. For $\overline{R}_{t+1} > R_{t+1}$, we have seen in the proof of Proposition 1 that $\lambda_t = 0$ and $\mu_t > 0$. Thus, the last two equations of system (31) can be rewritten as follows:

$$\begin{cases} -u'(c_{0,t}) + \theta \overline{R}_{t+1} u'(c_{1,t+1}) = 0, \\ -u'(c_{1,t+1}) + \theta R_{t+2} u'(c_{2,t+2}) = 0. \end{cases} \tag{36}$$

Thus, we have:

$$\frac{u'(c_{0,t})}{u'(c_{1,t+1})} \geq \frac{u'(c_{1,t+1})}{u'(c_{2,t+2})} \Leftrightarrow \overline{R}_{t+1} \geq R_{t+2}, \tag{37}$$

which, using that fact that $\overline{R}_{t+1} > R_{t+1}$, allow us to write (12). To obtain (13), we us the fact that if u is homogeneous of degree $\kappa + 1$, u' is homogeneous of degree κ, then (37) can be rewritten as follows:

$$\left(\frac{c_{1,t+1}}{c_{0,t}}\right)^{\kappa} \geq \left(\frac{c_{2,t+2}}{c_{1,t+1}}\right)^{\kappa} \Leftrightarrow \overline{R}_{t+1} \geq R_{t+2} \;.\square \tag{38}$$

<u>Proof of Lemma 1.</u> An intertemporal equilibrium is a collection:

$$\{c_{0,t}, c_{1,t}, c_{2,t}, b_t, a_{0,t}, a_{1,t}, \pi_{t+1}, R_{t+1}, w_{t+1}, k_{t+1}\}_{t\geq 0},$$

which satisfies the budget constraints (6), (8) and (9), the optimality conditions (15) and (31), the complementary slackness conditions (32), the zero-profit condition (4) and the equilibrium condition (15). At steady state, the equilibrium is the solution of the following system:

$$\left\{ \begin{array}{l} c_0 = f(k) - kf'(k) - a_0 - b, \\ c_1 = a_0 \frac{f'(k)}{p_1} + b\left(f'(k)\frac{1+n+p_1}{p_1(1+n+p_2)} - 1\right) - a_1, \\ c_2 = a_1\frac{f'(k)}{p_2} + bf'(k)\frac{1+n+p_1}{p_1(1+n+p_2)}, \\ k = \frac{a_0+b}{1+n} + \frac{p_1(a_1+b)}{(1+n)^2}, \\ 0 = [u'(c_0) - \theta f'(k) u'(c_1)]\, a_0, \\ 0 = \left[u'(c_0) - \theta p_1\left(f'(k)\frac{1+n+p_1}{p_1(1+n+p_2)} - 1\right)u'(c_1) - \theta^2 p_2 f'(k)\frac{1+n+p_1}{1+n+p_2}u'(c_2)\right] b, \\ 0 = u'(c_1) - \theta f'(k) u'(c_2), \end{array} \right. \tag{39}$$

as well as (4) and (15).

Let us first consider the steady-state such that $a_0 = 0$ and $b > 0$. It reduces to a system in (c_0, k) that writes:

$$\left\{ \begin{array}{c} -u'(c_0) + \frac{\theta[f'(k)(1+n+p_1)-(p_1-p_2)(1+n)]}{(1+n+p_2)}u'(c_1) = 0, \\ -u'(c_1) + \theta f'(k) u'(c_2) = 0, \end{array} \right. \tag{40}$$

where:

$$c_1 = \frac{[f(k) - kf'(k) - c_0]}{p_1}\left(\frac{f'(k)(1+n+p_1)}{1+n+p_2} + (1+n)\right) - \frac{(1+n)^2 k}{p_1}, \tag{41}$$

$$c_2 = \frac{(1+n)^2}{p_1 p_2}kf'(k) - \frac{[f(k) - kf'(k) - c_0]\, f'(k)(1+n+p_1)(1+n)}{p_1 p_2 (1+n+p_2)}. \tag{42}$$

Consider also the steady-state such that $b = 0$ and $a_0 > 0$, which reduces to:

$$\left\{ \begin{array}{l} -u'(c_0) + \theta f'(k) u'(c_1) = 0, \\ -u'(c_1) + \theta f'(k) u'(c_2) = 0, \end{array} \right. \tag{43}$$

where:

$$c_1 = [f(k) - kf'(k) - c_0] \frac{f'(k) + (1+n)}{p_1} - \frac{(1+n)^2 k}{p_1}, \tag{44}$$

$$c_2 = [(1+n)k - [f(k) - kf'(k) - c_0]] \frac{(1+n) f'(k)}{p_1 p_2}. \tag{45}$$

To prove the existence of a steady-state, we consider a more general problem parametrized by $z \in \left[1, \frac{1+n+p_1}{1+n+p_2}\right]$ that encompasses system (40) for $z = 1$ and system (43) for $z = \frac{1+n+p_1}{1+n+p_2}$. This general system can be written as:

$$\begin{cases} F(k, c_1) = 0, \\ G(k, c_1) = 0, \end{cases}$$

where:

$$\begin{aligned} F(k, c_1) &= -u'\left(f(k) - kf'(k) - \frac{p_1 c_1 + (1+n)^2 k}{[f'(k) z + (1+n)]}\right) \\ &\quad + \theta [f'(k) z - (1+n)(z-1)] u'(c_1), \end{aligned} \tag{46}$$

$$\begin{aligned} G(k, c_1) &= -u'(c_1) \\ &\quad + \theta f'(k) u'\left(\left[(1+n)k - \frac{p_1 c_1 + (1+n)^2 k}{[f'(k) z + (1+n)]} z\right] \frac{(1+n) f'(k)}{p_1 p_2}\right). \end{aligned} \tag{47}$$

Let us define

$$c_1(k, \varepsilon) = \frac{[f(k) - kf'(k) - \varepsilon][f'(k) z + (1+n)] - (1+n)^2 k}{p_1}, \tag{48}$$

where $0 < \varepsilon \ll 1$, which corresponds to $c_0 = \varepsilon$. Let k_ε^F be the solution of $F(k, c_1(k, \varepsilon)) = 0$,and k_ε^G be the solution of $G(k, c_1(k, \varepsilon)) = 0$. As $F(k, c_1(k, \varepsilon)) > G(k, c_1(k, \varepsilon))$, we conclude that $k_\varepsilon^F < k_\varepsilon^G$.

Let us now define

$$c_1(k, \varepsilon') = \frac{\frac{[(1+n)k - \varepsilon'][f'(k) z + (1+n)]}{z} - (1+n)^2 k}{p_1}, \tag{49}$$

where $0 < \varepsilon' \ll 1$, which corresponds to $c_2 = \varepsilon'$. Let $k_{\varepsilon'}^F$ be the solution of $F(k, c_1(k, \varepsilon')) = 0$,and $k_{\varepsilon'}^G$ be the solution of $G(k, c_1(k, \varepsilon')) = 0$. As $F(k, c_1(k, \varepsilon')) < G(k, c_1(k, \varepsilon'))$, we conclude that $k_{\varepsilon'}^F > k_{\varepsilon'}^G$. □

Proof of Proposition 3. From Proposition 1, we know that the possible portfolios at steady-state are: (1) $a_0^* > 0$ and $b^* > 0$, (2) $a_0^* = 0$ and $b^* > 0$, (3) $a_0^* > 0$ and $b^* > 0$. Let us consider those three cases successively.

For $a_0^* > 0$ and $b^* > 0$, the last three equations of (39) can be rewritten as:

$$\left[f'(k) + p_1 - (f'(k) + p_2) \frac{1+n+p_1}{1+n+p_2}\right] u'(c_1) = 0, \tag{50}$$

which is satisfied for $f'(k) = 1 + n$, i.e. when the capital is at the Golden Rule.

For $a_0^* = 0$ and $b^* > 0$, the last three equations of (39) can be rewritten as:

$$\begin{cases} u'(c_0) - \theta f'(k)\, u'(c_1) \geq 0, \\ u'(c_0) - \theta \left(f'(k) \frac{1+n+p_1}{1+n+p_2} - p_1 + p_2 \frac{1+n+p_1}{1+n+p_2} \right) u'(c_1) = 0, \end{cases} \tag{51}$$

which are satisfied only if $f'(k) \geq 1+n$. Using what has been shown just above, we conclude that if $f'(k) > 1+n$, one has $a_0^* = 0$.

Finally, for $a_0^* > 0$ and $b^* = 0$, the last three equations of (39) can be rewritten as:

$$\left[f'(k) \left(1 - \frac{1+n+p_1}{1+n+p_2} \right) + p_1 - p_2 \frac{1+n+p_1}{1+n+p_2} \right] u'(c_1) \geq 0, \tag{52}$$

which is satisfied only if $f'(k) \leq 1+n$. As above, $f'(k) < 1+n$, implies $b^* = 0$. $\square$

Proof of Proposition 4. The proof proceeds in two steps. In step 1, we show that the capital stock is higher in an economy where flexible and illiquid annuities are proposed than in an economy where only flexible annuities are proposed. In step 2, we show that the utility increases with the capital stock.

Step 1. Let us consider first an economy where flexible and illiquid annuities are proposed. If $f'(k) > 1+n$, we can use the proof of Proposition 3 to state that the steady-state solves:

$$\begin{cases} c_0 = f(k) - kf'(k) - b, \\ c_1 = b \left(f'(k) \frac{1+n+p_1}{p_1(1+n+p_2)} - 1 \right) - a_1, \\ c_2 = a_1 \frac{f'(k)}{p_2} + bf'(k) \frac{1+n+p_1}{p_1(1+n+p_2)}, \\ k = \frac{b}{1+n} + \frac{p_1(a_1+b)}{(1+n)^2}, \\ 0 = u'(c_0) - \theta \left([f'(k) + p_2] \frac{1+n+p_1}{1+n+p_2} - p_1 \right) u'(c_1), \\ 0 = u'(c_1) - \theta f'(k)\, u'(c_2). \end{cases} \tag{53}$$

System (53) reduces to a system in (c_0, k) that reads as:

$$\begin{cases} -u'(c_0) + \frac{\theta[f'(k)(1+n+p_1) - (p_1-p_2)(1+n)]}{(1+n+p_2)} u'(c_1) = 0, \\ -u'(c_1) + \theta f'(k)\, u'(c_2) = 0, \end{cases} \tag{54}$$

where:

$$\begin{cases} c_1 = \frac{[f(k) - kf'(k) - c_0]}{p_1} \left(\frac{f'(k)(1+n+p_1)}{1+n+p_2} + (1+n) \right) - \frac{(1+n)^2 k}{p_1}, \\ c_2 = \frac{(1+n)^2}{p_1 p_2} kf'(k) - \frac{[f(k) - kf'(k) - c_0] f'(k)(1+n+p_1)(1+n)}{p_1 p_2 (1+n+p_2)}. \end{cases} \tag{55}$$

Let us consider now an economy where only flexible annuities are proposed. The steady-state of such an economy solves:

$$\begin{cases} c_0 = f(k) - kf'(k) - a_0, \\ c_1 = a_0 \frac{f'(k)}{p_1} - a_1, \\ c_2 = a_1 \frac{f'(k)}{p_2}, \\ k = \frac{a_0}{1+n} + \frac{p_1 a_1}{(1+n)^2}, \\ 0 = u'(c_0) - \theta f'(k) u'(c_1), \\ 0 = u'(c_1) - \theta f'(k) u'(c_2). \end{cases} \tag{56}$$

System (56) reduces to a system in (c_0, k) that reads as:

$$\begin{cases} -u'(c_0) + \theta f'(k) u'(c_1) = 0, \\ -u'(c_1) + \theta f'(k) u'(c_2) = 0, \end{cases} \tag{57}$$

where:

$$\begin{cases} c_1 = [f(k) - kf'(k) - c_0] \frac{f'(k)+(1+n)}{p_1} - \frac{(1+n)^2 k}{p_1}, \\ c_2 = \{(1+n) k - [f(k) - kf'(k) - c_0]\} \frac{(1+n) f'(k)}{p_1 p_2}. \end{cases} \tag{58}$$

The objective is thus to compare the steady-state capital that is the solution of (54) with the one that is the solution of (57). To do so, we set up, for $z \in [1, (1+n+p_1)/(1+n+p_2)]$, a more general system that writes:

$$\begin{cases} -u'(f(k) - kf'(k) - x) + \theta [f'(k) z - (1+n)(z-1)] u'(c_1) = 0, \\ -u'(c_1) + \theta f'(k) u'(c_2) = 0, \end{cases} \tag{59}$$

where

$$\begin{cases} x = f(k) - kf'(k) - c_0, \\ c_1 = \frac{x[f'(k) z + (1+n)]}{p_1} - \frac{(1+n)^2 k}{p_1}, \\ c_2 = [(1+n) k - xz] \frac{(1+n) f'(k)}{p_1 p_2}. \end{cases} \tag{60}$$

We notice that for $z = 1$, system (59) reduces to system (57) while for $z = (1+n+p_1)/(1+n+p_2)$, system (59) reduces to system (54). To prove our claim, we hence aim at showing that:

$$\frac{dk^*}{dz} > 0, \tag{61}$$

where k^* is the capital stock that is the solution of (59). Let us rewrite the first equation in (59) as $F(x, k; z) = 0$ and the second as $G(x, k; z) = 0$. One has:

$$\frac{dk}{dz} = -\frac{F'_z - \frac{G'_z}{G'_x} F'_x}{F'_k - \frac{G'_k}{G'_x} F'_x}. \tag{62}$$

Consider first the numerator of (62). Let $\sigma(c) := -u'(c)/cu''(c)$. Simple computations give that the sign of $F'_z - G'_z F'_x / G'_x$ is the same as the one of:

$$\begin{aligned}
&\frac{f'(k)-(1+n)}{f'(k)z-(1+n)(z-1)} \\
&\times\left\{\frac{1}{\sigma(c_1)}\frac{f'(k)z+(1+n)}{x[f'(k)z+(1+n)]-(1+n)^2k}+\frac{1}{\sigma(c_2)}\frac{z}{(1+n)k-xz}\right\} \\
&+\frac{1}{c_0\sigma(c_0)}\frac{1}{\sigma(c_1)}\frac{xf'(k)}{x[f'(k)z+(1+n)]-(1+n)^2k} \\
&+\frac{1}{c_0\sigma(c_0)}\frac{1}{\sigma(c_2)}\frac{x}{(1+n)k-xz} \\
&+\frac{1}{\sigma(c_1)}\frac{1}{\sigma(c_2)}\frac{x}{(1+n)k-xz}\frac{(1+n)}{x[f'(k)z+(1+n)]-(1+n)^2k},
\end{aligned} \tag{63}$$

which is positive as we supposed that $f'(k) > (1+n)$. To determine the sign of the denominator, we use the assumption of the existence of a unique equilibrium. System (59) can be written as a single dimension problem: $F(\phi(k;z), k; z) = 0$ where $\phi(.)$ is the implicit function obtained using $G(x, k; z) = 0$. The derivative of $F(\phi(k;z), k; z)$ with respect to k is given by $F'_k - G'_k F'_x / G'_x$. As $F(\phi(0;z), 0; z) > 0$, we conclude that the derivative, computed at the equilibrium k^* is negative. Using (62), we finally conclude that $dk/dz > 0$.

Step 2. We now compute the derivative of the intertemporal utility function with respect to capital, such as

$$u'(c_0)\frac{dc_0}{dk}+\theta p_1 u'(c_1)\frac{dc_1}{dk}+\theta^2 p_1 p_2 u'(c_2)\frac{dc_2}{dk}, \tag{64}$$

In steady-state of an economy where both flexible and illiquid annuities are proposed, we use (54) and (55) to obtain that the sign of (64) is the same as the one of:

$$[f'(k)-(1+n)]\left[k(1+n)-[f(k)-2kf'(k)-c_0]\left(\frac{1+n+p_1}{1+n+p_2}\right)\right]. \tag{65}$$

Using the fact that c_2, whose expression is given in (55), is positive we conclude that (65) is positive. Hence, an increase in capital increases the welfare of the agent in the steady-state. □

Proof of Proposition 5. The objective is to prove that if there exists T such that $b_{T-1} = 0$ then $b_{T+i} = 0$ for all $i = 0, 1, 2, ...$ To prove it, we consider the yield of the investment in illiquid annuities made at time T. Replacing (4) and $b_{T-1} = 0$ in (10), we obtain:

$$\overline{R}_{T+1} = R_{T+1} - p_1\left(1-\frac{p_2}{p_1}\frac{(1+n)b_{T+1}+p_1 b_T}{(1+n)b_{T+1}+p_2 b_T}\right). \tag{66}$$

For $b_{T+1} > 0$, we obtain that $\overline{R}_{T+1} > R_{T+1}$, which implies, using Proposition 1, that $b_T = 0$. □

Proof of Proposition 6. The proof is similar to the one of Proposition 1. We denote $(\mu_t, \lambda_t, \gamma_t)$ as the Kuhn-Tucker multipliers associated with the non-negativity constraints:

$a_{0,t} \geq 0$, $b_t \geq 0$ and $a_{1,t} \geq 0$. The first order conditions of the optimization problem can be written as:

$$\begin{cases} u'(c_{0,t}) - \theta R_{t+1} u'(c_{1,t+1}) = \mu_t, \\ u'(c_{0,t}) - \theta p_1 \left(\frac{R_{t+1}}{\pi_{t+1}} - 1\right) u'(c_{1,t+1}) - \theta^2 p_1 p_2 \frac{R_{t+2}}{\pi_{t+2}} u'(c_{2,t+2}) = \lambda_t, \\ u'(c_{1,t+1}) - \theta R_{t+2} u'(c_{2,t+2}) = \gamma_{t+1}, \end{cases} \tag{67}$$

while the complementary slackness conditions are:

$$\mu_t a_{0,t} = 0, \lambda_t b_t = 0 \ and \gamma_{t+1} a_{1,t+1} = 0. \tag{68}$$

By rearranging equations in system (67), we obtain:

$$\left(\overline{R}_{t+1} - R_{t+1}\right) \theta u'(c_{1,t+1}) + \lambda_t - \mu_t - \gamma_{t+1} \frac{p_2}{\pi_{t+2}} = 0, \tag{69}$$

where $\overline{R}_{t+1}$ is defined in (10).

Let us consider the various configurations that are possible. As in the proof of Proposition 1, the case $\lambda_t > 0$ and $\mu_t > 0$ is not optimal as it implies that the sign of $c_{1,t+1}$ is the opposite of the one of $c_{2,t+2}$. Similarly, the case $\lambda_t > 0$ and $\gamma_{t+1} > 0$ is neither optimal as it implies $c_{2,t+2} = 0$. We now use equation (69) to establish that:
- for $\overline{R}_{t+1} > R_{t+1}$, one has $\lambda_t < \mu_t + \gamma_{t+1} p_2/\pi_{t+2}$, which necessarily implies: $\lambda_t = 0$ and $\mu_t + \gamma_{t+1} p_2/\pi_{t+2} > 0$. Condition $\lambda_t b_t = 0$ implies that $b_t \geq 0$. However, $b_t = 0$ is not possible as the positivity of $c_{1,t+1}$ would thus imply $a_{0,t} > 0$ (and $\mu_t = 0$) while the positivity of $c_{2,t+2}$ would imply $a_{1,t+1} > 0$ (and $\gamma_{t+1} = 0$). Thus, $b_t > 0$. Moreover, $\mu_t \geq 0$ and $\gamma_{t+1} \geq 0$, with at least one of the two inequalities being strict.
- for $\overline{R}_{t+1} = R_{t+1}$, one has $\lambda_t = \mu_t + \gamma_{t+1} p_2/\pi_{t+2}$, which necessarily implies: $\lambda_t = \mu_t = \gamma_{t+1} = 0$.
- for $\overline{R}_{t+1} > R_{t+1}$, one has $\lambda_t > \mu_t + \gamma_{t+1} p_2/\pi_{t+2}$, which necessarily implies: $\lambda_t > 0$ and $\mu_t = \gamma_{t+1} = 0$. Due to (68) we conclude that $b_t = 0$ while the positivity of $c_{1,t+1}$ implies $a_{0,t} > 0$ and the positivity of $c_{2,t+2}$ implies $a_{1,t+1} > 0$. □

Proof of Proposition 7. Following the same derivations as those made in the proof of Proposition 6, we obtain:

$$\left(\overline{R}_{t+1} - R_{t+1}\right) \theta E u'(\tilde{c}_{1,t+1}) + \lambda_t - \mu_t - \gamma_{t+1} \frac{p_2}{\pi_{t+2}} = 0, \tag{70}$$

which is the counterpart of (69). The reasonning made after (69) also applies here. □

Proof of Proposition 8. The first order conditions of the agent's problem are given by (31), (32) and:

$$\begin{cases} -u'(c_{0,t}) + \theta p_1 R_{t+1} u'(c_{1,t+1}) + (1 - p_1) R_{t+1} R_{t+2} v'(R_{t+1} R_{t+2} h_{0,t}) = 0, \\ -\theta u'(c_{1,t+1}) + \theta^2 p_2 R_{t+2} u'(c_{2,t+2}) + (1 - p_2) R_{t+2} v'(R_{t+2} h_{1,t+1}) = 0, \\ -\theta^2 u'(c_{2,t+2}) + v'(h_{2,t+2}) = 0. \end{cases} \tag{71}$$

Consequently, (33) and (35) still hold. Moreover, by replacing the first and the third equations of (31) in (71), we obtain:

$$\begin{cases} -\frac{\mu_t}{(1-p_1)R_{t+1}} - \theta u'\left(c_{1,t+1}\right) + R_{t+2}v'\left(R_{t+1}R_{t+2}h_{0,t}\right) = 0, \\ -\theta u'\left(c_{1,t+1}\right) + R_{t+2}v'\left(R_{t+2}h_{1,t+1}\right) = 0, \\ -\theta u'\left(c_{1,t+1}\right) + R_{t+2}v'\left(h_{2,t+2}\right) = 0. \end{cases} \tag{72}$$

This allow us to conclude that:

$$\begin{cases} R_{t+1}R_{t+2}h_{0,t} = R_{t+2}h_{1,t+1} = h_{2,t+2} \ \textit{if}\, \mu_t = 0, \\ R_{t+1}R_{t+2}h_{0,t} < R_{t+2}h_{1,t+1} = h_{2,t+2} \ \textit{if}\, \mu_t > 0. \ \square \end{cases}$$

<u>Proof of Proposition 9.</u> As in the proof of Proposition 1, the first order conditions of the optimization problem can be written as:

$$\begin{cases} u'\left(c_{0,t}\right) - \theta R_{t+1}\frac{\hat{p}_1}{p_1}u'\left(c_{1,t+1}\right) = \mu_t, \\ u'\left(c_{0,t}\right) - \theta \hat{p}_1\left(\frac{R_{t+1}}{\pi_{t+1}} - 1\right)u'\left(c_{1,t+1}\right) - \theta^2\hat{p}_1\hat{p}_2\frac{R_{t+2}}{\pi_{t+2}}u'\left(c_{2,t+2}\right) = \lambda_t, \\ u'\left(c_{1,t+1}\right) - \theta R_{t+2}\frac{\hat{p}_2}{p_2}u'\left(c_{2,t+2}\right) = 0, \end{cases} \tag{73}$$

while the complementary slackness conditions are given by (32). By rearranging equations in system (73), we obtain:

$$\left(\tilde{R}_{t+1} - R_{t+1}\right)\theta\frac{\hat{p}_1}{p_1}u'\left(c_{1,t+1}\right) + \lambda_t - \mu_t = 0, \tag{74}$$

where $\overline{R}_{t+1}$ is defined in (10). The rest of the proof is similar to the one of Proposition 1. $\square$

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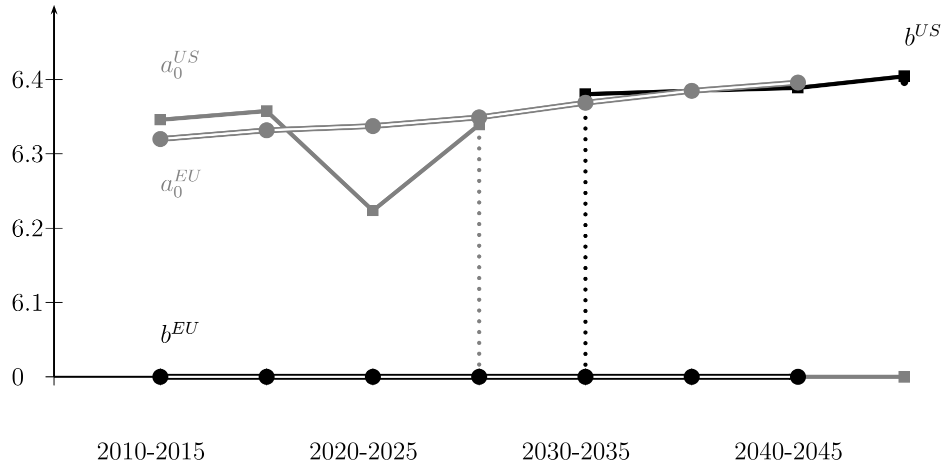

Figure 1: Annuity demands at period 0

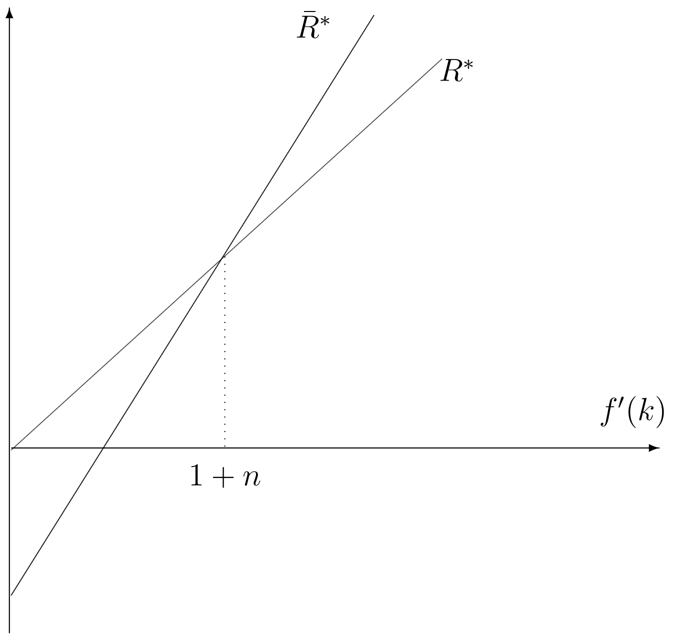

Figure 2. Spread in yields at steady state